Technical Appendix to Urban Decline and Durable Housing

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We start with a Rosen-Roback spatial equilibrium. A person who chooses to live in a given house receives each period a city-specific wage level, "w", a city-specific amenity level, "A", a location-specific amenity level "a", and pays rent "r".[1] Free mobility implies that $w + A + a - r = \underline{U}$, where $\underline{U}$ refers to the reservation flow of utility. This equality implies that rent equals $w + A + a - \underline{U}$, or $r = \theta + a$, where $\theta \equiv w + A - \underline{U}$. The composite term θ combines wages, city-specific amenities, and reservation utility to form an overall demand for the city. The location-specific amenities might capture distance to the city center in a conventional monocentric model, but they can also reflect any exogenous characteristic that makes one place more desirable than another. We further assume that each city is small relative to the overall economy so that general equilibrium effects can be ignored.[2]

We assume that θ follows a classic Brownian motion: $d\theta = \alpha dt + \sigma dz$. Location-specific amenities "a" are fixed over time. These amenities differ across the N lots that exist in the city and the distribution of 'a' is characterized by a cumulative distribution F(a) and a density f(a). Each lot can only contain one house, and we assume that all houses are physically identical. Houses cost 'C' to build and 'm' to use (or maintain). If the resident pays 'm', then the house does not depreciate. If the owner does not pay 'm', then the house is not usable, but it becomes usable as soon as the resident pays 'm' again. This can be understood as an endogenous depreciation model in which houses only leave the market when r<m, but even then, exit need not be permanent. Consequently, at any point in time, houses with values of 'a' for which $m - \theta > a$ will not be in use. We let $\hat{a}(t) = m - \theta(t)$ denote the amenity level of the lowest quality occupied housing in the city in period t.

[1] It is easiest to think of everyone as a renter, but equivalently, every resident can be thought of as an owner-occupier where "r" represents the user cost of owner-occupied housing.
[2] General equilibrium effects would prevent us from treating the reservation utility as an exogenous parameter.

Developers make construction decisions to maximize the expected value of future net revenues, discounted continuously at a rate ρ.[3] Using the tools of Dixit and Pindyck (1994), we know that a developer, who owns a site with amenity level 'a', will build when θ equals a threshold value of $\theta^* = \lambda + m - a$, where λ satisfies the condition $\phi(\lambda - \rho C) + e^{-\phi\lambda} = 1$ with $\phi = \frac{2\rho}{\sqrt{\alpha^2 + 2\rho\sigma^2} - \alpha}$. For values of θ above θ^*, building is strictly preferred, and for values of θ below θ^*, not building is strictly preferred.

Housing value is determined by the continuous time Bellman equation, $\rho V(\theta) = \pi + \frac{1}{dt} E(dV(\theta))$, where E(.) represents the expectations operator, $V(\theta)$ represents the value of the house and $\pi = Max(\theta + a - m, 0)$. The value of the house itself only changes with θ, which follows the stochastic process $d\theta = \alpha dt + \sigma dz$. Therefore, Ito's Lemma tells us that $E(dV) = (\alpha V'(\theta) + .5\sigma^2 V''(\theta))dt$. The Bellman equation then can be rewritten as $\rho V(\theta) = \pi + \alpha V'(\theta) + .5\sigma^2 V''(\theta)$. Solving this differential equation tells us that $V(\theta)$ takes the form $V(\theta) = k + k_0\theta + k_1 e^{\frac{-\alpha - \sqrt{\alpha^2 + 2\rho\sigma^2}}{\sigma^2}\theta} + k_2 e^{\frac{-\alpha + \sqrt{\alpha^2 + 2\rho\sigma^2}}{\sigma^2}\theta}$, where k, k_0, k_1 and k_2 are constants of integration.

[3] Their maximand is $\int_{t=0}^{\infty} e^{-\rho t} n(t) dt$, where n(t) refers to net revenues in each time period.

In the region where $\theta + a > m$, the value function must take the form $V(\theta) = \frac{\theta + a - m}{\rho} + \frac{\alpha}{\rho^2} + k_1^a e^{\frac{-\alpha - \sqrt{\alpha^2 + 2\rho\sigma^2}}{\sigma^2}\theta} + k_2^a e^{\frac{-\alpha + \sqrt{\alpha^2 + 2\rho\sigma^2}}{\sigma^2}\theta}$, and in the region where $\theta + a < m$, the value function takes the form $V(\theta) = k_1^b e^{\frac{-\alpha - \sqrt{\alpha^2 + 2\rho\sigma^2}}{\sigma^2}\theta} + k_2^b e^{\frac{-\alpha + \sqrt{\alpha^2 + 2\rho\sigma^2}}{\sigma^2}\theta}$, where k_j^i are constants. As the value of $V(\theta)$ should go to zero as θ approaches $-\infty$, it must be that $k_1^b = 0$. The option value of leaving the house empty should go to zero as θ approaches $+\infty$, so $k_2^a = 0$. Thus, we have the solution that $V(\theta)$ equals $\frac{\theta + a - m}{\rho} + \frac{\alpha}{\rho^2} + k_1^a e^{\frac{-\alpha - \sqrt{\alpha^2 + 2\rho\sigma^2}}{\sigma^2}\theta}$ when $\theta + a > m$ and $k_2^b e^{\frac{-\alpha + \sqrt{\alpha^2 + 2\rho\sigma^2}}{\sigma^2}\theta}$ otherwise.

To solve for k_1^a and k_2^b, we use that fact that there can be no jumps in either $V(\theta)$ or $V'(\theta)$ at $\theta + a = m$ (the smooth pasting property). The first equality requires that

$$\frac{\alpha}{\rho^2} + k_1^a e^{\frac{-\alpha - \sqrt{\alpha^2 + 2\rho\sigma^2}}{\sigma^2}(m-a)} = k_2^b e^{\frac{-\alpha + \sqrt{\alpha^2 + 2\rho\sigma^2}}{\sigma^2}(m-a)}.$$

The second equality requires that

$$\frac{1}{\rho} - \frac{\alpha + \sqrt{\alpha^2 + 2\rho\sigma^2}}{\sigma^2} k_1^a e^{\frac{-\alpha - \sqrt{\alpha^2 + 2\rho\sigma^2}}{\sigma^2}(m-a)} = \frac{-\alpha + \sqrt{\alpha^2 + 2\rho\sigma^2}}{\sigma^2} k_2^b e^{\frac{-\alpha + \sqrt{\alpha^2 + 2\rho\sigma^2}}{\sigma^2}(m-a)}.$$

or $\dfrac{\left(\sqrt{\alpha^2+2\rho\sigma^2}-\alpha\right)^2}{4\rho^2\sqrt{\alpha^2+2\rho\sigma^2}}e^{\frac{\alpha+\sqrt{\alpha^2+2\rho\sigma^2}}{\sigma^2}(m-a)}=k_1^a$. Solving these two equations yields:

$\dfrac{\left(\sqrt{\alpha^2+2\rho\sigma^2}-\alpha\right)^2}{4\rho^2\sqrt{\alpha^2+2\rho\sigma^2}}e^{\frac{\alpha+\sqrt{\alpha^2+2\rho\sigma^2}}{\sigma^2}(m-a)}=k_1^a$ and $\dfrac{\left(\sqrt{\alpha^2+2\rho\sigma^2}+\alpha\right)^2}{4\rho^2\sqrt{\alpha^2+2\rho\sigma^2}}e^{\frac{\alpha-\sqrt{\alpha^2+2\rho\sigma^2}}{\sigma^2}(m-a)}=k_2^b$. This implies that the value function is

$\dfrac{\left(\sqrt{\alpha^2+2\rho\sigma^2}+\alpha\right)^2}{4\rho^2\sqrt{\alpha^2+2\rho\sigma^2}}e^{\frac{-\alpha+\sqrt{\alpha^2+2\rho\sigma^2}}{\sigma^2}(\theta+a-m)}$ when rents are negative and $\dfrac{\theta+a-m}{\rho}+\dfrac{\alpha}{\rho^2}+\dfrac{\left(\sqrt{\alpha^2+2\rho\sigma^2}-\alpha\right)^2}{4\rho^2\sqrt{\alpha^2+2\rho\sigma^2}}e^{\frac{-\alpha-\sqrt{\alpha^2+2\rho\sigma^2}}{\sigma^2}(\theta+a-m)}$ when rents are positive.

