Amplitude Sampling

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Abstract— The theoretical basis for conventional acquisition of bandlimited signals typically relies on uniform time sampling and assumes infinite-precision amplitude values. In this paper, we explore signal representation and recovery based on uniform amplitude sampling with assumed infinite precision timing information. The approach is based on applying a one-level level-crossing detector to the result of adding a sawtooth waveform to the source signal. The source signal is then represented by the level-crossing times. For analysis purposes, the output of the level-crossing detector is interpreted as the result of applying a multi-level level crossing detector to the monotonic function consisting of the sum of the source signal and an appropriate linear ramp. This monotonic function is then uniformly sampled in amplitude with the source signal again represented by the level crossing times of the monotonic function. We refer to this technique as amplitude sampling. The approach can equivalently be viewed as nonuniform time sampling of the original source signal or uniform amplitude sampling of the monotonic function to which it is transformed. In this paper we explore this approach and, in particular, present duality and frequency-domain properties for the functions involved in the transformation and develop and compare two iterative algorithms for recovery of the source signal.

I. INTRODUCTION

The traditional sampling theorem [1], [2], [3], which forms the basis for most conventional signal acquisition systems, describes the class of bandlimited signals by infinite-precision amplitude values taken at uniform instants of time. In this representation the sampling times are integer multiples of the sampling interval and the amplitude samples may assume values on a continuum.

In this paper, we consider uniform sampling in amplitude corresponding to restricting amplitude information to integer multiples of a basic step size and allow the time values of the crossings of these levels to be on the continuum. One class of examples of this type of representation is that characterized by Logan's theorem [4]. In that work, it is shown theoretically that a class of bandpass signals can be completely described within a scaling factor by its zero crossings. Tractable recovery techniques based on Logan's theorem currently exist only for periodic signals [5]. Zero-crossings in the context of wavelet transforms have also been considered in [6]. Some extensions to level-crossing sampling which assumes equally-spaced time instants has been considered in [7] in the context of signal compression with a continuous-time version later proposed in [8].

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In this paper, we explore signal representation and recovery based on uniform amplitude sampling of a signal derived from the source signal with assumed infinite precision timing information. Our approach is based on the concept of transforming the source signal to a monotonic signal $g(t)$ which is then represented by the sequence of times $\{t_n\}$ at which it crosses successive equally spaced amplitude levels $\{n\Delta\}$. In principle, a nonmonotonic function can always be represented by the monotonic function which results from addition of a known ramp of sufficient slope which can then later be subtracted as part of the reconstruction process. As we discuss below, this time sequence $\{t_n\}$ is equivalent to the time sequence representing the successive times at which the source signal plus an appropriately constructed bounded sawtooth waveform reaches the amplitude value Δ. For conceptual simplicity in the analysis in this paper, we view the time sequence $\{t_n\}$ in terms of ramp addition to the source signal although a more practical implementation is suggested by the alternative representation with sawtooth waveform addition. We refer to either description as amplitude sampling since as we show, the sequence $\{t_n\}$ represents sample values of an underlying function at equally-spaced amplitude values. In this paper, we explore this concept of amplitude sampling and, in particular, present duality and frequency-domain properties for the underlying associated functions. As we discuss, the sequence $\{t_n\}$ can also be viewed as nonuniformly spaced time samples of the original source signal. This suggests that reconstruction can alternatively be based on utilizing nonuniform time-sampling reconstruction methods. We propose a specific approximate algorithm for reconstruction of the source signal from $\{t_n\}$ based on the interpretation of ramp addition and compare that algorithm to the use of a specific iterative algorithm for reconstruction from non-uniform time samples.

II. PRINCIPLE OF AMPLITUDE SAMPLING

Our analysis of amplitude sampling is based on the concept of reversibly transforming a source signal $f(t)$ into a strictly monotonic function $g(t)$ by the addition of a linear ramp and representing the source signal through the sequence of time instants $\{t_n\}$ at which the monotonic function crosses a set of equally spaced amplitude values $\{n\Delta\}$. For the strictly monotonic function $g(t)$ there is a one-to-one correspondence between its amplitude values $\{n\Delta\}$ and the sequence $\{t_n\}$, i.e. (see Fig. 1) and since the ramp is known, the time sequence $\{t_n\}$ also implicitly contains information about the source signal $f(t)$. In particular, we have that

$$g(t_n) = n\Delta = \alpha t_n + f(t_n). \tag{1}$$

Equivalently, it is possible to obtain the same sequence of time instants by considering the block diagram in Fig. 2. The Level Detector outputs an impulse whenever its input takes the value Δ. In the feedback loop, the ramp-segment generator takes as a reference the ramp αt that is shifted by an amount $-\Delta$ at times where the level detector generates an impulse. In other words,

