What can symplectic geometry tell us about Hamiltonian dynamics?

Dan Cristofaro-Gardiner

Institute for Advanced Study

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Thank you for inviting me to give this talk!

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- 3 Refinements of the Weinstein conjecture
- The restricted three-body problem
- 5 Non-autonomous Hamiltonians
- 6 Future directions

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Phase space

We will be primarily talking about \mathbb{R}^{2n} , with *position* coordinates x_1, \ldots, x_n and *momentum* coordinates y_1, \ldots, y_n .

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We call any function $H : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ a (autonomous) *Hamiltonian*. Our Hamiltonians will generally be smooth.

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Preliminaries Weinstein's conjecture Refinements of the Weinstein conjecture The restricted three-body problem Non-autonomous Hamiltonians Future directions	
Hamilton's ODEs	

The basic object of study in this talk will be trajectories (x(t), y(t)) such that

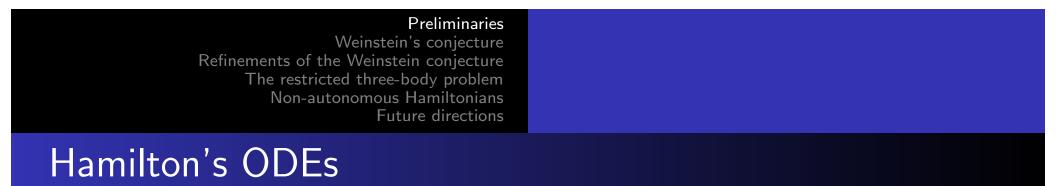
$$rac{dx_i}{dt} = rac{\partial H}{\partial y_i}, \qquad rac{dy_i}{dt} = -rac{\partial H}{\partial x_i}.$$

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These are the equations of Hamilton's reformulation of classical mechanics. We call them *Hamilton's equations of motions*, and we call a solution a *Hamiltonian trajectory*.

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Note that the x_i and the y_i in Hamilton's equations of motion are "intertwined". Symplectic (which means intertwined) geometry is a way of capturing this.

Periodic trajectories and conservation of energy

We will specifically be discussing *periodic trajectories*, i.e. Hamiltonian trajectories such that $(x(t_0), y(t_0)) = (x(0), y(0))$ for some positive t_0 .

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We will specifically be discussing *periodic trajectories*, i.e. Hamiltonian trajectories such that $(x(t_0), y(t_0)) = (x(0), y(0))$ for some positive t_0 .

A basic fact about Hamilton's equations are that they preserve H. Specifically, if (x(t), y(t)) solves Hamilton's equations, then H(x(t), y(t)) is always constant. Hence, Hamiltonian trajectories always travel along level sets of H.

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Does a Hamiltonian have a closed orbit along any level set?

Does a Hamiltonian have a closed orbit on every level set?

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Theorem 1

(Hofer-Zehnder, Steuwe, 1990). Let $H : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ be a proper smooth function. Then there is a closed periodic Hamiltonian trajectory along $H^{-1}(E)$ for almost every $E \in \mathbb{R}$ such that $H^{-1}(E) \neq \emptyset$.

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While the proof of this theorem uses some ideas from symplectic geometry, it will not be the focus of this talk.

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Does a Hamiltonian have a closed orbit along any level set? (cont.)

An essential point is that the word "almost" in the statement of Theorem 1 *can not be removed*, even for "regular" level sets.

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Does a Hamiltonian have a closed orbit along any level set? (cont.)

An essential point is that the word "almost" in the statement of Theorem 1 *can not be removed*, even for "regular" level sets.

Specifically, Ginzburg-Gurel (2003) found a proper C^2 Hamiltonian H on \mathbb{R}^4 with a regular level set with no closed Hamiltonian orbits. C^{∞} counter examples are also known in \mathbb{R}^{2n} for n > 2.

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When does a Hamiltonian have a closed orbit along a level set?

Basic calculation: if Y is a hypersurface in \mathbb{R}^{2n} that is a reguar level set of two different Hamiltonians H and K, then the existence of a closed Hamiltonian trajectory depends *only on* Y and not on H and K.

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Weinstein, late 1970s: If Y is compact and convex (meaning it bounds a convex subset of \mathbb{R}^{2n}), then any Hamiltonian with Y as a level set has a closed orbit along Y.