To calculate the point at which construction makes sense requires calculating the value of an empty lot. We use $Q(\theta)$ to capture this value and note that the differential equation $\rho Q(\theta)=\alpha Q'(\theta)+.5\sigma^2 Q''(\theta)$ again must hold. This tells us that

$Q(\theta)=k_1^c e^{\frac{-\alpha-\sqrt{\alpha^2+2\rho\sigma^2}}{\sigma^2}\theta}+k_2^c e^{\frac{-\alpha+\sqrt{\alpha^2+2\rho\sigma^2}}{\sigma^2}\theta}$. As the option to build must approach 0 as θ approaches $-\infty$, it must be that $k_1^c=0$. We now use the notation that $\theta*$ is the level of θ at which construction is optimal. As it would never make sense to build when $\theta+a<m$, we use the fact that $Q(\theta*)=V(\theta*)-C$ and $Q'(\theta*)=V'(\theta*)$ (i.e., the smooth pasting property) to solve for the constants. Together, these two conditions give us the equations

$\dfrac{\theta+a-m}{\rho}+\dfrac{\alpha}{\rho^2}+\dfrac{\left(\sqrt{\alpha^2+2\rho\sigma^2}-\alpha\right)^2}{4\rho^2\sqrt{\alpha^2+2\rho\sigma^2}}e^{\frac{-\alpha-\sqrt{\alpha^2+2\rho\sigma^2}}{\sigma^2}(\theta*+a-m)}-C=k_2^c e^{\frac{-\alpha+\sqrt{\alpha^2+2\rho\sigma^2}}{\sigma^2}\theta*}$ and

$$\frac{1}{\rho}-\frac{\left(\sqrt{\alpha^2+2\rho\sigma^2}-\alpha\right)}{2\rho\sqrt{\alpha^2+2\rho\sigma^2}}e^{\frac{-\alpha-\sqrt{\alpha^2+2\rho\sigma^2}}{\sigma^2}(\theta^*+a-m)}=\frac{\sqrt{\alpha^2+2\rho\sigma^2}-\alpha}{\sigma^2}k_2^c e^{\frac{-\alpha+\sqrt{\alpha^2+2\rho\sigma^2}}{\sigma^2}\theta^*}.$$

Solving these equations yields:

$\theta^*+a-m-\rho C=\frac{\left(\sqrt{\alpha^2+2\rho\sigma^2}-\alpha\right)}{2\rho}\left(1-e^{\frac{-2\rho(\theta^*+a-m)}{\sqrt{\alpha^2+2\rho\sigma^2}-\alpha}}\right)$, which if $\lambda=\theta^*+a-m$ and $\phi=\frac{2\rho}{\sqrt{\alpha^2+2\rho\sigma^2}-\alpha}$ can be rewritten

$\phi(\lambda-\rho C)+e^{-\phi\lambda}=1$. Differentiating this equation yields: $\frac{\partial\lambda}{\partial C}=\frac{\rho}{1-e^{-\phi\lambda}}$, and for other variables denoted "x" (except

for ρ): $\frac{\partial\lambda}{\partial x}=\frac{\partial\phi}{\partial x}\frac{e^{-\phi\lambda}}{\phi^2(1-e^{-\phi\lambda})}\left(1+\lambda\phi-e^{\phi\lambda}\right)$. For any positive constant, ϕ, $e^{\phi\lambda}>1+\lambda\phi$. Thus, the sign of $\frac{\partial\lambda}{\partial x}$ is the opposite of the sign of

$\frac{\partial\phi}{\partial x}$, and simple differentiation gives us $\frac{\partial\phi}{\partial\sigma}<0$ and $\frac{\partial\phi}{\partial\alpha}>0$.

The equation $\theta^*=\lambda+m-a$ can be inverted, so we can say that for any given value of θ, there exists a value of 'a', denoted a^*, which represents the least attractive lot-specific amenity level that will be developed. As such, optimal development implies that the number of built-up lots in a growing city equals $N(1-F(a^*))$. The value of λ, and, hence, θ^* (for a given a) or a* (for a given θ) rises with C and σ and falls with α. As construction costs (C) rise or as the city's growth trend (α) falls, the threshold for construction increases and the level of development falls. As the degree of randomness (σ) rises, the option value of leaving a house empty rises and this also reduces the amount of development.

We then let $\underline{a}^* = \lambda + m - \overline{\theta}$ denote the lowest amenity value lot that has been developed in the city, where $\overline{\theta}$ is the highest value that $\theta(t)$ has reached. If $\theta(t)$ reaches a value greater than $\overline{\theta}$, new housing is built and the city grows. However, if $\theta(t)$ falls below $\overline{\theta}$, the city does not necessarily shrink. All units remain occupied until $\overline{\theta} - \lambda > \theta(t)$ (or, equivalently, until $\hat{a}(t) > \underline{a}^*$). The size of λ determines the extent of the buffer zone in which demand can fall before the city loses population.

Because we can derive proofs in the case of the deterministic model, we proceed assuming that $\sigma^2 = 0$, and then return to the stochastic case for our simulations.[4] In deterministically growing cities, $\lambda = \rho C$ so $a^* + \theta - m = \rho C$, so that construction occurs to the point where the interest rate times construction costs equals net rents. In deterministically declining cities, λ satisfies $\rho(\lambda - \rho C) = -\alpha(1 - e^{\rho\lambda/\alpha})$, and $a^* = \lambda + m - \theta$. Net rents for new construction must exceed the interest rate times construction cost when rents are expected to decline.

We define time zero so that $\theta(0)$ is the historical maximum value that θ has reached (i.e., $\theta(0) = \overline{\theta}$). We also use the following linearizing approximation throughout the analysis:

Approximation One: $\dfrac{F(a+x) - F(a)}{1 - F(a)} = f_{\min} x$.

Proposition 1 then follows (see Appendix for the proofs).

[4] Given that urban growth rates are remarkably correlated over time (see Table 2 below), treating them as being characterized primarily by different growth trends may not be a bad approximation in reality.

Proposition 1: If $\theta(0)=\bar{\theta}$ *then the growth rate of the city between period 0 and period t will equal* $f_{\min}\alpha t$ *if* $\alpha \geq 0$, *0 if* $0>\alpha \geq \frac{-\rho^2 C}{\rho t + e^{-\rho t}-1}$ *, and* $f_{\min}(\alpha t+\lambda)$ *if* $\frac{-\rho^2 C}{\rho t + e^{-\rho t}-1} \geq \alpha$.

The proposition divides cities into three groups. Cities with positive growth trends grow at a rate equal to the trend rate times the time period times $f_{\min}$, where $f_{\min}$ reflects the number of lots close to the construction margin. Cities with negative values of α which are small in absolute value have no change in population. Rents and prices fall in these places, but there is no change in the number of occupied units because rents remain above maintenance costs. In cities with more substantial negative trends, the rents of some houses no longer cover maintenance costs, so they are left vacant. While these cities do have population losses, these losses are much smaller than the gains of growing cities' whose trends are comparable in absolute value.

This sluggish response to negative shocks leads to Proposition 2:

Proposition 2: If $\theta(0)=\bar{\theta}$, $f_{\min}$ *is constant across cities, but* α *varies across cities then:*

(a) If the distribution of α *is symmetric and single peaked around* $\hat{\alpha}>0$ *, then the mean growth rate is greater than the median, but the difference between the mean and median growth rates falls with* $\hat{\alpha}$.

(b) If $\alpha=\alpha_0+\beta z+\varepsilon$, *where z is an observable urban characteristic, and* ε *is symmetric, mean zero, and single peaked, then for any positive constant k, the derivative of growth with respect to z will be greater when* $z=(k-\alpha_0)/\beta$ *than when* $z=-(\alpha_0+k)/\beta$.

The first part of the proposition states that if differences across cities are the result of different trends in demand, and if these trends are symmetrically distributed across cities, then population growth rates will be skewed to the left. The second part of the proposition tells us that observable factors which impact city growth will have a greater impact on population growth when these observable characteristics predict a positive growth rate than they will when they predict negative growth. This result reflects the fact that durable housing restricts the tendency of cities to lose population, even if the level of demand for the city has fallen.

We now consider growth rates starting from time τ , so that for declining cities, $\theta(\tau)$ will not equal $\overline{\theta}$. Proposition 3 describes the persistence of growth rates:

Proposition 3: The population growth rate between time τ and time 2τ equals the growth rate between time 0 and time τ if the city is growing. If the city lost population between time 0 and time τ , then the growth rate between time τ and time 2τ will equal $\frac{1}{1-(\lambda/-\alpha\tau)}$ times the growth rate between time zero and time τ .

Both growing and declining cities have persistent growth rates because we assumed that the urban dynamics are the result of deterministic trends. However, the relationship between past growth and future growth is stronger for declining cities. This strong persistence of decline occurs because the durability of housing delays the onset of population loss. As a result, past population loss understates the degree that demand for the city truly has fallen.

For declining cities, the value of $\overline{\theta}-\theta(\tau)$ can be measured using the share of the housing stock that is valued at less than the price of new construction (denoted S). If $\lambda. > \overline{\theta}-\theta(\tau)$, no homes have yet fallen into disuse. Using Approximation One, we know that the share of homes with prices less than the price of new construction (S) equals $f_{\min}\left(\overline{\theta}-\theta(\tau)\right)$. If we then consider urban decline between time τ and time t (where t > τ), once homes start to decay, the share of housing priced less than construction costs rises to $f_{\min}\lambda$ and the rate of population decline equals $f_{\min}\alpha(t-\tau)$. Further calculations produce:

Proposition 4a: If there is variation in α across cities but $f_{\min}$ is constant, then the expected growth rate of a city between time τ and time t falls discontinuously as S rises from zero, is independent of S when S is greater than zero but less than $\frac{f_{\min}\tau\rho^2 C}{\rho t + e^{-\rho t} - 1}$, and then declines monotonically with S for higher values of S.

