Voltage-induced pitting pattern of constrained dielectric elastomer membranes

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Local instabilities have been observed when a large DC voltage is applied to a dielectric elastomer film, attached to a rigid and conducting substrate on one side and coated with a compliant electrode on the other side. Threshold voltage and wavelength of these local instabilities are analyzed in the following using the energy minimization method.

1 ENERGY MINIMIZATION

1.1 STRAIN ENERGY

Objective:

> Find out the change of strain energy associated with a small perturbation on the top surface.

Assumptions:

- Linear elasticity
- Plane strain
- The specific shape of the perturbation on the surface: either no lateral displacement on the top surface $u(x, z = h) = 0$ or no lateral stress on the top surface $\sigma_{xx}(x, z = h) = 0$

Procedure:

- Assume a $\psi(x, z) = g(z)\cos kx$ form for the Airy stress function and solve $\nabla^4\phi = 0$ to get $g(z) = C_1e^{-kz} + C_2ze^{-kz} + C_3e^{kz} + C_4ze^{kz}$
- Calculate the stresses $\sigma_{xx} = \partial^2\psi/\partial z^2$, $\sigma_{zz} = \partial^2\psi/\partial x^2$, $\sigma_{yy} = \nu(\sigma_{xx} + \sigma_{zz})$, $\sigma_{xz} = -\partial^2\psi/\partial x\partial z$, and strains $\epsilon_{ij} = \frac{1}{Y}\Big((1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\Big)$, and displacements $u = \int \epsilon_{xx}dx$ and w= $\int \epsilon_{zz}dz$.
- Use the boundary conditions, described in terms of u, w, and σ_{ij}, to solve for the constants C_i in $g(z)$.
- Integrate over one wavelength to calculate the total strain energy $\Delta U_{\mathrm{strain}} = \int_0^h \int_{-\pi/k}^{+\pi/k} \frac{1}{2}\sigma_{ij}\epsilon_{ij}dx\,dz$.

1.2 ELECTROSTATIC ENERGY

Objective:

Find out the change of total electrostatic energy associated with a small perturbation on the top surface.

Assumptions:

The electric field is along the z direction.

Procedure:

- The total electrostatic energy includes the electrostatic energy stored in the capacitor (the elastomer membrane can be viewed as a soft capacitor) and the electrostatic energy of the external battery. The change of electrostatic energy of the capacitor and battery comes from the change of the total charge Q due to perturbation of the surface (voltage ϕ is constant):

$$\Delta U_{\text{battery}} = -\phi \Delta Q, \qquad U_{\text{capacitor}} = \frac{1}{2}\phi Q \xrightarrow{\phi=\text{const.}} \Delta U_{\text{capacitor}} = \frac{1}{2}\phi \Delta Q$$

- The change of total charge is

$$Q = \int_{-\pi/k}^{+\pi/k} \rho dx = \int_{-\pi/k}^{+\pi/k} [\![D]\!] dx = \int_{-\pi/k}^{+\pi/k} \epsilon E_n dx \cong \int_{-\pi/k}^{+\pi/k} \epsilon \frac{\phi}{z} dx$$
$$\rightarrow \Delta \mathrm{Q} = \epsilon\phi\, \Delta\left(\int_{-\pi/k}^{+\pi/k} \frac{dx}{z(x)}\right)$$

- The total change in electrostatic energy is the sum of these two

$$\Delta U_{\text{electric}} = \Delta U_{\text{capacitor}} + \Delta U_{\text{battery}} = -\frac{1}{2}\epsilon\phi^2 \Delta\left(\int_{-\pi/k}^{+\pi/k} \frac{dx}{z(x)}\right)$$

Equivalently this is $\Delta U_{\text{electric}} = -\frac{1}{2}(C_2 - C_1)\phi^2$ where C_2 and C_1 are the capacitance of the dielectric elastomer membrane after and before perturbing the surface, respectively.