$$\tilde{g}(t) = f(t) + \alpha t - k\Delta \tag{2}$$

for $t \in (t_k, t_{k+1}]$ and $k \in \mathbb{Z}$. Thus,

$$\tilde{g}(t_{k+1}) = \Delta = f(t_{k+1}) + \alpha t_{k+1} - k\Delta \tag{3}$$

which gives $(k+1)\Delta = g(t_{k+1})$ for all $k \in \mathbb{Z}$. Since there is a one-to-one correspondence between amplitude values and time instants due to the monotonicity of $g(t)$, this implies that $\{t_k\} = \{t_n\}$. In summary, the procedure shown in Fig. 2, which only involves bounded signals, generates impulses at the same time instants at which $g(t)$ in Fig. 1 crosses the set of amplitude levels $\{n\Delta\}$.
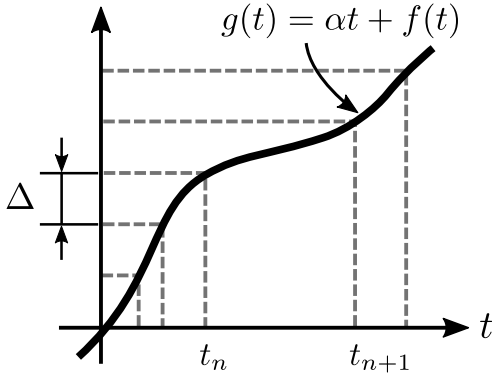

Fig. 1. Principle of amplitude sampling based on a transformation by ramp addition of the source signal f resulting in a monotonic function $g(t) = \alpha t + f(t)$ with a sufficiently large value $|\alpha|$.
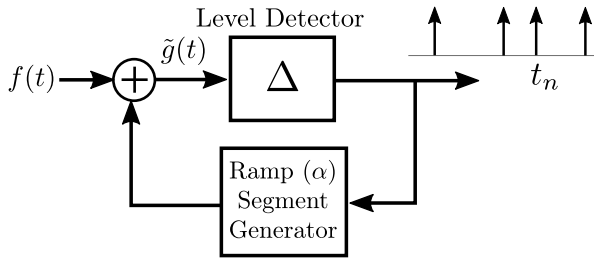

Fig. 2. Equivalent representation of the amplitude sampling process.

III. TRANSFORMATION BY RAMP ADDITION

Assume that for a continuous and bounded input signal $f(t)$ and appropriate choice of $\alpha \in \mathbb{R}$, we form the variable u which is a strictly monotonic function of t :

$$u = g(t) = \alpha t + f(t). \tag{4}$$

We also consider the inverse function of $g(t)$ denoted by $\hat{g}(u)$. The independent variable u then represents the amplitude of $g(t)$ which contains the amplitude of $f(t)$ plus

the values of the ramp. It will also be convenient to define the function $h(u)$ representing the result of subtracting the inverse of the ramp from $\hat{g}(u)$ i.e. $h(u) = \hat{g}(u) - u/\alpha$.

A. Mapping Between $f(\cdot)$ and $h(\cdot)$

As illustrated in Fig. 3 the invertibility of $g(\cdot)$ implies an invertible relationship between $f(\cdot)$ and $h(\cdot)$. In effect, we can view the procedure of obtaining $h(\cdot)$ from $f(\cdot)$ as an invertible transformation $M_\alpha f(t) = h(u)$ parameterized by α. In fact, the inverse transformation of M_α, denoted by $(M_\alpha)^{-1}$, is given by substituting the parameter α by its reciprocal, i.e. $(M_\alpha)^{-1} = M_{1/\alpha}$.
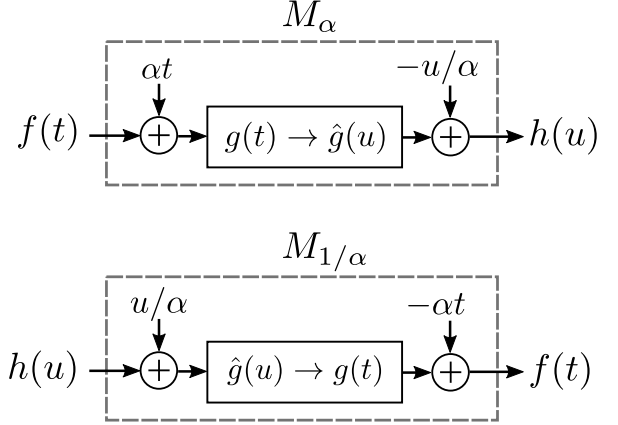

Fig. 3. Illustration of the invertibility of the transformation between f and h when $g(t) = \alpha t + f(t)$ and $\hat{g}(u) = u/\alpha + h(u)$.