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Weinstein, late 1970s: If Y is compact and convex (meaning it bounds a convex subset of \mathbb{R}^{2n}), then any Hamiltonian with Y as a level set has a closed orbit along Y.

However, as I will explain very shortly, the existence of a closed orbit is a "symplectic" condition, while convexity *is not*.

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Symplectic geometry

As remarked earlier, Hamilton's equations are in some sense intertwined. We would like to make this precise.



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As remarked earlier, Hamilton's equations are in some sense intertwined. We would like to make this precise.

There is a bilinear product b on \mathbb{R}^{2n} that captures this intertwinedness. It is given for n = 2 by

$$b((x_1, x_2, y_1, y_2), (x'_1, x'_2, y'_1, y'_2)) = x_1y'_1 - x'_1y_1 + x_2y'_2 - x'_2y_2,$$

and extended for any n by this pattern.

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and extended for any *n* by this pattern. Unlike the dot product, this product is *anti*-symmetric, hence not *positive-definite*.

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Symplectic geometry (cont.)

A symplectic transformation

$$T:\mathbb{R}^{2n}\longrightarrow\mathbb{R}^{2n},$$

is a C^{∞} transformation that preserves *b*. (This means that the Jacobian of *T* preserves *b*).

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Many interesting symplectic transformations. Example: product of two area preserving maps is a symplectic transformation of \mathbb{R}^4 . Symplectic geometry is essentially the geometry of symplectic transformations.

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Weinstein's result re-examined

Here is the relevance of all this to Weinstein's 1970s result.



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Weinstein's result re-examined

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The point: by a simple calculation, if Y is a hypersurface carrying a closed Hamiltonian orbit, and T is a symplectic transformation, then T(Y) also has a closed orbit. Moreover, it is easy to construct examples (e.g. n = 1) where Y is convex but T(Y) is not.

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Weinstein therefore sought a condition for Y to carry a closed orbit that is invariant under symplectic transformations.

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Weinstein therefore sought a condition for Y to carry a closed orbit that is invariant under symplectic transformations. He conjectured that Y should be of "contact type".

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Weinstein's conjecture

The definition of contact is not the focus of this talk. However, Weinstein's conjecture is so central to symplectic geometry that I will write it out:

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Conjecture 2

(Weinstein (1979)) If Y is a contact type hypersurface in \mathbb{R}^{2n} , then any Hamiltonian with Y as a level set carries a closed orbit.

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(Weinstein (1979)) If Y is a contact type hypersurface in \mathbb{R}^{2n} , then any Hamiltonian with Y as a level set carries a closed orbit.

This was proved by Viterbo in 1987, but there are many important phases spaces, called *symplectic manifolds*, that are not \mathbb{R}^{2n} . The analogue of Weinstein's conjecture for symplectic manifolds remains open, except for dimensions 2 and 4.

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Global surfaces of section

For the remainder of the talk, I want to discuss situations where one can find much more structure than one periodic orbit. In dimension 4, there is a beautiful body of work by Hofer, Wysocki, and Zehnder on finding *global surfaces of section*.

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Let Y be a compact hypersurface in \mathbb{R}^4 that is the level set of some Hamiltonian H. A global surface of section for Y is an embedded compact surface $\Sigma \subset Y$ such that

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Let Y be a compact hypersurface in \mathbb{R}^4 that is the level set of some Hamiltonian H. A global surface of section for Y is an embedded compact surface $\Sigma \subset Y$ such that

- The boundary components of Σ are periodic Hamiltonian trajectories
- Every trajectory is transverse to the interior Σ^o and intersects the interior both forwards and backwards in time (other than the boundary components).

Poincare return map

If we have a global surface of section then we can define a *Poincare return map*

$$\Psi: \Sigma^o \longrightarrow \Sigma^o.$$

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It is defined by following a point $p \in \Sigma^o$ along its trajectory until the first time it hits Σ^o again. We can use the Poincare return map to reduce the study of our four-dimensional Hamiltonian system to studying an area preserving map of Σ^o and its iterates.

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HWZ's theorem

It is therefore advantageous to know when a four-dimensional Hamiltonian system admits a global surface of section.

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HWZ's theorem

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Theorem 3

(Hofer, Wysocki, Zehnder 1998) Any Hamiltonian on \mathbb{R}^4 possesses a global surface of section along any strictly convex energy hypersurface.