1.3 ENERGY MINIMIZATION

The total change in energy due to the infinitesimal perturbation $w_0 \cos kx$ is a function of voltage ϕ, permittivity ϵ, Young's modulus Y, Poisson's ratio ν, elastomer thickness h, pattern wavelength $\lambda = 2\pi/k$, and perturbation amplitude w_0:

$$\Delta U_{\text{total}} = \Delta U_{\text{electric}} + \Delta U_{\text{strain}} = -\frac{1}{2}\epsilon\phi^2 \Delta\left(\int_{-\pi/k}^{+\pi/k} \frac{dx}{z(x)}\right) + \int_0^h \int_{-\pi/k}^{+\pi/k} \frac{1}{2}\sigma_{ij}\epsilon_{ij} dx\, dz$$

Taylor expansion of the total energy has the form of

$$\Delta U_{\text{total}} = f(\phi, \epsilon, Y, \nu, h, \lambda) w_0^2 + O(w_0^4)$$

Which means that the change in energy at $w_0 = 0$ is zero as expected, the first derivative $\frac{\partial}{\partial w_0}\Delta U_{\text{total}}$ is zero indicating $w_0 = 0$ is an equilibrium point, and the second derivative $\frac{\partial^2}{\partial w_0^2}\Delta U_{\text{total}}$ can be either positive or negative, showing stable equilibrium or unstable equilibrium, respectively. Therefore, in the following analysis we will use $\frac{\partial^2}{\partial w_0^2}\Delta U_{\text{total}} = 0$ as the transition from stable equilibrium to instability.

2 RESULTS

2.1 INCOMPRESSIBLE MATERIAL ($\nu = 0.5$)

Assuming an incompressible material, i.e. $\nu = 0.5$, the homogeneous strains are all zero in the base state, i.e. the state prior to instability threshold. The strain energy due to a perturbation on the top surface $w(x,y,h) = w_0 \cos kx$ and $u(x,y,h) = 0$ (and $v(x,y,z) = 0$, i.e. plane strain case) is

$$\Delta U_{\text{strain}} = \frac{\pi\left(e^{4hk} + 4hk\, e^{2hk} - 1\right)}{3(e^{4hk} - 2(2h^2k^2 + 1)e^{hk} + 1)} Y w_0^2$$

Where Y is the Young's modulus of the elastomer. The total change in electrostatic energy is

$$\Delta U_{\text{electric}} = -\frac{1}{2}\epsilon\phi^2 \Delta\left(\int_{-\pi/k}^{+\pi/k} \frac{dx}{z(x)}\right) = -\frac{1}{2}\epsilon\phi^2\left(\int_{-\pi/k}^{+\pi/k} \frac{dx}{h + w_0\cos kx} - \int_{-\pi/k}^{+\pi/k}\frac{dx}{h}\right) =$$
$$= -\frac{1}{2}\epsilon\phi^2\left(\frac{2\pi}{k\sqrt{h^2 - w_0^2}} - \frac{2\pi}{kh}\right) = -\frac{\pi}{k}\epsilon\phi^2\left(\frac{1}{\sqrt{h^2 - w_0^2}} - \frac{1}{h}\right)$$

Therefore, the total energy, the first derivative and the second derivative are

$$\Delta U_{\text{total}} = \Delta U_{\text{electric}} + \Delta U_{\text{strain}} = \frac{\pi\left(e^{4hk} + 4hk\, e^{2hk} - 1\right)}{3(e^{4hk} - 2(2h^2k^2 + 1)e^{hk} + 1)} Y w_0^2 - \frac{\pi}{k}\epsilon\phi^2\left(\frac{1}{\sqrt{h^2 - w_0^2}} - \frac{1}{h}\right)$$
$$\rightarrow \Delta U_{\text{total}}\big|_{w_0=0} = 0$$
$$\frac{\partial}{\partial w_0}\Delta U_{\text{total}} = \frac{2\pi\left(e^{4hk} + 4hk\, e^{2hk} - 1\right)}{3(e^{4hk} - 2(2h^2k^2 + 1)e^{hk} + 1)} Y w_0 - \frac{\pi}{k}\epsilon\phi^2 \frac{w_0}{(h^2 - w_0^2)^{3/2}}$$
$$\rightarrow \frac{\partial}{\partial w_0}\Delta U_{\text{total}}\bigg|_{w_0=0} = 0$$
$$\frac{\partial^2}{\partial w_0^2}\Delta U_{\text{total}} = \frac{2\pi\left(e^{4hk} + 4hk\, e^{2hk} - 1\right)}{3(e^{4hk} - 2(2h^2k^2 + 1)e^{hk} + 1)} Y - \frac{\pi}{k}\epsilon\phi^2 \frac{h^2 + 2w_0^2}{(h^2 - w_0^2)^{5/2}}$$
$$\rightarrow \frac{\partial^2}{\partial w_0^2}\Delta U_{\text{total}}\bigg|_{w_0=0} = \frac{2\pi\left(e^{4hk} + 4hk\, e^{2hk} - 1\right)}{3(e^{4hk} - 2(2h^2k^2 + 1)e^{hk} + 1)} Y - \frac{\pi}{k}\epsilon\phi^2 \frac{1}{h^3}$$