Fig. 3 can also be described in terms of the equations [9]

$$
\begin{aligned}
f(t) &= -\alpha h(f(t) + \alpha t) \\
h(u) &= -\frac{1}{\alpha} f(h(u) + \frac{u}{\alpha}).
\end{aligned}
\tag{5}
$$

This set of equations suggests a signal-dependent time warping so as to obtain $f(t)$ from $h(u)$ or the reverse. Equivalently, we summarize these equations in matrix form as

$$
\begin{pmatrix} f(t) \\ t \end{pmatrix} = \begin{pmatrix} -\alpha & 0 \\ 1 & 1/\alpha \end{pmatrix} \begin{pmatrix} h(u) \\ u \end{pmatrix}. \tag{6}
$$

We have focused so far on the transformation from the point of view of $f(t)$, i.e. addition of a ramp with a sufficiently large slope and subsequent inversion of the resulting function. However, a duality in the transformation is evident in either (5) or by inverting (6) to obtain

$$
\begin{pmatrix} h(u) \\ u \end{pmatrix} = \begin{pmatrix} -1/\alpha & 0 \\ 1 & \alpha \end{pmatrix} \begin{pmatrix} f(t) \\ t \end{pmatrix}. \tag{7}
$$

If we begin from knowledge of $h(u)$, the function $f(t)$ is obtained by inverting $h(u)$ and adding a ramp of appropriate slope. This is conceptually dual to starting with $f(t)$ and constructing $h(u)$ (see Fig. 3). In summary, this duality in the transformation implies that any properties of $h(u)$ inherited by assumptions made on $f(t)$ hold, by duality, for $f(t)$ if such assumptions are first imposed on $h(u)$.

B. The Sampling Process

The sampling process corresponds to sampling the amplitude of $g(t)$ at equally spaced amplitude intervals to obtain the sequence of times t_n. This is equivalent to sampling $h(u)$ at equally spaced intervals $u = n\Delta$ i.e.

$$h(n\Delta) = t_n - n\Delta/\alpha. \qquad (8)$$

.

Clearly, if the function $h(u)$ belongs to a class of functions that can be described by their samples on a uniform grid, for example if $h(u)$ is bandlimited, then perfect reconstruction is possible if the sampling interval Δ is sufficiently small. Furthermore, since there is a one-to-one relation between $f(t)$ and $h(u)$, reconstruction of $h(u)$ provides $f(t)$ through the transformation $M_{1/\alpha}h(u) = f(t)$. This amplitude sampling technique can of course also be interpreted as a form of time encoding, i.e. the information about the source signal is implicitly contained in the time instants $\{t_n\}$ since the quantizer levels are known.

C. Sampling Density

The times associated with the level crossings of $g(t)$ can in effect be interpreted as providing amplitude samples of the source signal $f(t)$ non-uniformly spaced in time. If we assume that $f(t)$ is bandlimited, there are well known results on the required density of the nonuniformly-spaced time samples to permit perfect reconstruction. To explore the sampling density implied by our sampling process, assume that the source signal $f(t)$ is differentiable with bounded derivative B, i.e. $|f'(t)| \leq B$ for $B > 0$. The slope α of the ramp is set such that $|\alpha| > B$. For increasing and positive values of the derivative of $f(t)$ the slope of $g(t)$ increases, thus generating an increasing number of level crossings per unit of time. When $f'(t)$ is close to $-B$ the slope of $g(t)$ is reversed. As a result, the signal $f(t)$ is naturally sampled more densely as its derivative becomes more positive and less densely as the derivative becomes more negative. Furthermore, reducing the value of Δ leads to an increase in the overall level-crossing density.

Specifically, it can be shown that the time intervals between level crossings can be bounded as [10]:

$$\frac{\Delta}{|\alpha| + B} \leq |t_{n+1} - t_n| \leq \frac{\Delta}{|\alpha| - B} \qquad (9)$$

for all $n \in \mathbb{Z}$. Equation (9) emphasizes the role that is played by the slope α of the ramp and the quantizer step size Δ. It can be observed that if $|\alpha|$ increases significantly, the bounds imply an almost uniform time spacing. Indeed, as the value of $|\alpha|$ increases, the contribution of $f(t)$ in amplitude to the value of $g(t)$ becomes less significant. In the extreme case, sampling $g(t)$ would be close to the situation of sampling a straight line (that, clearly, would generate a uniform time sequence) with the amplitude values of $f(t)$ encoded in very slight modulation of the values of t_n.

