Contextually Private Mechanisms

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Motivation

In many settings, it is important that the mechanism designer does not learn "too much" about participants.

- Participants have privacy concerns?
- Designer has political worries or legal constraints?

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Google's Alleged Scheme to Corner the Online Ad Market

A newly unredacted legal filing sheds light on internal programs that antitrust enforcers argue advantaged Google at the expense of advertisers and publishers.

Google induced advertisers to bid their true value, only to override pre-set AdX floors and … generate unique and custom per-buyer floors **depending on what a buyer had bid in the past.**

State of Texas v. Google

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> "In one extreme case, a firm that bid NZ$100,000 paid the second-highest bid of NZ$6… **Politically embarrassing** newspaper headlines resulted, as winners paid prices far below their bids… By revealing the high bidder's willingness to pay, the auction **exposed the government to criticism**, because after the auction everyone knew that the firm valued the license at more than it paid."
>
> **McMillan (1994)**

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- **"purpose limitation"**

  Personal data shall be collected for specified, explicit and legitimate purposes and **not further processed in a manner that is incompatible with those purposes**;

- **"data minimisation"**

  Personal data shall be adequate, relevant and **limited to what is necessary in relation to the purposes** for which they are processed;

EU GDPR Article 5(1)

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How much is "too much"?

In a **contextually private** mechanism, information revelation is justified by the choice rule.

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In a **contextually private** mechanism, information revelation is justified by the choice rule.

Setting.

- Standard mechanism design environments, with and without transfers.
- Dynamic protocols for eliciting agents' reports.

Outline

1. **Definitions**

   Protocols, contextual privacy

2. **Fully contextually private choice rules**

   A necessary condition
   SPA is not contextually private

3. **Maximally contextually private protocols**

   Representation theorem: bi-monotonic protocols
   Maximally contextually private choice rules for SPA

4. **Brief discussion of other results**

   Settings without transfers, characterization for general protocols, incentives, variants.

Set Up

- $N < \infty$ agents
- Private information $\theta_i \in \Theta_i$, $|\Theta_i| < \infty$, profiles $\boldsymbol{\theta} \in \boldsymbol{\Theta}$
- Outcomes $x \in X$
- Preferences over outcomes $u_i : X \times \boldsymbol{\Theta} \to \mathbb{R}$
- Choice rule $\phi : \boldsymbol{\Theta} \to X$, deterministic

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The designer learns about the type space through a protocol.

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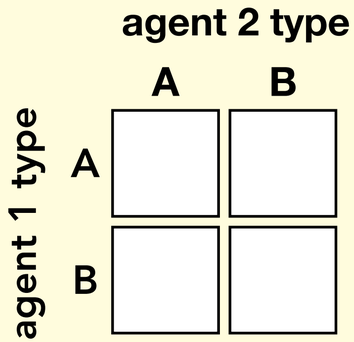

Example. A simple protocol.

- Two agents $N = \{1, 2\}$
- Binary type space $\boldsymbol{\Theta} = \{A, B\}^2$

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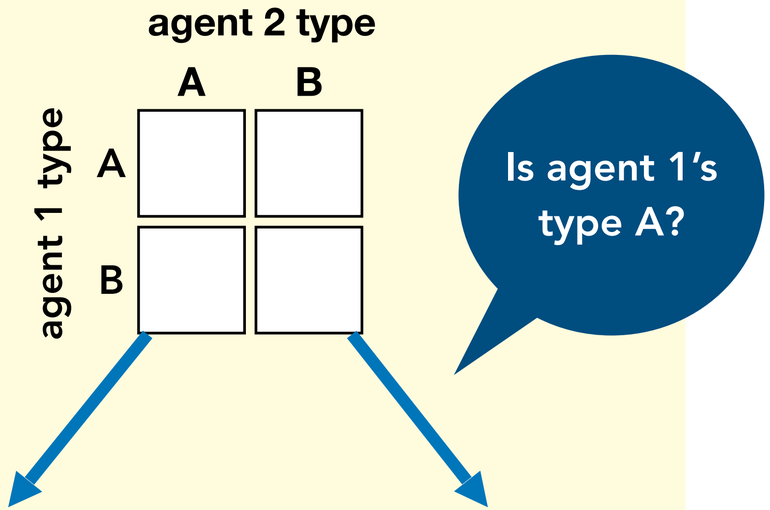

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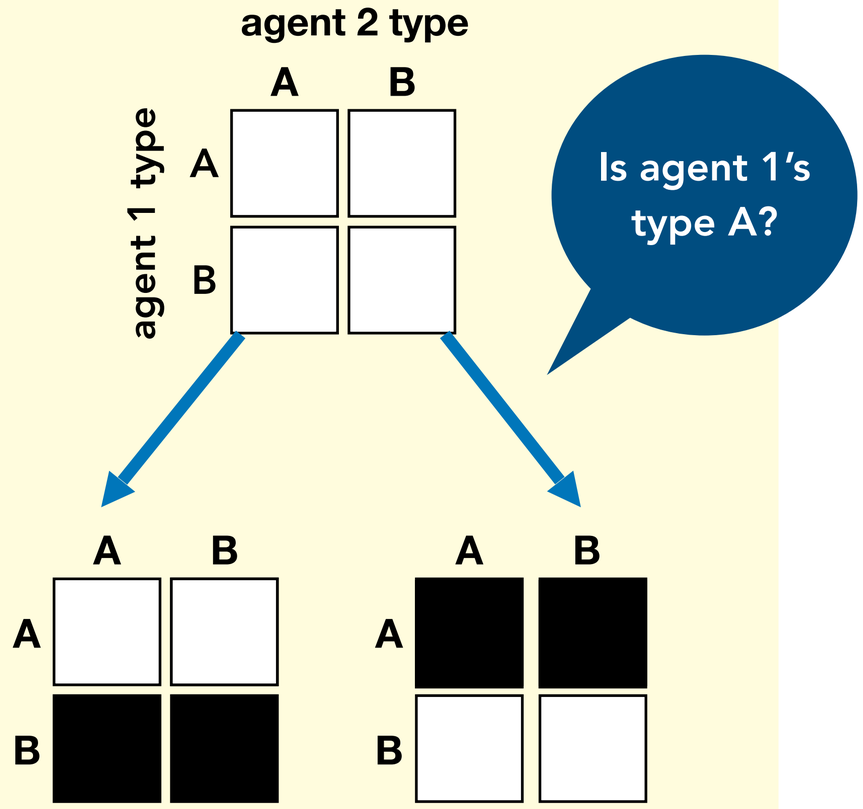

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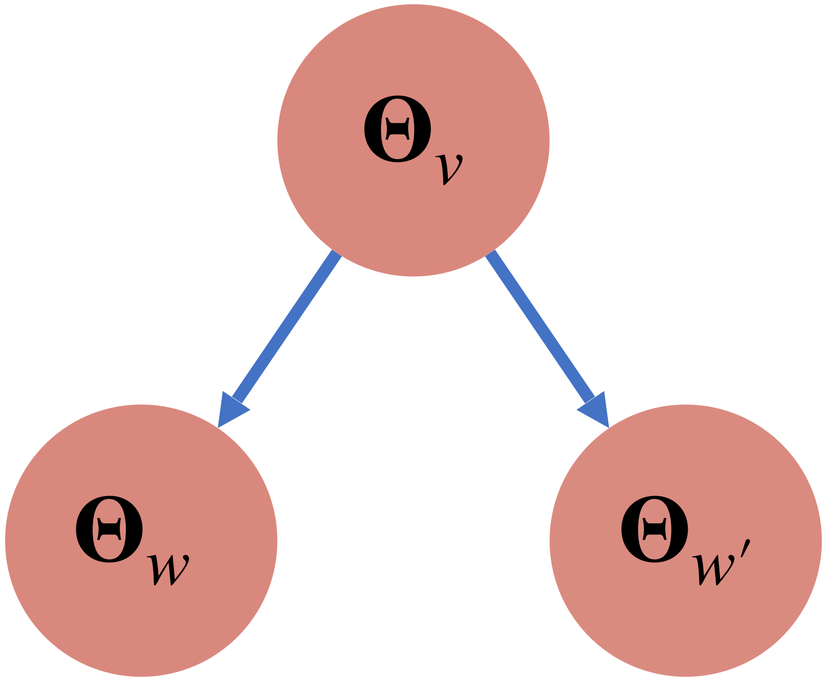

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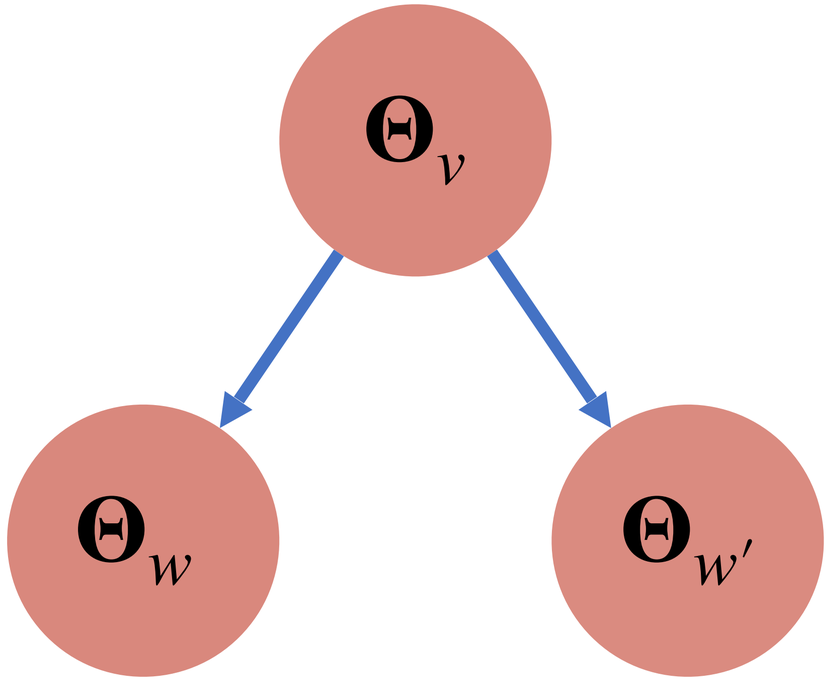
Definition.

A (direct) **protocol** is a directed rooted tree $P = (V, E, r)$ where each node v is labelled with a subset of the type space $\Theta_v \subseteq \Theta$ such that

$$\Theta_r = \Theta \quad \text{and} \quad \bigcup_{w:\,(v,w)\in E} \Theta_w = \Theta_v$$

At the root node all type profiles are possible.

The child nodes (w) form a partition of the parent node (v)



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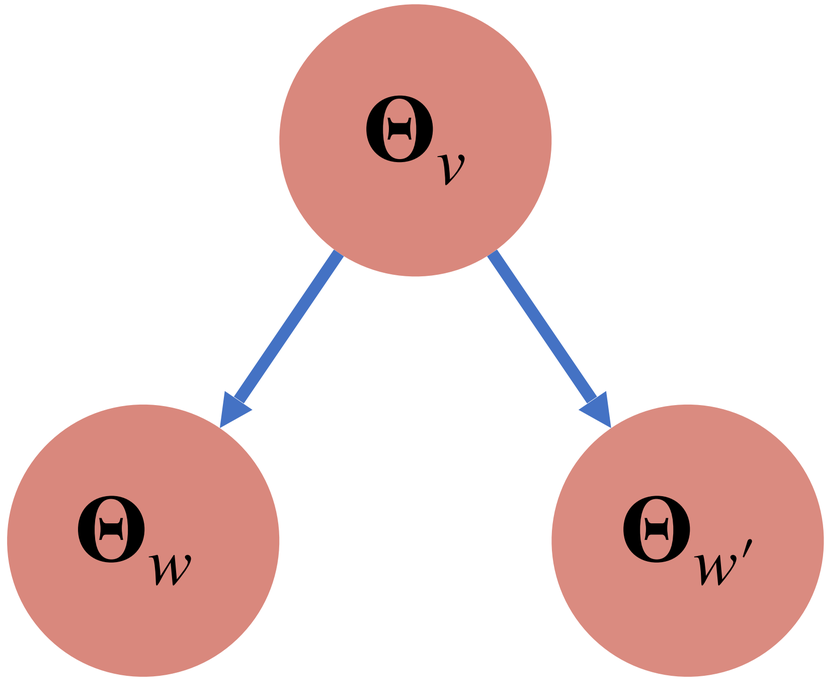

P is a **protocol for choice rule ϕ** if the terminal nodes of P yield enough information to compute the choice rule.

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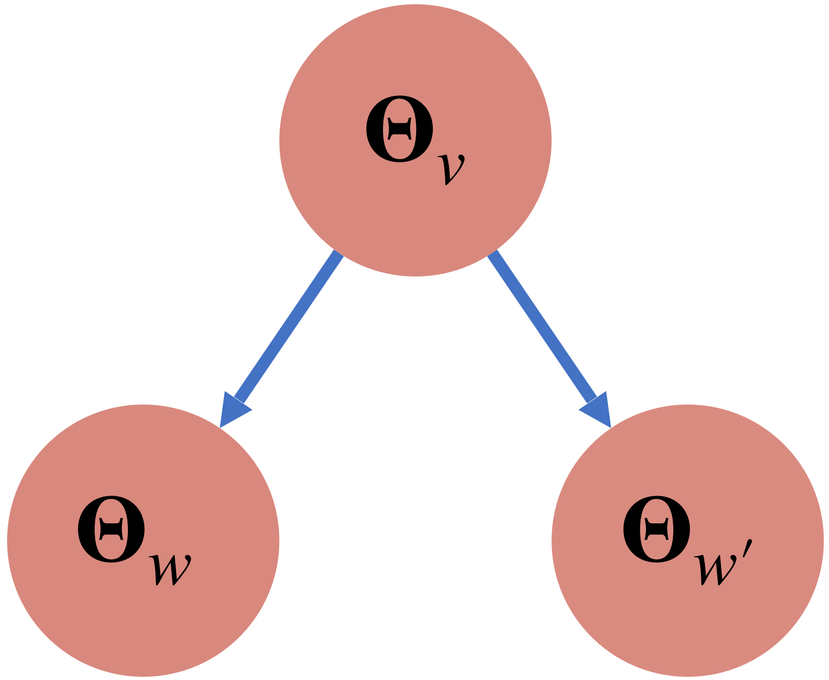

P is a **protocol for choice rule ϕ** if ϕ is measurable with respect to the partition induced by the terminal nodes of P.

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DEFINITIONS

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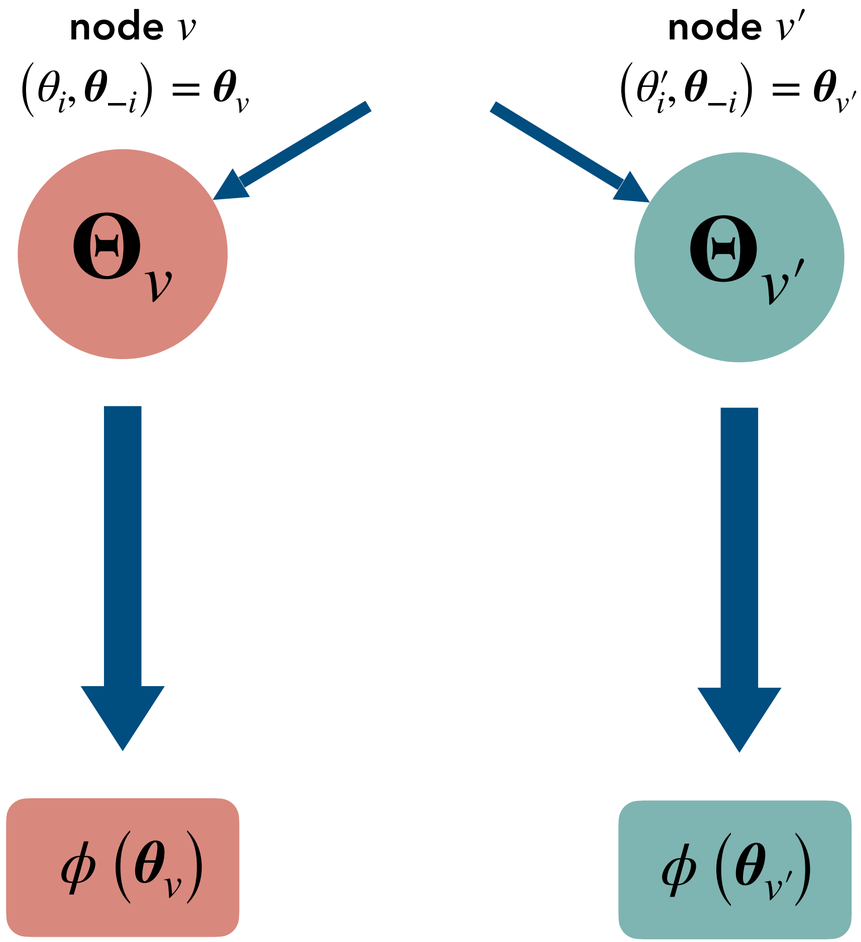

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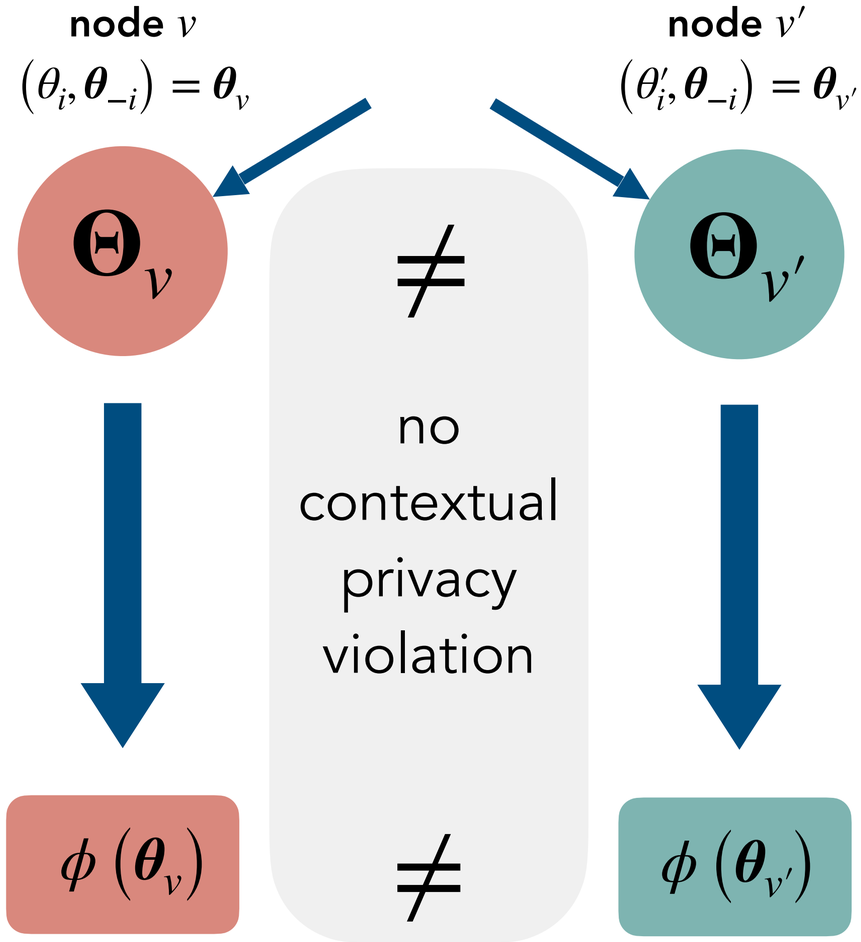

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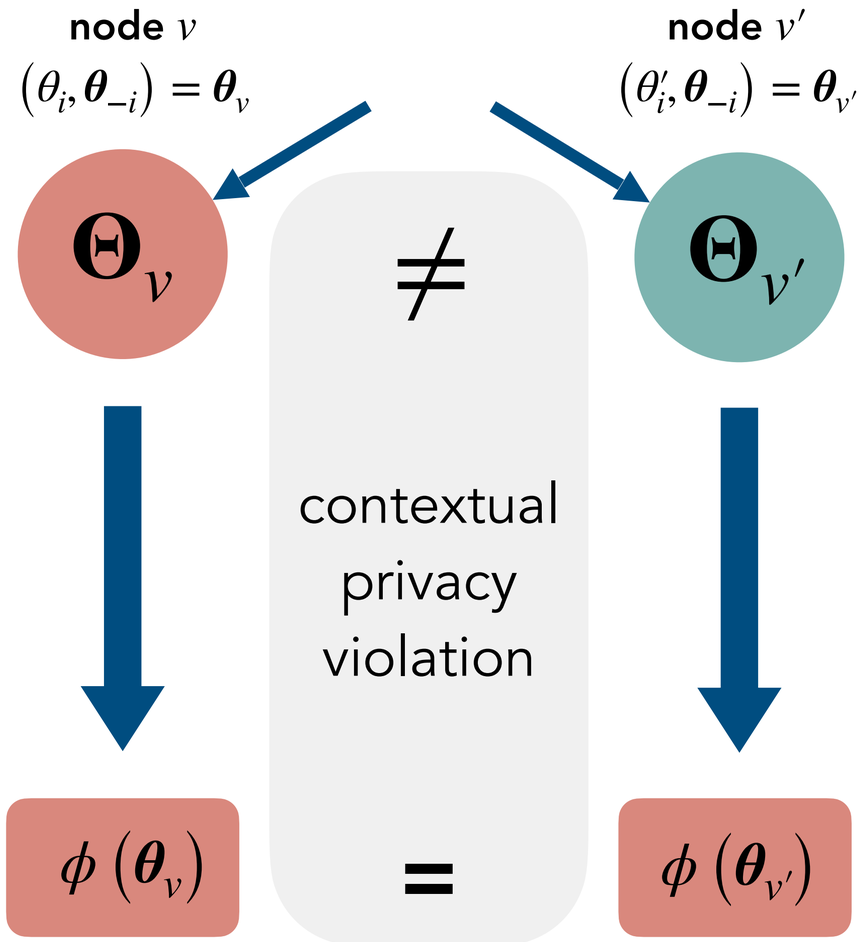

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1. **Privacy preservation.**
   Agents give up their private information for a reason.
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   Designer learns nothing more than what is contained in the outcome.

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Toward a statistical privacy-implementation frontier:

if designer distinguishes types **at all**

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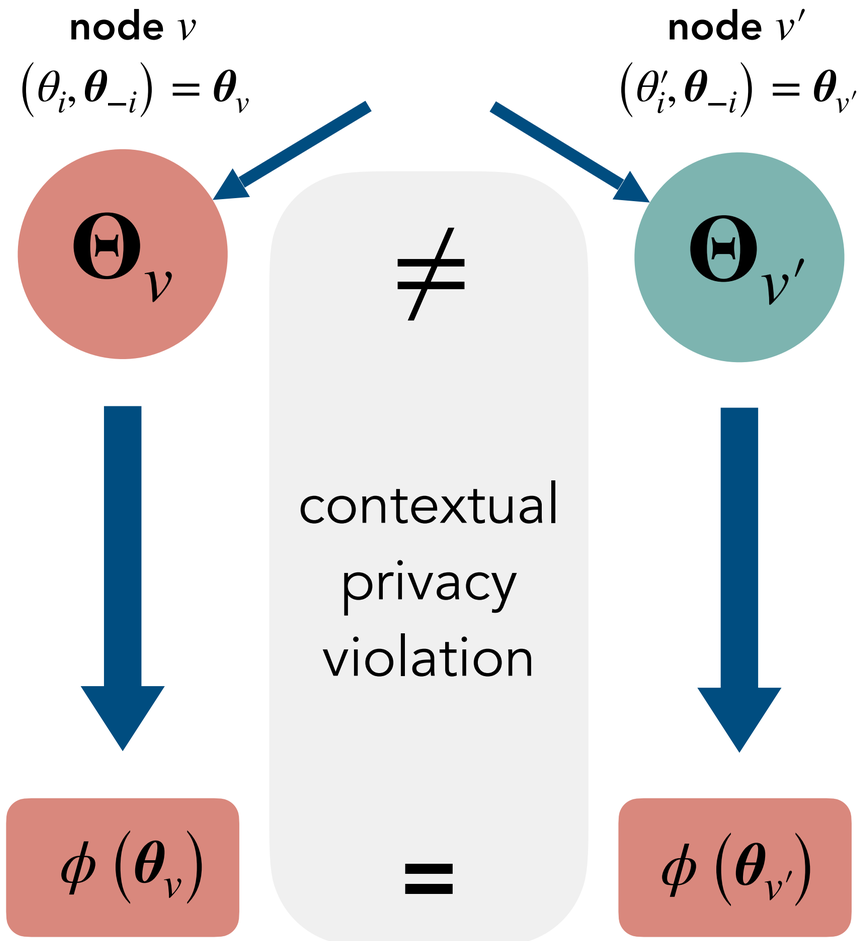
Denote the **set of contextual privacy violations** produced by P for ϕ, $\Gamma(P, \phi) \subseteq N \times \boldsymbol{\Theta}$.