Therefore, the transition from stable equilibrium to instability occurs when

$$\frac{\partial^2}{\partial w_0^2}\Delta U_{\text{total}}\bigg|_{w_0=0} = 0 \rightarrow \frac{2\pi\left(e^{4hk} + 4hk\, e^{2hk} - 1\right)}{3(e^{4hk} - 2(2h^2k^2 + 1)e^{hk} + 1)} Y - \frac{\pi}{k}\epsilon\phi^2\frac{1}{h^3} = 0$$
$$\rightarrow \boxed{\sqrt{\frac{\epsilon}{Y}}\frac{\phi}{h} = \left(\frac{2}{3}hk\frac{e^{4hk} + 4hke^{2hk} - 1}{e^{4hk} - 2(2h^2k^2 + 1)e^{2hk} + 1}\right)^{1/2}}$$

Figure 2.1.1 shows the plot of this transition line for $\sqrt{\frac{\epsilon}{Y}}\frac{\phi}{h}$ versus $\lambda/h = 2\pi/hk$.

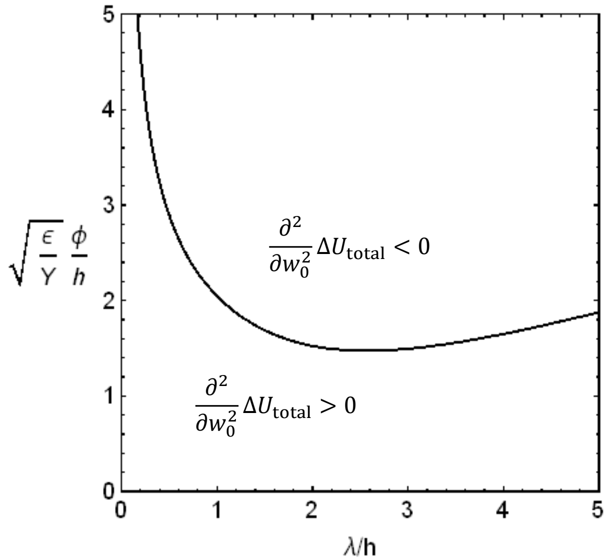

Figure 2.1.1. plot of the transition line from stable equilibrium to instability for an incompressible elastomer whose surface is perturbed by $w(x, y, h) = w_0 \cos kx$ and $u(x, y, h) = 0$ (and $v(x, y, z) = 0$, i.e. plane strain case)

For small voltage $\sqrt{\frac{\epsilon}{Y}}\frac{\phi}{h}$, the second derivative is positive, $\frac{\partial^2}{\partial w_0^2}\Delta U_{\text{total}}\Big|_{w_0=0} > 0$, showing that the base state is a stable equilibrium. As we increase the voltage to $\sqrt{\frac{\epsilon}{Y}}\frac{\phi}{h} = 1.47$, the second derivative becomes negative for $\lambda/h = 2.57$. Therefore, the critical voltage at which the system goes unstable and the corresponding pattern wavelength are

$$\boxed{\sqrt{\frac{\epsilon}{Y}}\frac{\phi_c}{h} = 1.47, \qquad \frac{\lambda_c}{h} = 2.57}$$

Further increasing the voltage leads to negative second derivative for a wider range of λ/h, as shown in figure 2.1.1.