The share of the initial stock of housing that is valued below construction costs reflects the size of the city's negative trend. When S is low, α is negative but small in absolute value, and the expected growth rate between time τ and time t is zero. For higher levels of S, the growth rate equals $f_{\min} \bullet (\lambda + \alpha(t-\tau)) - S$.

We can prove a result similar to Proposition 4a in the stochastic case, where we eliminate city-specific trends and assume that all cities have the same basic parameter values. At time τ, the city is again characterized by $\theta(\tau)$, the current value of θ, and $\overline{\theta}(\tau)$, the highest value that θ has reached by that time. The share of the housing stock that is priced below construction costs again equals $f_{\min} Max\left(\overline{\theta}(\tau) - \theta(\tau), \lambda\right)$ and Proposition 4b follows:[5]

[5] In this case, λ solves $(\lambda - \rho C)\sqrt{2\rho} = \sigma\left(1 - e^{-\lambda\sqrt{2\rho}/\sigma}\right)$.

Proposition 4b: Expected city population growth declines with S when S is less than λ, but then rises discontinuously at the point where S equals λ.

This proposition reiterates the conclusion of Proposition 4a that increases in the share of housing with values below the cost of new construction lower the expected growth rate, but in this case there is a reversal of the comparative static at the point where S equals λ. This reversal occurs because once S equals λ, the large supply of cheap housing makes it easier to grow because these houses can be reoccupied more cheaply than houses could be built from scratch.

We now turn to the implications of the model for prices. Data availability forces us to focus on median prices and rents, and we let $a_{med}(t)$ refer to the median occupied lot's amenity value as of time t. As demand for a city changes, two things happen which cause the median rent to shift. First, the price of any given lot is directly determined by θ so demand directly moves prices. Second, the identity of the median lot will change as the size of housing stock changes. As cities build more homes, the median lot will have a lower value of "a" because new construction has increased the number of fringe lots with low levels of site-specific amenities. This effect can help explain observations like Las Vegas (discussed above) where population soars and real rents actually fall. As the city grows, the median lot is becoming relatively less attractive.

We now treat $\frac{f(a^*)}{2f(a_{med})}$ as a constant and use the following approximation,

Approximation # 2: $a_{med}(t) - a_{med}(\tau) = \frac{f(a^*)}{2f(a_{med})} \bullet \left(a_{mar}(t) - a_{mar}(\tau)\right)$, where $a_{mar}(t)$ denotes the marginal lot in use, which equals $a^*(t)$ in growing cities and $\hat{a}(t)$ in declining cities. We also now return to the deterministic model where home prices equal $(\theta + a - m)/\rho + \alpha/\rho^2$ in a growing city and $(\theta + a - m)/\rho + \left(1 - e^{\frac{\rho}{\alpha}(\theta + a - m)}\right)\alpha/\rho^2$ in a declining city. In a growing city, θ is rising and the value of a* is falling over time because of new construction, so the overall change in the median housing price equals $\frac{\alpha t}{\rho}\left(1 - \frac{f(a^*)}{2f(a_{med})}\right)$. If "a" was distributed uniformly, then new supply would halve the positive impact of increasing demand on the median housing price. Following Proposition 1, in a declining city in which α is less than zero but greater than $\frac{-\rho^2 C}{\rho t + e^{-\rho t} - 1}$, there will be no supply response to urban decline. Hence, the change in demand will be fully reflected in price changes. For declining cities with values of α that are more negative, there will be a supply response that will mediate the price response and less attractive homes will leave the market.

These calculations and conclusions lead us to Proposition 5:

Proposition 5: (A) If $\alpha = \alpha_0 + \beta z + \varepsilon$*, where z is an observable urban characteristic, and* ε *is symmetric, mean zero and single peaked, then for any positive constant k, the derivative of rent growth with respect to z will be less when* $z = (k - \alpha_0)/\beta$ *than when* $z = -(\alpha_0 + k)/\beta$.
(B) If $\alpha = \alpha_0 + \beta z + \varepsilon$*, where z is an observable urban characteristic, then the derivative of housing price growth with respect to z will jump from* $\frac{\beta t}{\rho}\left(1 - \frac{f(a^*)}{2f(a^{med})}\right)$ *to* $\frac{\beta t}{\rho}$ *when* $\alpha = 0$.

(C) For any growing city, housing price growth will equal $\frac{1}{\rho f_{\min}}\left(1-\frac{f(a^{*})}{2f(a^{med})}\right)$ *times the percentage growth in population. For any declining city, the ratio of housing price decline to population decline will be greater than this quantity.*

Part (A) of Proposition 5 tells us that the impact of exogenous characteristics on rent growth will be smaller for growing cities than for declining cities. This comes from the fact that durability means that the supply adjustment will be less for declining cities. Part (B) gives us a similar result for housing price changes. Part (C) suggests that the relationship between changes in prices and changes in population will be concave. The slope of price change on population change will be less for growing cities than for declining cities. This result stems from the smaller supply response for declining cities. These conclusions formalize Figure 1, which provides the essential intuition that urban growth should show up most in rising numbers of people, but urban decline should show up most in declining home prices.

The different implications for price and quantity changes may be the most distinct implication of the durable housing model. If housing was putty, not clay, then rising demand for a city would create more houses that have higher prices; declining demand would reduce the housing stock and cause housing prices to fall. Durability generates the asymmetry between housing prices and population growth. In growing cities, rising prices and rising population levels go together; but in declining cities, prices fall first and far more steeply than population declines. We now turn to the empirical implications of the model, and then simulate the model in Section V.

Calibrating the Model

We now calibrate the model to see whether our framework can mimic the empirical features of urban decline. To do so, cities with the strongest negative trends should not depopulate even over long time spans, and declining cities generally should exhibit relatively small population changes and relatively large price declines. Because rents reflect current demand for the city rather than expectations about the future, we focus on them. Since most rented units are apartments, we use figures pertaining to them for the calibration exercise.

The key parameters of the model are as follows: (1) C, the cost of construction, (2) m, the cost of maintenance, (3) ρ, the interest rate, (4) α, a city-specific growth trend, (5) the distribution of a's, or lot-specific amenity values within the city, and (6) σ^2, the variance of city-specific demand shocks. Given these parameter values, and given an initial value for θ, we will be able to examine in more detail the implications of the model. We will assume each time period is a year and choose parameter values that reflect conditions between 1950 and 2000. We then simulate changes in population and prices over a fifty year period. All figures discussed in this section are in 2000 dollars.

The dynamics of median annual rents are used to impute the values of α and σ^2. We were able to collect a consistent set of rent data from the six decennial censuses dating back to 1950 for 206 cities.[6] As $r = \theta - a$, the change in median rents (denoted r_{50}) over a decade will have two components: $r_{50, t+10} - r_{50, t} = (\theta_{t+10} - \theta_t) + (a_{50, t+10} - a_{50, t})$. The $\theta_{t+10} - \theta_t$ term represents the change in demand for the city over the decade, and the $a_{50, t+10} - a_{50, t}$ term reflects the change in the identity of the median unit. Because of the supply response, the change in median rents understates the true change in demand for a city.

[6] More specifically, the data are as reported in various *County and City Data Books*, except for the 2000 figures, which were obtained from the Census web site. Median home price data also are available, but using rent data is more straightforward for the reason noted above and because it obviates the need to estimate a housing service flow based on the asset price.

The value of $a_{50, t+10}$ - $a_{50, t}$ is determined by estimating how much the median rent should have changed based on the change in the rental stock and the initial distribution of rents. For example, if the rental stock in a city increased by 10 percent over the decade, then the new median rental would have been the 45th percentile rental unit in the initial time period. More generally, $a_{50, t+10} - a_{50, t} = a_{50, t} - a_{50-.5x, t}$, with 'x' representing the percentage change in units over the decade. The value of $a_{50, t} - a_{50-.5x, t}$ is the difference in rents between the median rent and the rent at the 50+.5x percent. We use this difference, estimated from the initial period distribution of rents, to correct rent changes so they only reflect $\theta_{t+10} - \theta_t$, the true change in demand for the city.[7]

We then decompose $\theta_{t+10} - \theta_t$ into three terms: (a) the city-specific trend α_i , which does not change over time for any city i; (b) a common national effect, μ_t , which can vary over time; and (c) a city-specific idiosyncratic shock, $\varepsilon_{i,t}$, that also can vary across decades, and is mean zero with variance σ^2. In effect, $\theta_{t+10} - \theta_t = \alpha_i + \mu_t + \varepsilon_{i,t}$. This representation is useful because it allows us to isolate individual city trends separately from national changes. While we had no national changes in the model (because everything was relative to the national average), for simulation purposes it is impossible to sensibly treat post-war rent data without addressing the nation-wide increase in housing rents. Hence, we compute μ_t for each decade as the mean change in θ, weighted by beginning of period population.[8]

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[7] The results reported below are based on the rent distribution (from well-defined central cities) from the 1980 Census, which is the first year for which consistently good distributional data were available. The impact of adjusting for the change in the median unit can be very large for the most rapidly growing places in particular. In San Diego for example, median real rent growth over the 1960s and 1970s was only $57 and $23, respectively. However, the expansion of the city's housing stock in each decade was so large that the implied changes in theta (the true demand shock) were $690 and $1178, respectively. For a city such as Detroit that has been losing units on net, the decline in real median rents understates the true magnitude of the negative demand shock. For example, real median rents declined by about -$13 during the 1960s, but we estimate that the change in theta between 1960 and 1970 was -$117. We also experimented using the rent distribution from the 1990 Census, but none of our findings are sensitive to the choice of year.