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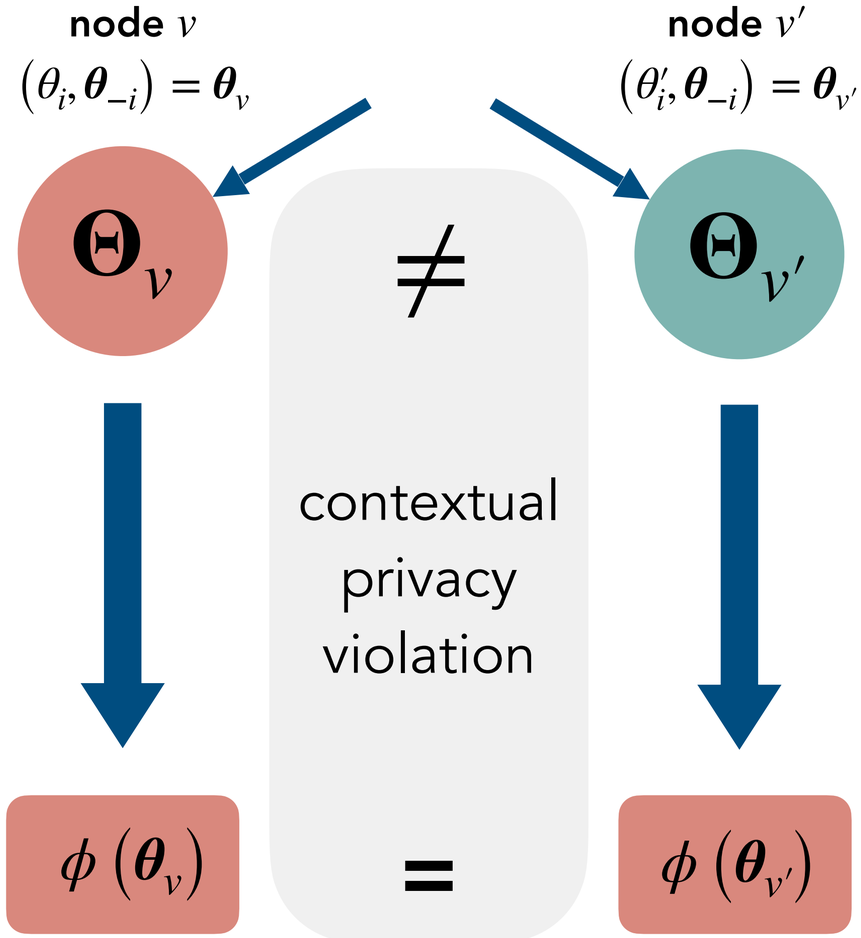

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A protocol P for choice rule ϕ is **at least as contextually private** as protocol P' if

$$\Gamma(P, \phi) \subseteq \Gamma(P', \phi).$$

A protocol P is **contextual-privacy equivalent** to P′ if

$$\Gamma(P, \phi) \subseteq \Gamma(P', \phi) \text{ and } \Gamma(P', \phi) \subseteq \Gamma(P, \phi).$$

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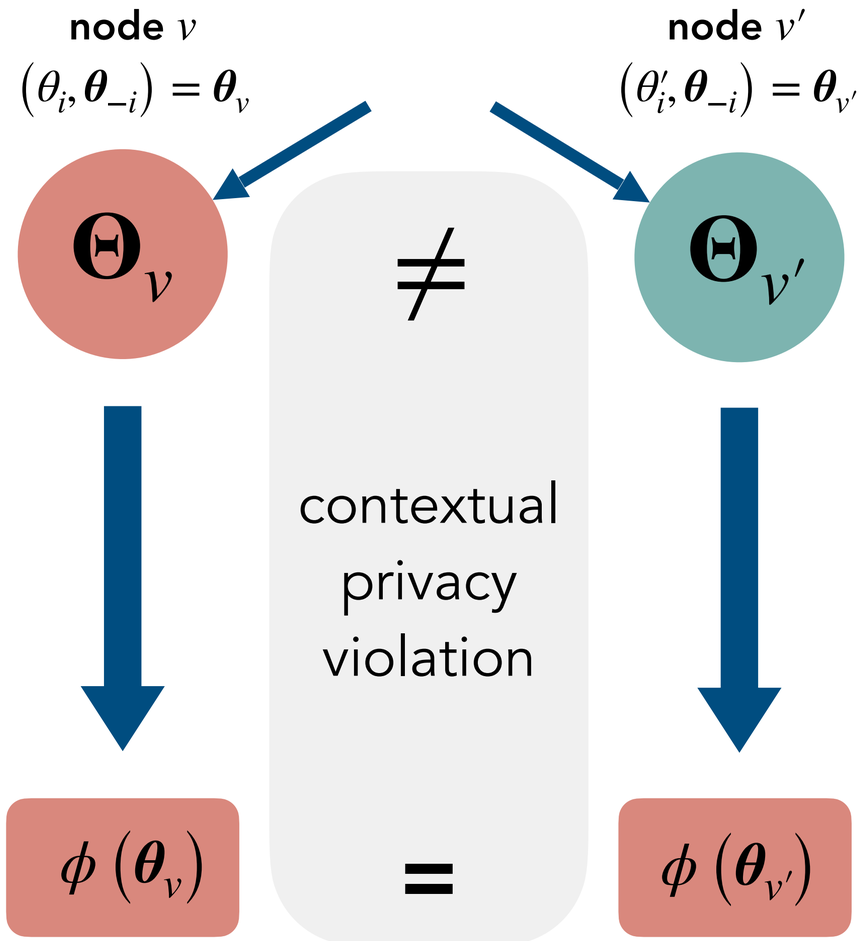

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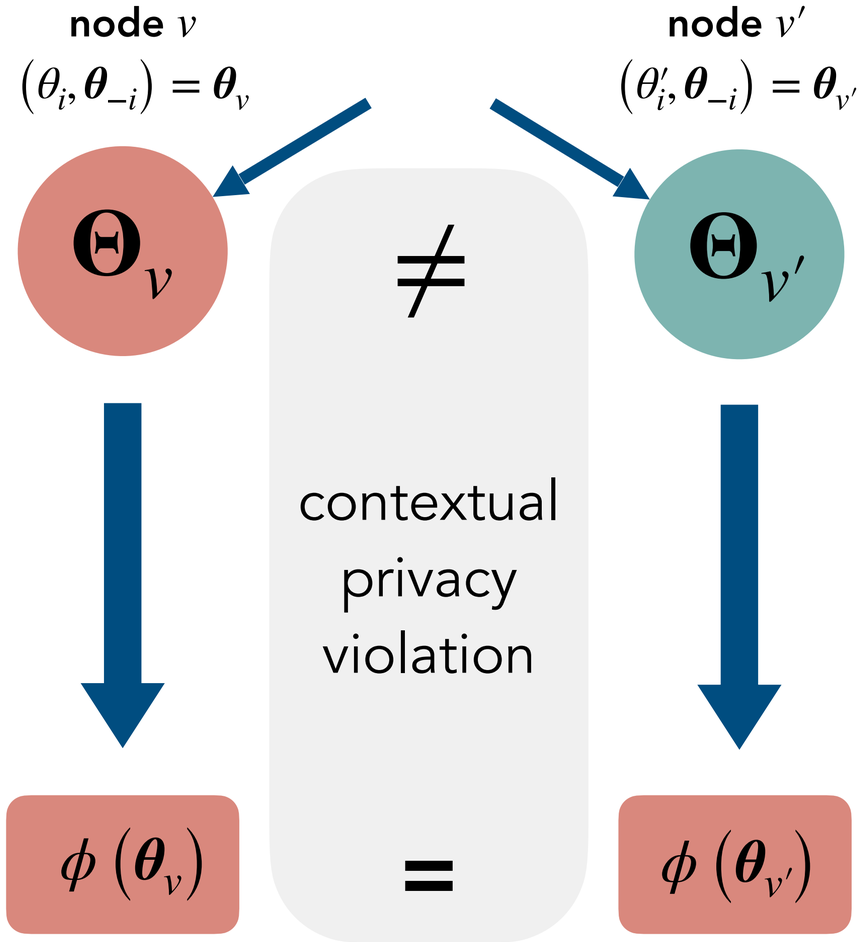

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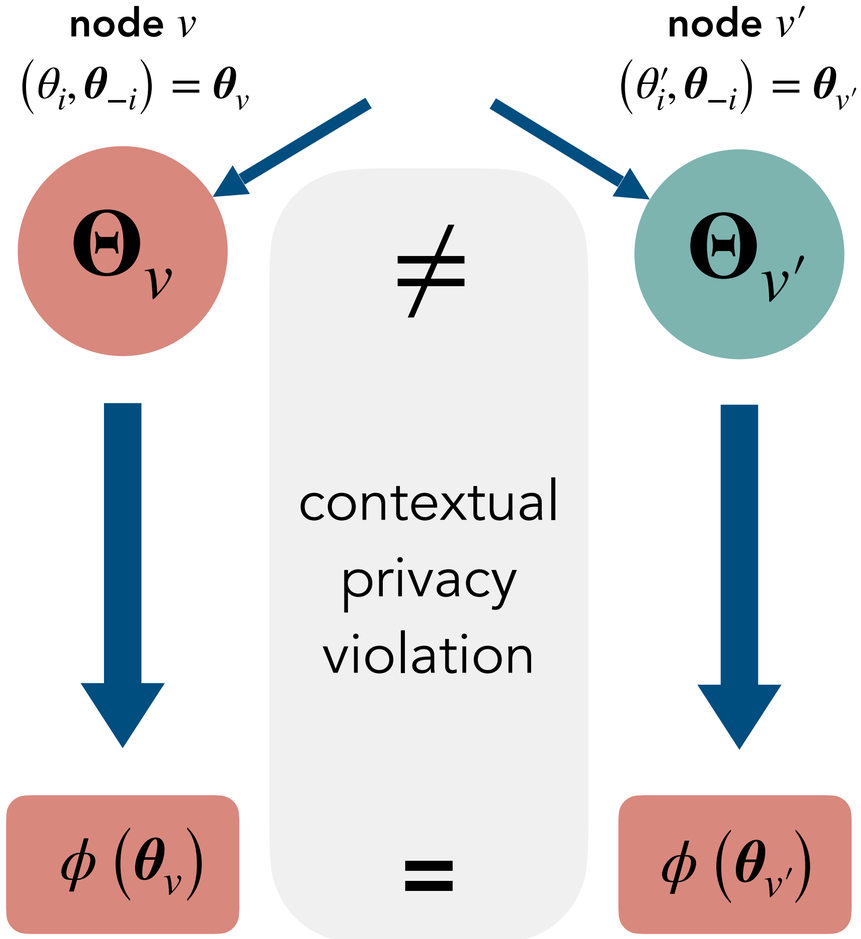

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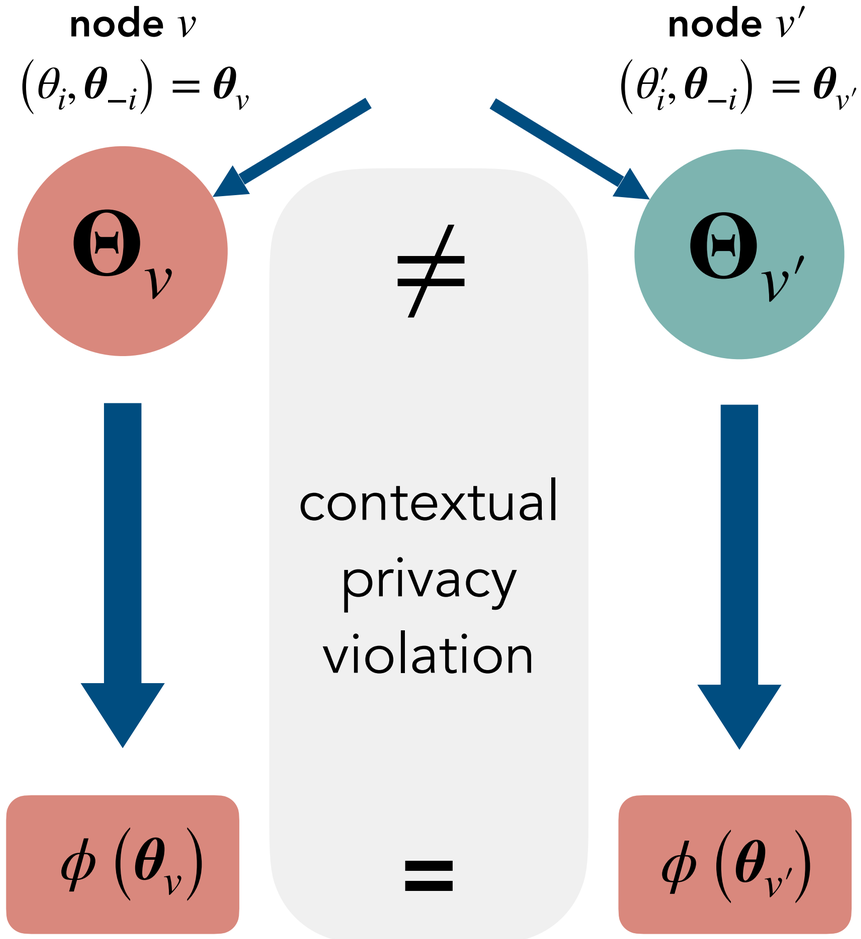

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How does the designer physically elicit agents' reports?

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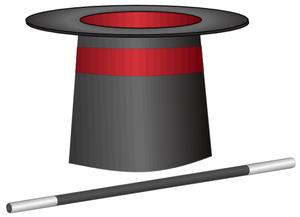

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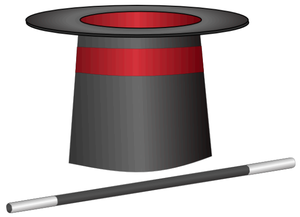

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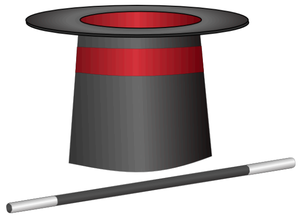

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A **choice rule ϕ is fully contextually private** if there is an **admissible** protocol P for ϕ that produces no violations.

The set of admissible protocols depends on the environment.

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2. Today: a restrictive class of admissible protocols that makes a **minimal assumption.**

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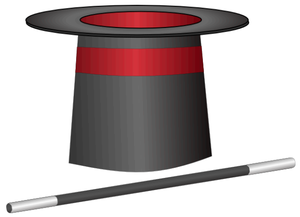

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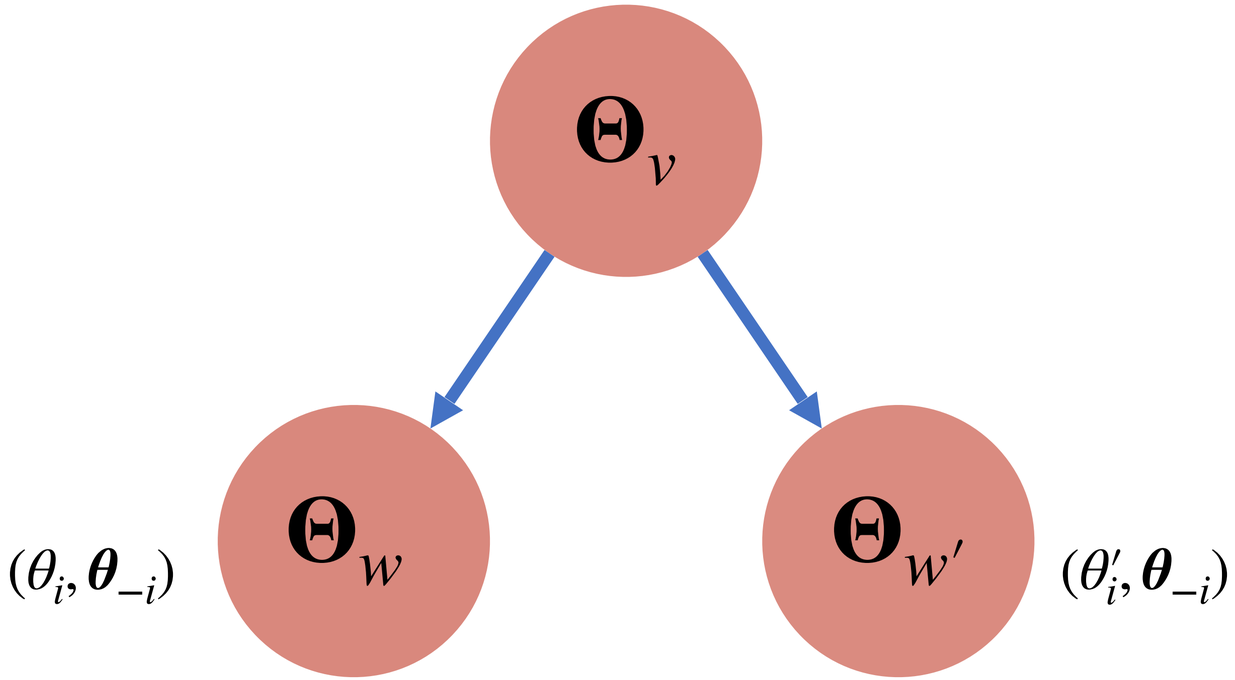
This Talk: Restriction on Trusted Protocols

Definition.

A protocol P is **a sequential elicitation protocol** if for all $(v, w), (v, w') \in E$, there is at most one agent i such that $(\theta_i, \boldsymbol{\theta}_{-i}) \in \boldsymbol{\Theta}_w$ and $(\theta_i', \boldsymbol{\theta}_{-i}) \in \boldsymbol{\Theta}_{w'}$.



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Related Literature

Unconditional privacy.

Chor and Kushilevitz (1989), Chor, Gerèb-Graus and Kushilevitz (1994), Ausubel (2004), Brandt (2006), Brandt and Sandholm (2005, 2008), Milgrom and Segal (2020).

We bring unconditional privacy into mechanism design; study settings with and without transfers; study a more general class of protocols.

Auditability, simplicity and credibility.

Li (2017), Akbarpour and Li (2020), Hakimov and Raghavan (2022), Pycia and Ünver (2022), Mackenzie and Zhou (2022), Pycia and Troyan (2023), Grigoryan and Möller (2023).

We introduce an axiom and an order based on how much superfluous information is revealed to the designer.

Communication requirements.

Kushilevitz and Nisan (1997), Nisan and Segal (2006), Segal (2007), Blumrosen, Nisan and Segal (2007), Gonczarowski et al. (2019).

We focus on minimizing superfluous information disclosure whereas these approaches focus on minimizing the number of "bits" communicated.

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Other approaches to privacy.

Our criterion is sensitive to context: It is…

1. not just about whether or how much info revealed, but how the info revealed is used.

Bayesian privacy.

Eilat, Eliaz and Mu (2021).

Differential privacy.

Dwork (2006), Pai and Roth (2013).

2. agnostic about set of admissible protocols.

Secure multi-party computation.

Bogetoft et al. (2009).

Zero-knowledge proofs.

Canetti, Fiat and Gonczarowski (2023).

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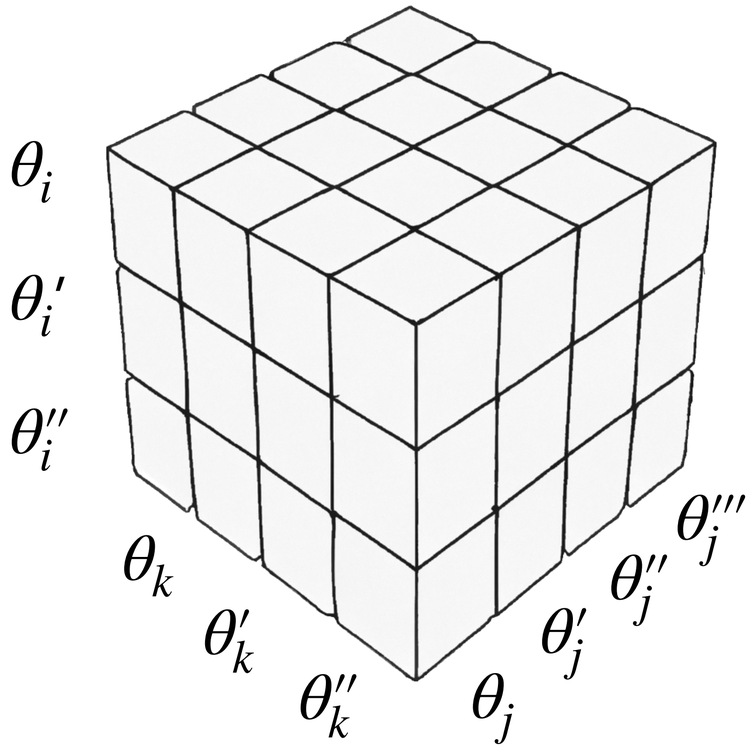
A Necessary Condition for Contextual Privacy

Consider a Θ^n matrix in which:

- rows are labelled with possible agent types
- entries correspond to the outcome $\phi(\boldsymbol{\theta})$
- rows labels need not be ordered



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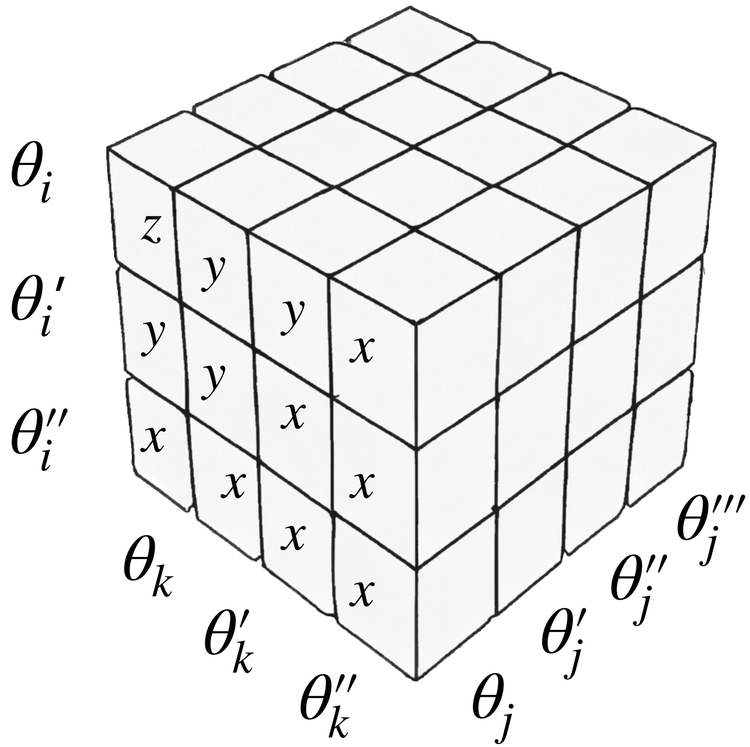
- rows are labelled with possible agent types
- entries correspond to the outcome $\phi(\boldsymbol{\theta})$
- rows labels need not be ordered



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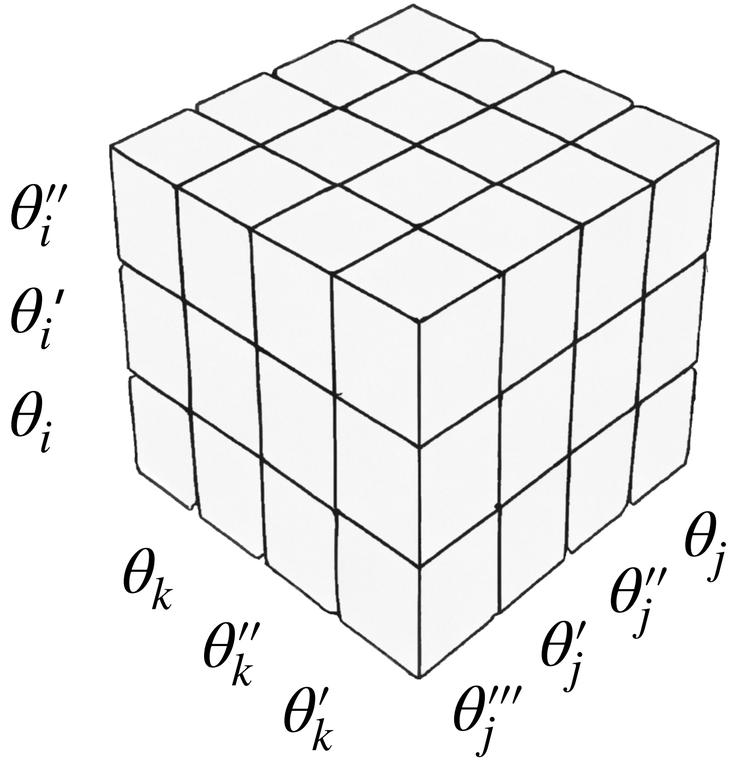

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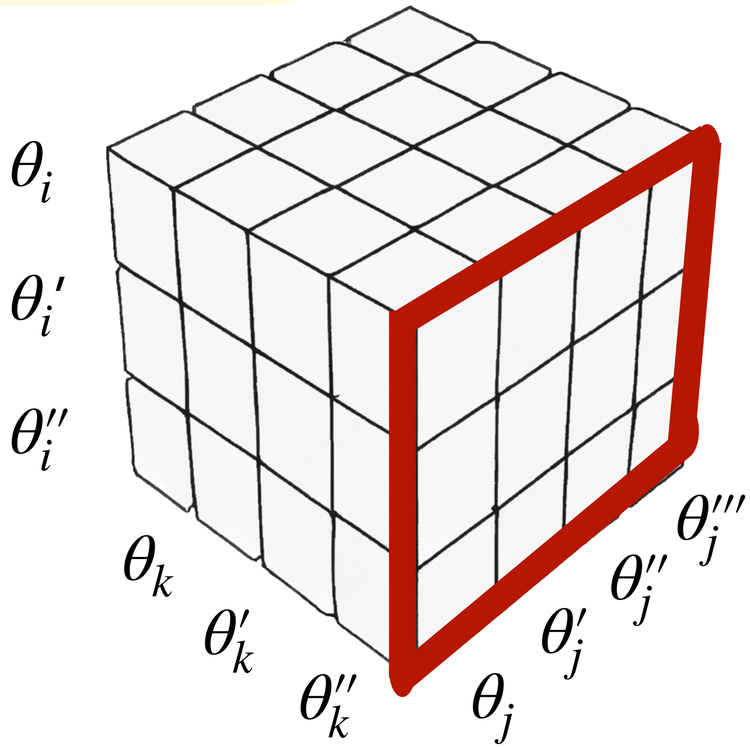

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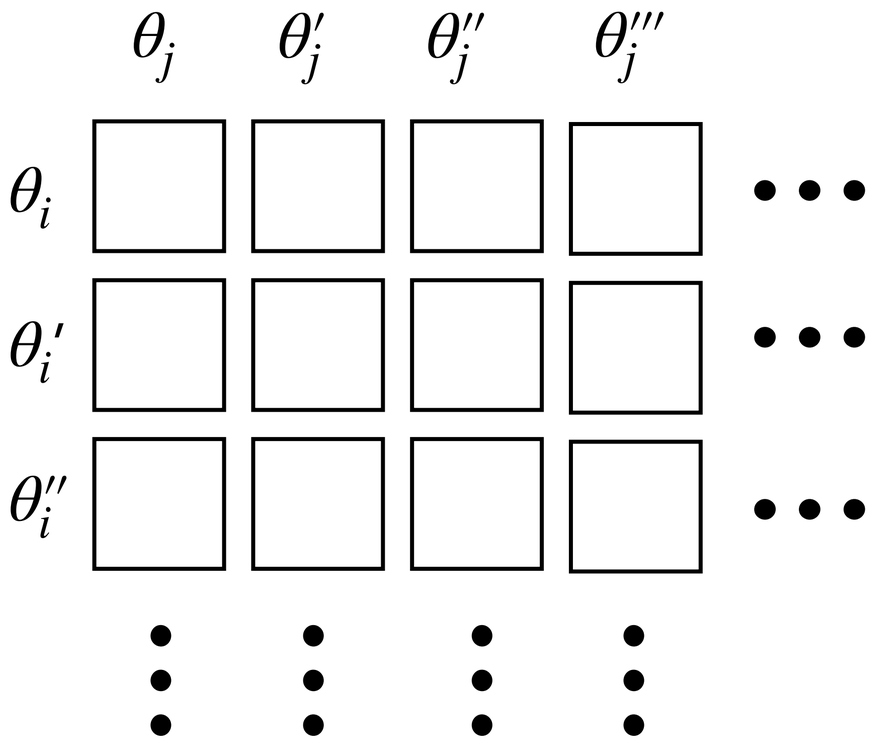