In fact, they show that one can always take this surface of section to be a disc.

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Implication for Hamiltonian dynamics

HWZ were able to use their theorem to prove the following:

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Corollary 4

Any Hamiltonian on \mathbb{R}^4 carries either 2 or ∞ -ly many closed orbits along any strictly convex energy hypersurface.

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The proof very heavily uses the global surface of section. The idea is that it is known, by work of Franks, that an area preserving map of an annulus has either no, or ∞ -ly many periodic points.

Similarity with Weinstein conjecture: strictly convex condition not a symplectic condition. HWZ find a symplectic condition, called "dynamical convexity", which yields the same results.

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Pseudoholomorphic curves

A novel feature of HWZ's proof is that it is not exactly variational. Instead it uses the theory of "pseudoholomorphic curves", introduced by Gromov, to produce the desired surface of section.

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Pseudoholomorphic curves

A novel feature of HWZ's proof is that it is not exactly variational. Instead it uses the theory of "pseudoholomorphic curves", introduced by Gromov, to produce the desired surface of section.

This is beyond the scope of this talk, but these are basically surfaces in \mathbb{R}^4 that are quite similar to images of holomorphic functions from

 $\mathbb{C}\longrightarrow\mathbb{C}^{2},$

but are more flexible. They are central to modern symplectic geometry.

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The setup

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where $\mu \in [0, 1]$ is the mass ratio $\frac{m_S}{m_E + m_S}$. This is \approx .999997 for the actual sun/earth.

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Lagrange points

The Hamiltonian H has five critical points L_1, \ldots, L_5 ordered by increasing value of H, called the *Lagrange points*. Our example will primarily involve the first Lagrange point, which intersects the earth-sun axis.

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If energy $c < H(L_1)$, then $H^{-1}(c)$ has three connected components: one near earth, one near sun, and one near ∞ .

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If energy $c < H(L_1)$, then $H^{-1}(c)$ has three connected components: one near earth, one near sun, and one near ∞ . Components *not* compact (because of collisions), but can be "regularized", i.e. noncompactness can be removed.

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We will focus on the component closest to the earth.



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Convexity?

We will focus on the component closest to the earth. For sufficiently low c, Albers, Fish, Frauenfelder, Hofer, and Van Koert show: $H^{-1}(c)$ is *strictly convex*. Hence, Hofer's result applies to show that $H^{-1}(c)$ admits a global surface of section, and 2 or ∞ -ly many periodic orbits.

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They also show that as c approaches the first Lagrange point from below, the component of $H^{-1}(c)$ fails to be strictly convex.

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Convexity?

We will focus on the component closest to the earth. For sufficiently low c, Albers, Fish, Frauenfelder, Hofer, and Van Koert show: $H^{-1}(c)$ is *strictly convex*. Hence, Hofer's result applies to show that $H^{-1}(c)$ admits a global surface of section, and 2 or ∞ -ly many periodic orbits.

They also show that as c approaches the first Lagrange point from below, the component of $H^{-1}(c)$ fails to be strictly convex. They conjecture, however, that $H^{-1}(c)$ is dynamically convex, which would still imply the existence of a global surface of section.

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What about above the first Lagrange point?

The level sets for $c < H(L_1)$ are almost "three-spheres"; they are examples of what is called *real projective three-space*, \mathbb{RP}^3 .

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This has the effect that the level sets for c just above the first Lagrange point are a "connect sum" of two \mathbb{RP}^3 s.

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This has the effect that the level sets for c just above the first Lagrange point are a "connect sum" of two \mathbb{RP}^3 s. For topological reasons, these can not carry a global surface of section. However, Fish and Siefring conjecture that they should carry a "finite energy foliation", which is a closely related idea.

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Back to hypersurfaces of contact type

As mentioned previously, Taubes recently proved the Weinstein conjecture for hypersurfaces in any four-dimensional symplectic manifold. Michael Hutchings and I proved a slight refinement of Taubes' result:

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Back to hypersurfaces of contact type

As mentioned previously, Taubes recently proved the Weinstein conjecture for hypersurfaces in any four-dimensional symplectic manifold. Michael Hutchings and I proved a slight refinement of Taubes' result:

Theorem 5

(CG., Hutchings) Any contact type hypersurface in a symplectic 4-manifold must carry at least 2 closed orbits for any Hamiltonian.