[8] Nothing of importance is changed if we weight by housing units.

Subtracting this value from $\theta_{t+10} - \theta_t$ provides an estimate of $\alpha_i + \varepsilon_{i,t}$ for each city in each decade. Our estimate of α_i is then obtained by averaging this value over the five decades of data we have for every city. The weighted mean of α is zero by construction (median=-$9), with a standard deviation of $40 per year. While there are many places with strong positive annual trends, (e.g., the 75th percentile value for α equals +$22 per year over the 1950-2000 time span), we are most interested in cities with declining trends. Des Moines (IA) is the 25th percentile city with a negative trend of $-30 per year, Buffalo is the 10th percentile city with a negative trend of -$49 per year, and Youngstown (OH) is the first percentile city with a trend of -$71 per year.

We can then estimate the value of σ^2 by examining the deviation of adjusted rent changes (or $d\theta$) from their trend for each city. The overall standard deviation of this value is $750 for decadal differences, which implies an annual standard deviation of $75.[9] Thus, our estimate of σ^2 is $5,625 (or 75*75). As for the discount rate, Dixit and Pindyck (1994) show that the risk-free model using a risk-adjusted rate is equivalent to a model with risk-averse investors, so the appropriate calibration for ρ is a risk-adjusted rate. We use .08 as a value for the annual rate of interest ρ, a figure that approximates the real return on equities.[10]

The R.S. Means Company provides consistent data on construction costs per square foot for single-family homes dating back to 1950, but to match the rent data we must translate this into a production cost for apartments. We use the *American Housing Surveys* to gain insight into the relative size of rented versus owned units. According to the 1985 national file of the *AHS*, the living space of the median apartment was almost

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[9] This standard deviation hides a break between the 1950-1980 period and the 1980-2000 period. During the early period, standard deviations are about $500 dollars per decade. In the later period, they are approximately double that amount.
[10] We have experimented with different rates ranging from .06 to .10. None of our basic conclusions are altered by these changes in ρ.

exactly half that of the median owner-occupied home ($810ft^2$ versus $1,612ft^2$).[11] If we assume that the standard home in 1950 contained $1,400ft^2$ of living area, then a typical apartment of that era was about $700ft^2$ in size.[12] Multiplying this by the cost per square foot in 1950 for a modest quality home yields an estimate of $30,800 (in 2000 dollars) for C.[13]

These parameters help us to simplify the model discussed above. The key construction condition implied by the Dixit and Pindyck-based model is

$$(4)\quad \theta^* + a - m - \rho C = \frac{\left(\sqrt{\alpha^2 + 2\rho\sigma^2} - \alpha\right)}{2\rho}\left(1 - e^{\frac{-2\rho(\theta^* + a - m)}{\sqrt{\alpha^2 + 2\rho\sigma^2} - \alpha}}\right).$$

Given our parameter values, or indeed any reasonable perturbation of those values, the $e^{\frac{-2\rho C}{\sqrt{\alpha^2 + 2\rho\sigma^2} - \alpha}}$ option value term is essentially zero (even for cities with strong negative trends). This means that we can work with the construction condition that $\theta^* + a - m$ (net rents) equals $\rho C + \left(\sqrt{\alpha^2 + 2\rho\sigma^2} - \alpha\right)/2\rho$. Given our parameters, the value of ρC is $2,464. The cutoff level for net rents is $2,516 for growing cities with an alpha of +$50, $2,652 for cities with no trend, and $3,141 for declining cities with α=-$50. Thus, the net rents needed to justify construction must be about 25 percent higher in a declining city than in a growing city.

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[11] This size relationship held throughout the 1990s, but we do not have similar data for prior years.

[12] There are no good data on house or apartment size prior to the 1985 *AHS*. Data on the living area of newly constructed units are available from 1973-on, and the size of new homes has been increasing over the past quarter century. Thus, there is no doubt that the typical residence was somewhat smaller in 1950 than it is today.

[13] Given the uncertainty surrounding the true value of C, we experimented with different values for it. Changes of 10-20 percent (plus or minus) do not change the nature of the results reported in Table 5.

Another immediate implication of our parameter values is that the component of housing prices that comes from the option of leaving the house empty (described fully in Appendix 2) is almost always less than one percent of the value of the home. As such, housing prices can be approximated by the formula from the deterministic version of the model, $\frac{\theta + a - m}{\rho} + \frac{\alpha}{\rho^2}$.[14]

We look to data on the bottom end of the rent distribution for guidance on minimum required maintenance expenditures. Required costs must be at least as high as the cheapest rents we see being charged. Data on apartments in urban areas from the 1950 Census indicates that 97 percent of all units rented for at least $860 per year (in 2000 dollars).[15] Only five percent of units were rented for under $990 annually.[16] Given that the vast majority of units are not rented for under $1,000 per year, we assume that 'm' equals $1,000 in our base simulation.[17]

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[14] For example, in a city where demand is declining annually by an extreme -$100 per year, the option component is worth only $3,274 for a unit with annual net rents of $2,000, $7,157 for a unit with annual net rents of $1,000, and $10,583 for a unit with annual net rents of only $500.

[15] More specifically, these figures were taken from the table entitled "Dwelling Units—Contract and Gross Monthly Rent of Non-Farm Renter-Occupied Units: 1940 and 1950" (No. 905) in the 1956 *Statistical Abstract of the United States* (p. 784). The underlying data source is the decennial census.

[16] The underlying data actually are reported in interval form, so this calculation was made assuming a uniform density of units across dollar values within any given interval.

[17]This implies that 'm' amounts to 29 percent of median annual rent in 1950 and is a little more than 3 percent of C, or production costs. Data from other sources confirms the reliability of this parameter value and these ratios to median rent and asset value. For example, more recent data on rents from the 2000 *Census* documents that 95 percent of rents were at least $2,400 per year, with the latest *American Housing Survey* telling us that 95 percent of apartments rented for more than $1,775. These numbers suggest that something like $1,800 per year would be a good estimate for 'm' if we wanted to simulate fifty years beginning in 2000. Mean and median rents in 2000 for our 206 city sample were $6,672 and $6,492, respectively, so that the minimum rent required before it is left vacant is about 25 percent of median rent in the 2000. This is slightly below, but still quite close to the ratio of 'm' to median rent that we are using for our 1950-based starting point.

Data from the 2001 *Consumer Expenditure Survey(CES)* on spending by homeowners provide added support that our assumption regarding required costs is a sound one. Given an average house value of approximately $150,000, the *CES* data show that owners spend about $8,850 annually across five relevant categories: Property Taxes, Maintenance/Repairs/Insurance, Utilities/Fuels/Public Services, Housekeeping Supplies, and Household Furnishings and Equipment. While this amounts to nearly 6 percent of value, much of it clearly is not required for continued occupancy. If we assume that 80 percent of Property Taxes and Utilities/Fuels/Public Services costs are required for the marginal unit, along with 20 percent of the other costs (on the assumption that one can really skimp on maintenance, housekeeping, and furniture), estimated spending falls to about $4,800 or 3.2 percent of asset value.

The last parameter we need to pin down before running our calibration exercise involves the distribution of "a" or the site-specific amenity value. Conceptually, this should be inferred from the distribution of rents across sites, holding constant the quality of the structure. To approximate this, we estimated a rent hedonic on apartments (including MSA dummies) using the national files of the *American Housing Survey*, and then use the residuals from that estimation to represent site quality.[18] The simulation results discussed below are based on a distribution of 'a' from the 1985 *AHS*, with the figures scaled down to account for the change in median rents between 1950 and 1985. Naturally, the mean of the residuals is zero (the median is -$126, with all figures representing annual rents in 2000 dollars to be comparable with the other parameter values discussed above). The standard deviation of the distribution is $987, indicating that there are some very high and low quality sites at the extremes of the distribution.[19]

There are a fixed number of developable sites in our calibration exercise, and we set the initial value of θ=$4,000 to reflect an initially attractive city which still has room to grow before its bumps into our artificial, binding constraint on new residences. This initial value generates a decent-sized city in the initial period and is consistent with the typical 1950 rent.[20] We think of this starting point as characterizing the many

[18] The hedonic itself is a standard one, with rents regressed on a host of physical characteristics reported in the *AHS*. These include the type of heating system, the age of the structure, the presence of full kitchen and plumbing facilities, the number of bathrooms, the number of bedrooms, the number of other rooms, the presence of central air conditioning, and MSA dummies. Experimentation with different functional forms does not alter the results in any significant way.

[19] A relatively large city with (say) 75 percent of its potential sites developed would have the site quality associated with the value from the 25th percentile of the rent residual distribution, or -$624, in our calibration. Conversely, a relatively small city with only one-quarter of all its potential sites developed would have the site quality given by the value from the 75th percentile of the rent residual distribution, or +$495. Real median rents averaged $304 per month across our 206 cities in 1950. Hence, site quality (as opposed to the physical structure) accounts for no more than two months of rent for places in the middle of the distribution. In the tails (i.e., for really large and really small places), this obviously is not the case.