IV. SPECTRAL PROPERTIES

Since the function $h(u)$ is sampled uniformly, it is important to understand its frequency domain characteristics in relation to those of the source signal $f(t)$. In this section, we consider the spectral content of $h(u)$ when the source signal $f(t)$ is bandlimited. By duality, the corresponding conclusions apply to $f(t)$ when $h(u)$ is assumed to be bandlimited.

Consider the example in Fig. 4 in which $f(t)$ is a sinc function—hence bandlimited—and α is positive. The shape of the function $h(u)$ can be viewed as $f(t)$ with a nonlinear warping of the independent variable. It presents shear-like parts that may manifest themselves as high-frequency content. These tilted regions in $h(u)$ are due to the regions in $f(t)$ with large negative values of the slope. Obviously, analogous conclusions can be drawn if α is negative. Locally, the derivative of the inverse function is inversely proportional to that of the original function. Thus it is reasonable to expect that the spectral content of $h(u)$ is, in some sense, controlled by the difference $|\alpha| - B$, where $|f'(t)| \leq B$ for all $t \in \mathbb{R}$.
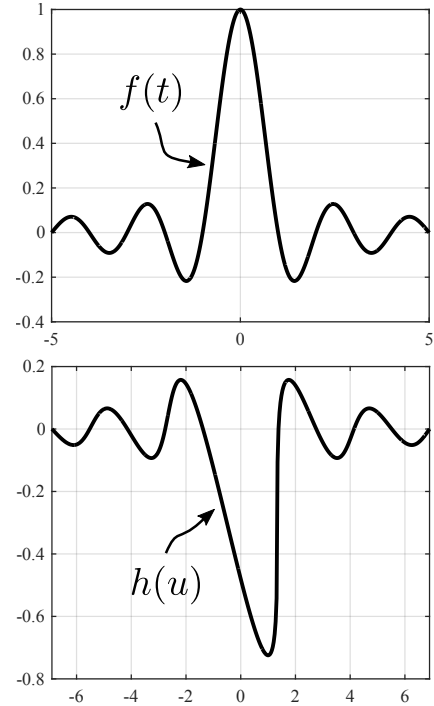

Fig. 4. Example of the transformation in amplitude sampling where $f(t) = \mathrm{sinc}(t)$, $\alpha = 1.38$, and $h = M_\alpha f$.

We formalize this notion in Theorem 1. In particular, it is assumed that $f(t)$ is a bounded function and bandlimited to σ rad/s. In order to choose the slope of the ramp in the transformation M_α, we assumed previously a bound for the first derivative of the input function. Bernstein's inequality [11] provides a bound on the real line for the derivative of a bounded bandlimited function. Specifically, if $f(t)$ is bounded by $A > 0$, its derivative satisfies $|f'(t)| \leq A\sigma$ for all $t \in \mathbb{R}$. Thus, it is sufficient to choose $|\alpha| > A\sigma$ to construct the invertible transformation M_α. In doing so, we

implicitly form the invertible function

$$u = g(t) = f(t) + \alpha t \qquad (10)$$

to derive $h(u) = \hat{g}(u) - u/\alpha$. As a result, it can be shown that the Fourier transform of $h(u)$, denoted by $H(\omega)$ for $\omega \in \mathbb{R}$ in rad/s, satisfies $|H(\omega)| = \mathcal{O}(e^{-|\omega|b})$, where $b > 0$ is dependent on the difference $|\alpha| - A\sigma$.

The precise statement is presented in the following theorem. Its detailed proof can be found in [10]. For clarity in the proofs but without loss of generality, the assumption about the decay on the real line of $h(u)$ is made. However, the theorem can be extended in a straightforward manner to square-integrable functions.

Theorem 1: Let $f(t) : \mathbb{R} \to \mathbb{R}$ be a continuous function bandlimited to $\sigma > 0$ rad/s. Assume further that $|f(t)| \leq A/(1 + t^2)$ for all $t \in \mathbb{R}$ and some $A > 0$. Construct the function

$$u = g(t) = \alpha t + f(t) \qquad (11)$$

for $|\alpha| > A\sigma$. Then, there exists $\hat{g}(u)$, the inverse function of g, for all $u \in \mathbb{R}$ and a constant $C > 0$ such that the Fourier transform of $h(u) = \hat{g}(t) - u/\alpha$ satisfies $|H(\omega)| \leq Ce^{-|\omega|b}$ for any $0 \leq b < a$ such that

$$a = \frac{|\alpha|}{\sigma} \log\left(\frac{|\alpha|}{A\sigma}\right) - \frac{|\alpha| - A\sigma}{\sigma}. \qquad (12)$$

and $\omega \in \mathbb{R}$.