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If a choice rule ϕ is contextually private, then for all "squares" in the type space,

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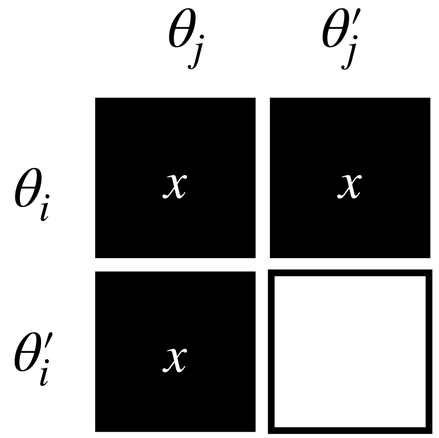

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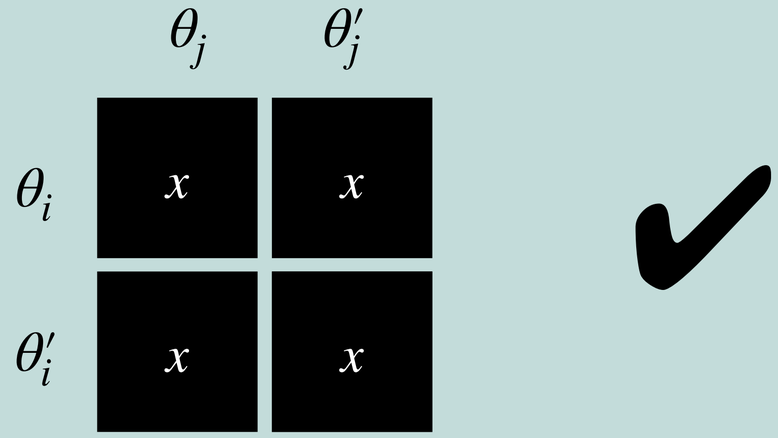

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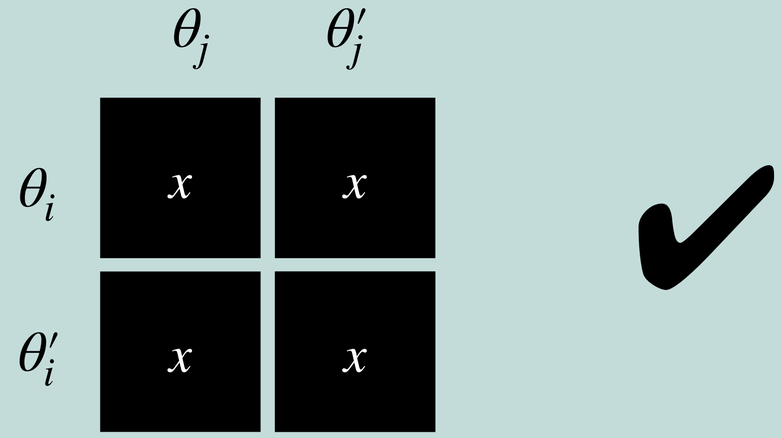

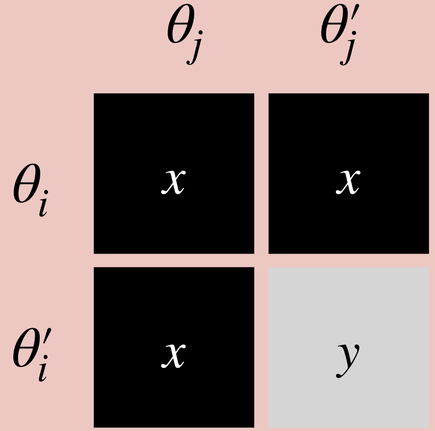

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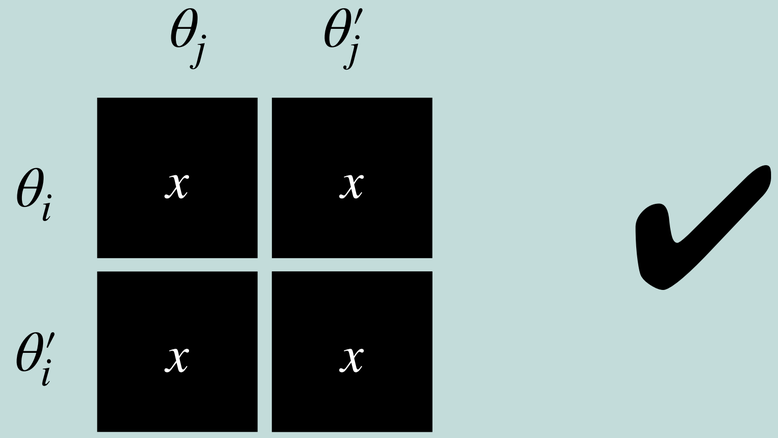

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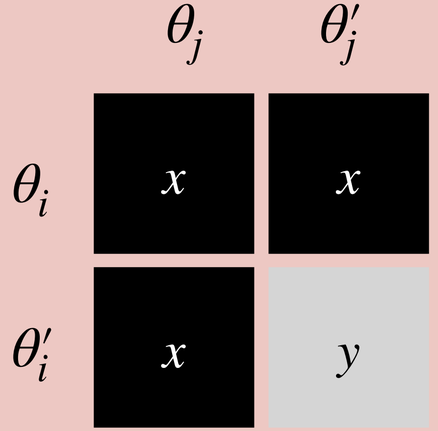

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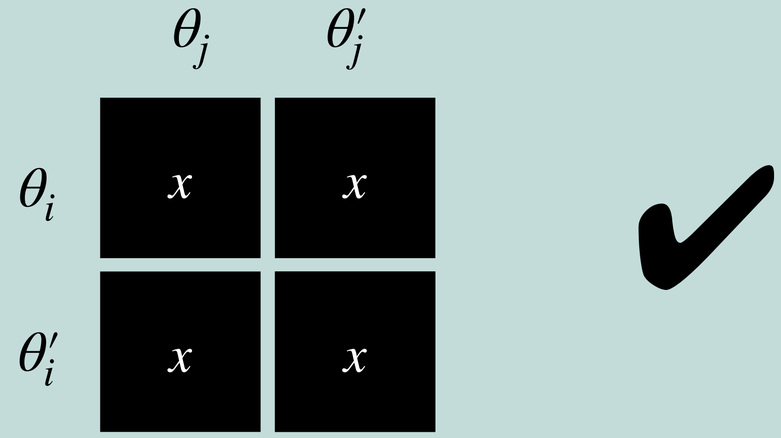

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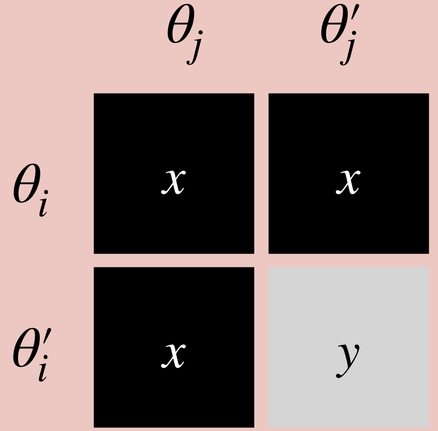

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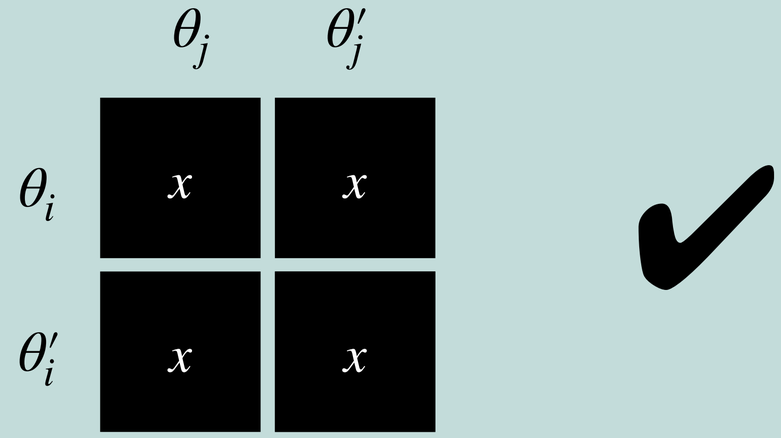

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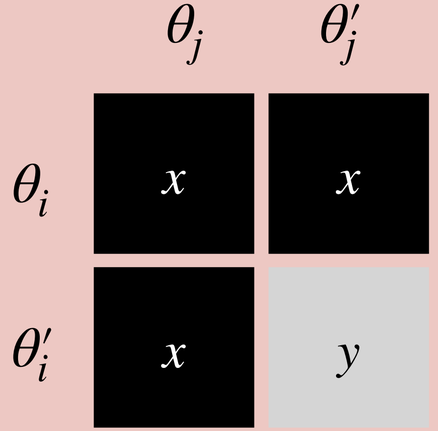

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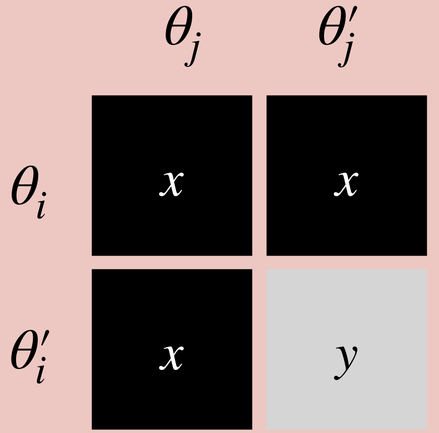
Simplifies search for counterexamples to full contextual privacy.



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SPA Is Not Contextually Private

Proposition.

The second-price auction choice rule is not contextually private under sequential elicitation protocols.

Hold fixed the winner.

Consider the second and third highest bids $\underline{\theta} < \overline{\theta}$.

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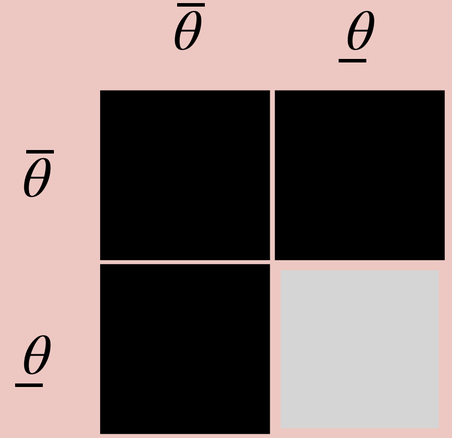

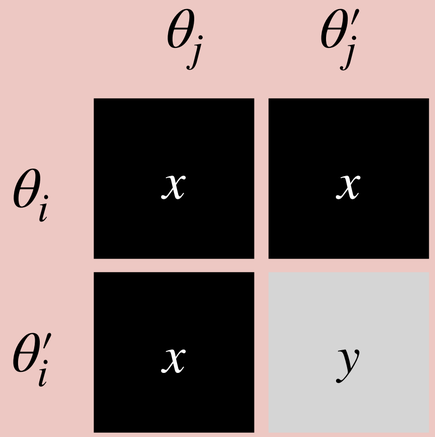

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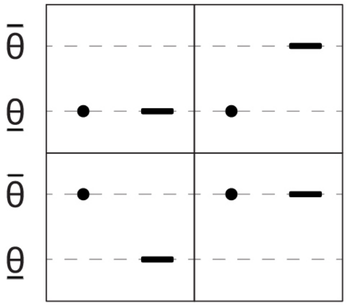
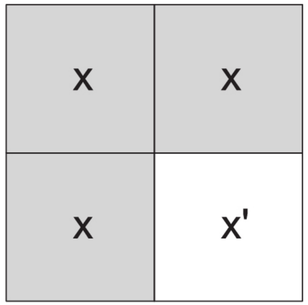
The power of the Corners Lemma

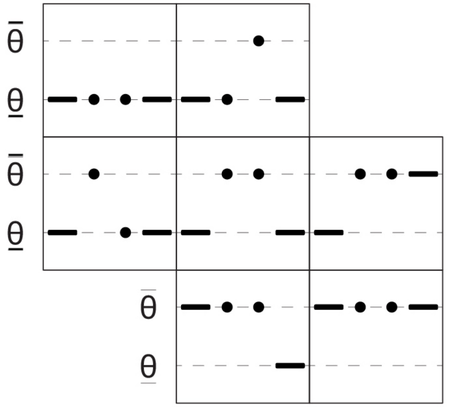
Auctions*

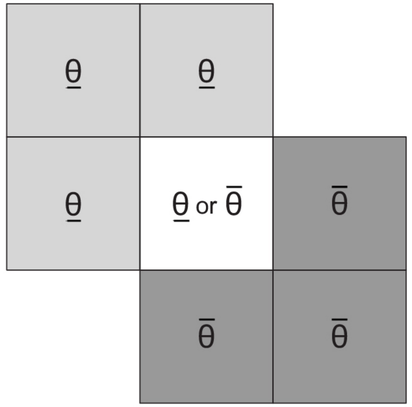
Second Price Auction





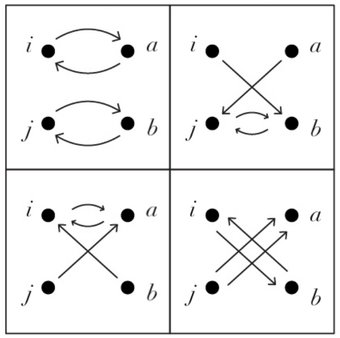
Efficient Double Auction

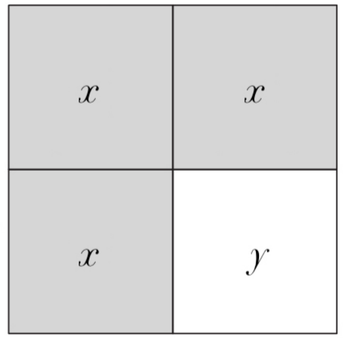




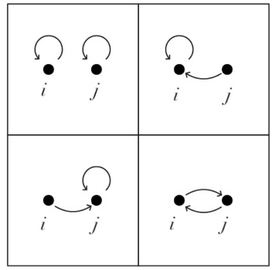
Matching

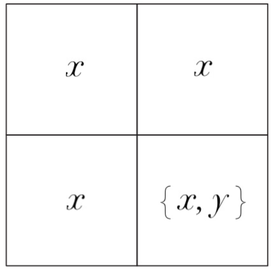
Stability

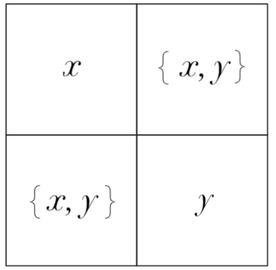


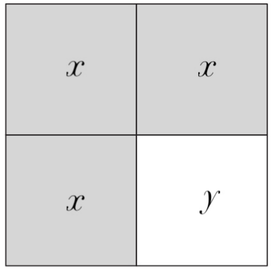


House Assignment









*See paper for more general proofs that don't rely on ties. This counterexample also works for gen. median voting rules.

Outline

1. **Definitions**

   Protocols, contextual privacy

2. **Fully contextually private choice rules**

   A necessary condition
   SPA is not contextually private

3. **Maximally contextually private protocols**

   Representation theorem: bi-monotonic protocols
   Maximally contextually private choice rules for SPA

4. **Brief discussion of other results**

   Settings without transfers, characterization for general protocols, incentives, variants.

Maximal Contextual Privacy in Auctions

Preview of the next result.

When searching for maximally contextually private protocols, we can restrict attention to a much smaller space of protocols.

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When searching for maximally contextually private protocols, we can restrict attention to a much smaller space of protocols.

This result will help us show, for instance…

that for the **second-price auction rule**, the

ascending-join and **overdescending-join**

protocols are maximally contextually private.

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Theorem.

If a choice rule ϕ satisfies the **"interval pivotality property"**, then any protocol for ϕ is contextual-privacy equivalent to a **"bi-monotonic"** protocol.

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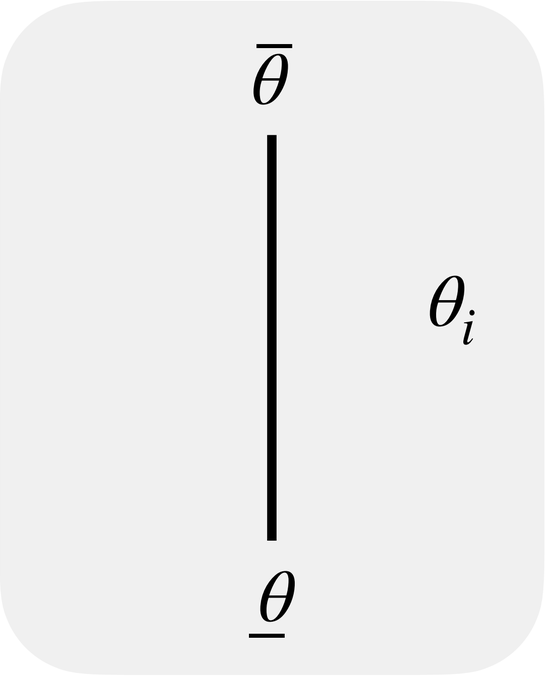

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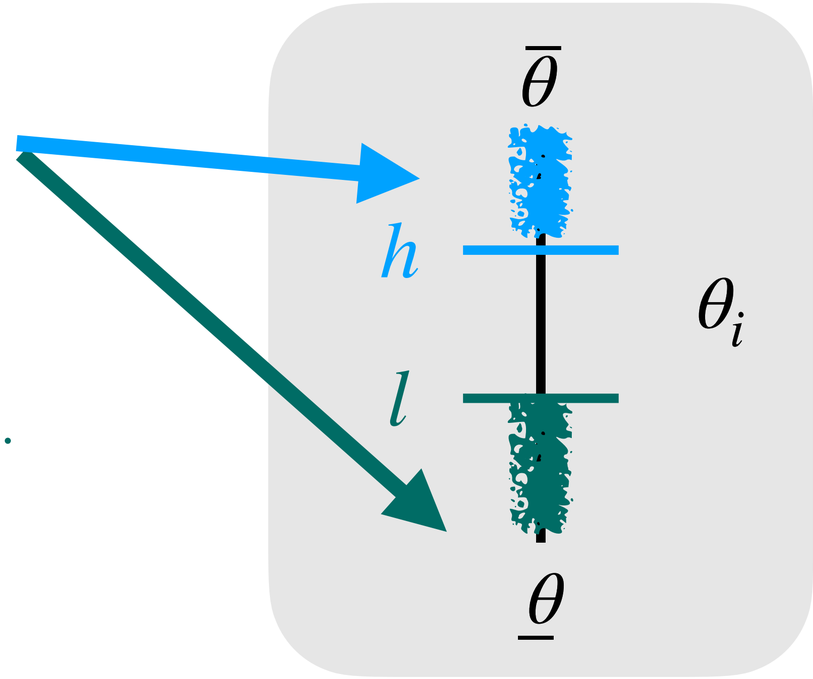

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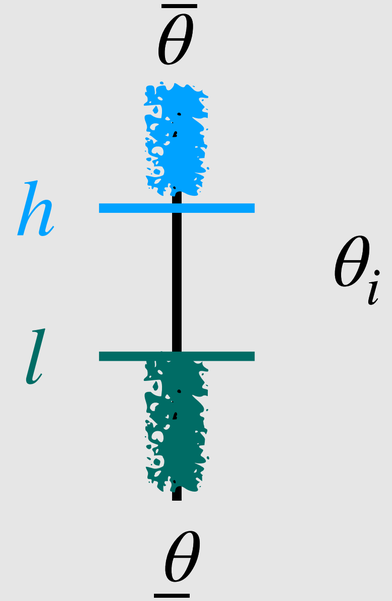

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Definition.

A protocol P is **bi-monotonic** if

1. all queries are "threshold queries", and
2. for each agent, the answer to the first query determines whether subsequent queries are monotonically increasing or decreasing in the threshold.

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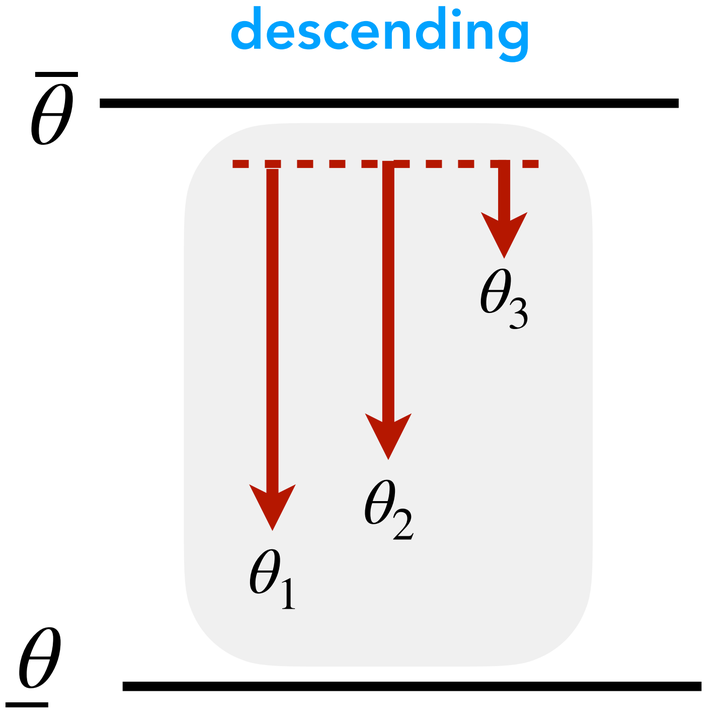

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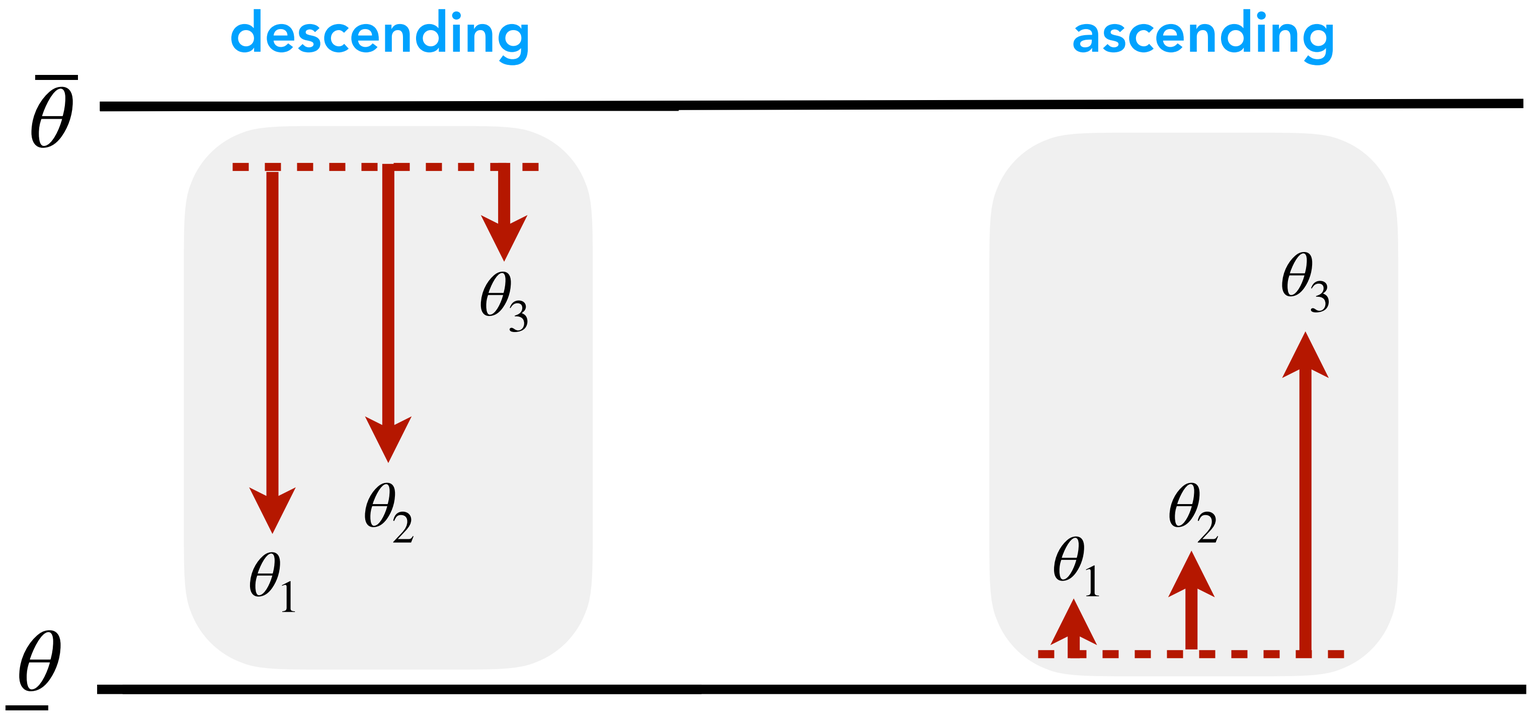

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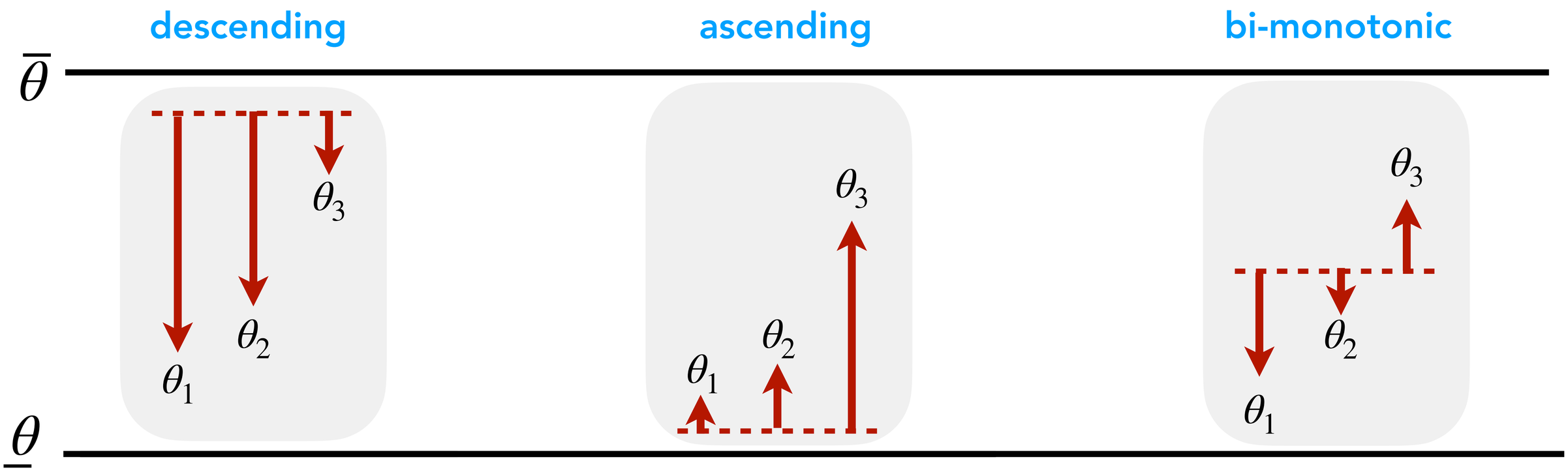