[20] The mean of the median rents in 1950 across the 206 cities in our sample was $3,650 (in 2000 dollars). Hence, the initial attractiveness of the city is being driven by city-level forces (wages or amenities), not site-specific quality values, which we think is appropriate.

manufacturing cities that were thriving following the Second World War, but that were soon beset by the long-term negative shock associated with the de-urbanization of manufacturing.

We then computed 1,000 simulations of cities with different city-level trends. We suspect that our inability to control fully for increasing unit quality over time (e.g., rising unit size, better materials, etc.) leads the estimates of α discussed above to understate the true extent of trend decline and overstate the amount of trend growth. If they are systematically misstated by $25, then we should use an alpha of -$100 to represent the cities with the most severely negative trends in the data (as α's<-$75 are rarely observed in the data), and so forth. Table 5 reports actual and simulated population changes for cities with differing levels of trend decline.[21]

Among the 206 central cities in our sample, there literally are only a handful (Detroit, Johnston, Saginaw, St. Louis and Youngstown) with average annual declines in demand of around -$75 over the entire second half of the 20th Century. As the top panel of Table 5 documents, actual population loss among this group over the 1950-2000 time period averaged a little over 50 percent. The simulation results for this group of cities, which are reported in the adjacent panel, yielded slightly higher losses, with the mean run losing 68 percent of its population (the median was -72 percent), and with the inter-quartile range of simulated population losses running from -83 percent to –58 percent. While this overestimates actual losses, the correspondence with reality seems close given that we are dealing with such extreme cases. Our simulations also suggest that the probability of complete depopulation is zero for these cities with the strongest negative trends.[22]

[21] We also provide simulation results for cities with a zero trend. However, we do not report findings for cities with clear positive trends. Because our calibration exercise works with a fixed number of developable sites, cities with strong positive trends quickly build out and then are constrained. While such a set-up well may have useful implications for the analysis of attractive areas with binding growth controls, that is a topic for future research. Our focus here is on decline.

[22] That is, there is no case among the 1,000 simulation runs in which all housing units were abandoned.

A larger group of 26 cities was estimated to have somewhat less negative trends in demand of about -$50. We include cities with alphas as low as -$60 (e.g., Flint and Cleveland) and as high as -$40 (e.g., Toledo and Baltimore). Once again, because our estimates of alpha understate the full extent of the negative trend in demand for these cities, the second panel of Table 5 reports on how our simulations for a city with α=-$75 matches their history over 1950-2000.

The mean actual population loss between 1950 and 2000 for these 26 cities was -25 percent. The median city lost 31 percent of its people, and the inter-quartile range runs from –39 percent to –17 percent. The mean population loss among the 1,000 simulation runs was –26 percent, which almost perfectly matches actual experience. And, we only do slightly less well when we look at the inter-quartile range of simulation results which ran from –47 percent to 0 percent (i.e., no population loss).[23] The model captures the long-run history of these cities quite well.

Our third set of cities includes 51 places with modestly negative alphas of about -$25 per year. We include all cities with estimated annual trends ranging from -$15 to -$35 (this would include Chicago, Philadelphia, Des Moines and New Orleans). For the reason discussed above, we try to simulate their experience using α=-$50. Actual experience for this group shows relatively little population loss or gain. The mean percentage change in population over the 1950-2000 time span was –3 percent and the median was -9 percent, with a standard deviation of 24 percent.

[23] Five of the 26 cities in this group actually experienced a population increase between 1950 and 2000. Thus, a 25 percent probability of having no loss or a population gain is not at odds with actual experience for this group of cities. Of these, Toledo (OH), Peoria (IL), and Waterloo (IA) grew from 1-5 percent over the fifty years. Five percent of our simulation runs yielded population gains of from 2-7 percent, with none generating more than 10 percent growth. Only Decatur (GA) and Davenport (IA), which each grew by over 20 percent, are outside the range of our simulation findings, as only one percent of the simulations generated values of that magnitude.

The simulation results reported in the third panel of Table 5 largely are consistent with this type of outcome, as the mean and median simulations yield virtually no population change. The inter-quartile range also is tightly centered on no change, with the 25th percentile result being no change and the 75th percentile result showing a 3 percent increase. Ten percent of the simulations generated growth of more than 10 percent, with one percent of the runs resulting in more than a 25 percent population gain. Less than 10 percent of the simulation runs resulted in population loss. These simulations suggest that a productive, attractive place with a high initial θ that experiences only a moderate negative trend in demand will not decline or grow very much.

Our fourth set of cities have measured city trends that are close to zero, and we include each place with an annual alpha ranging from -\$10 to +\$10. As discussed above, we then try to simulate their experience using α=-\$25 in the model. It is noteworthy that the 40 cities in this group almost always (in 36 cases) had larger populations in 2000 than in 1950. The mean growth rate was 61 percent (standard deviation of 49 percentage points), with the inter-quartile range running from 29 percent to 86 percent. The fourth panel in Table 5 shows that we under predict the extent of population growth for this group, as mean simulated growth is 8 percent and the 75th percentile growth is 11 percent. Hence, we do not capture the magnitude of growth that actually occurs among this group of cities, but we are successful in that we never predict decline (i.e., there are no cases of population decline in any of the 1,000 simulation runs).

The fifth set of cities examined includes 29 places with estimated local trends around \$25 per year that we try to simulate using α=\$0 for the reason discussed above. Included in this group are places such as Tuscaloosa, Columbus, Boston, Abilene, and Memphis. These cities experienced strong population gains over the second half of the 20th Century. The mean growth rate is 127 percent (standard deviation of 85), with the 25th percentile growth being 68 percent. The simulation results in the adjacent panel do systematically predict growth for this type of city, but once again we understate the actual change. Our mean simulation run generates a 23 percent population increase, and there are only 6 percent of the runs yielding increases of over 50 percent (the maximum is 59 percent). Hence, once again we are able to correctly model that a trendless city will grow, not decline, but we are not able to predict the strength of growth that actually occurs.[24]

Our simulation results also yield patterns of price changes that are consistent with the implications of our durable housing model of urban decline. For example, plotting the relationship between the percentage change in price and population across all our simulations would find it strongly concave. The more negative the trend in demand, the greater the percentage decline in price; and, positive growth generally is not associated with strongly increasing prices.[25] If we look over one period intervals (versus the fifty year spans considered in Table 5), the variability in price change is greater when there is no population change than it is when there is population growth. Idiosyncratic local shocks can lead to price increases even for cities not growing or can magnify price drops in places with negative trends, but adding units in growing places naturally reduces the price impact.

While these results are consistent with the findings of the previous section, we are unable to match the precise extent of price changes the way we can with population. In places with negative trends in demand, our simulations generally predict greater real price declines that are indicated in the data. This was not unexpected, as our simulations do not permit changes in unit quality. While increasing quality for the typical

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[24] As mentioned in footnote 52, we cannot hope to accurately mimic the behavior of rapidly growing cities. Among the 29 cities in our sample with clear positive trends (e.g., measured alphas in excess of $40 per year), the mean population gain was 239 percent (standard deviation of 141 percent). The interquartile range is from 146-303 percent, so the majority of these places at least doubled in size. In our framework, cities with strong positive local trends quickly expand to occupy all available sites, so that we are constrained to underpredict their growth. Their behavior is best characterized and explained by the urban growth studies discussed in the Introduction.

[25] That is the case until the city is completed developed. By not allowing the set of developable sites to expand indefinitely, it is as if we have imposed a binding growth control. With a positive alpha, prices can appreciate greatly in that context. While there appear to be some places with binding restrictions on development, they are rare. Hence, the statement regarding concavity generally applies.

unit leads to mismeasurement (i.e., underestimation) of constant quality price declines in places such as Detroit, we have no way of accurately measuring the bias.[26]

In sum, calibrating the model tell us that, given reasonable parameter values, the gap between the rents needed to justify new construction and the rents at which units are left empty is extremely large. As a result, among cities that once were very productive in the sense they provided high wages (or some valuable local amenity), there is no trend decline in demand within the range of the data that leads us to predict these places will significantly depopulate (i.e. lose more than 75 percent of their population) even over periods as long as 50 years. Highly productive cities that suffer only small or modestly negative trends in demand, should not expect any significant long-run loss of population. Productive cities paying high wages that are not experiencing any trend in demand will grow over time, as long as builders are free to respond to positive idiosyncratic shocks. An added implication is that for the vast majority of homes in most places, the option value component is not relevant in order to understand the slow nature of decline.

Reference

Dixit, Avinash and Robert Pindyck. *Investment Under Uncertainty*. Princeton University Press, 1994.

[26] The decennial census for 1950 does not report house value or house traits, so we cannot estimate a hedonic model from which constant quality prices could be computed. In fact, micro data on house prices and characteristics are not reported for geographic areas below the state level until 1970. More modern repeat sales indices do not go back in time before 1975.

Appendix : Proofs of Propositions

Proof of Proposition 1: If $\alpha \geq 0$, then the lowest amenity lot that is developed at time zero has a value of 'a' equal to $m + \rho C - \theta(0)$ and the lowest amenity lot developed at time t has a value of 'a' equal to $m + \rho C - \theta(t) = m + \rho C - \theta(0) - \alpha t$. Using Approximation One, this implies that the growth rate of the city equals $f_{\min} \alpha t$.