We expected the decay rate to be a function of $|\alpha| - A\sigma$. In effect, the tilted regions in $h(u)$, which are also the highest slope portions, correspond to more horizontal regions in $g(t)$. This is inherited from the properties of the inverse function. Indeed, note that $g'(t) \neq 0$ for all $t \in \mathbb{R}$, thus we can write $(\hat{g}(u))' = 1/g'(\hat{g}(u))$. Loosely speaking, the slope of the inverse function $\hat{g}$ is the reciprocal of the original function g. Therefore, if for example, $\alpha > 0$, regions with large negative values of f' create regions in h with large positive slopes. The latter are in turn responsible for the high-frequency content of $H(\omega)$. The influence of this effect is clearly controlled by the bound on the derivative of $f(t)$, or more precisely, by the difference $|\alpha| - A\sigma$.

In (12), we can clearly observe that the decay of the energy present in high frequencies depends on $|\alpha| - A\sigma$. The difference is logarithmic in the first term and linear in the second. Since we have chosen $|\alpha| > A\sigma$, it always holds that $a > 0$. This is consistent with our previous interpretation: the larger the difference the faster the eventual decay.

It is important to note that the exponential decay of the spectrum of $h(u)$ does not preclude $h(u)$ from being bandlimited, i.e. any bandlimited signal has a Fourier transform that belongs to $\mathcal{O}(e^{-|\omega|b})$ as $|\omega| \to \infty$ for any $b \neq 0$. The next theorem states that, when the signal $f(t)$ is bandlimited, the function $h(u)$ cannot also be bandlimited unless f is a constant. The proof is presented in [10].

Theorem 2: Under the conditions of Theorem 1 and unless $f(t)$ is constant, the function $h(u)$ is nonbandlimited for every $\alpha > A\sigma$ with at most one exception.

This theorem establishes that if $f(t)$ is a nonconstant bandlimited function, then $h(u)$ is essentially guaranteed to be nonbandlimited. Strictly speaking, there may exist a value of α for which we cannot make such an assertion. This is due to the fact that in the proof we have used a theorem [12] that states that an analytic function on the whole complex plane assumes every value in $\mathbb{C}$ with the possible exception of a single point. For practical purposes, we ignore this singular case.

In summary, if the function $f(t)$ is assumed to be bandlimited and nonconstant, then $h(u)$ is nonbandlimited and with an exponential decay in its spectrum. In some sense, the function $h(u)$ becomes closer to a bandlimited function as $|\alpha|$ increases. Recall from (5) that $f(t) = -\alpha h(f(t) + \alpha t)$. This implies that $f(t)$ is close to $h(u)$ for sufficiently large $|\alpha|$ and after appropriate scaling of the axes.

V. NONUNIFORM SAMPLING OF A CLASS OF NONBANDLIMITED SIGNALS

As has been mentioned earlier, amplitude sampling as discussed in this paper, can be viewed as uniformly sampling the function $h(u) = M_\alpha f(t)$ for a source signal $f(t)$ and an appropriate choice of α. If we are able to reconstruct $h(u)$ from its uniform samples, then the input signal $f(t)$ can of course also be recovered due to the invertibility of M_α.

Suppose that $h(u)$ is bounded by some $B > 0$ and bandlimited to σ rad/s. Assume now that we perform the amplitude sampling procedure on $f(t)$ with $f(t)$ and $h(u)$ related through $f(t) = M_{1/\alpha} h(u)$ for a fixed $1/\alpha > B\sigma$. Note that $f(t)$ will not be bandlimited since $h(u)$ is assumed to be. In particular, $f(t)$ belongs to a subset of nonbandlimited signals whose Fourier transform decays at least exponentially fast. Then, amplitude sampling generates the pair of samples $\{t_n, f(t_n)\}_{n \in \mathbb{Z}}$ where $n\Delta = \alpha t_n + f(t_n)$. Clearly, this mechanism generates a nonuniform sampling set for the function $f(t)$. Essentially, we are nonuniformly sampling a nonbandlimited signal.

Moreover, we are performing simultaneously uniform sampling on $h(u)$ with a sample spacing of Δ. Obviously, if we choose $\Delta < \pi/\sigma$ to satisfy the Nyquist criterion, we can then reconstruct $h(u)$ with sinc interpolation, i.e. $f(t)$ can be perfectly recovered through the interpolation of $h(u)$.