2.2 THREE-DIMENSIONAL ANALYSIS

It is straightforward to extend the plane strain analysis of section 1.1 to three-dimensional case. The linear elastic stress and strain fields for the 3D case is essentially superimposition of the stress and strain fields of two plane strain problems with perpendicular zero strain directions, i.e.

$$\sigma_{ij} = \sigma_{ij}^x + \sigma_{ij}^y, \qquad \epsilon_{ij} = \epsilon_{ij}^x + \epsilon_{ij}^y,$$

Where σ_{ij}^x and σ_{ij}^y are the plane strain stress fields for the cases where $\epsilon_{xx} = 0$ and $\epsilon_{yy} = 0$ (the case discussed in section 1.1), respectively. Similarly, for the strain fields ϵ_{ij}^x and ϵ_{ij}^y

are the strains for the plane strain cases where $\epsilon_{xx} = 0$ and $\epsilon_{yy} = 0$, respectively. The governing equations are checked to make sure that the stress, strain, and displacement fields obtained from the superimposition of the corresponding plane strain fields satisfy the governing equations. For the plane strain surface perturbations of $w^x(x,y,h) = w_0 \cos k_y y$ and $w^y(x,y,h) = w_0 \cos k_x x$, the surface perturbation in 3D case is $w(x,y,h) = w^x + w^y = w_0(\cos k_y y + \cos k_x x)$, represented in figure 2.2.1.

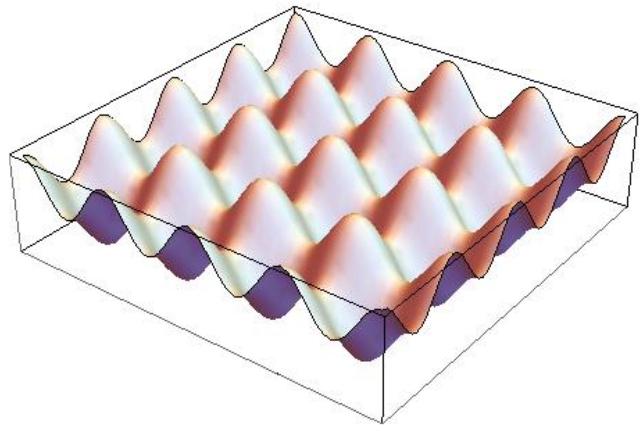

Figure 2.2.1. Schematic representation of the 3D surface perturbation $w(x,y,h) = w_0(\cos k_x x + \cos k_y y)$ for $k_x = k_y = k$.

The strain energy for the 3D case when $k_x = k_y = k$ is

$$\Delta U_{\text{strain}} = \int_0^h \int_{-\pi/k}^{+\pi/k} \int_{-\pi/k}^{+\pi/k} \frac{1}{2}\sigma_{ij}\epsilon_{ij} dx\, dy\, dz = \frac{4\pi^2(e^{4hk} + 4hk\, e^{2hk} - 1)}{3k(e^{4hk} - (4h^2k^2 + 2)e^{hk} + 1)} Y w_0^2$$

The total electrostatic energy can be estimated using Taylor expansion when calculating the integrals:

$$\begin{aligned}
\Delta U_{\text{electric}} &= -\frac{1}{2}\epsilon\phi^2 \Delta\left(\int_{-\pi/k}^{+\pi/k}\int_{-\pi/k}^{+\pi/k} \frac{1}{z(x)} dx\, dy\right) \\
&= -\frac{1}{2}\epsilon\phi^2\left(\int_{-\pi/k}^{+\pi/k}\int_{-\pi/k}^{+\pi/k} \frac{1}{h + w_0(\cos kx + \cos ky)} dx\, dy - \int_{-\pi/k}^{+\pi/k}\int_{-\pi/k}^{+\pi/k} \frac{1}{h} dx\, dy\right) \\
&= -\frac{1}{2}\epsilon\phi^2\left(\frac{4\pi^2(h^2 + w_0^2)}{h^3k^2} + O(w_0^4) - \frac{4\pi^2}{hk^2}\right) \cong -\frac{2\pi^2\epsilon\phi^2}{h^3k^2} w_0^2
\end{aligned}$$