If $\alpha < 0$, then no new lots will be developed. The least attractive lot, characterized by $a = \underline{a}$ that has been developed satisfies $\rho(\lambda - \rho C) = -\alpha\left(1 - e^{\frac{\rho\lambda}{\alpha}}\right)$ or $\rho(\theta(0) + \underline{a} - m - \rho C) = -\alpha\left(1 - e^{\frac{\rho(\theta(0) + \underline{a} - m)}{\alpha}}\right)$. As long as $\underline{a} + \theta(t) > m$ or $\lambda(\alpha) + \alpha t > 0$, the lot will remain occupied. Let $\tilde{\alpha}$ solve $\lambda(\tilde{\alpha}) + \tilde{\alpha} t = 0$, which has the unique solution $\tilde{\alpha} = \dfrac{-\rho^2 C}{\rho t + e^{-\rho t} - 1} < 0$. As the derivative of $\lambda(\alpha) + \alpha t > 0$ with respect to α equals $t + \dfrac{\lambda\left(e^{\frac{\rho\lambda}{\alpha}} - 1\right) + \rho C}{\rho(\lambda - \rho C)}$ (which is always positive), for all cities with values of α above $\tilde{\alpha}$, $\lambda(\alpha) + \alpha t > 0$ and no units will be left unoccupied.

For all cities with values of α below $\tilde{\alpha}$, $\lambda(\alpha)+\alpha t<0$ and $\underline{a}+\theta(0)+\alpha t<m$, so some lots will be left unoccupied. The number of lots left unoccupied will equal $N\left(F(m-\theta(0)-\alpha t)-F(\underline{a})\right)$ or $N\left(F(\underline{a}+\lambda-\alpha t)-F(\underline{a})\right)$, and using Approximation One, this implies that the rate of decline will be $f_{\min}(\alpha t+\lambda)$.

Proof of Proposition 2: (a) We let $\alpha=\hat{\alpha}+\varepsilon$ and use $g(\varepsilon)$ to reflect the density of ε, which is mean zero and symmetrically distributed. The median growth rate will equal $f_{\min}\hat{\alpha}t$. The mean growth rate averaging over cities equals:

$$\text{(A1)}\quad f_{\min}\left(\hat{\alpha}t-\int_{\varepsilon=-\hat{\alpha}\frac{-\rho^2C}{\rho t+e^{-\rho t}-1}}^{-\hat{\alpha}}(\hat{\alpha}+\varepsilon)tg(\varepsilon)d\varepsilon+\int_{\varepsilon=-\infty}^{-\hat{\alpha}\frac{-\rho^2C}{\rho t+e^{-\rho t}-1}}\lambda g(\varepsilon)d\varepsilon\right),$$

which is obviously less than $f_{\min}\hat{\alpha}dt$. The difference between the mean and the median equals $f_{\min}$ times $\left(-\int_{\varepsilon=-\hat{\alpha}\frac{-\rho^2C}{\rho t+e^{-\rho t}-1}}^{-\hat{\alpha}}(\hat{\alpha}+\varepsilon)tg(\varepsilon)d\varepsilon+\int_{\varepsilon=-\infty}^{-\hat{\alpha}\frac{-\rho^2C}{\rho t+e^{-\rho t}-1}}\lambda g(\varepsilon)d\varepsilon\right)$, and the derivative of this with respect to $\hat{\alpha}$ equals $-\int_{\varepsilon=-\hat{\alpha}\frac{-\rho^2C}{\rho t+e^{-\rho t}-1}}^{-\hat{\alpha}}tg(\varepsilon)d$, which is negative.

(b) The expected growth rate for a city with characteristic z is:

(A1') $$f_{\min}\left((\alpha_0+\beta z)t-\int_{\varepsilon=-\alpha_0-\beta z-\frac{-\rho^2 C}{\rho t+e^{-\rho t}-1}}^{-\alpha_0-\beta z}(\alpha_0+\beta z+\varepsilon)tg(\varepsilon)d\varepsilon+\int_{\varepsilon=-\infty}^{-\alpha_0-\beta z-\frac{-\rho^2 C}{\rho t+e^{-\rho t}-1}}\lambda g(\varepsilon)d\varepsilon\right).$$

The derivative of this growth rate with respect to z is $f_{\min}\beta t\left(1-\int_{\varepsilon=-\alpha_0-\beta z-\frac{\rho^2 C}{\rho t+e^{-\rho t}-1}}^{-\alpha_0-\beta z}g(\varepsilon)d\varepsilon\right)$.

The difference in this derivative between when $\alpha_0+\beta z=k$ and when $\alpha_0+\beta z=-k$ if k>0 equals $f_{\min}\beta dt\left(\int_{\varepsilon=k-\frac{\rho^2 C}{\rho dt+e^{-\rho dt}-1}}^{k}g(\varepsilon)d\varepsilon-\int_{\varepsilon=-k-\frac{\rho^2 C}{\rho dt+e^{-\rho dt}-1}}^{-k}g(\varepsilon)d\varepsilon\right)$,

which by the symmetry of ε equals $f_{\min}\beta dt\left(\int_{\varepsilon=k-\frac{\rho^2 C}{\rho dt+e^{-\rho dt}-1}}^{k}g(\varepsilon)d\varepsilon-\int_{\varepsilon=-k}^{k+\frac{\rho^2 C}{\rho dt+e^{-\rho dt}-1}}g(\varepsilon)d\varepsilon\right)$. This term is strictly positive by the fact that g(.) is single peaked.

Proof of Proposition 3: If the city is growing, then the growth rate for any time period of length τ equals $f_{\min}\tau$. If the city has lost population between time zero and time τ, the overall growth during that period must equal $f_{\min}(\lambda+\alpha\tau)$. Once a city begins to lose population, the future decline rate will be $f_{\min}\alpha\tau$, so that future decline will equal $\frac{\alpha\tau}{\lambda+\alpha\tau}$ times past decline.

Proof of Proposition 4a: All cities for which α is positive will have a value of S equal to zero. Thus, if S=0, the expected growth rate equals $f_{\min} tE(\alpha|\alpha > 0)$, where $E(\alpha|\alpha > 0)$ denotes the expectation of α conditional upon α being positive.

When S is greater than zero but less than $\frac{f_{\min}\tau\rho^2 C}{\rho t + e^{-\rho t} - 1}$, S will equal $-\alpha\tau$ and there will be no population change between time τ and time t. As $E(\alpha|\alpha > 0)$ is a positive number, there must be a jump down in the expected growth rate as S rises above zero. There will be no relationship between S and later decline during this range.

When S is greater than $\frac{f_{\min}\tau\rho^2 C}{\rho t + e^{-\rho t} - 1}$, but less than $\frac{f_{\min}\tau\rho^2 C}{\rho\tau + e^{-\rho\tau} - 1}$, then the expected growth rate equals $-S\frac{t-t^*}{\tau}$. Differenting finds that $\frac{\partial Growth}{\partial S} = -\frac{t-t^*}{\tau} + \frac{S}{\tau}\frac{\partial t^*}{\partial S}$, where t* is defined by $\frac{\rho C}{\alpha} = \frac{1-e^{-\rho t^*}}{\rho} - t^*$, which implies $\frac{\partial t^*}{\partial \alpha} = \frac{\rho C\left(1-e^{-\rho t^*}\right)}{\alpha^2}$ or $\frac{\partial t^*}{\partial S} = -\frac{\rho C\left(1-e^{-\rho t^*}\right)}{\alpha^2 \tau f_{\min}}$.

Thus, $\frac{\partial Growth}{\partial S} = -\frac{t-t^*}{\tau} - \frac{\left(1-e^{-\rho t^*}\right)}{\tau}\left(\frac{\rho C}{-\alpha}\right)$, which is negative.

When S is greater than $\frac{f_{\min}\tau\rho^2 C}{\rho\tau + e^{-\rho\tau} - 1}$, the expected growth rate equals $f_{\min}\alpha t$, so the impact of S on growth will equals $f_{\min} t \frac{\partial\alpha}{\partial S}$. The value of S equals $f_{\min}\lambda(\alpha(S))$, so $f_{\min}\frac{\partial\alpha}{\partial S} = \frac{1}{\frac{\partial\lambda}{\partial\alpha}}$ or $\frac{\alpha\left(1-e^{\frac{\rho\lambda}{\alpha}}\right)}{\lambda - \rho C + \lambda e^{\frac{\rho\lambda}{\alpha}}}$, which is always negative.