VI. RECONSTRUCTION IN AMPLITUDE SAMPLING

A. Amplitude Sampling as Nonuniform Sampling

Amplitude sampling can be regarded as a form of nonuniform time sampling of the source signal $f(t)$. Specifically, it generates the sampling pairs $\{(t_n, f(t_n))\}_{n \in \mathbb{Z}}$ where $f(t_n) = n\Delta - \alpha t_n$ for an appropriately chosen $\alpha \neq 0$. Assuming $f(t)$ is bandlimited to σ rad/s and bounded by $A > 0$, we can always choose $|\alpha| > A\sigma$.

One approach to recovering $f(t)$ is to utilize well known nonuniform sampling reconstruction algorithms to recover $f(t)$ from its samples. Nonuniform sampling theory is based in one way or another on the basic concept of sampling density. In this paper we use the concept of sampling density as stated by Beurling [13] and later used by Landau to derive necessary density conditions for stable reconstruction

of bandlimited signals [14], i.e. for our purposes the sampling density $D^-(\{t_n\})$ is defined as

$$D^-(\{t_n\}) = \liminf_{r \to \infty} \inf_{a \in \mathbb{R}} \frac{\mathfrak{n}([a, a+r])}{r} \qquad (13)$$

where $\mathfrak{n} : \mathbb{R} \supseteq \Lambda \to \mathbb{N}$ is a counting function that indicates the number of elements of $\{t_n\}_{n \in \mathbb{Z}}$ contained in Λ. Additionally, the set $\{t_n\}$ is said to be *separated* if $|t_n - t_m| \geq \delta$ for some $\delta > 0$ and all $n \neq m$. Weakening this notion, $\{t_n\}$ is *relatively separated* if it can be expressed as the finite union of separated sets.

It is then sufficient that $\{t_n\}$ is relatively separated and $D^-(\{t_n\}) > \pi/\sigma$ for f to be completely described by the pairs $\{(t_n, f(t_n))\}_{n \in \mathbb{Z}}$ [15]. In the amplitude sampling setting in this paper, it is clearly seen in (9) that the sequence $\{t_n\}_{n \in \mathbb{Z}}$ is separated, i.e. $|t_n - t_m| \geq \Delta/(|\alpha| + A\sigma) > 0$ for all $n \neq m$. Moreover, it is straightforward to observe that

$$D^-(\{t_n\}) \geq \frac{\Delta}{(|\alpha| - A\sigma)} > \frac{\pi}{\sigma} \qquad (14)$$

providing $\Delta < \pi(|\alpha| - A\sigma)/\sigma$. Thus, the conclusions about recovery from nonuniform samples also apply to amplitude sampling for an appropriate choice of the parameters.

B. Approximate Reconstruction

Amplitude sampling also generates uniform samples of $h(u) = M_\alpha f(t)$ given by

$$\{(n\Delta, h(n\Delta) = t_n - n\Delta/\alpha)\}_{n \in \mathbb{Z}}. \qquad (15)$$

We have previously shown that $f(t)$ and $h(u)$ cannot be simultaneously bandlimited. With $f(t)$ assumed to be bandlimited, we consider the approximate reconstruction of $h(u)$ through bandlimited interpolation of these uniform samples, i.e.

$$h_\Delta(u) = \sum_{k \in \mathbb{Z}} h(k\Delta)\text{sinc}(u/\Delta - k). \qquad (16)$$

Since $h(u)$ is not bandlimited, aliasing error results, as has been characterized in [10].
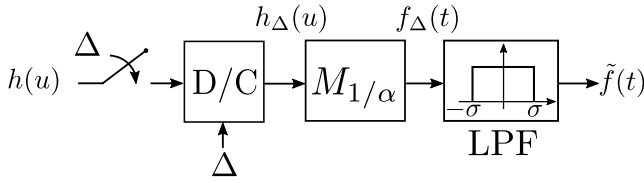

Fig. 5.   Approximate reconstruction procedure for a bandlimited source signal f such that $h = M_\alpha f$. The block D/C is a discrete-to-continuous operation involving sinc interpolation with period Δ.

As expected, the exponent in the bound has two contributions. First, the difference $|\alpha| - A\sigma$ reduces the error when the difference becomes more pronounced. We intuitively interpret this result by noting that $h(u)$ becomes closer to a bandlimited function as $|\alpha| - A\sigma$ increases. Therefore, the bandlimited interpolation is a better approximation for large values of $|\alpha|$. In the same way, reducing the value of Δ

decreases the aliasing error resulting in improvement of this reconstruction.