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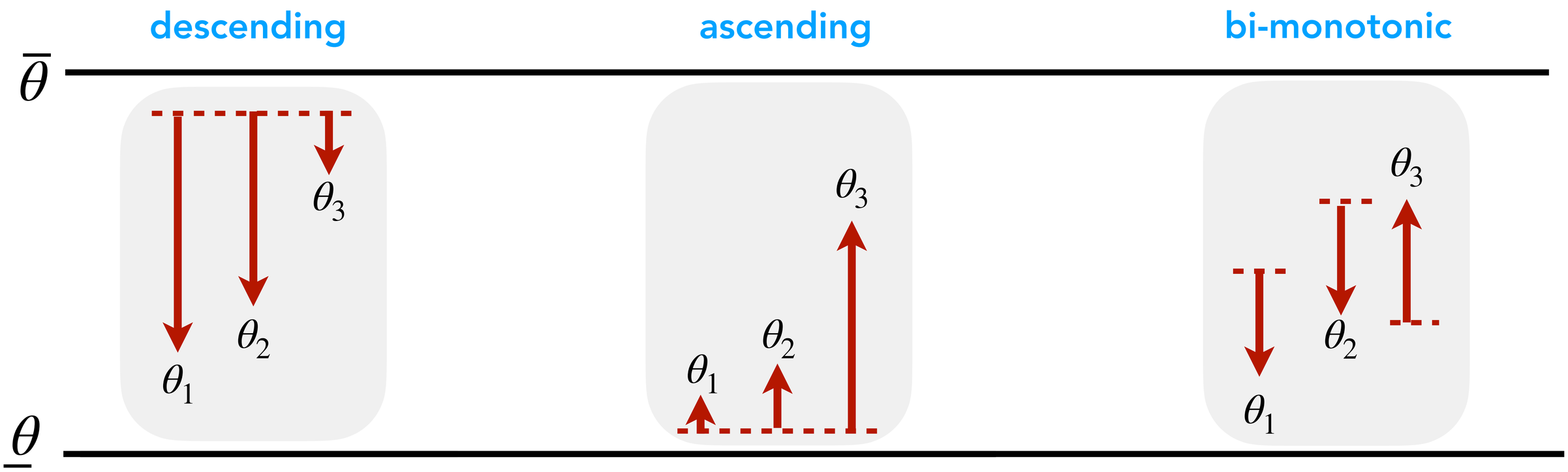

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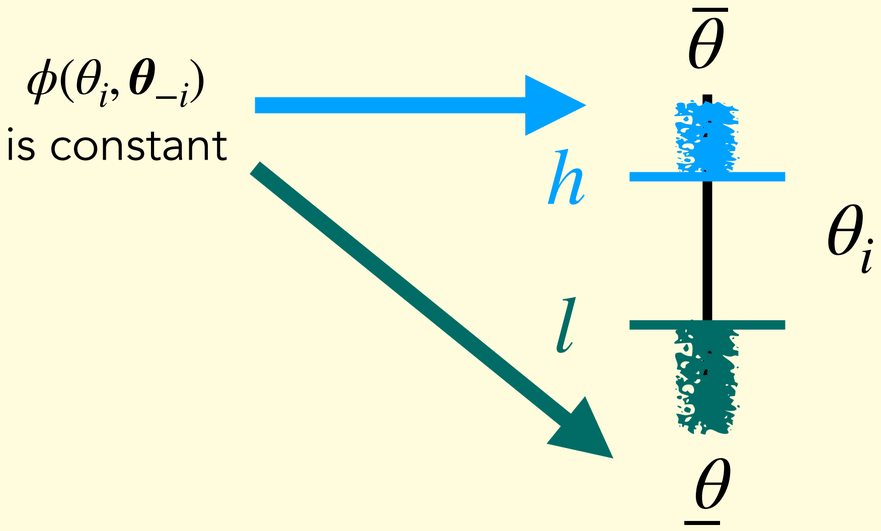
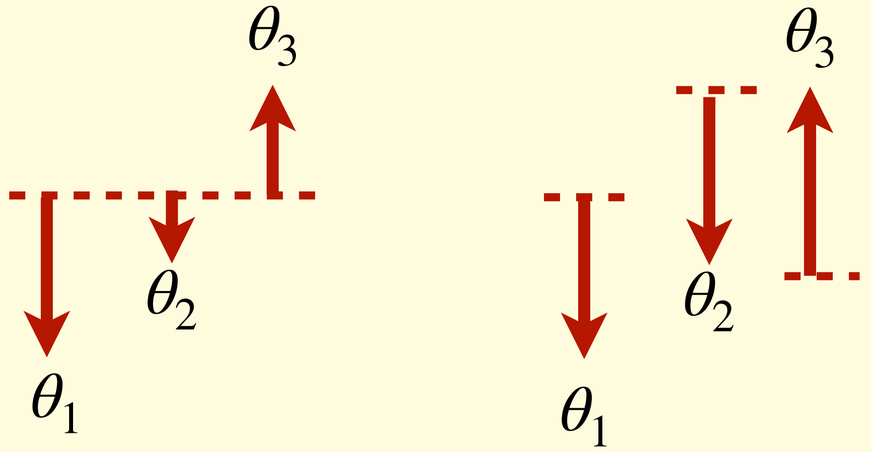
choice rule with interval pivotality



bi-monotonic protocol



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If a choice rule ϕ satisfies the **interval pivotality property**, then any protocol for ϕ is contextual privacy equivalent to a **bi-monotonic protocol**.

Proof idea.

Show that any protocol P for ϕ that is not bi-monotonic can be transformed into a protocol that is at least as contextually private as P and also bi-monotonic.

Step 1. **Inject threshold queries** between highest and lowest types separated at node v.

Step 2. **Fill in gaps** between threshold queries.

Step 3. **Delete redundant queries**.

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The **ascending-join** protocol and the **overdescending-join** protocol are **maximally contextually private** protocols for the second-price auction.

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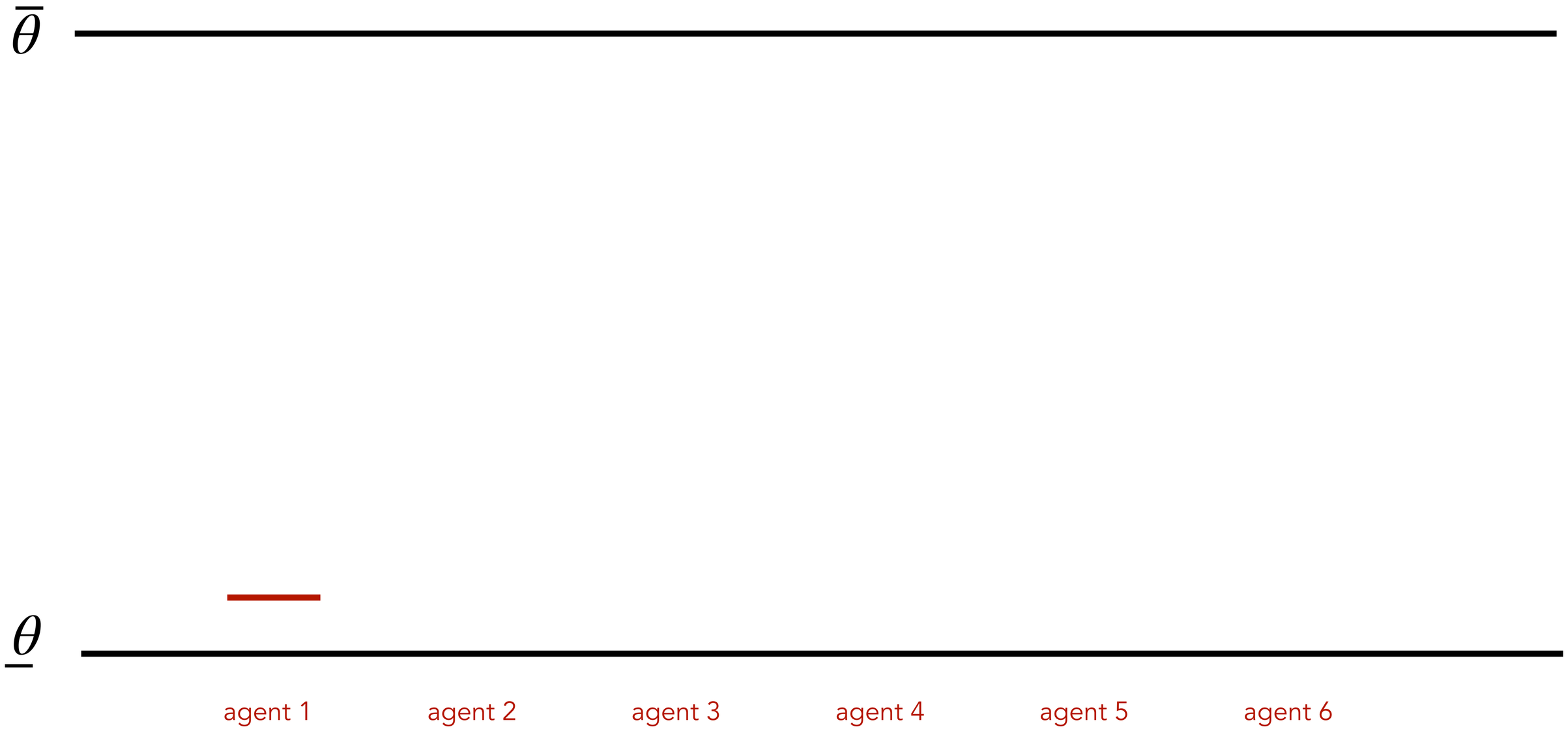
Maximally contextually private protocols trade off privacy of different agents.

For choice rules that have a "price," delay asking as much as possible.

Ascending Protocol Is Not Maximally Contextually Private



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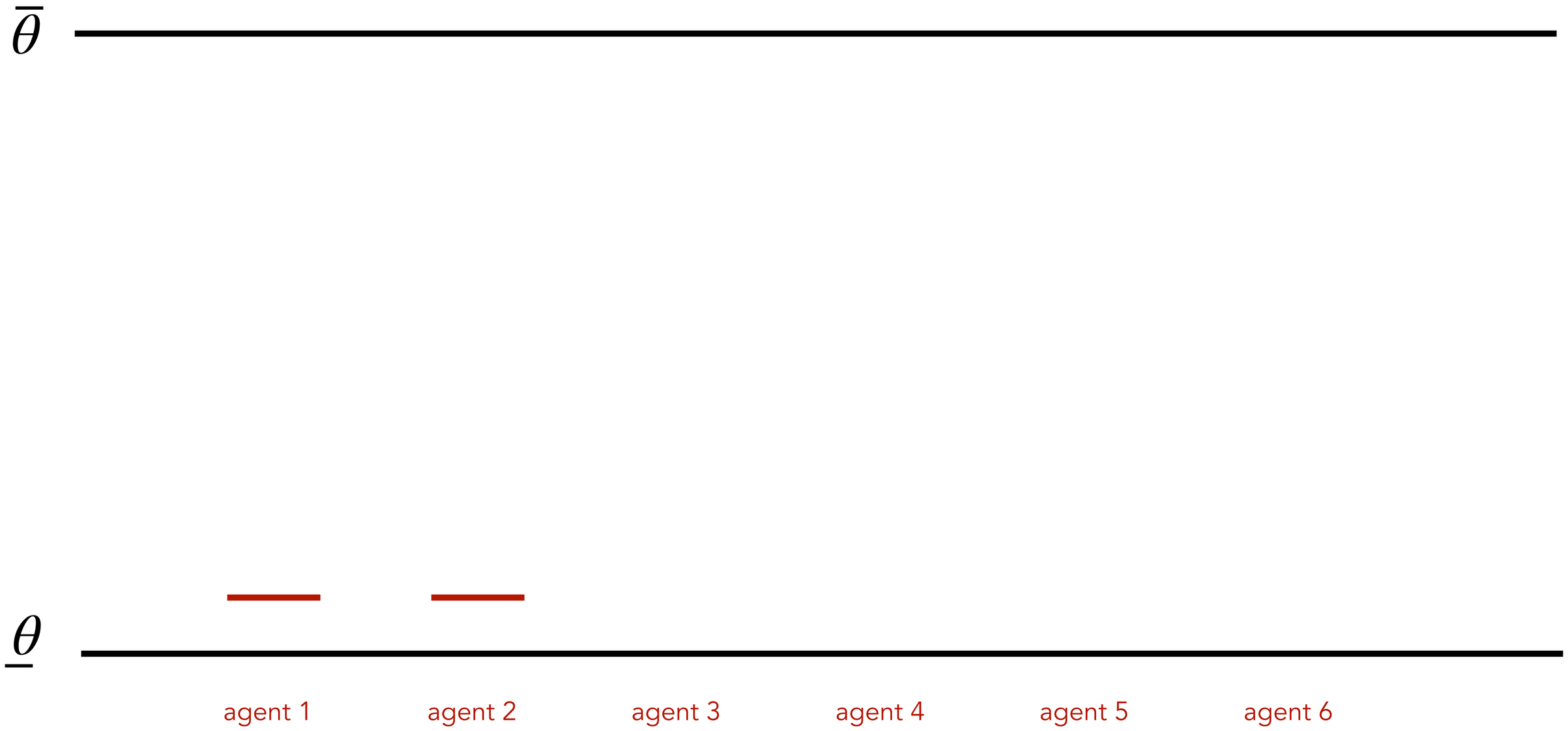

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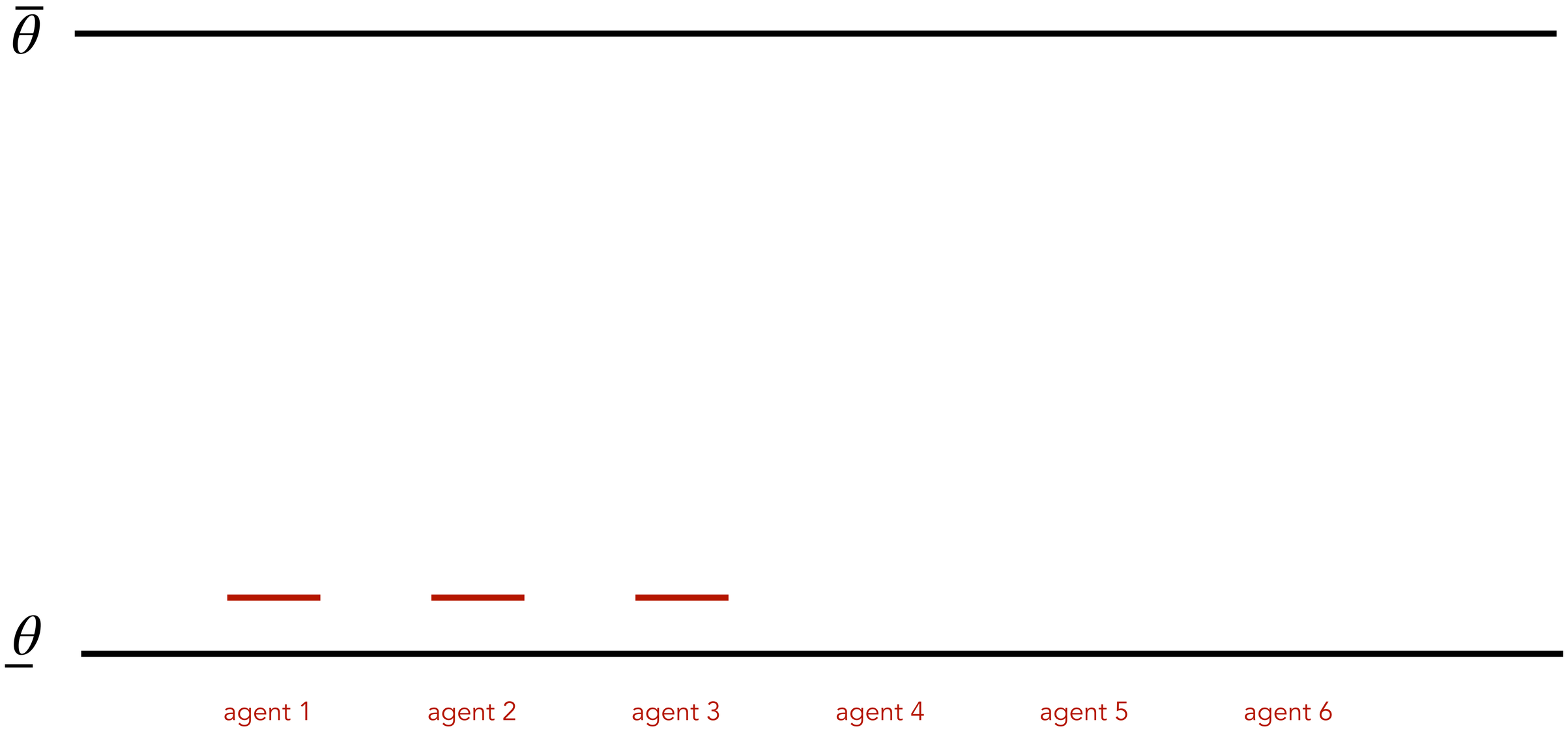

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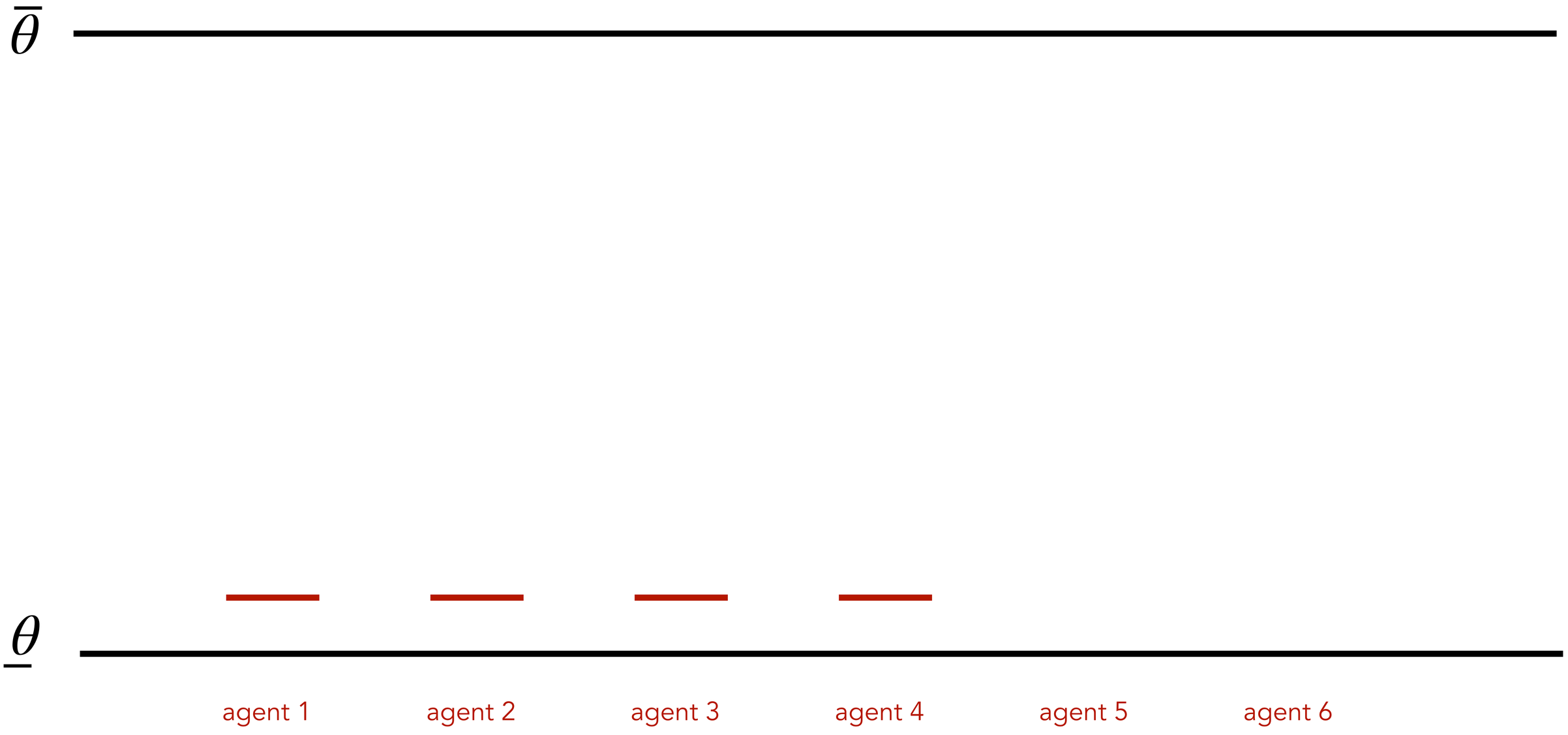

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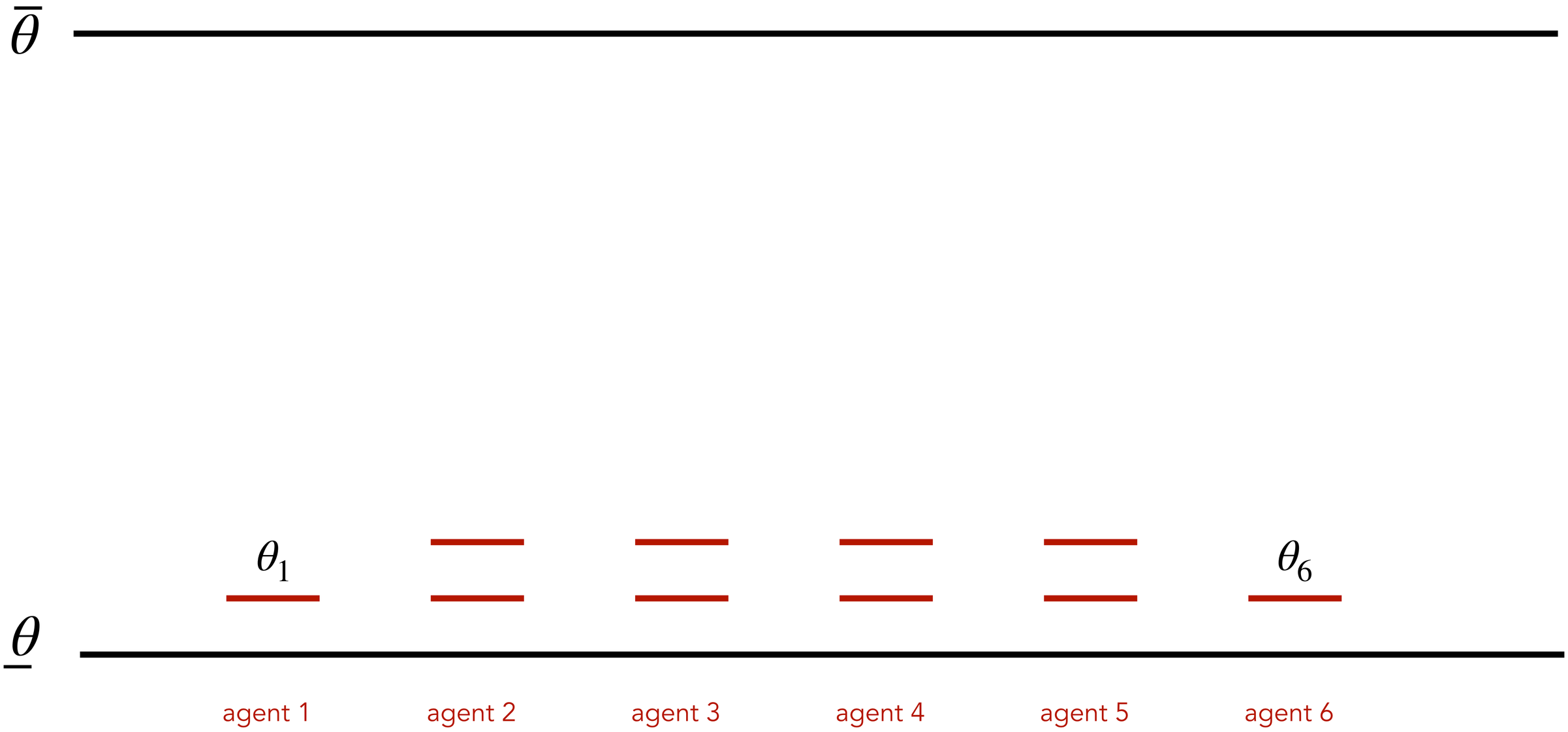

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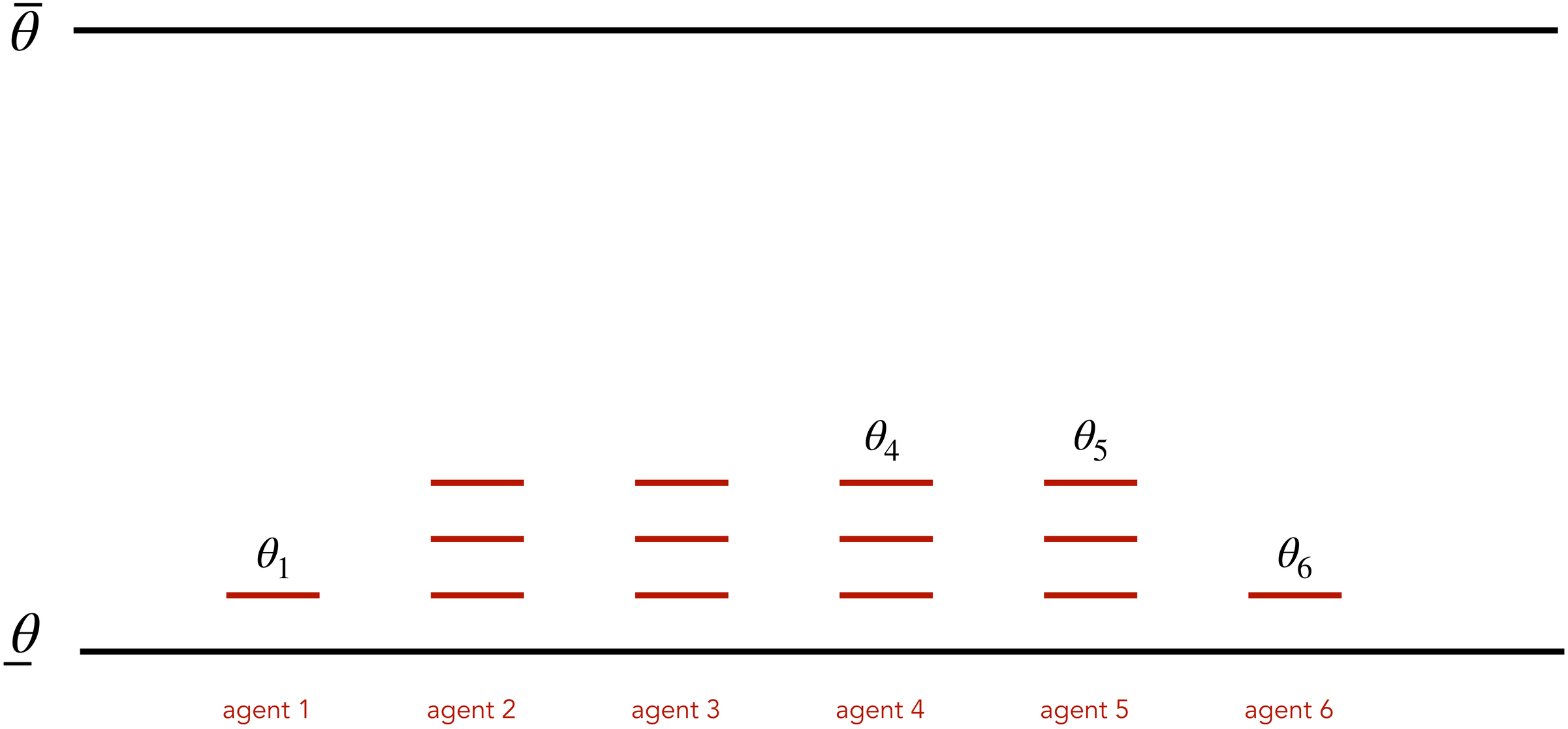