Proof of Proposition 4b: The value of S equals $f_{\min} Max(\overline{\theta}(\tau) - \theta(\tau), \lambda)$. Using the notation that $\overline{\theta}(t)$ is the highest value that θ has reached as of time t, we know that if both $\overline{\theta}(t) - \theta(t) > \lambda$ and $\overline{\theta}(\tau) - \theta(\tau) > \lambda$, the growth rate between time zero and time t equals $f_{\min}(\theta(t) - \theta(\tau))$. Alternatively, if $\overline{\theta}(t) - \theta(t) < \lambda$ and $\overline{\theta}(\tau) - \theta(\tau) < \lambda$, then urban development is determined by $f_{\min}(\overline{\theta}(t) - \overline{\theta}(\tau))$. In the case where $\overline{\theta}(\tau) - \theta(\tau) > \lambda$ and $\overline{\theta}(t) - \theta(t) < \lambda$ (which requires $\theta(t) > \theta(\tau)$), then the level of growth equals $f_{\min}(\overline{\theta}(t) - \lambda - \theta(\tau))$. Finally, when $\overline{\theta}(t) - \theta(t) > \lambda$ and $\overline{\theta}(\tau) - \theta(\tau) < \lambda$, the level of growth equals $f_{\min}(\theta(t) + \lambda - \overline{\theta}(\tau))$.

We now define a new variable $\tilde{\theta}(t)$, which equals the maximal value that $\theta(t)$ takes on between time τ and time t. The distribution of $\tilde{\theta}(t)$ and $\theta(t)$ depend only on $\theta(\tau)$, not on $\overline{\theta}(\tau)$. The density of $\theta(t)$ conditional upon $\theta(\tau)$ is $\frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(\theta(t)-\theta(\tau))^2}{2\sigma^2(t-\tau)}}$. The density of $\tilde{\theta}(t)$

conditional on both $\theta(\tau)$ and $\theta(t)$ is $\frac{(4\tilde{\theta}(t)-2\theta(t)-2\theta(\tau))}{\sigma^2(t-\tau)}e^{\frac{-2(\tilde{\theta}(t)-\theta(\tau))(\tilde{\theta}(t)-\theta(t))}{\sigma^2(t-\tau)}}$, and the cumulative distribution is $1-e^{\frac{-2(\tilde{\theta}(t)-\theta(\tau))(\tilde{\theta}(t)-\theta(t))}{\sigma^2(t-\tau)}}$

(following Harrison, 1988, p. 8). The value of $\overline{\theta}(t)=Max\left(\overline{\theta}(\tau),\tilde{\theta}(t)\right)$.

If $\overline{\theta}(\tau)-\theta(\tau)<\lambda$, then using the fact that $\overline{\theta}(\tau)=\theta(\tau)+\frac{S}{f_{\min}}$, the expected growth rate of the city equals $f_{\min}$ times

$$\int_{\theta(t)=\theta(\tau)+\frac{S}{f_{\min}}-\lambda}^{\infty}\left(\begin{array}{l}\left(\theta(t)+\lambda-\theta(\tau)-\frac{S}{f_{\min}}\right)e^{-\frac{2\lambda(\theta(t)+\lambda-\theta(\tau))}{\sigma^2(t-\tau)}}+\\ \displaystyle\int_{\tilde{\theta}(t)=Max\left(\theta(\tau)+\frac{S}{f_{\min}},\theta(t)\right)}^{\theta(t)+\lambda}\left(\tilde{\theta}(t)-\overline{\theta}(\tau)\right)\frac{\left(4\tilde{\theta}(t)-2\theta(t)-2\theta(\tau)\right)}{\sigma^2(t-\tau)}e^{-\frac{2(\tilde{\theta}(t)-\theta(t))(\tilde{\theta}(t)-\theta(\tau))}{\sigma^2(t-\tau)}}d\tilde{\theta}(t)\end{array}\right)\frac{e^{-\frac{(\theta(t)-\theta(\tau))^2}{2\sigma^2(t-\tau)}}}{\sigma\sqrt{2\pi}}d\theta(t)$$

$$+\int_{\theta(t)=-\infty}^{\theta(\tau)+\frac{S}{f_{\min}}-\lambda}\left(\theta(t)+\lambda-\theta(\tau)-\frac{S}{f_{\min}}\right)\frac{e^{-\frac{(\theta(t)-\theta(\tau))^2}{2\sigma^2(t-\tau)}}}{\sigma\sqrt{2\pi}}d\theta(t)$$

The derivative of this with respect to S equals

$$-\int_{\theta(t)=\theta(\tau)+\frac{S}{f_{\min}}-\lambda}^{\infty}\left(e^{-\frac{2\lambda(\theta(t)+\lambda-\theta(\tau))}{\sigma^2(t-\tau)}}+\int_{\tilde{\theta}(t)=Max\left(\theta(\tau)+\frac{S}{f_{\min}},\theta(t)\right)}^{\theta(t)+\lambda}\frac{\left(4\tilde{\theta}(t)-2\theta(t)-2\theta(\tau)\right)}{\sigma^2(t-\tau)}e^{-\frac{2(\tilde{\theta}(t)-\theta(t))(\tilde{\theta}(t)-\theta(\tau))}{\sigma^2(t-\tau)}}d\tilde{\theta}(t)\right)\frac{e^{-\frac{(\theta(t)-\theta(\tau))^2}{2\sigma^2(t-\tau)}}}{\sigma\sqrt{2\pi}}d\theta(t)$$

$$-\int_{\theta(t)=-\infty}^{\theta(\tau)+\frac{S}{f_{\min}}-\lambda}\frac{e^{-\frac{(\theta(t)-\theta(\tau))^2}{2\sigma^2(t-\tau)}}}{\sigma\sqrt{2\pi}}d\theta(t)$$

This is always negative. However, if $\overline{\theta}(\tau)-\theta(\tau)>\lambda$, then the expected growth rate of the city equals $f_{\min}$ times

$$\int_{\theta(t)=\overline{\theta}(\tau)}^{\infty}\left(\begin{array}{l}(\theta(t)-\theta(\tau))e^{-\frac{2\lambda(\lambda+\theta(t)-\theta(\tau))}{\sigma^2(t-\tau)}}\\ \displaystyle\int_{\tilde{\theta}(t)=\theta(t)}^{\theta(t)+\lambda}\left(\tilde{\theta}(t)-\theta(\tau)-\lambda\right)\frac{\left(4\tilde{\theta}(t)-2\theta(t)-2\theta(\tau)\right)}{\sigma^2(t-\tau)}e^{-\frac{2\left(\tilde{\theta}(t)-\theta(t)\right)\left(\tilde{\theta}(t)-\theta(\tau)\right)}{\sigma^2(t-\tau)}}d\tilde{\theta}(t)\end{array}\right)\frac{e^{-\frac{(\theta(t)-\theta(\tau))^2}{2\sigma^2(t-\tau)}}}{\sigma\sqrt{2\pi}}+$$

$$\int_{\theta(t)=\overline{\theta}(\tau)-\lambda}^{\overline{\theta}(\tau)}\int_{\tilde{\theta}(t)=\theta(t)}^{\theta(t)+\lambda}\left(\begin{array}{l}\left(Min\left(\theta(t),Max\left(\tilde{\theta}(t),\overline{\theta}(\tau)\right)-\lambda\right)\right)\bullet\\ \displaystyle\frac{\left(4\tilde{\theta}(t)-2\theta(t)-2\theta(\tau)\right)}{\sigma^2(t-\tau)}\end{array}\right)e^{-\frac{2\left(\tilde{\theta}(t)-\theta(t)\right)\left(\tilde{\theta}(t)-\theta(\tau)\right)}{\sigma^2(t-\tau)}}\frac{e^{-\frac{(\theta(t)-\theta(\tau))^2}{2\sigma^2(t-\tau)}}}{\sigma\sqrt{2\pi}}d\tilde{\theta}(t)d\theta(t)\ .$$

$$+\int_{\theta(t)=-\infty}^{\overline{\theta}(\tau)-\lambda}(\theta(t)-\theta(\tau))\frac{e^{-\frac{(\theta(t)-\theta(\tau))^2}{2\sigma^2(t-\tau)}}}{\sigma\sqrt{2\pi}}d\theta(t)$$

The derivative of this with respect to $\overline{\theta}(\tau)$ equals

$$\left(\overline{\theta}(\tau)-\lambda-\theta(\tau)\right)\int_{\tilde{\theta}(t)=\overline{\theta}(\tau)-\lambda}^{\infty}\left(\frac{\left(4\tilde{\theta}(t)-2\left(\overline{\theta}(\tau)-\lambda\right)-2\theta(\tau)\right)}{\sigma^2(t-\tau)}\right)e^{-\frac{2\left(\tilde{\theta}(t)-\left(\overline{\theta}(\tau)-\lambda\right)\right)\left(\tilde{\theta}(t)-\theta(\tau)\right)}{\sigma^2(t-\tau)}}d\tilde{\theta}(t)\frac{e^{-\frac{\left(\overline{\theta}(\tau)-\lambda-\theta(\tau)\right)^2}{2\sigma^2(t-\tau)}}}{\sigma\sqrt{2\pi}}+$$

$$\int_{\theta(t)=\overline{\theta}(\tau)-\lambda}^{\overline{\theta}(\tau)}\int_{\tilde{\theta}(t)=\theta(t)}^{\overline{\theta}(\tau)}\frac{\left(4\tilde{\theta}(t)-2\theta(t)-2\theta(\tau)\right)}{\sigma^2(t-\tau)}e^{-\frac{2\left(\tilde{\theta}(t)-\theta(t)\right)\left(\tilde{\theta}(t)-\theta(\tau)\right)}{\sigma^2(t-\tau)}}\frac{e^{-\frac{(\theta(t)-\theta(\tau))^2}{2\sigma^2(t-\tau)}}}{\sigma\sqrt{2\pi}}d\tilde{\theta}(t)d\theta(t)\ ,$$

$$+\left(\overline{\theta}(\tau)-\lambda-\theta(\tau)\right)\frac{e^{-\frac{(\theta(t)-\lambda-\theta(\tau))^2}{2\sigma^2(t-\tau)}}}{\sigma\sqrt{2\pi}}$$

which is positive. Thus, the expected growth rate rises with $\overline{\theta}(\tau)$ when $S = f_{\min}\lambda$, so that the expected growth rate jumps discontinuously from the point where S rises from less than $f_{\min}\lambda$ to the point where $S = f_{\min}\lambda$.