Therefore, the total energy, the first derivative and the second derivative are

$$\Delta U_{\text{total}} = \Delta U_{\text{electric}} + \Delta U_{\text{strain}} = \frac{4\pi^2(e^{4hk} + 4hk\, e^{2hk} - 1)}{3k(e^{4hk} - (4h^2k^2 + 2)e^{hk} + 1)} Y w_0^2 - \frac{2\pi^2\epsilon\phi^2}{h^3k^2} w_0^2 + O(w_0^4)$$

$$\rightarrow \Delta U_{\text{total}}\big|_{w_0=0} = 0$$

$$\frac{\partial}{\partial w_0}\Delta U_{\text{total}} = \frac{8\pi^2(e^{4hk} + 4hk\, e^{2hk} - 1)}{3k(e^{4hk} - (4h^2k^2 + 2)e^{hk} + 1)} Y w_0 - \frac{4\pi^2\epsilon\phi^2}{h^3k^2} w_0 + O(w_0^3)$$

$$\rightarrow \frac{\partial}{\partial w_0}\Delta U_{\text{total}}\bigg|_{w_0=0} = 0$$

$$\frac{\partial^2}{\partial w_0^2}\Delta U_{\text{total}} = \frac{8\pi^2\left(e^{4hk} + 4hk\, e^{2hk} - 1\right)}{3k(e^{4hk} - (4h^2k^2 + 2)e^{hk} + 1)}Y - \frac{4\pi^2\epsilon\phi^2}{h^3k^2} + O(w_0^2)$$

$$\rightarrow \frac{\partial^2}{\partial w_0^2}\Delta U_{\text{total}}\bigg|_{w_0=0} = \frac{8\pi^2\left(e^{4hk} + 4hk\, e^{2hk} - 1\right)}{3k(e^{4hk} - (4h^2k^2 + 2)e^{hk} + 1)}Y - \frac{4\pi^2\epsilon\phi^2}{h^3k^2}$$

Therefore, the transition from stable equilibrium to instability occurs when

$$\frac{\partial^2}{\partial w_0^2}\Delta U_{\text{total}}\bigg|_{w_0=0} = 0 \rightarrow \frac{8\pi^2\left(e^{4hk} + 4hk\, e^{2hk} - 1\right)}{3k(e^{4hk} - (4h^2k^2 + 2)e^{hk} + 1)}Y - \frac{4\pi^2\epsilon\phi^2}{h^3k^2} = 0$$

$$\rightarrow \boxed{\sqrt{\frac{\epsilon}{Y}}\frac{\phi}{h} = \left(\frac{2}{3}hk\frac{e^{4hk} + 4hke^{2hk} - 1}{e^{4hk} - 2(2h^2k^2 + 1)e^{2hk} + 1}\right)^{1/2}}$$

Which is the same as the critical voltage expression obtained for plane strain case. For the case where k_x is different from k_y, we get the critical voltage versus critical wavelength plot represented in figure 2.2.2, which shows that the minimum critical voltage occurs when $k_x = k_y = k$.

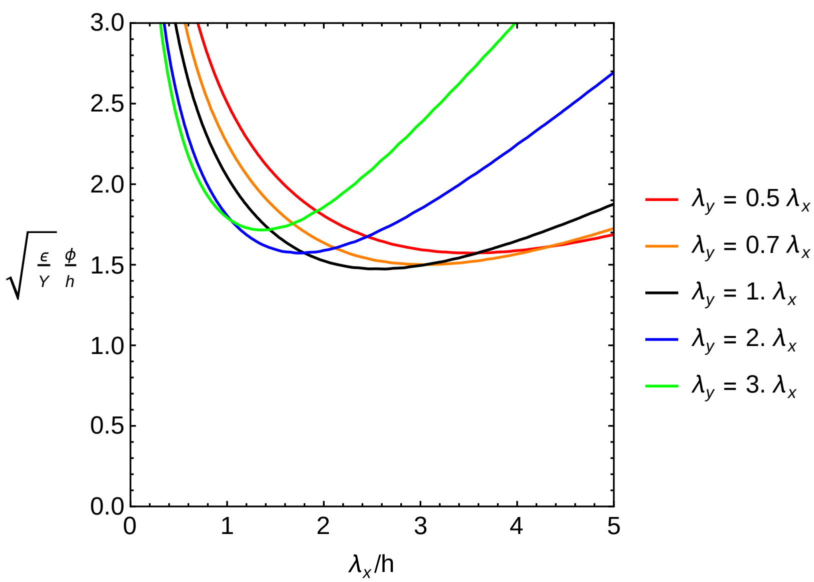

Figure 2.2.2. Plot of critical voltage versus critical pattern wavelength for the three-dimensional case where the surface is perturbed by $w(x, y, h) = w_0\left(\cos k_x x + \cos k_y y\right)$.