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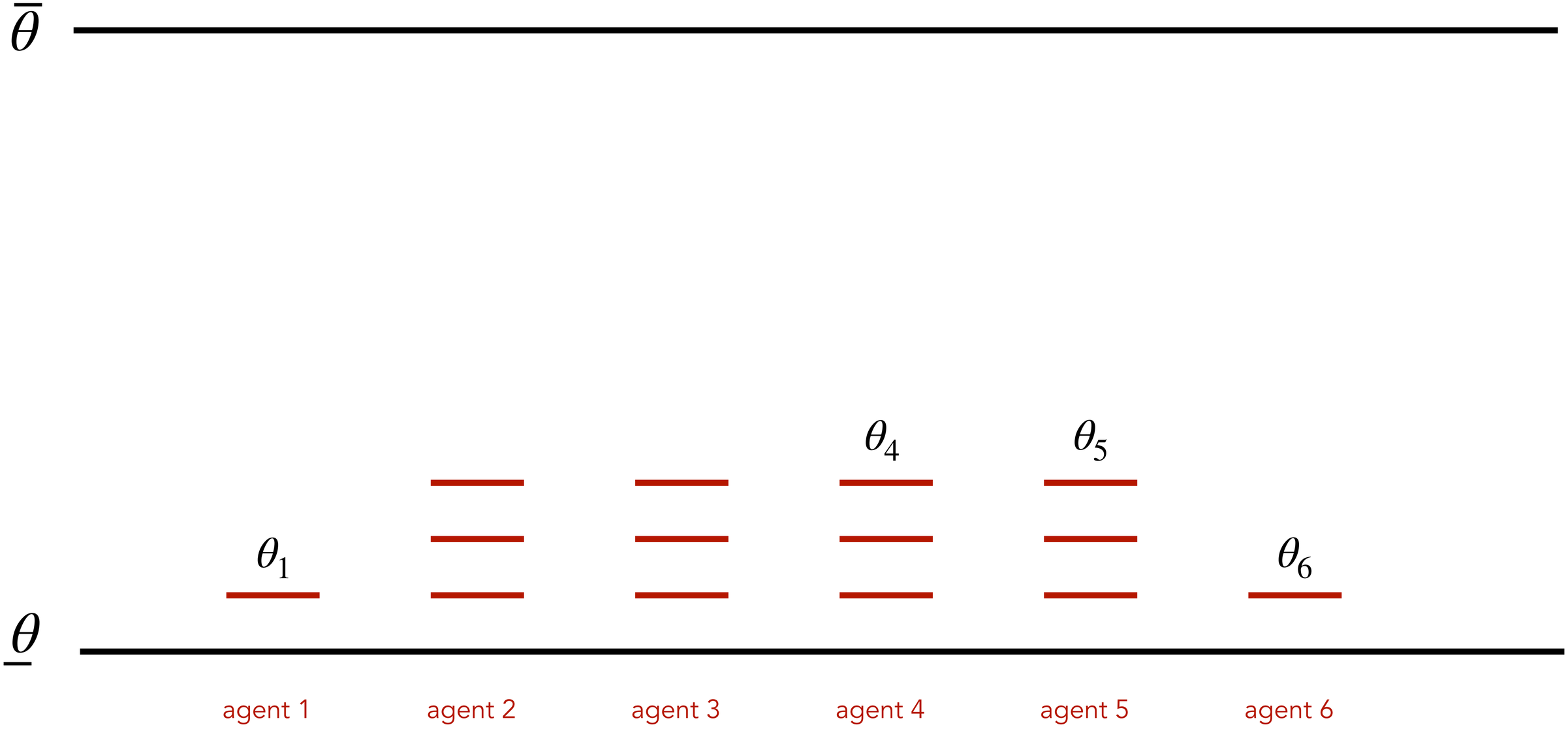

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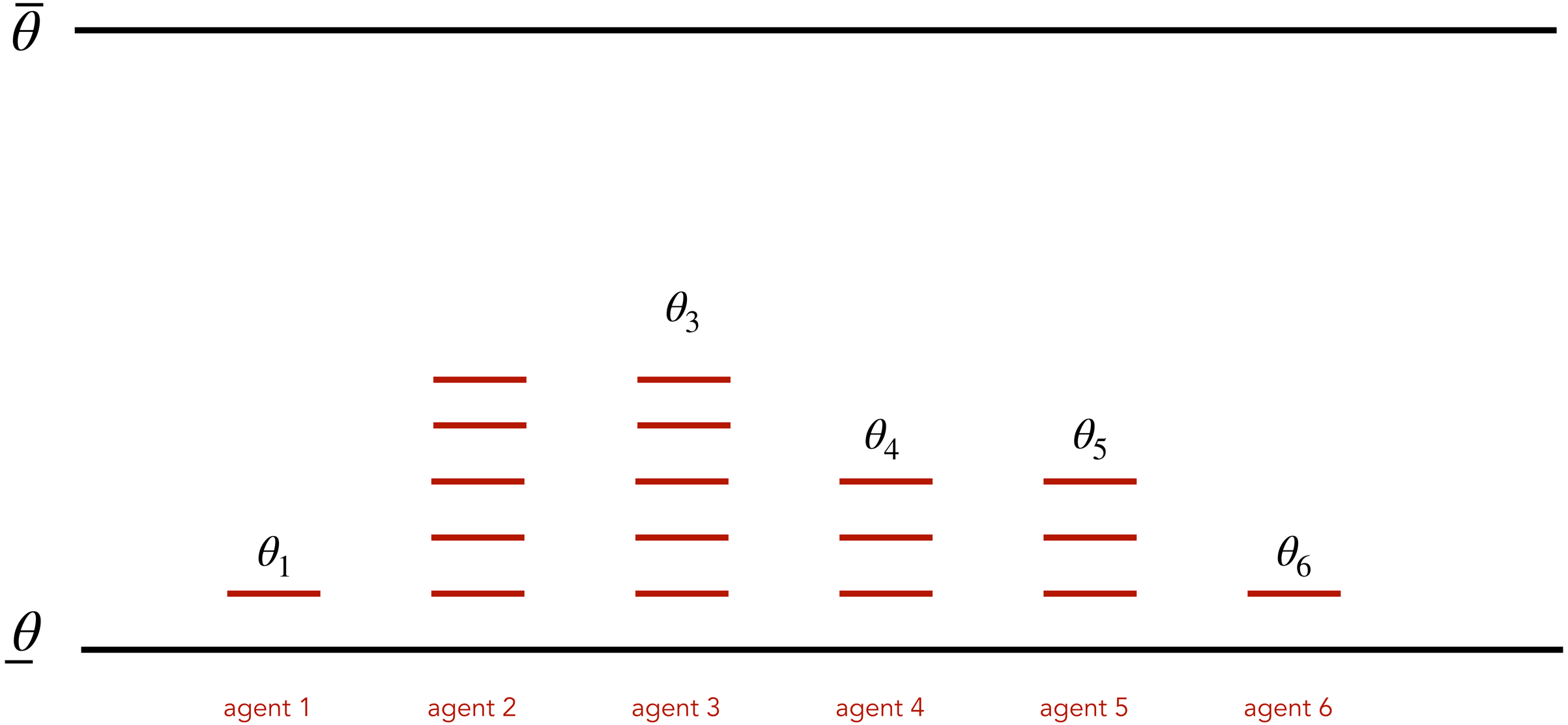

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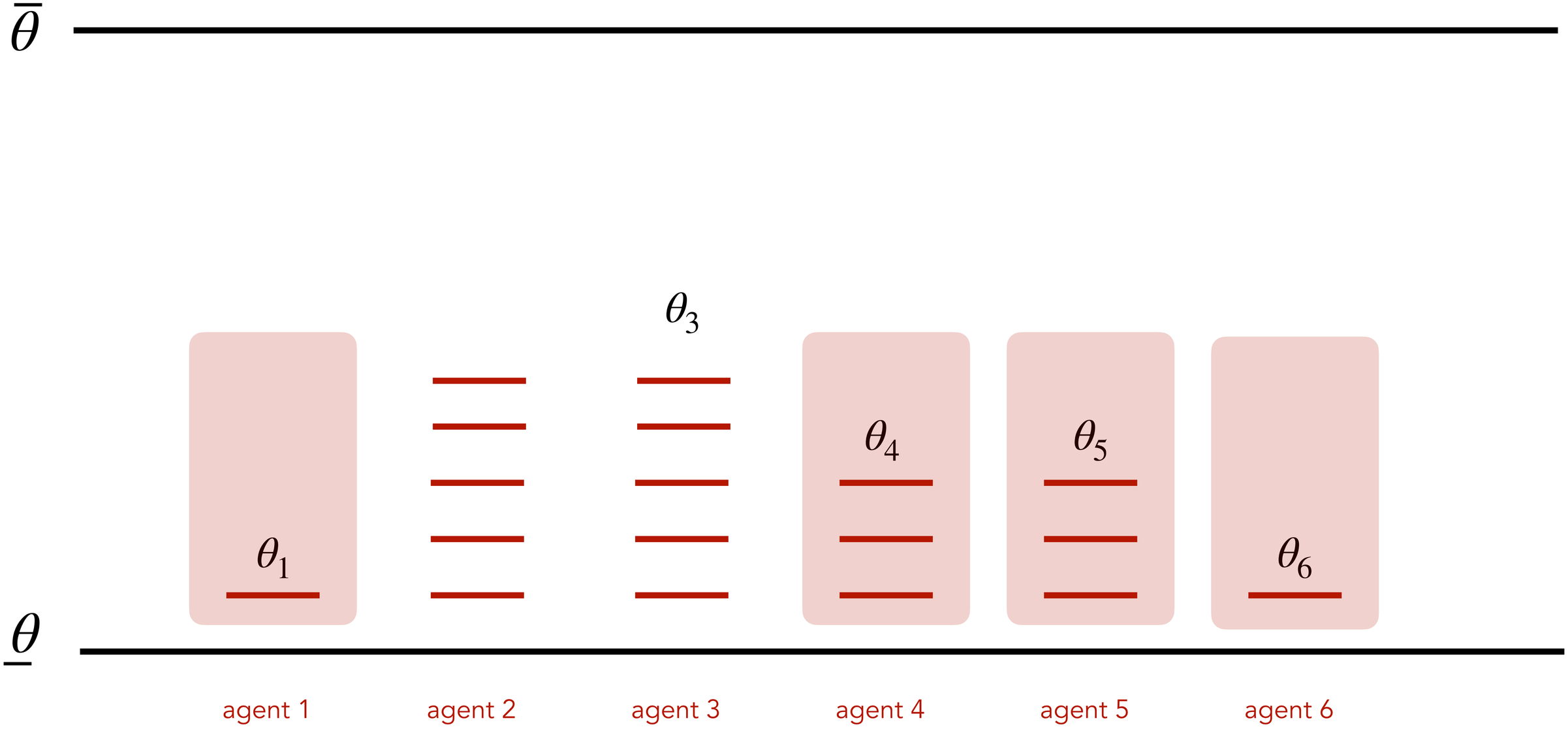

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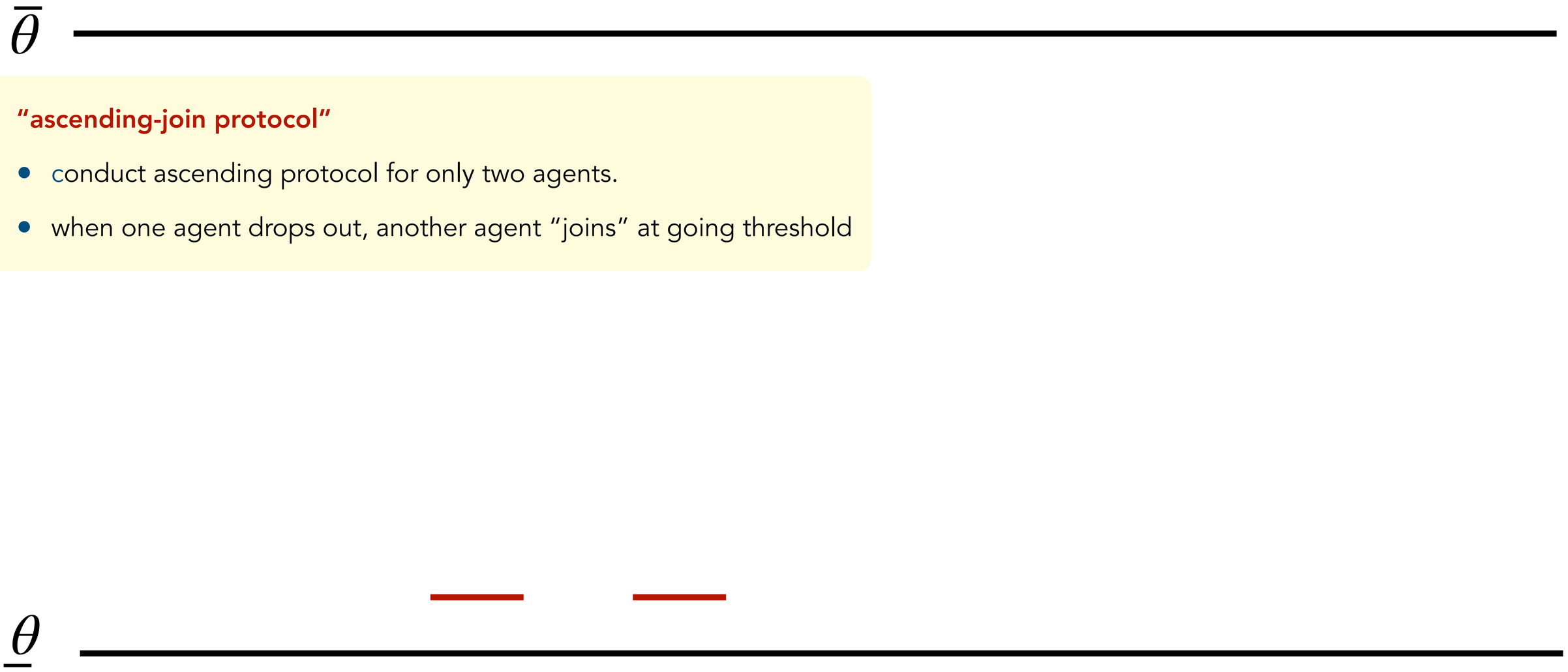

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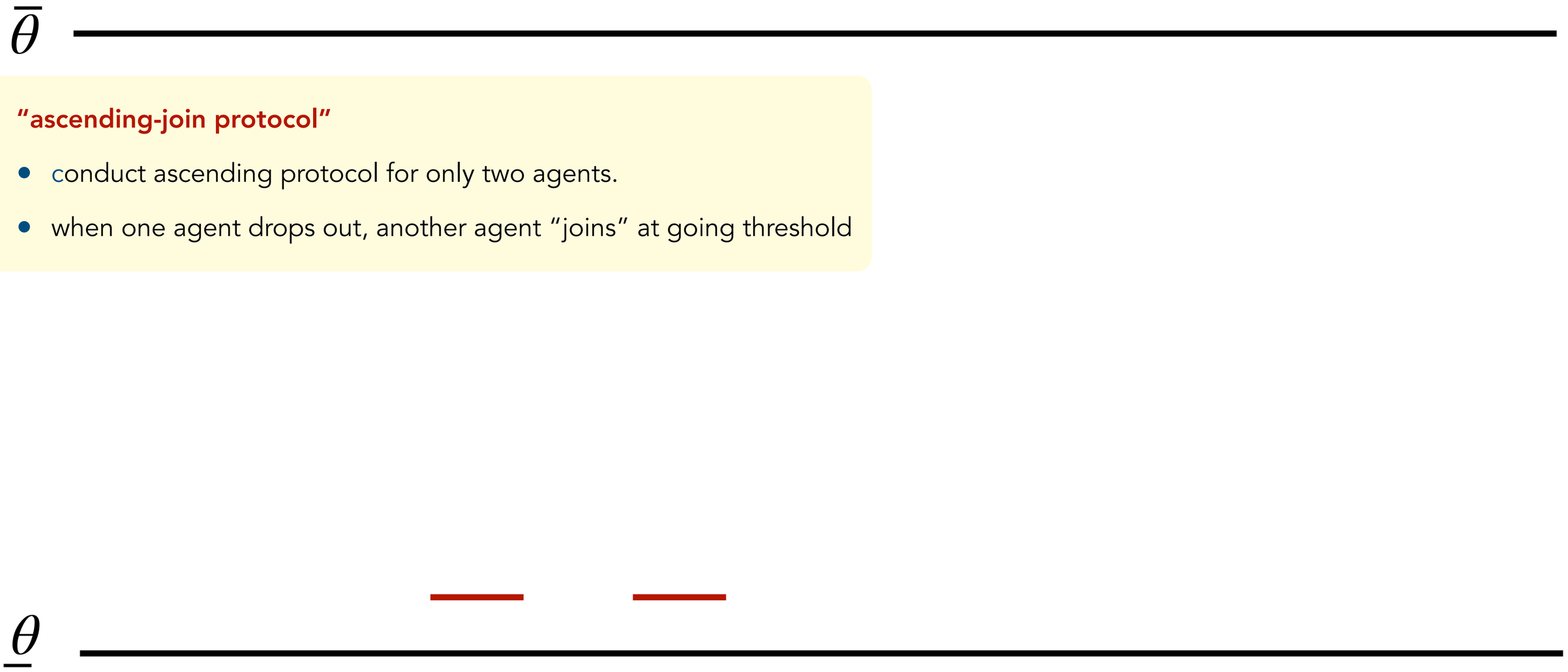

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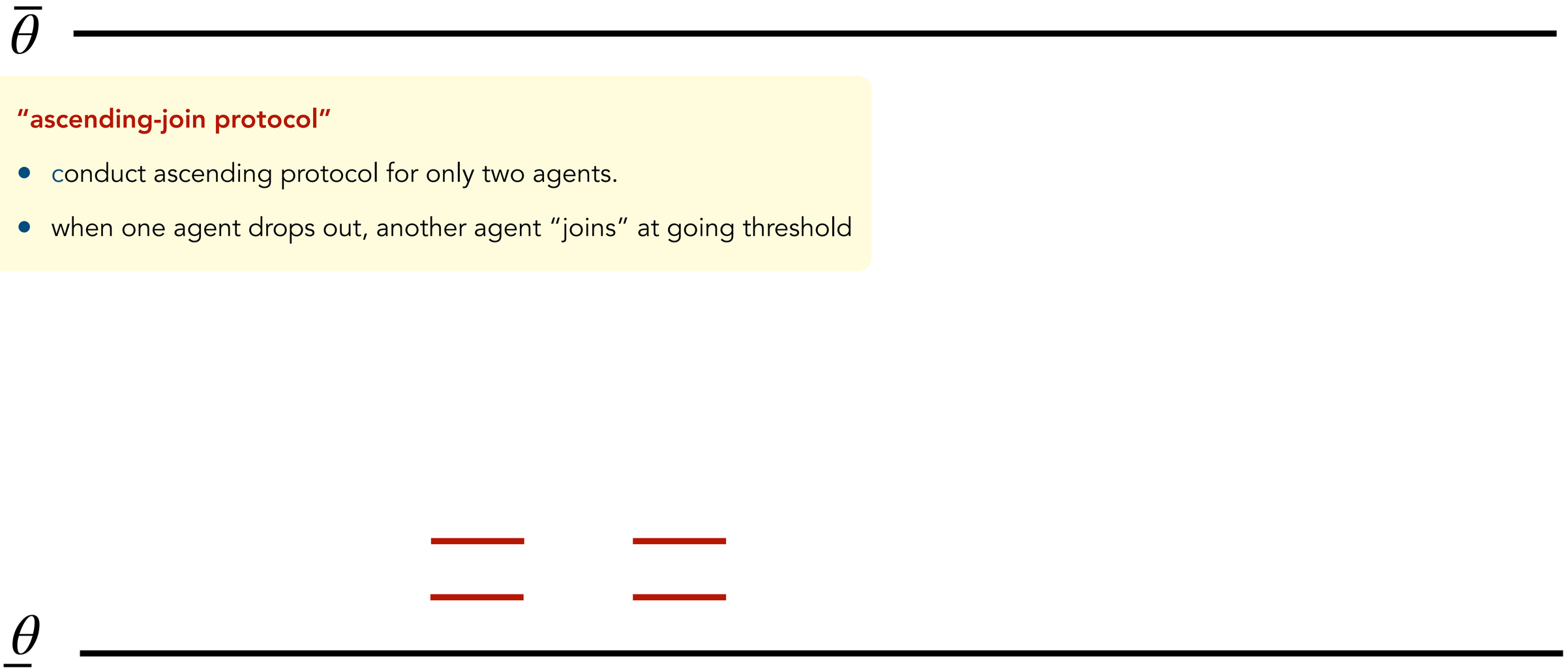

"ascending-join protocol"

- conduct ascending protocol for only two agents.
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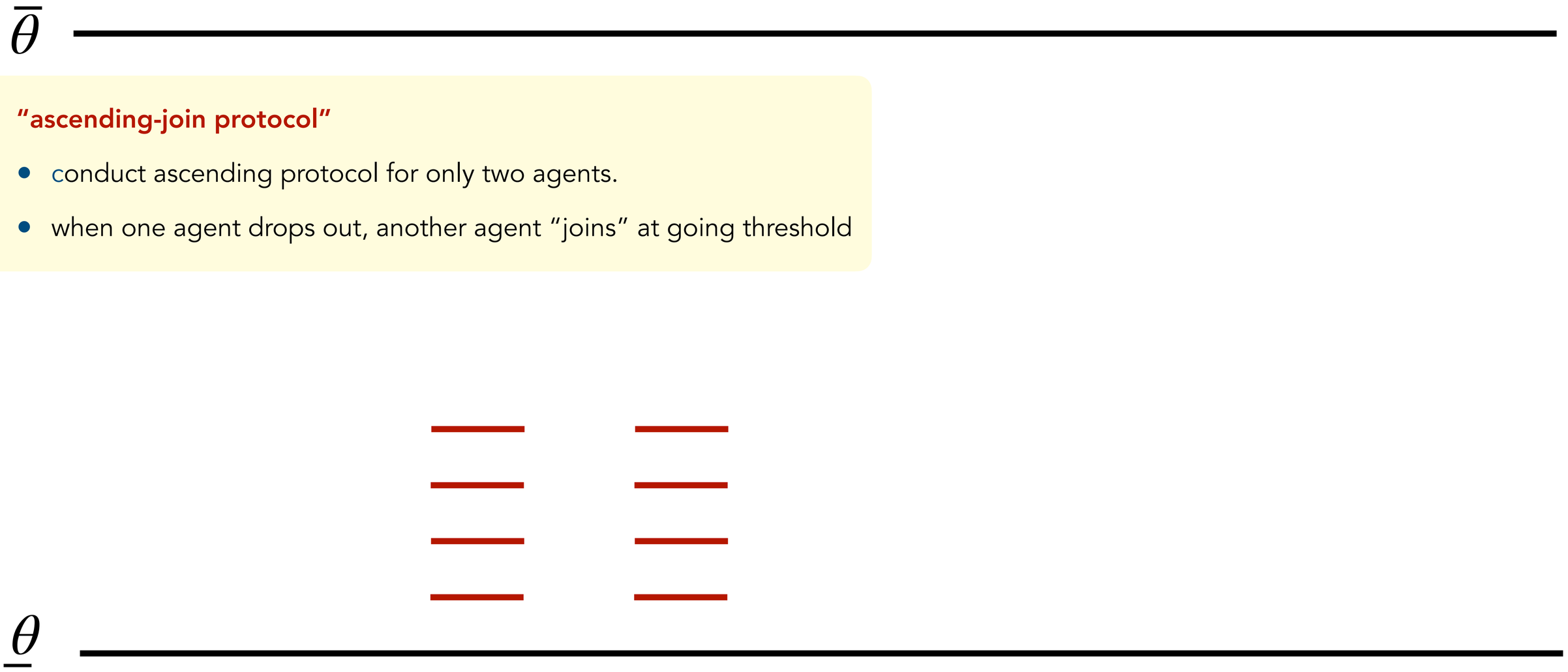