Proof of Proposition 5: (A) The expected growth in mean rents for a city with exogenous characteristic z equals $\alpha t\left(1-\frac{f(a^*)}{2f(a_{med})}\right)$ when $\alpha > 0$, αt when $0 > \alpha > \frac{-\rho^2 C}{\rho t + e^{-\rho t} - 1}$,

and $\alpha t\left(1-\frac{f(a^*)}{2f(a_{med})}\right) - \lambda\frac{f(a^*)}{2f(a_{med})}$ when $\frac{-\rho^2 C}{\rho t + e^{-\rho t} - 1} > \alpha$. Thus, if $\alpha = \alpha_0 + \beta z + \varepsilon$, then the expected change in rents for a city with characteristic z equals:

$$(\alpha_0 + \beta z)t\left(1-\frac{f(a^*)}{2f(a_{med})}\right) - \frac{\lambda f(a^*)}{2\rho f(a_{med})}G\left(-\alpha_0 - \beta z - \frac{\rho^2 C}{\rho t + e^{-\rho t} - 1}\right) + \int_{\varepsilon = -\alpha_0 - \beta z - \frac{\rho^2 C}{\rho t + e^{-dt} - 1}}^{-\alpha_0 - \beta z} (\alpha_0 + \beta z + \varepsilon)\frac{f(a^*)}{2f(a_{med})} g(\varepsilon) d\varepsilon,$$

and the derivative of this with respect to z equals: $\beta t\left(1-\frac{f(a^*)}{2f(a_{med})}\right)\left(1 - G(-\alpha_0 - \beta z) + G\left(-\alpha_0 - \beta z - \frac{\rho^2 C}{\rho t + e^{-\rho t} - 1}\right)\right)$.

The difference in this component if $\alpha_0 + \beta z = k$ and when $\alpha_0 + \beta z = -k$, if k>0, using the symmetry of ε equals

$$\frac{\beta t}{\rho}\frac{f(a^*)}{2f(a_{med})}\left(\left(G\left(k+\frac{\rho^2 C}{\rho t+e^{-\rho t}-1}\right)-G(k)\right)-\left(G(k)-G\left(k-\frac{\rho^2 C}{\rho t+e^{-\rho t}-1}\right)\right)\right)$$, which is always negative since G(.) is single peaked.

(B) If $\alpha=\alpha_0+\beta z+\varepsilon$, the derivative of the growth rate of housing prices with respect to z equals $\frac{\beta t}{\rho}\left(1-\frac{f(a^*)}{2f(a^{med})}\right)$ when $\alpha>0$. The growth in housing prices equals $\frac{\alpha}{\rho}\left(t-\frac{e^{\frac{\rho}{\alpha}(\theta+a_{med}(0)-m)}\left(e^{\rho t}-1\right)}{\rho}\right)$ when $0>\alpha>\frac{-\rho^2 C}{\rho t+e^{-\rho t}-1}$, and if $\alpha=\alpha_0+\beta z$, then the derivative of this expression with respect to z equals

$$\frac{\beta t}{\rho}-\frac{\beta\left(e^{\rho t}-1\right)}{\rho^2}\left(e^{\frac{\rho}{\alpha}(\theta+a_{med}(0)-m)}-\frac{(\theta+a_{med}(0)-m)}{\alpha}e^{\frac{\rho}{\alpha}(\theta+a_{med}(0)-m)}\right)$$. As α approaches zero, $\frac{(\theta+a_{med}(0)-m)}{\alpha}$ approaches negative infinity and therefore the first term in parentheses goes to zero. The limit of the second term equals the limit as x goes to infinity of $xe^{-\rho x}$ which is also zero. Thus the limit of the change in prices as α approaches zero equals $\frac{\beta t}{\rho}$.

(C) If $\alpha=0$, the city has no population or price change. For a growing city, the change in price equals $\frac{\alpha t}{\rho}\left(1-\frac{f(a^*)}{2f(a^{med})}\right)$ and the change in population equals $f_{\min}\alpha t$, thus the change in price equals $\frac{1}{\rho f_{\min}}\left(1-\frac{f(a^*)}{2f(a^{med})}\right)$ times the percentage growth of the cities, and the slope of any positive line will equal this quantity. For cities with values of α that are between zero and $\frac{-\rho^2 C}{\rho t + e^{-\rho t} - 1}$, the change in population will be zero and the change in price will be negative. For cities will levels of α below $\frac{-\rho^2 C}{\rho t + e^{-\rho t} - 1}$, the change in median housing prices between time zero and time t is $\frac{(\lambda + \alpha t)}{\rho}\left(1 - \frac{f(a^*)}{2f(a_{med})}\right) - \frac{\lambda}{\rho} - \frac{\alpha}{\rho^2}\left(e^{\frac{\rho}{\alpha}(\theta + a_{med}(t) - m)} - e^{\frac{\rho}{\alpha}(\theta + a_{med}(0) - m)}\right)$ and the change in population equals $(\lambda + \alpha t) f_{\min}$. Thus, we need to show that $\lambda > \frac{-\alpha}{\rho}\left(e^{\frac{\rho}{\alpha}(\theta + a_{med}(t) - m)} - e^{\frac{\rho}{\alpha}(\theta + a_{med}(0) - m)}\right)$. Using the fact that $\lambda = \rho C - \frac{\alpha}{\rho}\left(1 - e^{\frac{\rho\lambda}{\alpha}}\right)$, the inequality follows because $1 - e^{\frac{\rho\lambda}{\alpha}}$ is greater than $e^{\frac{\rho}{\alpha}(a_{med}(0) - a^*(0))}\left(e^{\frac{\rho}{\alpha}(\lambda + a_{med}(t) - a_{med}(0))} - e^{\frac{\rho\lambda}{\alpha}}\right)$.

Table 1: Simulating Long-Run Decline, 1950-2000

Actual Decline		Simulated Decline	
Category #1 (cities in most extreme decline: Detroit, Johnstown, Saginaw, St. Louis, Youngstown)		$\theta(0)$ = $4,000 α = -$100 1,000 Runs	
Mean Population Change:	-51%	Mean Population Change:	-68%
Standard Deviation	11%	Standard Deviation	20%
25th Percentile:	-59%	25th Percentile:	-83%
50th Percentile:	-51%	50th Percentile:	-73%
75th Percentile:	-49%	75th Percentile:	-58%
Category #2 (26 cities– e.g., Flint, Cleveland, Toledo, Baltimore, Buffalo, Dayton, Pittsburgh—with second most severe negative local trends)		$\theta(0)$ = $4,000 α = -$75 1,000 Runs	
Mean Population Change:	-25%	Mean Population Change:	-26%
Standard Deviation	22%	Standard Deviation	25%
25th Percentile:	-39%	25th Percentile:	-47%
50th Percentile:	-31%	50th Percentile:	-23%
75th Percentile:	-17%	75th Percentile:	0%
Category #3 (51 cities – e.g., Sioux City (IA), Des Moines, Reading (PA), Philadelphia, Chicago, Kansas City, Worcester (MA) – with third most severe negative local trends)		$\theta(0)$ = $4,000 α = -$50 1,000 Runs	
Mean Population Change:	-3%	Mean Population Change:	1%
Standard Deviation	24%	Standard Deviation	11%
25th Percentile:	-20%	25th Percentile:	0%

50th Percentile: -9% 75th Percentile: 12%	50th Percentile: 0% 75th Percentile: 3%
Category #4 (40 cities – e,g., Wichita (KS), Miami (FL), Mobile (AL), New York City, Little Rock, Shreveport (LA) – with modest, negative local trends) Mean Population Change: 61% Standard Deviation 49% 25th Percentile: 29% 50th Percentile: 54% 75th Percentile: 86%	$\theta(0)$ = \$4,000 α = -\$25 1,000 Runs Mean Population Change: 8% Standard Deviation 11% 25th Percentile: 0% 50th Percentile: 3% 75th Percentile: 11%

Table 1 (cont'd).

Category #5 (25 cities – e.g., Tuscaloosa, Boston, Columbus (OH), Abilene, Memphis – with no meaningfully positive or negative local trend) Mean Population Change: 127% Standard Deviation 85% 25th Percentile: 68% 50th Percentile: 112% 75th Percentile: 178%	$\theta(0)$ = \$4,000 α = \$0 1,000 Runs Mean Population Change: 23% Standard Deviation 16% 25th Percentile: 8% 50th Percentile: 22% 75th Percentile: 36%