2.3 DOUBLE PITTING

So far, we considered a sinusoidal perturbation of the surface of the form $w(x, y, h) = w_0 \cos kx$ for the plane strain case or $w(x, y, h) = w_0\left(\cos k_x x + \cos k_y y\right)$ for the three-dimensional case. In this section, we analyze a double pitting pattern where the surface perturbation has the form of $w(x, y, h) = -w_0\left((1 - w_r)\cos kx + w_r \cos 2kx\right)$ and we will assume plane strain deformation. Following the same procedure as in section 1.1. The critical voltage versus critical wavelength is calculated and plotted in figure 2.3.1. The minimum critical voltage occurs when there is no secondary pitting, i.e. $w_r = 0$ and

$w(x, y, h) = -w_0 \cos kx$, which corresponds to the sinusoidal surface perturbation of section 2.1.

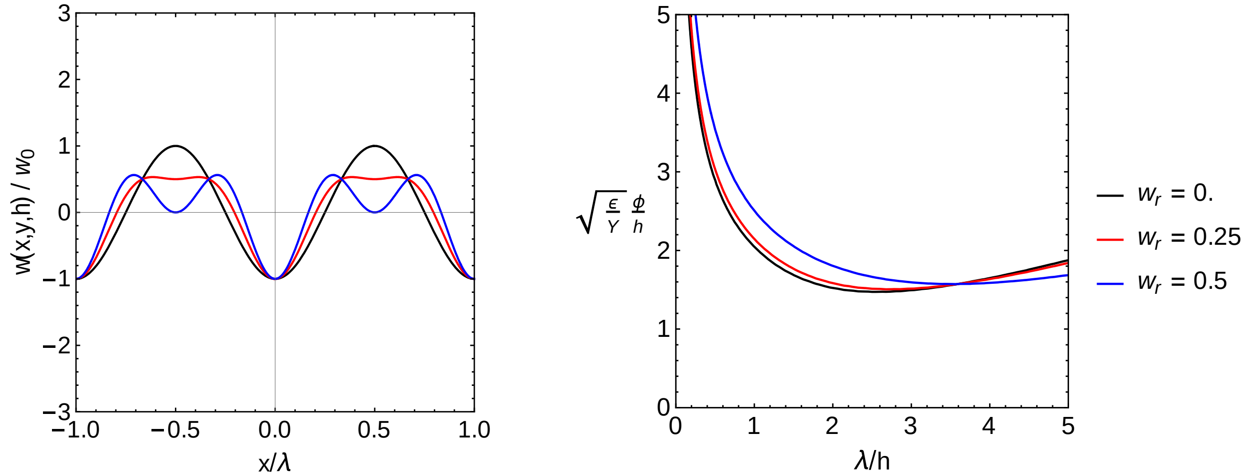

Figure 2.3.1. Plot of double pitting surface perturbation for different values of w_r (left) and critical voltage versus critical pattern wavelength (right).

2.4 Double layer

In the experimental setups, to prevent premature electrical breakdown we usually use two layers of elastomers: a stiff bottom elastomer of thickness h_1 which is attached to the rigid electrode, and a softer top elastomer of thickness h_2 which is attached to the stiff elastomer on one side and coated by a complaint electrode on the other side. For this case we need to solve the plane strain problem inside the stiff and the soft elastomer and evaluate the constants using both the boundary conditions and the following interface equations:

$$
\begin{aligned}
w^{\text{stiff}}(x, y, h_1) &= w^{\text{soft}}(x, y, h_1) \\
u^{\text{stiff}}(x, y, h_1) &= u^{\text{soft}}(x, y, h_1) \\
\sigma_{zz}^{\text{stiff}}(x, y, h_1) &= \sigma_{zz}^{\text{soft}}(x, y, h_1) \\
\epsilon_{xz}^{\text{stiff}}(x, y, h_1) &= \epsilon_{xz}^{\text{soft}}(x, y, h_1)
\end{aligned}
$$

Results are plotted in figure 2.4.1.