The above discussion suggests an approximate reconstruction procedure for $f(t)$ consisting of the bandlimited interpolation of $h(u)$ denoted by $h_\Delta(u)$ from which we then obtain the function $f_\Delta(t) = M_{1/\alpha} h_\Delta(u)$. However, based on the duality of the transformation M_α, the function f_Δ is nonbandlimited since h_Δ is bandlimited by construction. Thus, the reconstruction error in the L_2 sense is reduced by lowpass filtering f_Δ with unity gain and obtaining $\tilde{f}(t)$, i.e. $||f - \tilde{f}||_2 < ||f - f_\Delta||_2$. This reconstruction approach, which we call bandlimited interpolation approximation (BIA), is depicted in Fig. 5.

C. Simulations

In this section, we show the numerical performance of the bandlimited interpolation algorithm in comparison with a specific iterative algorithm, the Voronoi method [16], for recovery from nonuniform samples. This method has been shown to achieve empirically the best convergence rate in terms of the required number of iterations. The Voronoi method requires that $\sup_{n \in \mathbb{Z}} |t_n - t_{n+1}| < \pi/\sigma$ assuming an ordered sequence of time instants $\ldots < t_{n-1} < t_n < t_{n+1} < \ldots$ and $\lim_{|n| \to \infty} t_n = \infty$. It is easy to verify from (9) that the sequence of time instants generated by amplitude sampling satisfies such requirements for an appropriate choice of the parameters α and Δ.

In the simulations, we considered bandlimited synthetic signals with frequency components up to σ rad/s. We choose as a measure of approximation error the signal-to-error ratio (SER) given by

$$\text{SER} = 10 \log_{10}\left(\frac{||f||_2^2}{||f - f_a||_2^2}\right) \qquad (17)$$

(t) where $f_a(t)$ is the corresponding approximation to $f(t)$, i.e. $f_a = \tilde{f}(t)$ for BIA and the k-th iteration $f_a(t) = f_k(t)$ in the Voronoi method.
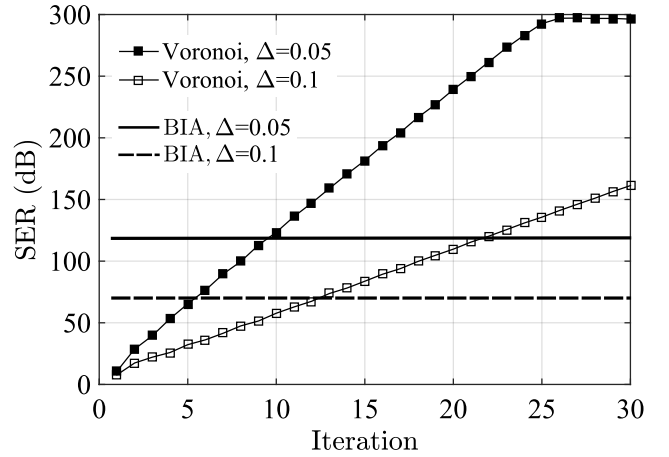

Fig. 6.   Performance comparison between BIA and the Voronoi method for fixed α and varying Δ.

As discussed, with fixed input bandwidth, the sampling density can be modified by changing the parameters α and

Δ appropriately. Fig. 6 and Fig. 7 show the numerical results for both methods as the parameters α and Δ are varied.

Fig. 6 shows the performance improvement when Δ is reduced. Theoretically, the Voronoi method has a geometric rate of convergence that depends in some sense on the maximal gap between consecutive sampling instants. Clearly, the increase in sampling density is caused by reducing the separation between amplitude levels. The effect in the BIA is connected to the bandlimited interpolation of the samples $h(n\Delta)$, i.e. the maximum deviation of h_Δ from h is reduced when Δ decreases. This empirically has an impact on the reduction of the approximation error of $\tilde{f}$ in terms of energy.
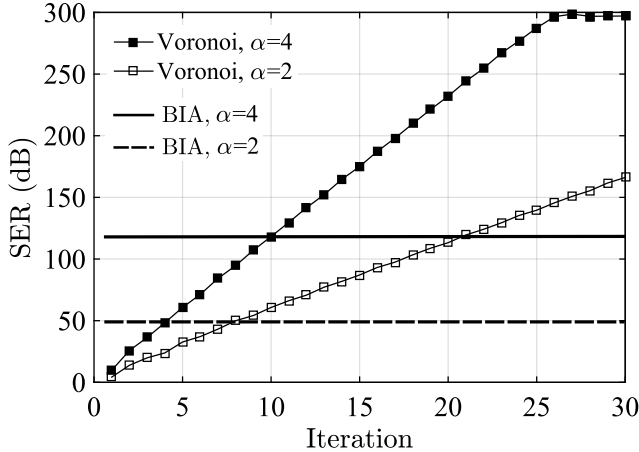

Fig. 7. Performance comparison between BIA and the Voronoi method for fixed Δ and varying α.