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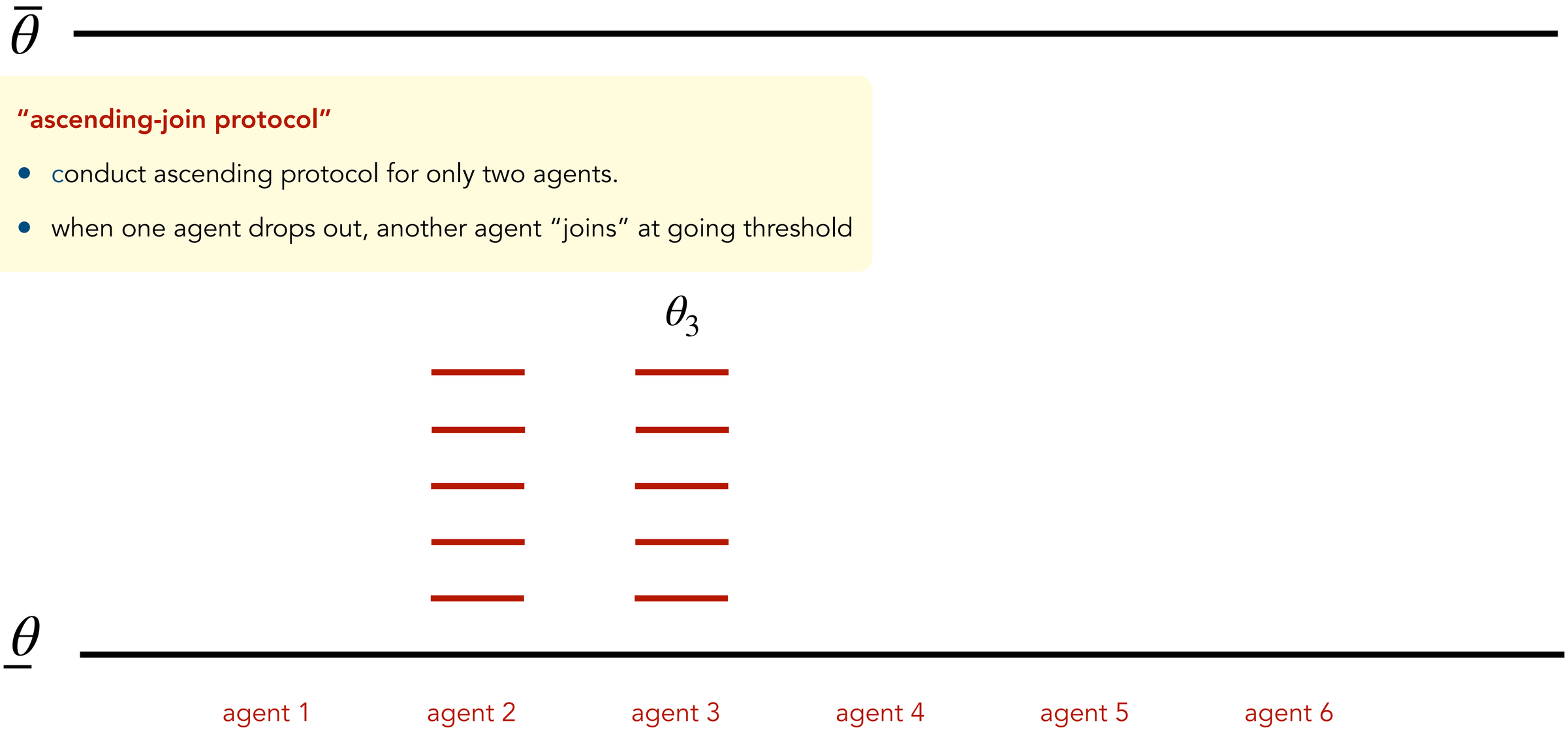

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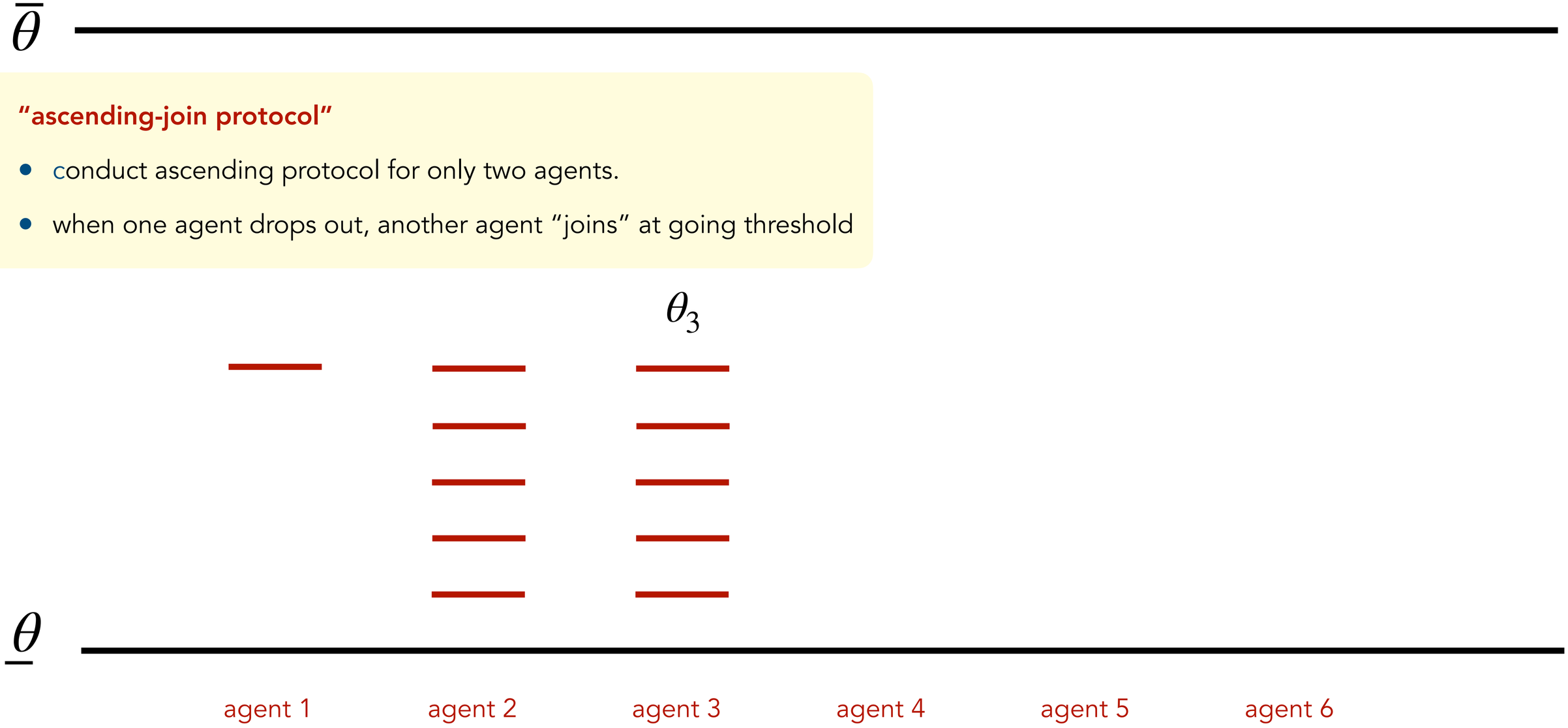

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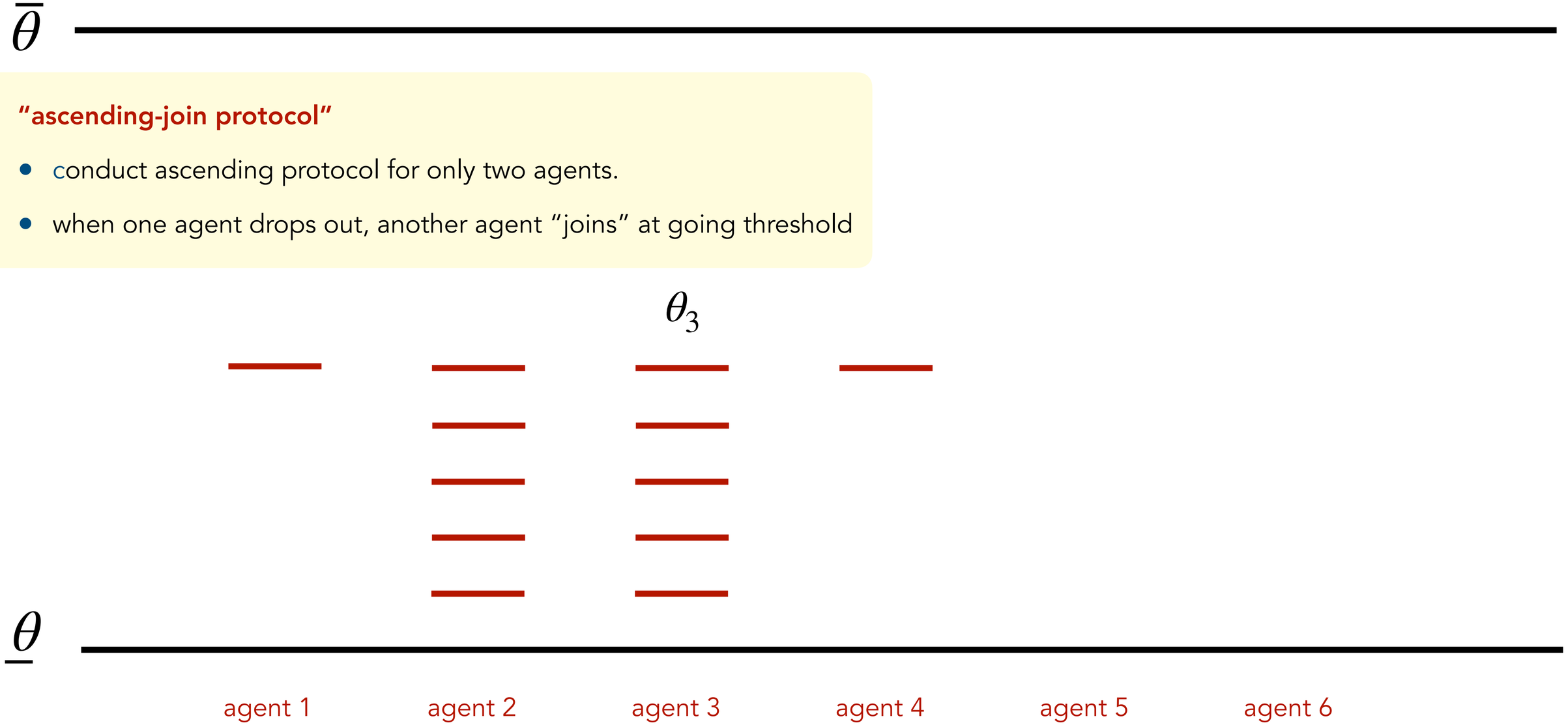

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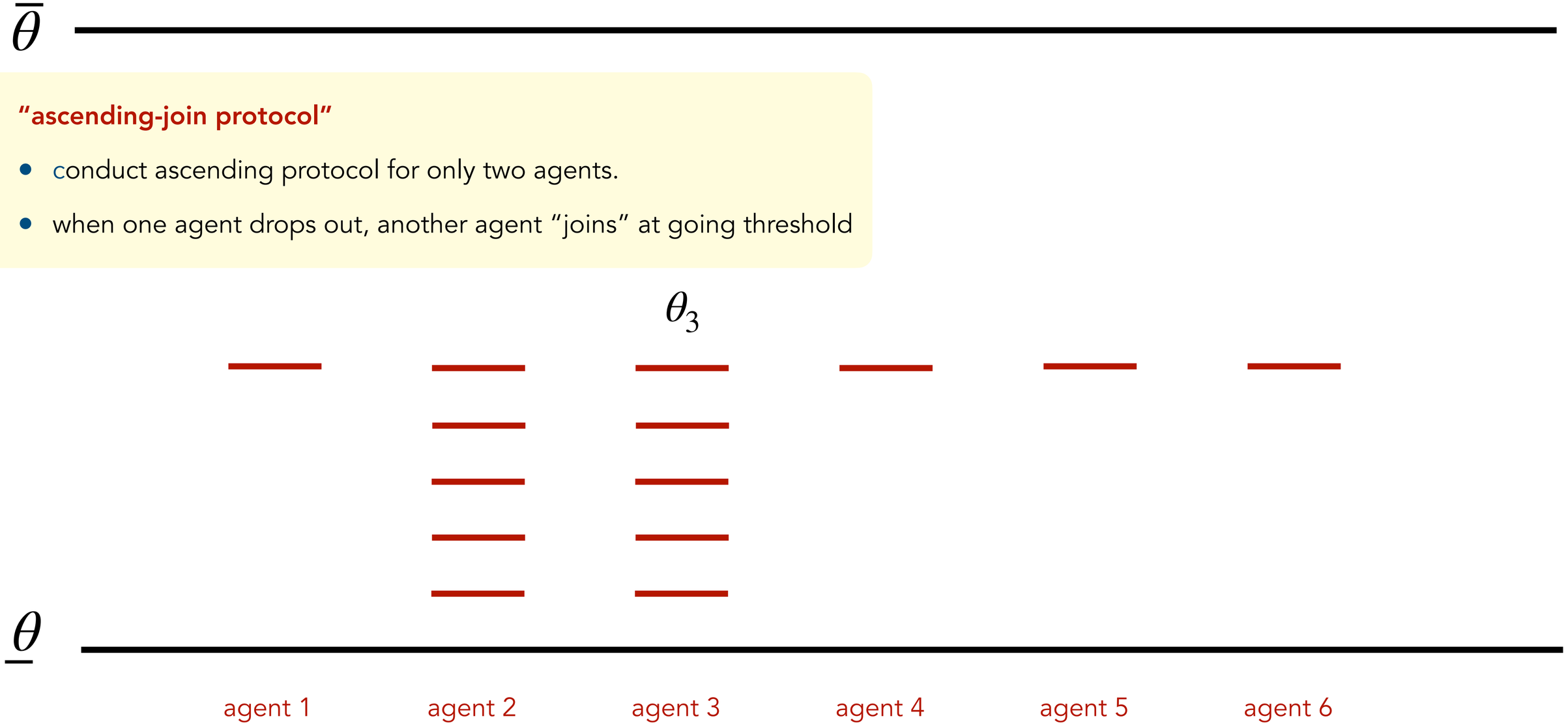

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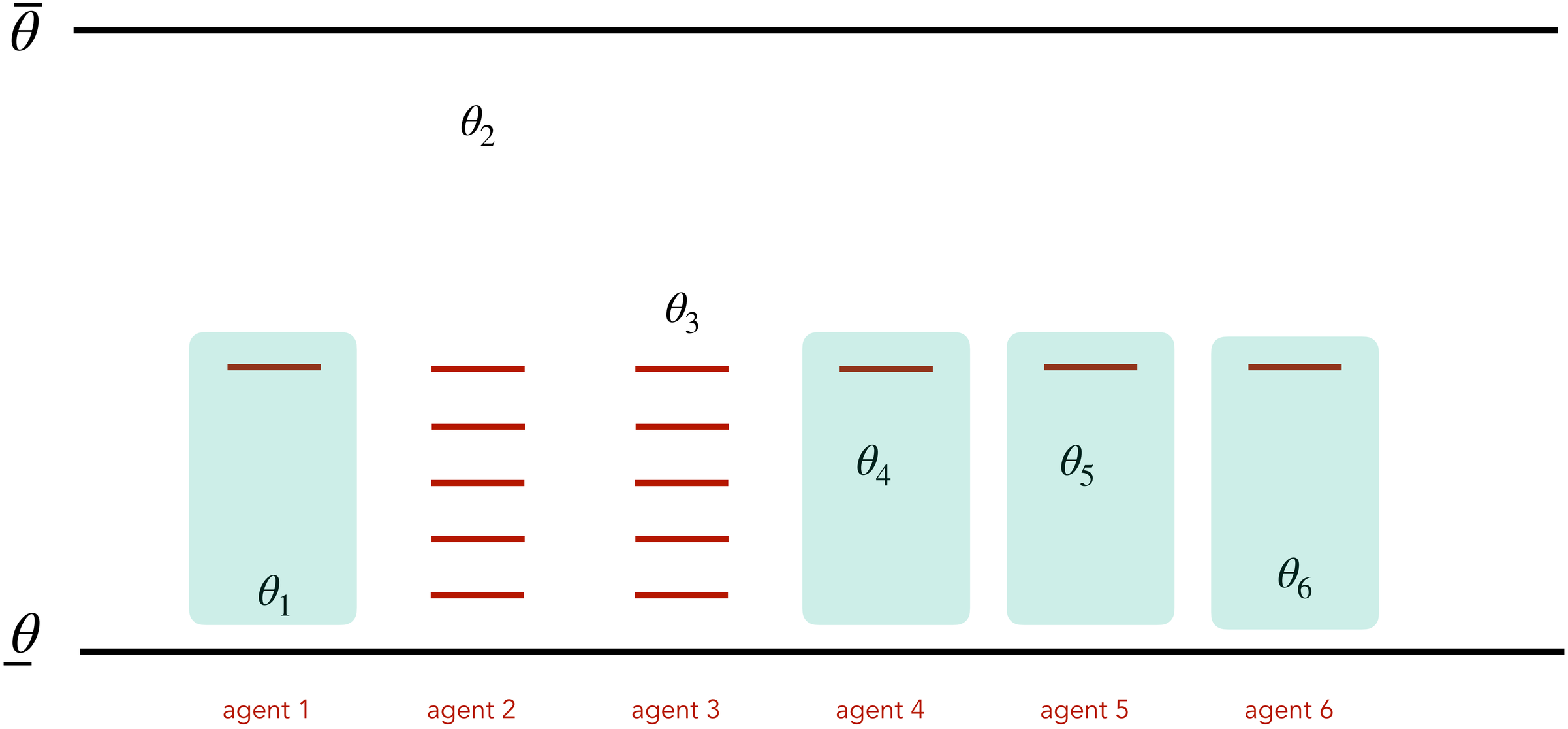


Ascending Protocol Improvement?

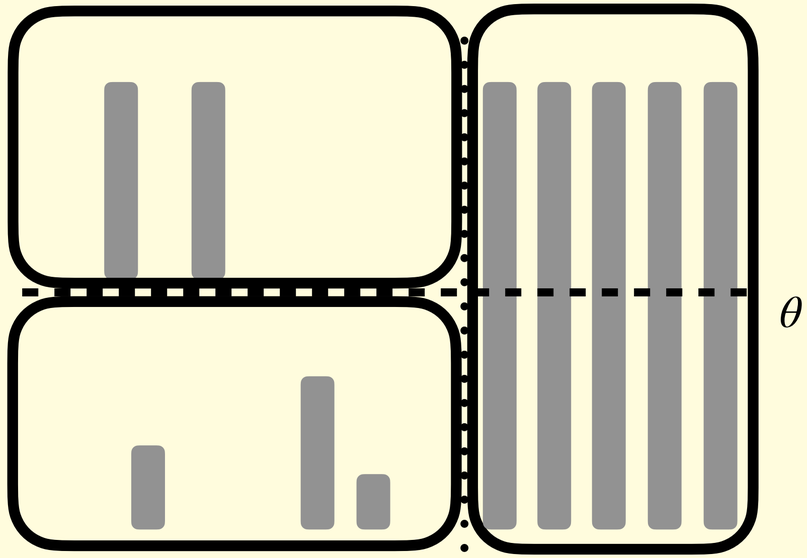
Theorem.

The **ascending-join** protocol is a **maximally contextually private** protocol for the second-price auction. It protects the winner and low-priority losers.

Proof idea.

Construct the protocol by induction.

- Bimonotonicity reduces to a choice of (i) threshold to start each agent ask and (ii) order in which to ask agents.
- Show that at every stage there is exactly one query that either leads into a "protective information state" or termination.



protective information state

Outline

1. **Definitions**

   Protocols, contextual privacy

2. **Fully contextually private choice rules**

   A necessary condition
   SPA is not contextually private

3. **Maximally contextually private protocols**

   Representation theorem: bi-monotonic protocols
   Maximally contextually private choice rules for SPA

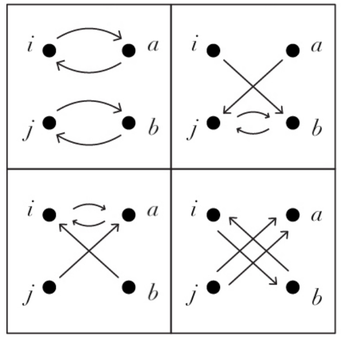
4. **Brief discussion of other results**

   Settings without transfers, characterization for general protocols, incentives, variants.

Impossibility Results In Assignment Domains

Matching

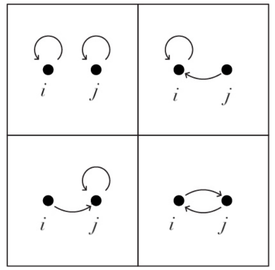
Stability



x	x
x	y

House Assignment

Efficiency + IR



x	x
x	$\{x, y\}$

x	$\{x, y\}$
$\{x, y\}$	y

x	x
x	y

Proposition.

There is no stable matching rule that is contextually private.

Proposition.

There is no efficient, individually rational and contextually private choice rule.

Additional Results

Theorem. Characterization of contextually private choices rules.

For any "non-adaptive protocol", a social choice function ϕ is contextually private

if and only if

there is no **generalized corner** $\tilde{\Theta} \subseteq \Theta$ such that any "non-trivial" query could separate two type profiles that lead to the same outcome.

Proposition. Incentives in (maximally)-contextually private auction protocols.

- **Ascending** and **ascending-join** protocols for second-price auction rule are obviously DSIC implementable.
- **Over-descending** protocol for second-price auction rule is DSIC implementable.
- **Descending** protocol for first-price auction rule has an equilibrium equivalent to the BNE in a static auction.

Conclusion

Today.

- **Defined contextual privacy.**
  - The designer only learns what they need to know to compute the choice rule.
- **Looked for contextually private choice rules.**
  - SPA ✗
  - Individual vs Collective Pivotality (Corners Lemma).
- **Characterized maximally contextually private protocols.**

  Maximally contextually private protocols…
  - are bi-monotonic.
  - trade off the privacy of different agents.

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Tomorrow?

Contextual privacy and IC?

Privacy-implementation frontier?

Other restrictions on protocols?

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Thanks!

zhitzig@g.harvard.edu

That's It

Comments?

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zhitzig@g.harvard.edu

Indirect Protocols

- $N < \infty$ agents
- Private information $\theta_i \in \Theta$, $|\Theta| < \infty$, profiles $\boldsymbol{\theta} \in \boldsymbol{\Theta}$
- Outcomes $x \in X$
- Preferences over outcomes $u_i : X \to \mathbb{R}$
- Choice rule $\phi : \boldsymbol{\Theta} \to X$, deterministic
- Universal message set M
- Set of **elicitation technologies** $S \subseteq \{f : M^n \to \tilde{M}\}$

Learn everything

$\text{id} : M^n \to \tilde{M} := M^n$

$\mathbf{m} \mapsto \mathbf{m}$

Learn agent i's msg

$\text{proj}_i : M^n \to \tilde{M} := M$

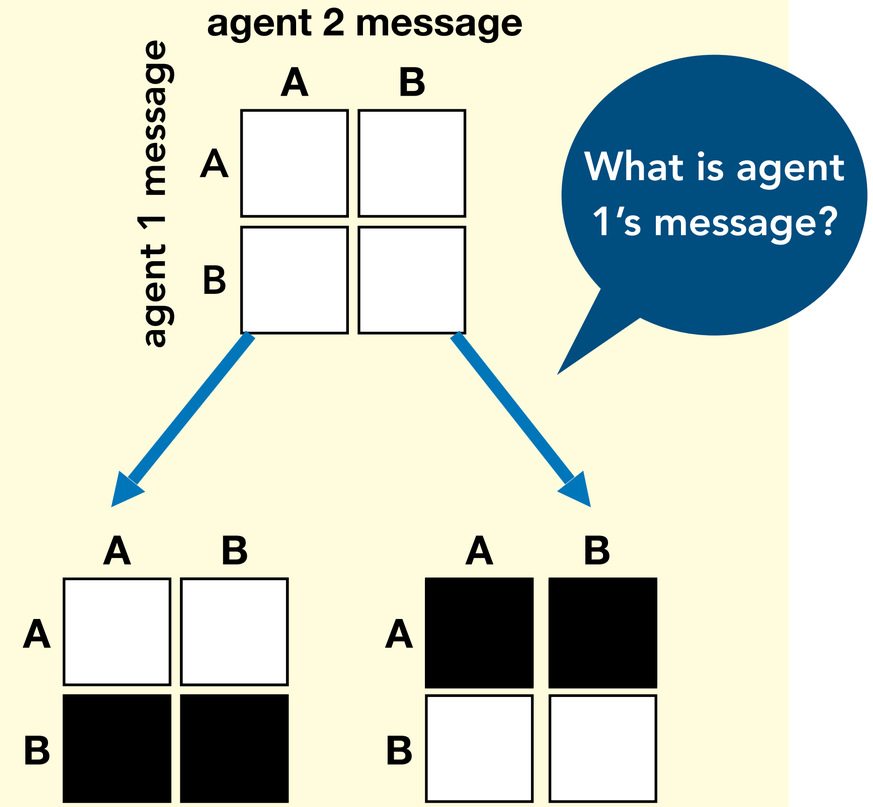
$\mathbf{m} \mapsto m_i$

Count msgs

$\text{count} : M^n \to \tilde{M} := \mathbb{N}^M$

$\mathbf{m} \mapsto (|\{i : m_i = m\}|)_{m \in M}$

↪ back



Example. A simple protocol.

- Two agents $N = \{1, 2\}$
- Type space $\Theta = \{1,2,3\}$
- Message space $M = \{A, B\}$

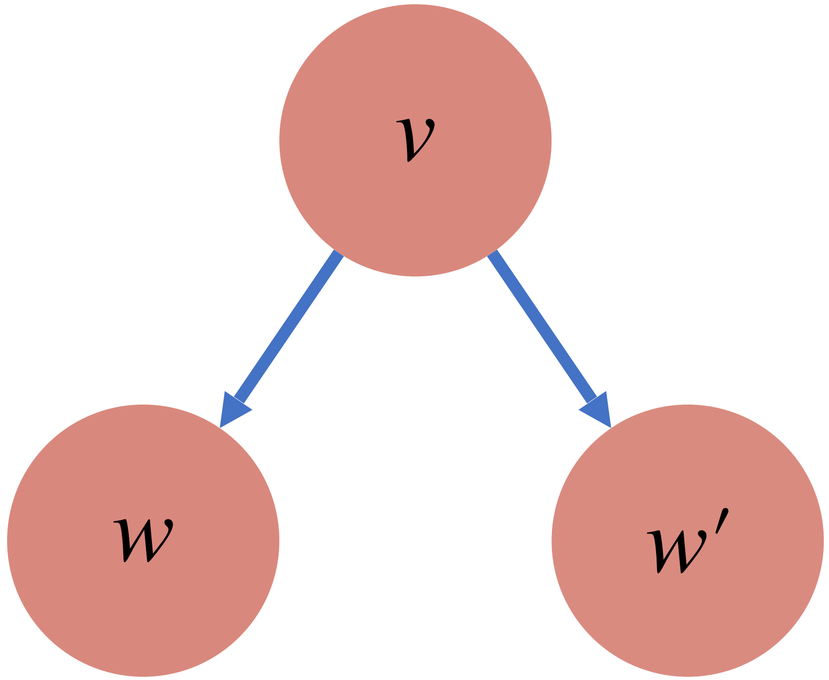
TECHNOLOGIES

Indirect Protocols

Definition.

A (direct) **protocol** is an extensive-form game in which agents repeatedly submit **messages** $\mathbf{m} \in M^n$ from a set M according to reporting strategies $(\sigma_1, \sigma_2, \ldots, \sigma_n)$. A node v must be a function of the responses of the technology,

$$(f_w(\mathbf{m}_w))_{w \in \text{ancestor}(v)}.$$



The designer learns about the types through a protocol.