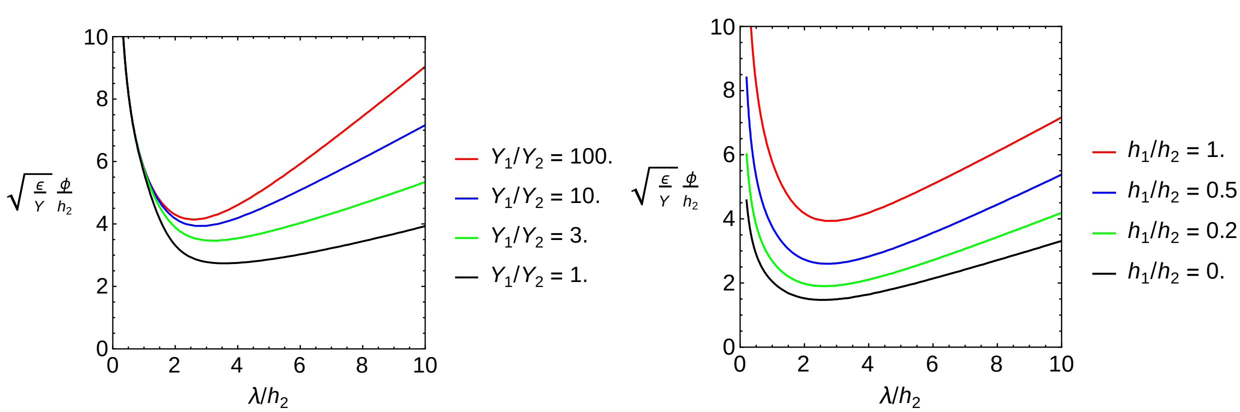

Figure 2.4.1. Critical voltage for double layer system when $h_1 = h_2 = 1$, for different stiffness ratio of the two layers (left figure), and for $Y_1/Y_2 = 10$ and different h_1/h_2.

2.5 Compressible material

For a compressible material, the initial state prior to instability is not strain free anymore, but has a displacement field of

$$u = 0, \qquad w(z) = -\frac{(1+\nu)(1-2\nu)}{1-\nu}\sqrt{\frac{\epsilon}{Y}}\frac{\phi}{h^2}z$$

Which corresponds to

$$\sigma_{zz} = -\frac{\epsilon\phi^2}{2h^2}, \qquad \sigma_{xx} = \sigma_{yy} = -\frac{\nu}{1-\nu}\frac{\epsilon\phi^2}{2h^2}$$

To include this in our analysis using Airy stress function, we should rewrite the Air stress function in the following form:

$$\psi(x,z) = g(z)\cos kx - \frac{\epsilon\phi^2}{2h^2}x^2 - \frac{\nu}{1-\nu}\frac{\epsilon\phi^2}{2h^2}z^2$$

Following the procedure in section 1 with this new Airy stress function, we get a critical voltage that decreases with decreasing Poisson's ratio, and a critical wavelength that increases with Poisson's ratio.

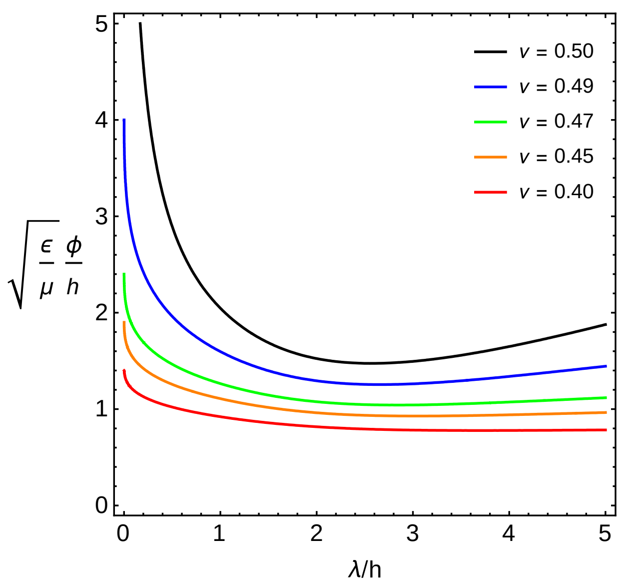

Figure 2.5.1. Critical voltage and wavelength for different Poisson ratios.