Figure 7 illustrates the improvement when the slope of the ramp is increased. This increase results in an increase of the level crossings per unit of time, hence the sampling density is increased. Similarly, increasing α causes the spectrum of h to decay faster—the bound in (12) increases—reducing the aliasing error. Similarly, the simulation results also show a decrease in $||f - \tilde{f}||_2$.

It is important to emphasize that the Voronoi method requires multiple iterations until it reaches the SER level of the respective BIA reconstruction. Therefore, iterative algorithms successively incorporating the chain of operations of Fig. 5 are being explored.

VII. CONCLUSIONS

A sampling and reconstruction technique was presented in which amplitude is quantized and time is allowed to take arbitrary real values. This approach is in contrast to conventional sampling schemes where time is quantized. Amplitude sampling is based on the concept of transforming the source signal into an invertible monotonic function. This perspective allows the sampling process to be viewed in terms of an associated amplitude-time function.

Approximate recovery is performed through bandlimited interpolation. In a noisy scenario, this may potentially represent an advantage over nonuniform sampling reconstruction algorithms which are known to be highly unstable in such

circumstances. The performance of this approximation suggests the possibility of extension to iterative reconstruction algorithms. In a general sense amplitude sampling represents a signal by a sequence of analog time instants that implicitly contain quantized amplitude information. Amplitude sampling can potentially be a promising alternative to time-sampling regimes when fine time resolution is available and amplitude is coarsely quantized.

REFERENCES

[1] E. T. Whittaker, "XVIII.—On the functions which are represented by the expansions of the interpolation-theory," *Proceedings of the Royal Society of Edinburgh*, vol. 35, pp. 181–194, 1915.

[2] C. E. Shannon, "Communication in the presence of noise," *Proceedings of the IRE*, vol. 37, no. 1, pp. 10–21, 1949.

[3] V. A. Kotelnikov, "On the carrying capacity of the ether and wire in telecommunications," in *Material for the First All-Union Conference on Questions of Communication, Izd. Red. Upr. Svyazi RKKA, Moscow*, 1933.

[4] B. F. Logan, "Information in the zero crossings of bandpass signals," *Bell System Technical Journal*, vol. 56, no. 4, pp. 487–510, 1977.

[5] S. Roweis, S. Mahajan, and J. Hopfield, "Signal reconstruction from zero-crossings," August 1998, draft (accessed May 2016) https://www.cs.nyu.edu/ roweis/papers/logan.ps.

[6] S. Mallat, "Zero-crossings of a wavelet transform," *IEEE Transactions on Information Theory*, vol. 37, no. 4, pp. 1019–1033, 1991.

[7] J. W. Mark and T. D. Todd, "A nonuniform sampling approach to data compression," *IEEE Trans. Commun.*, vol. 29, no. 1, pp. 24–32, 1981.

[8] Y. Tsividis, "Continuous-time digital signal processing," *Electron. Lett.*, vol. 39, no. 21, pp. 1551–1552, 2003.

[9] H. Lai, "Analysis and realization of amplitude sampling," Master's thesis, Massachusetts Institute of Technology, 2016.

[10] P. Martínez-Nuevo, "Amplitude sampling for signal representation," Ph.D. dissertation, Massachusetts Institute of Technology, 2016.

[11] S. N. Bernstein, *Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réelle*. Paris, 1926.

[12] É. Picard, "Sur une propriété des fonctions entières," *CR Acad. Sci. Paris*, vol. 88, pp. 1024–1027, 1879.

[13] A. Beurling, *The collected works of Arne Beurling. Vol. 2*, ser. Contemporary Mathematicians. Birkhäuser Boston, Inc., Boston, MA, 1989, harmonic analysis, Edited by L. Carleson, P. Malliavin, J. Neuberger and J. Wermer.

[14] H. J. Landau, "Necessary density conditions for sampling and interpolation of certain entire functions," *Acta Mathematica*, vol. 117, no. 1, pp. 37–52, July 1967.

[15] S. Jaffard *et al.*, "A density criterion for frames of complex exponentials." *The Michigan Mathematical Journal*, vol. 38, no. 3, pp. 339–348, 1991.

[16] H. G. Feichtinger and K. Gröchenig, "Theory and practice of irregular sampling," *Wavelets: mathematics and applications*, pp. 305–363, 1994.