Indirect Protocols

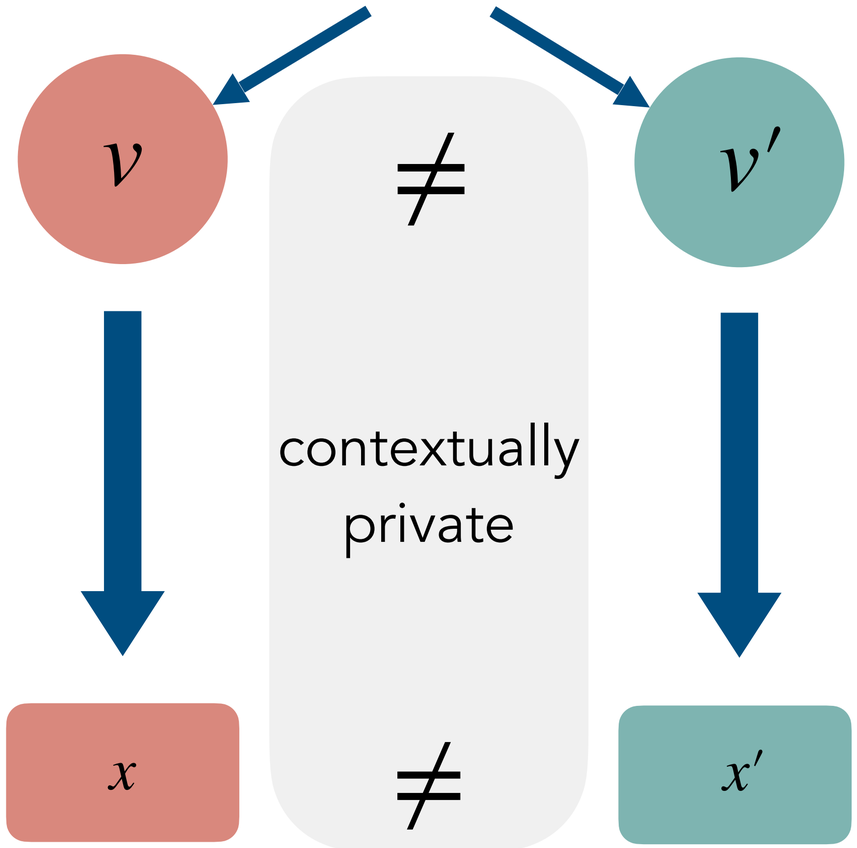
Definition.

A **protocol P is contextually private** if a unilateral change in messages $(m_i, \mathbf{m}_{-i})$, $(m_i', \mathbf{m}_{-i})$ that P "distinguishes" lead to different outcomes

$$x(m_i, \mathbf{m}_{-i}) \neq x(m_i, \mathbf{m}_{-i}).$$

Definition.

A **social choice function ϕ is contextually private** if there are reporting strategies $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ such that $(P, \boldsymbol{\sigma})$ "implement" ϕ and P is contextually private.



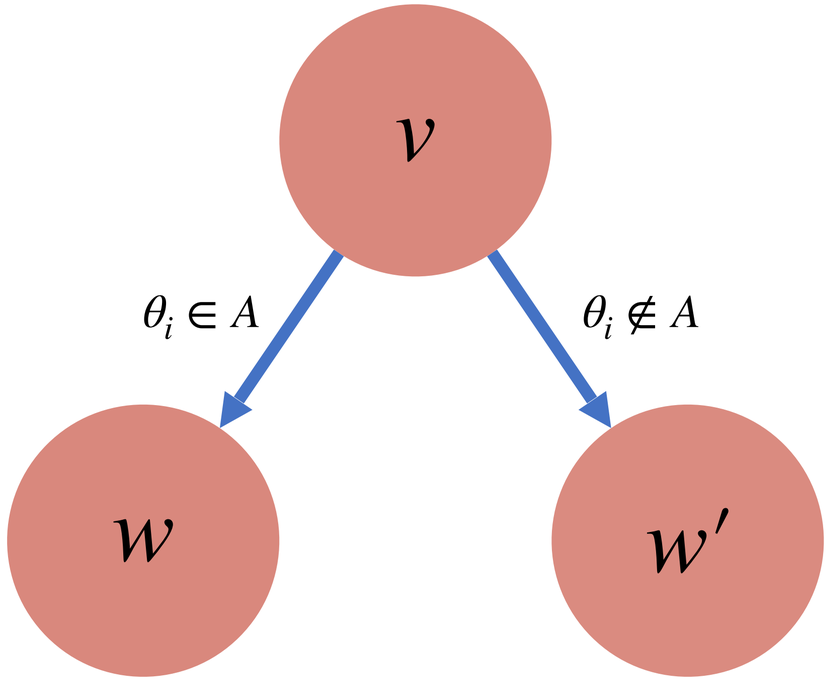
Indirect Protocols

Definition.

Let $S_{\mathrm{SE}} = \{\pi \circ \mathrm{proj}_i \mid \pi\colon \Theta_i \to \Theta_i\}$, the sequential elicitation technology.

Proposition.

Let $M \supseteq \Theta$. For every S_{SE}-protocol P for ϕ with strategy $\boldsymbol{\sigma}$ there exists a contextual privacy equivalent protocol P' for ϕ with strategy $\sigma_i\colon (\theta_i, h) \mapsto \theta_i$.



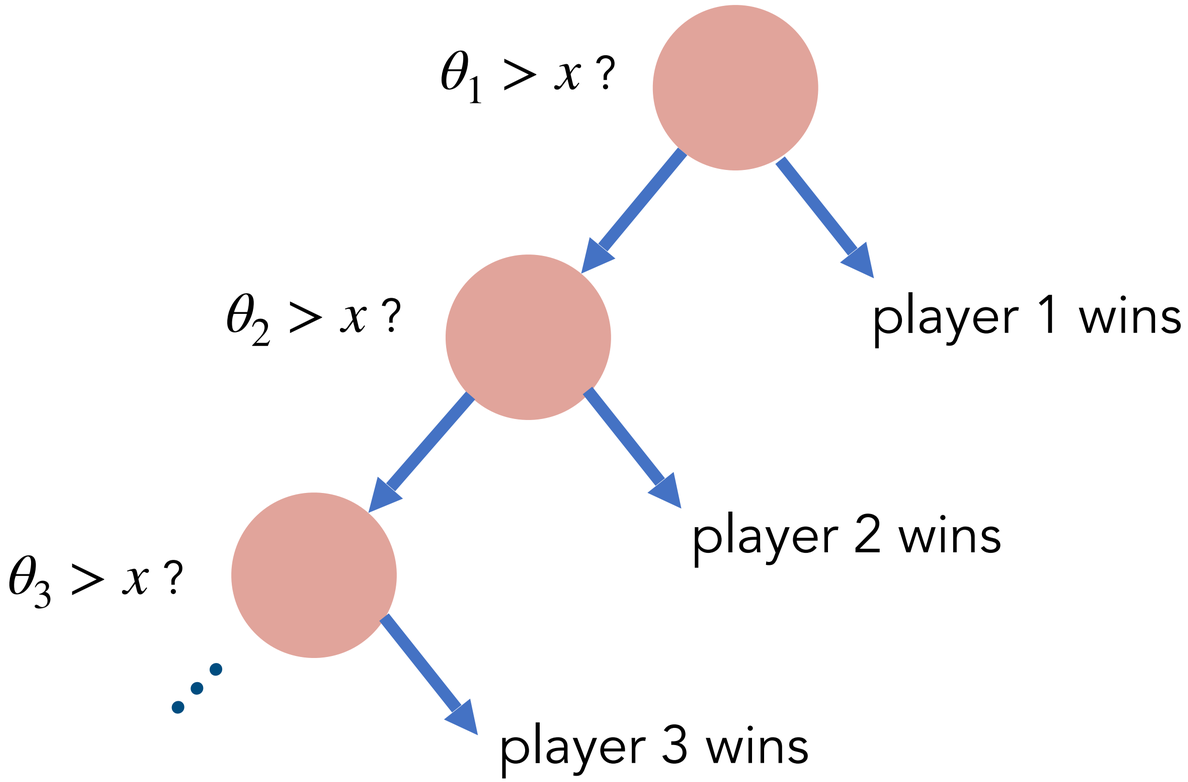
S_{SE} is necessary for this result to hold. Consider, e.g., counting.

FPA Is Contextually Private

Proposition.

The first-price auction choice rule is contextually private under sequential elicitation protocols, with a descending ("Dutch") protocol.

Illustration of proof.



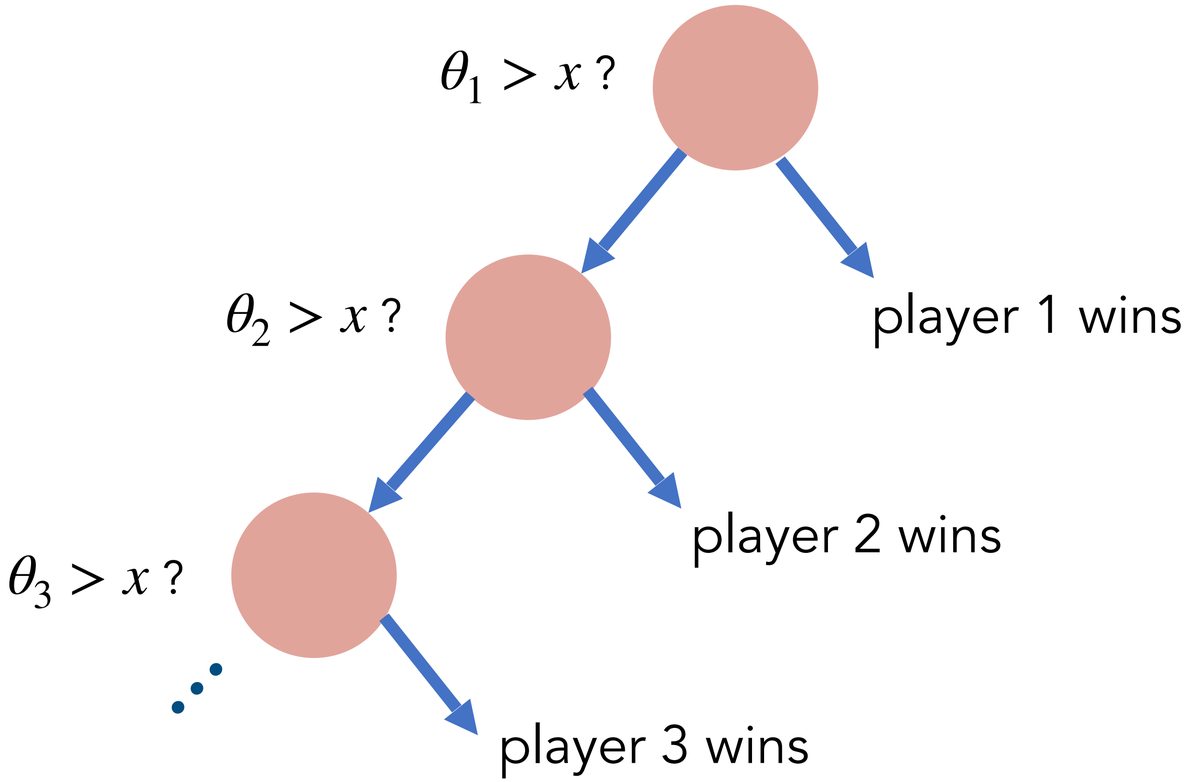
FPA Is Contextually Private

Proposition.

The first-price auction choice rule is contextually private under sequential elicitation protocols, with a descending ("Dutch") protocol.

Illustration of proof.

If at any point, one agent had answered "yes" instead of "no," the outcome would be different.



VARIANTS

Definition.

A protocol P is **individually** contextually private if all type profiles at distinct terminal nodes v, v' that differ for agent i

$$(\theta_i, \boldsymbol{\theta}_{-i}) = \boldsymbol{\theta}_v \in \boldsymbol{\Theta}_v \,, \;\; (\theta'_i, \boldsymbol{\theta}_{-i}) = \boldsymbol{\theta}_{v'} \in \boldsymbol{\Theta}_{v'}$$

lead to different outcomes for agent i

$$\phi_i(\boldsymbol{\theta}_v) \neq \phi_i(\boldsymbol{\theta}_{v'}).$$

Proposition.

ϕ is individually contextually private if and only if it is non-bossy and contextually private.

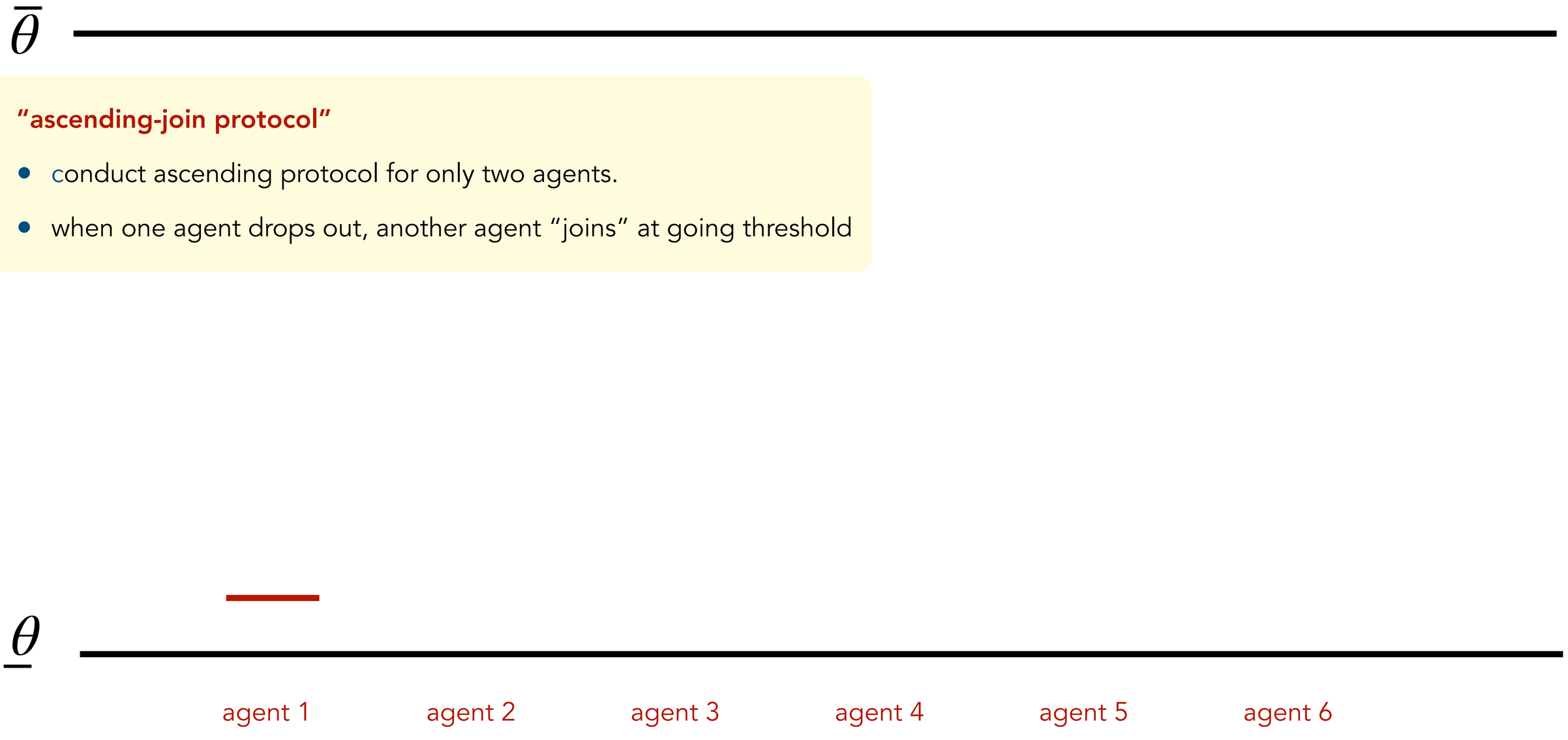
ϕ is non-bossy if

$$\phi_i(\theta_i, \boldsymbol{\theta}_{-i}) = \phi_i(\theta'_i, \boldsymbol{\theta}_{-i})$$

implies

$$\phi(\theta_i, \boldsymbol{\theta}_{-i}) = \phi(\theta'_i, \boldsymbol{\theta}_{-i}) \,.$$

A Maximally Contextually Private Protocol



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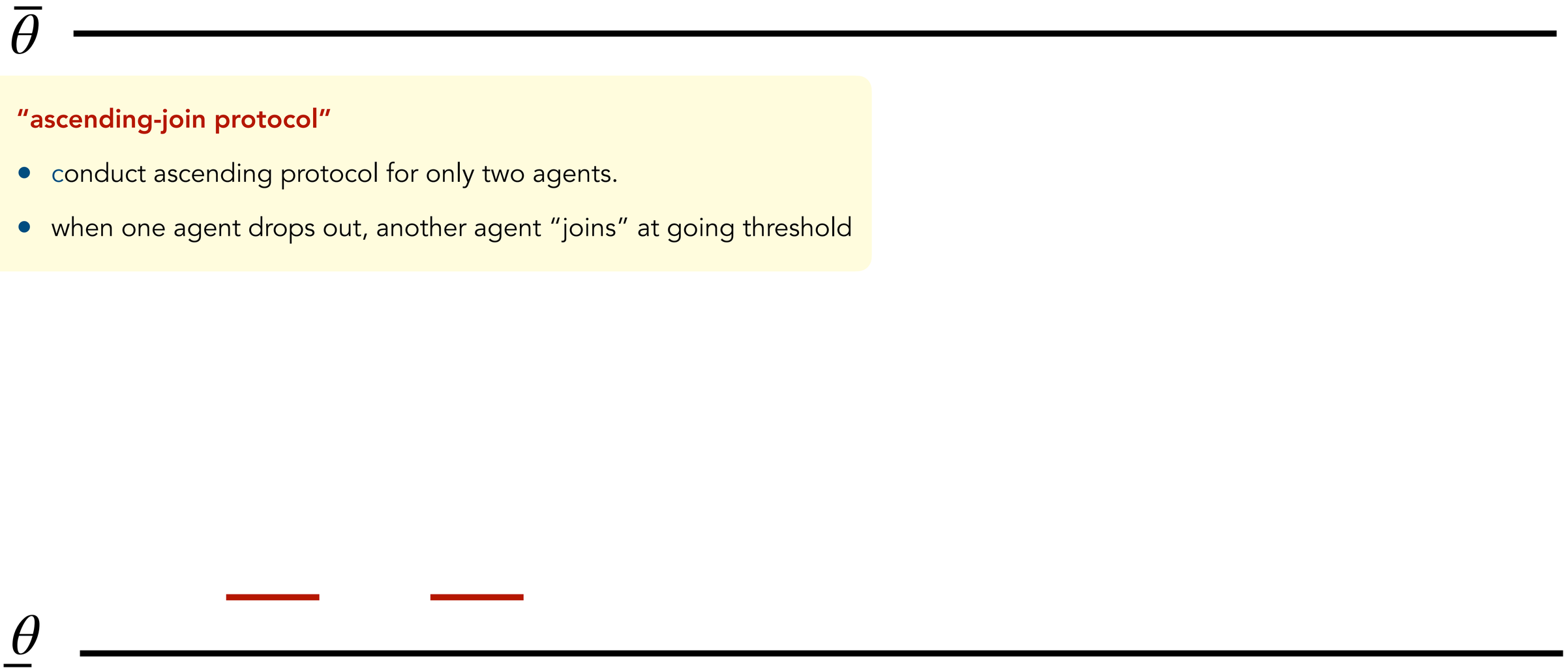

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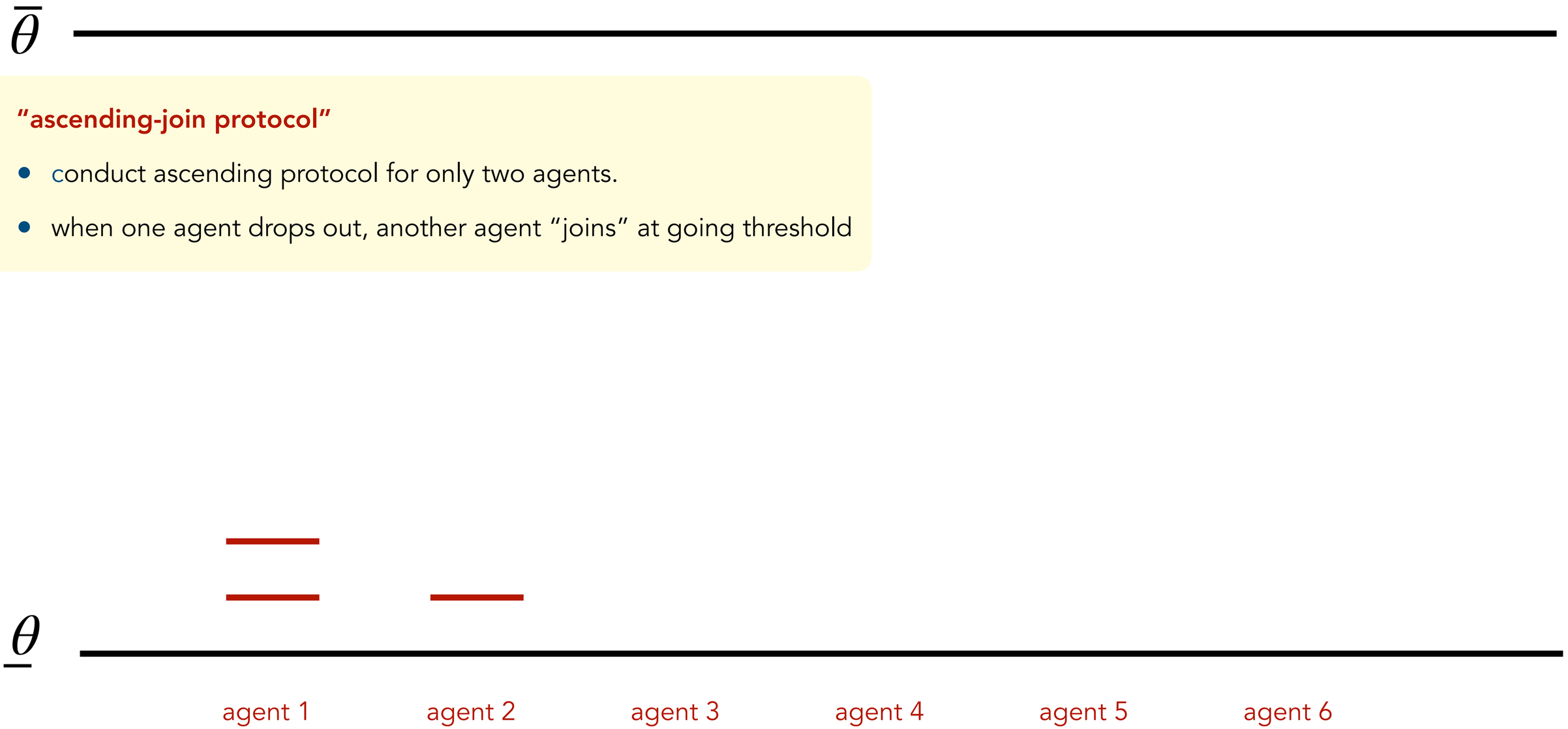

A Maximally Contextually Private Protocol

$\overline{\theta}$

"ascending-join protocol"

- conduct ascending protocol for only two agents.
- when one agent drops out, another agent "joins" at going threshold

$\underline{\theta}$

agent 1 agent 2 agent 3 agent 4 agent 5 agent 6

A Maximally Contextually Private Protocol

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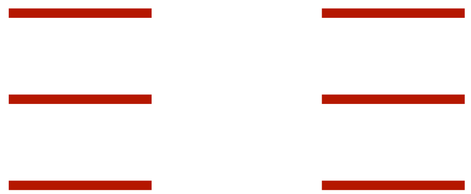

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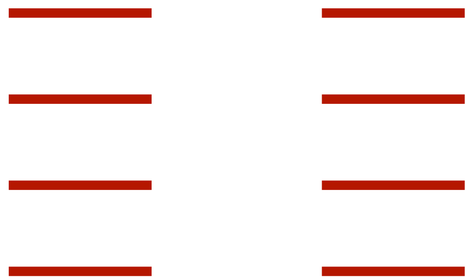

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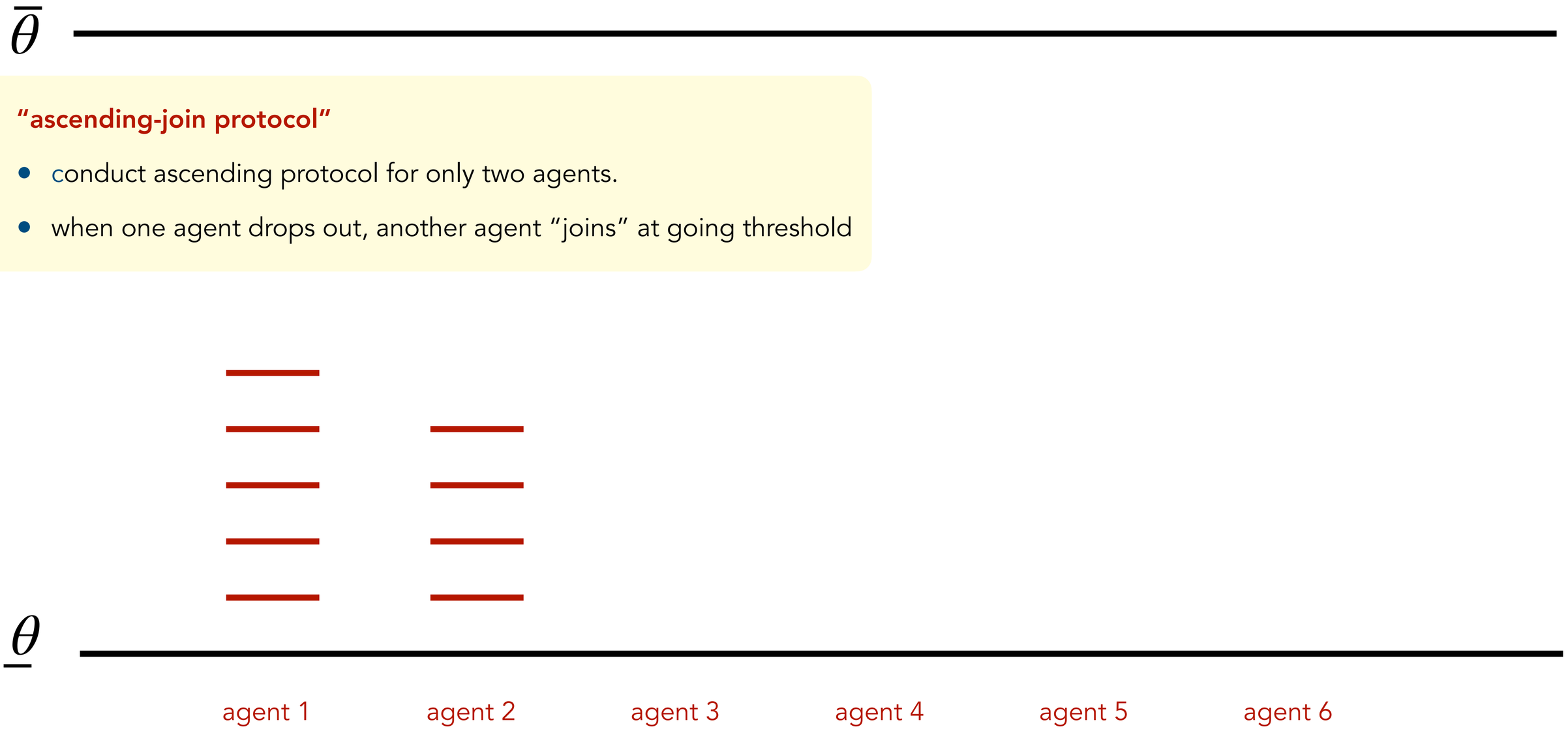