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Implications for the restricted three-body problem

Albers, Frauenfelder, Van Koert, and Paternain: for (circular) planar restricted three-body problem, $H^{-1}(c)$ is always a hypersurface of contact type for c below $H(L_1)$ and also for c just slightly above $H(L_1)$ (they also conjecture that this should hold for all energy levels).

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My result with Hutchings therefore applies to show that these hypersurfaces carry at least two closed orbits. Actually I believe that the connect sum of two \mathbb{RP}^3 s should always carry infinitely many closed orbits for any Hamiltonian for which it is a contact-type hypersurface.

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Plan



- 2 Weinstein's conjecture
- 3 Refinements of the Weinstein conjecture
- 4 The restricted three-body problem
- 5 Non-autonomous Hamiltonians
- 6 Future directions

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The Conley conjecture

Symplectic geometry can also be used to study Hamilton's ODEs for *non-autonomous* Hamiltonians, i.e.

$$H: \mathbb{R}^{2n} \times \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}.$$

Here, the dynamics no longer take place along a fixed energy level. However, much is known. Here are two highlights:

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 Hein has shown that 1-periodic Hamiltonians on cotangent bundles of closed manifolds have infinitely many periodic orbits, provided they are "quadratic at infinity".

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The Arnold conjecture

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Dan Cristofaro-Gardiner What can symplectic geometry tell us about Hamiltonian dynami

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The Arnold conjecture

 Floer and others (essentially) proved the Arnold conjecture. This gives a lower bound on the number of 1-periodic orbits for any 1-periodic Hamiltonian on a compact symplectic manifold in terms of the topology of the manifold, assuming all periodic orbits are "nondegenerate".

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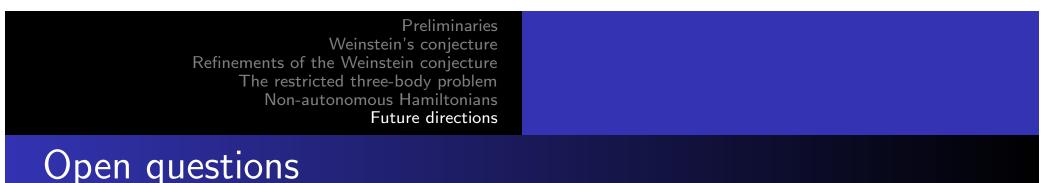


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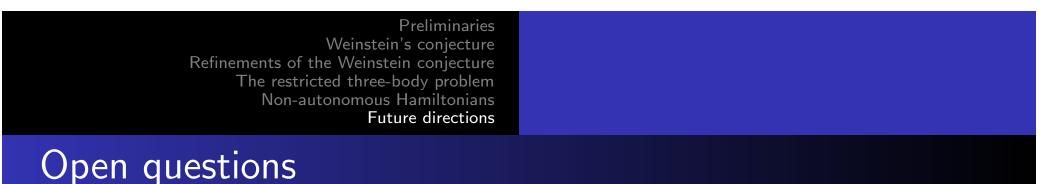


Here are three open questions (all for autonomous case) I am interested in:

 Does every compact contact type hypersurface in a 4-dimensional symplectic manifold carry a "short" Hamiltonian trajectory?

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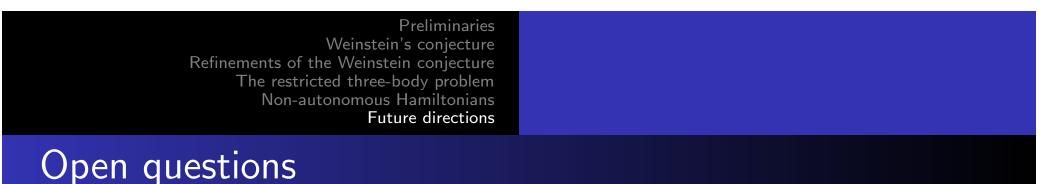
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- Does every compact contact type hypersurface in a 4-dimensional symplectic manifold carry a "short" Hamiltonian trajectory?
- What do the local dynamics look like around the periodic orbits that do appear?
- O all "topologically complicated" contact type hypersurfaces carry infinitely many periodic orbits?

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