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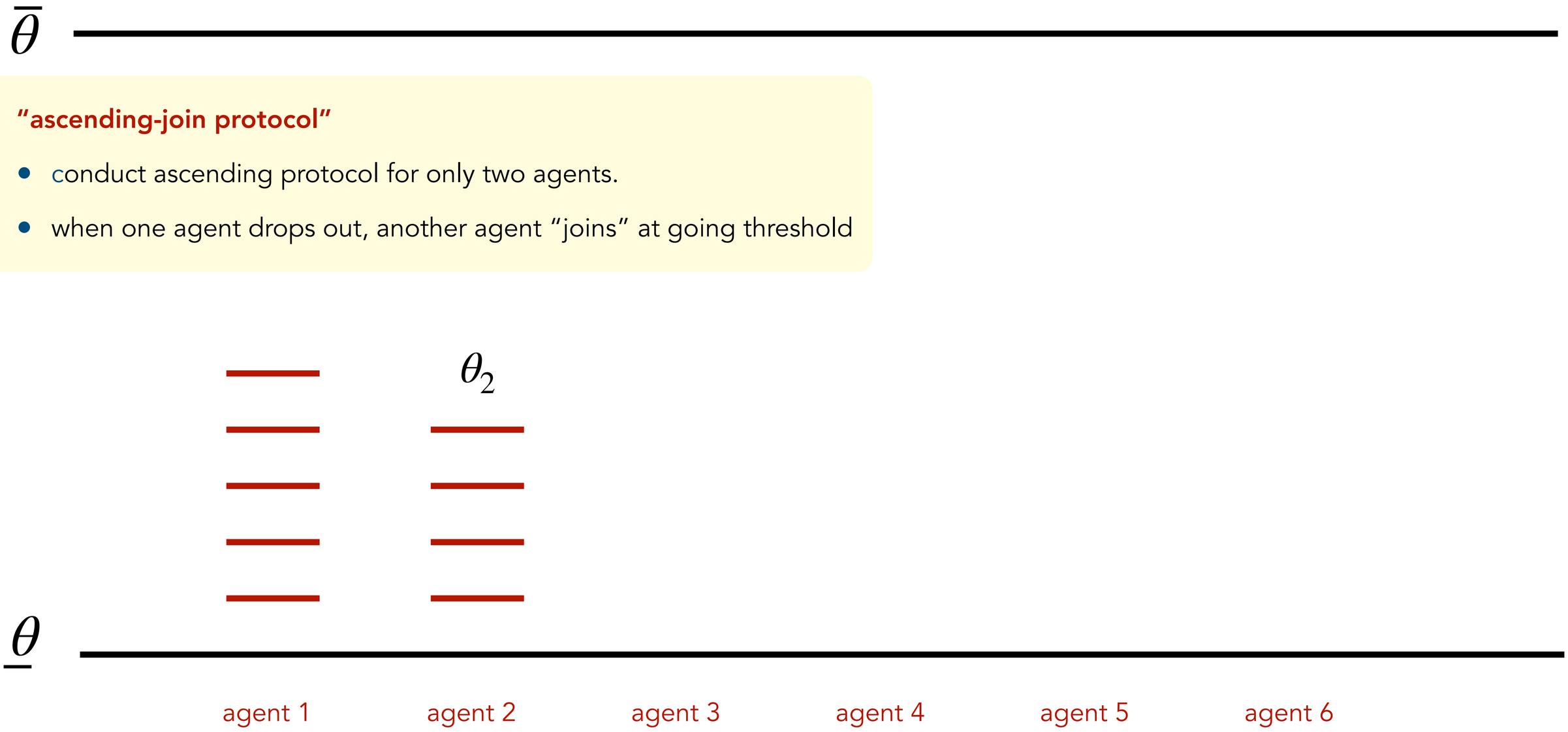

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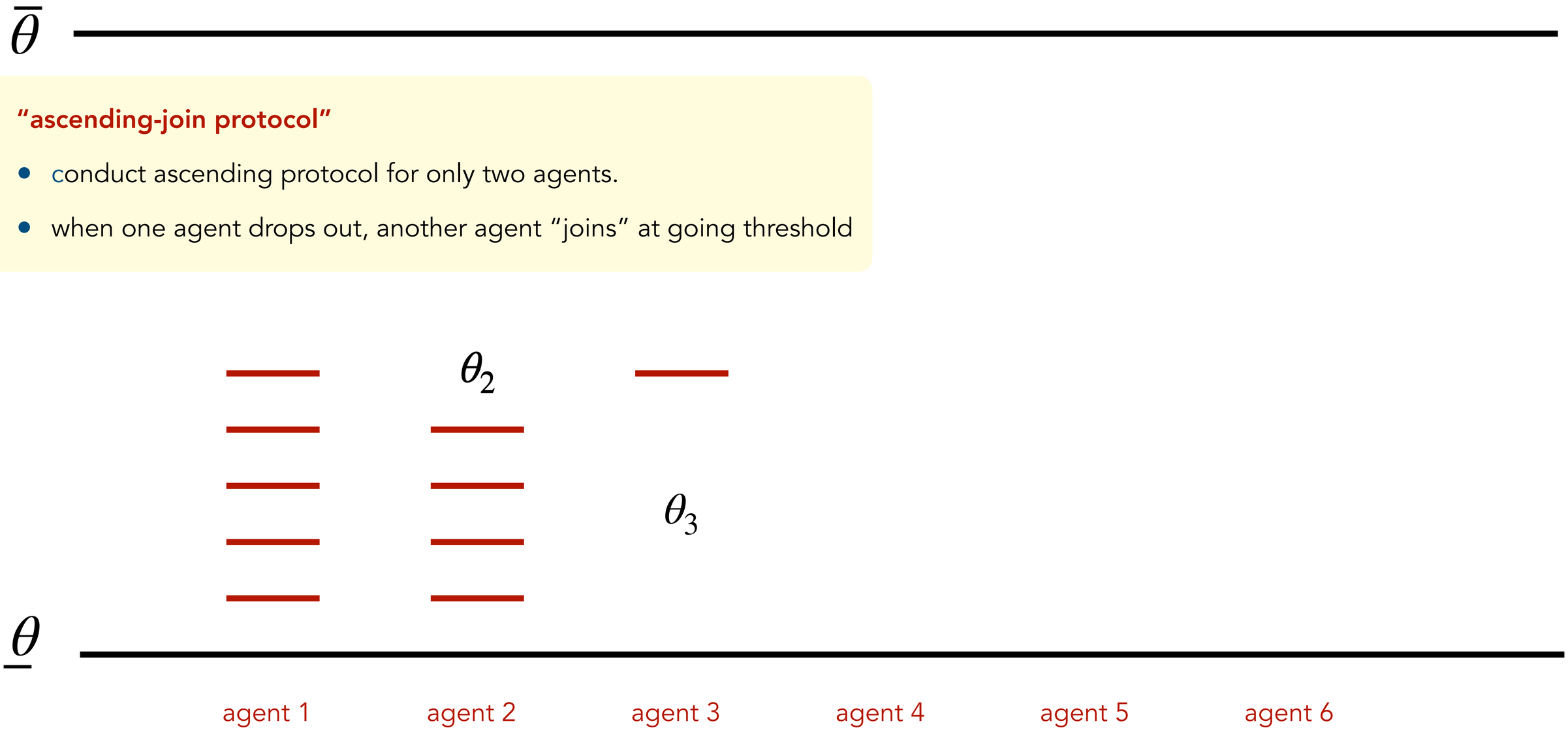

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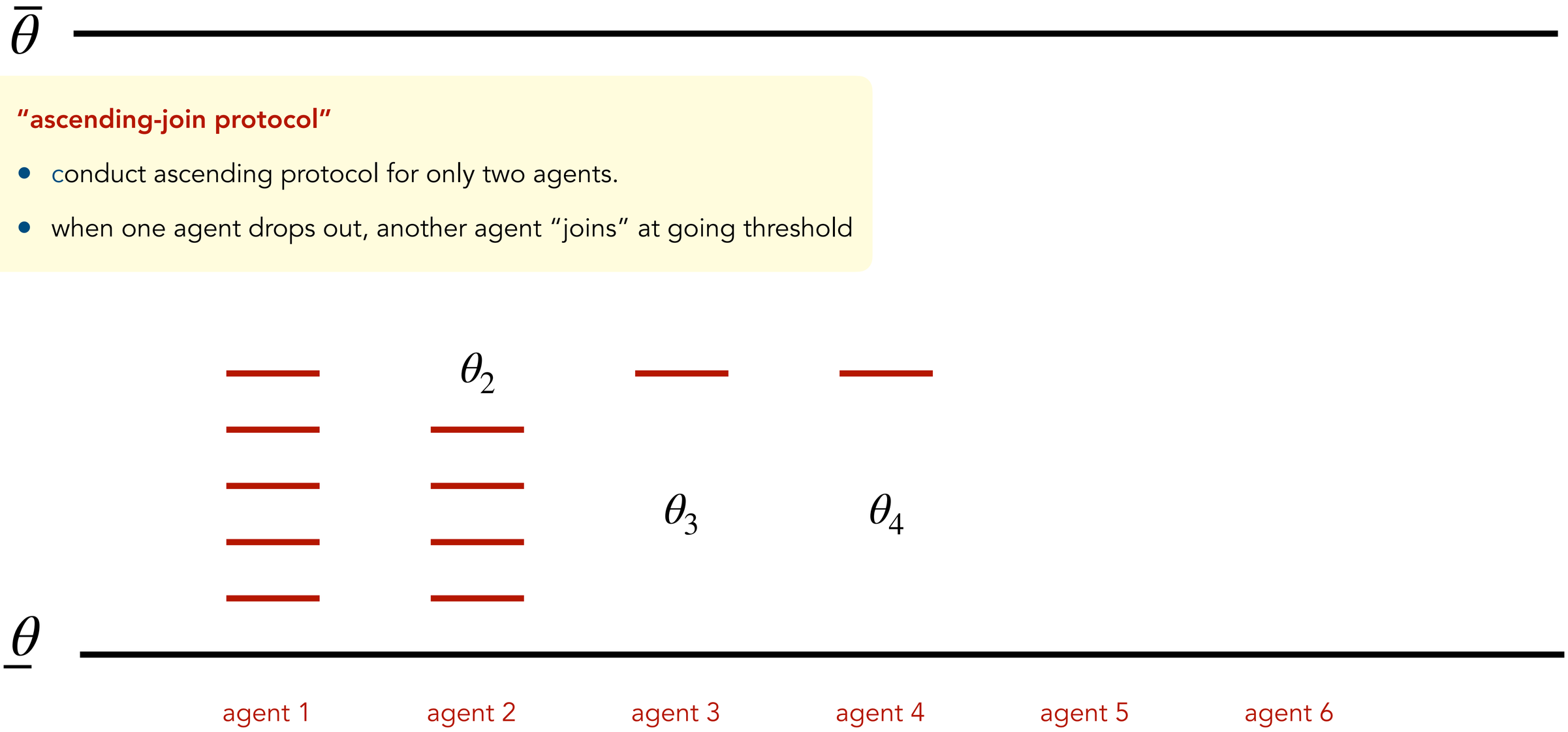

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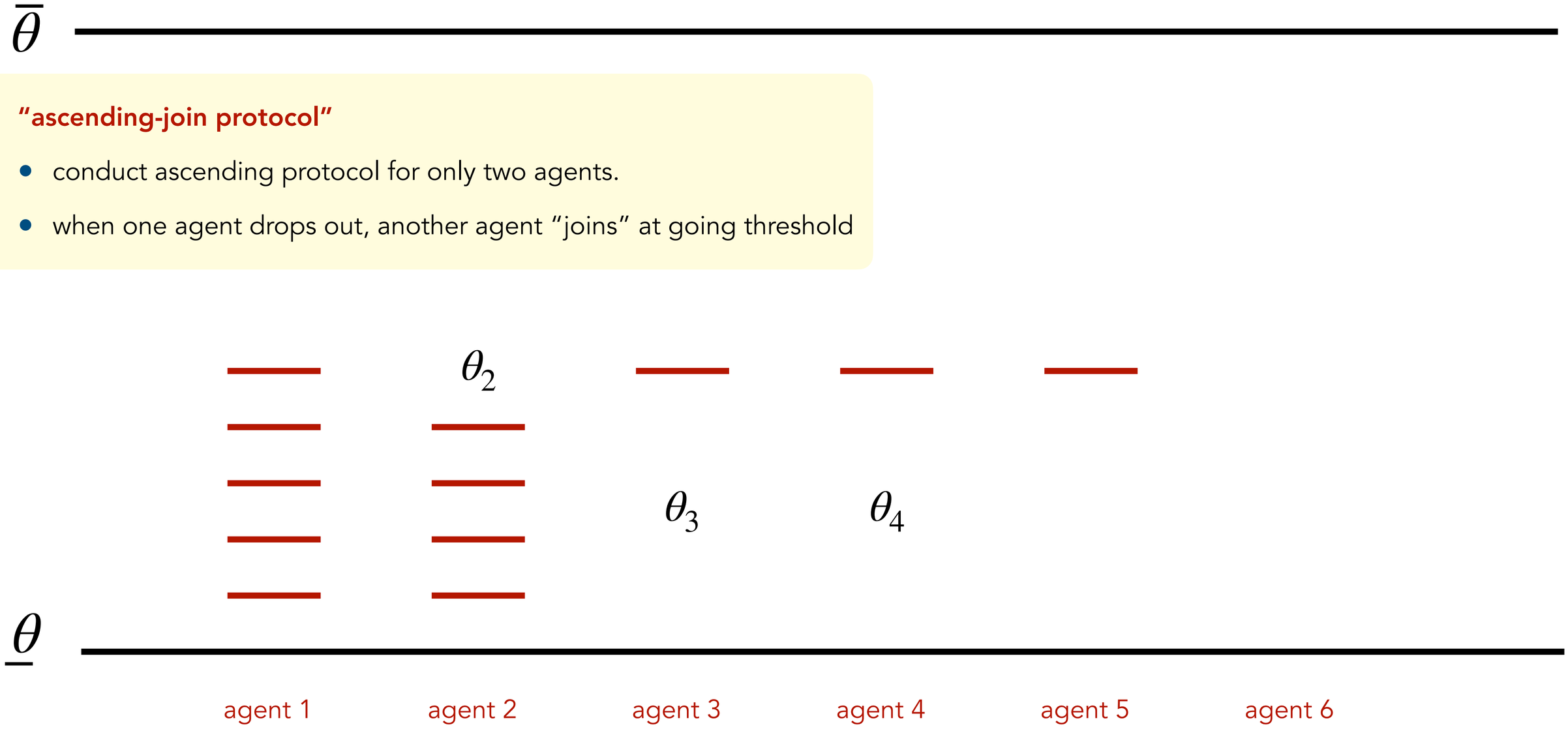

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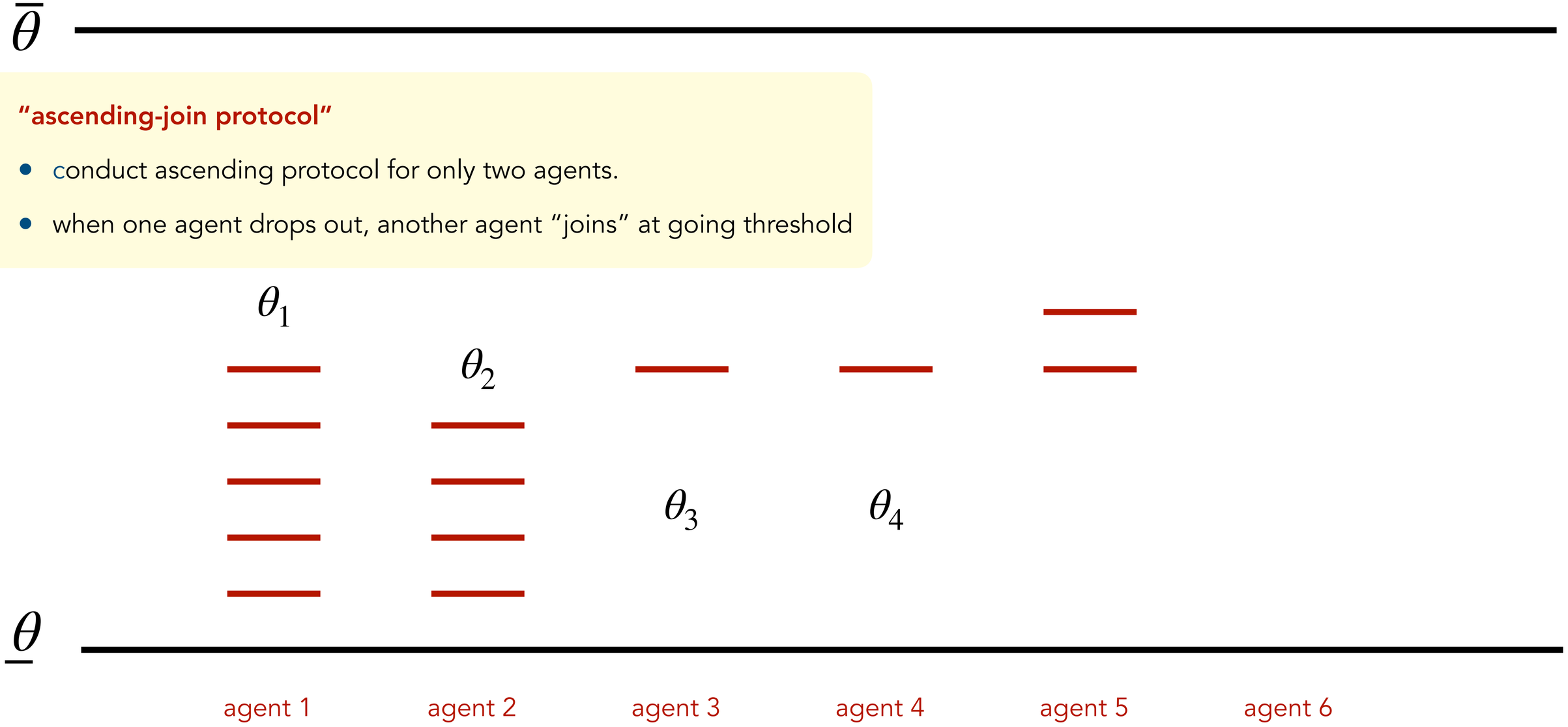

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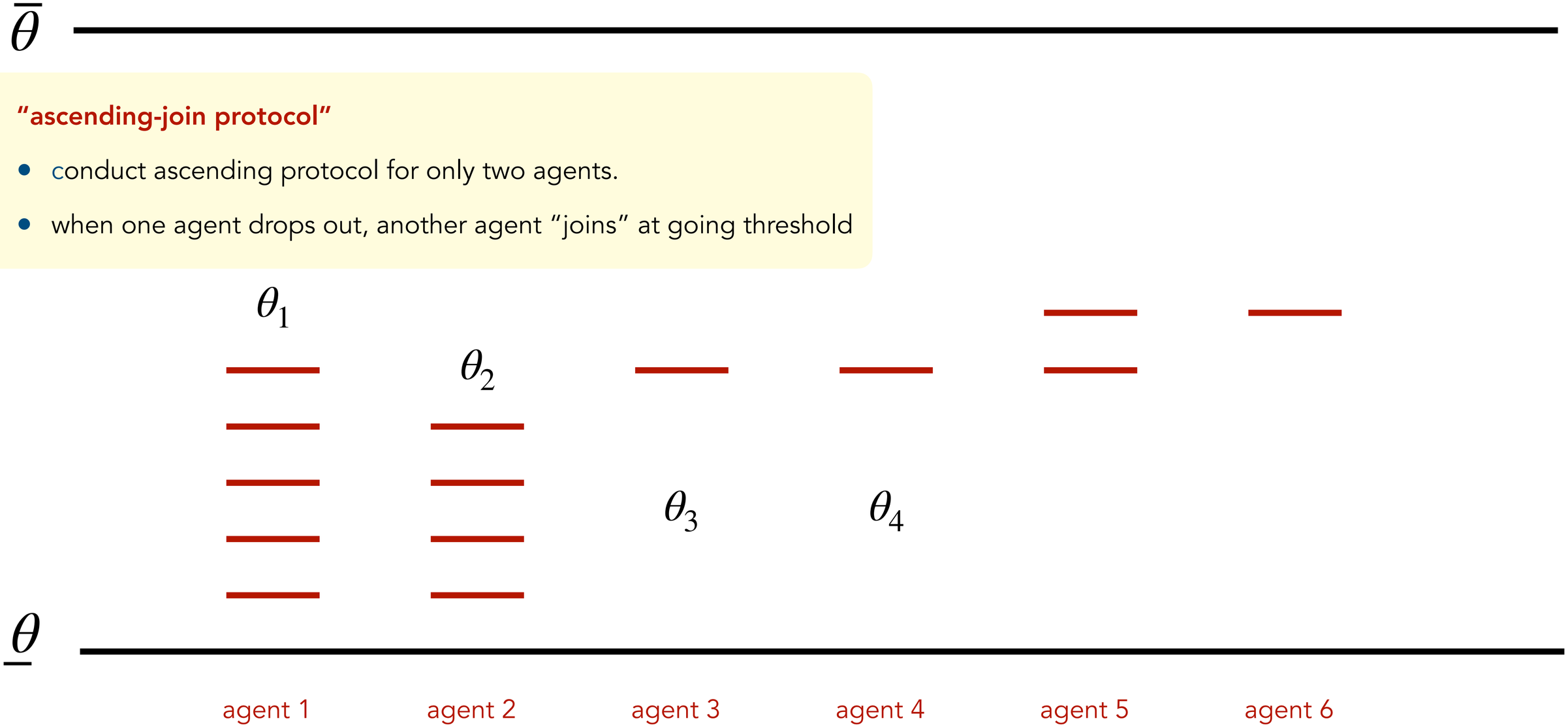

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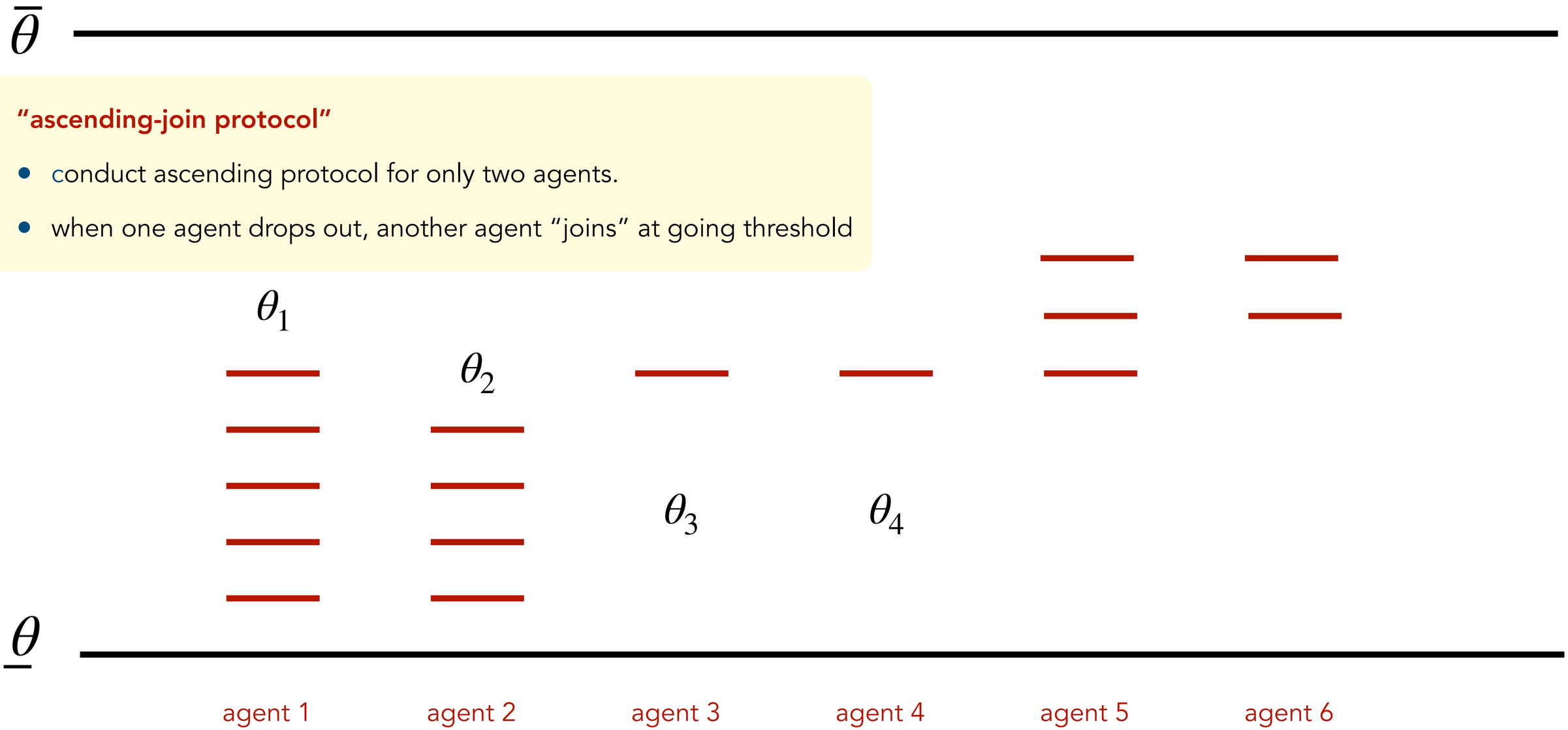

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