Implementation by vote-buying mechanisms*

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Abstract

Simple majority voting does not allow preference intensities to be expressed, and hence fails to implement choice rules that take them into account. A vote-buying mechanism, instead, permits preference intensities to be revealed since each agent can buy any quantity of votes x to cast for an alternative of her choosing at a cost $c(x)$ and the outcome is the most voted alternative. In the context of binary decisions, we characterize the class of choice rules implemented by vote-buying mechanisms. Rules in this class can assign any weight to preference intensities and to the number of supporters for each alternative.

Keywords: implementation; mechanism design; vote-buying; social welfare; utilitarianism; quadratic voting.
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1 Introduction

Consider a binary collective choice problem: a society must choose one of two alternatives. Which alternative is socially preferable depends on the normative principles we have in mind. Political philosophers such as Locke (1689) argue that society should follow majority rule, disregarding the intensity of individual preferences. Alternatively, society might be utilitarian (Mill 1863) –thus, paying attention both to the number of supporters of each alternative and their preference intensities– or it could choose according to the desire of the individual with the most intense preference. These three rules are only examples within a wide class of choice rules that take into account the number of supporters of each alternative and/or their preference intensities. Each society may weigh individual preference intensities differently, and hence each society may wish to follow a different choice rule from this class.

If society wishes to fully disregard intensity of preferences, it can reach a decision by majority rule. Otherwise, an optimal mechanism needs to weigh more heavily the preferences of agents with more intense preferences. However, practical constraints may influence which mechanism is used: some collective entities may have adopted majority rule not because they wish to ignore preference intensities, but because "one-person one-vote" majority rule is easier to run than more sophisticated mechanisms. If so, to the extent that technological advances in the field of encryption and data management (e.g. blockchain voting[1]) make it easier to use more flexible forms of democracy, we expect calls for institutional innovation to enhance the opportunities to express preference intensities. Indeed, a number of organizations such as Google (Hardt and Lopes 2015) and some political parties in Europe (Blum and Zuber 2016) have recently experimented with procedures that endogenously redistribute political power among the concerned agents. Theoretical results ought to anticipate these developments –or at least, to anticipate their expansion into the public arena– in order to inform any decisions on institutional reform.

A recent literature on preference aggregation has shown that preference intensities can be

[1] See for instance, Yermack (2017).

taken into account in an extreme way via *decentralized* markets for votes: such mechanisms lead to the redistribution of political power and implement the alternative that is preferred by the agent who cares most about the outcome (Casella, Llorente-Saguer and Palfrey 2012). We turn attention to *centralized* vote markets and in particular to *vote-buying mechanisms*: each agent can express her intensity of preference by acquiring any quantity of votes x for either alternative, at a pre-announced monetary amount $c(x)$ that is evenly distributed to the rest of the players, and the social choice is determined by the total number of votes cast for each alternative.

We show that for any weight that the society assigns to preference intensities relative to the number of supporters for each alternative, *there exists a vote-buying mechanisms that implements the desired social choice rule.* Moreover, we establish that well-behaved *vote-buying mechanisms only implement choice rules that are optimal with respect to some weight on intensity of preferences.* These results establish a complete two-way mapping between a simple kind of centralized vote markets and an intuitive class of choice rules, which differ in the weight that they assign to individuals' preference intensities. To our knowledge this is the first general class of mechanisms that allows preference intensities to be expressed in every possible degree.

To gain an intuition over our results, consider the following formalization. Suppose that each subject i would trade v_i units of real wealth to change the social decision from a random coin toss to A with certainty; that is, the valuation v_i measures how intensely subject i cares that society chooses A and not B (agents who prefer B have a negative valuation). Then, a possible normative choice rule for a given $\rho \in \mathbb{R}_{++}$, is to declare A a better choice if $\sum_{i=1}^{n} sgn(v_i)|v_i|^\rho > 0$, and B if $\sum_{i=1}^{n} sgn(v_i)|v_i|^\rho < 0$; where $sgn(v_i)$ is the sign (positive or negative) of the valuation v_i. The class, indexed by $\rho \in \mathbb{R}_{++}$, of all such choice rules is characterized by a collection of appealing axioms (Burk 1936; Roberts 1986; Moulin 1988; Eguia and Xefteris 2018a).[2]

[2]The axioms are: anonymity, neutrality, monotonicity, continuity, separability, and scale-invariance. See as well Miller's (2018) characterization of polynomial majority rules.

Majority rule is the lower limit of this class, $\rho = 0$. Utilitarianism corresponds to parameter $\rho = 1$: it declares alternative A socially preferred if $\sum_{i=1}^{n} v_i > 0$. At the higher limit of the class, the alternative socially preferred given $\rho = \infty$ is the alternative preferred by the agent whose valuation has the highest absolute value. Throughout the class of choice rules, the social preference according to a small ρ is highly influenced by the number of agents who support each alternative, and less so by their intensity, while if ρ is large the social preference better reflects the preferences of the individuals whose well-being is greatly affected by the decision.

We prove that every vote-buying mechanism with limit cost elasticity $\lim_{x\to 0} \frac{c'(x)x}{c(x)} = 1 + 1/\rho$ asymptotically implements the choice rule with intensity parameter ρ in the above class; and that no choice rule outside this class is asymptotically implementable by any vote-buying mechanism.[3] That is, we fully characterize the class of social choice rules that can be implemented by vote-buying mechanisms. This class is rich, yet neatly characterized by a single parameter. Indeed, we allow for arbitrary vote-buying costs, $c(x)$, which may not even scale like power or other polynomial functions, and we still find that the welfare optima that they implement are in the class we describe. Moreover, we observe that individual equilibrium transfers converge to zero as the society grows large. That is, centralized vote markets converge toward functioning like a simpler mechanism without transfers, such as a standard voting mechanism. This feature mitigates concerns about the role of individual budgets in social choice, and it is in stark contrast to decentralized vote markets, which require certain individuals to make substantial payments (Casella, Llorente-Saguer and Palfrey 2012).

Overall, our work contributes to the understanding of centralized markets for votes by pinning down: a) what determines how much weight will be given to intensity of preferences and how much weight will be attributed to the popularity of each alternative (i.e. the elasticity of the vote-buying costs near zero), and b) how these weights relate to a specific social

[3] A mechanism asymptotically implements a given choice rule if the probability that the mechanism chooses the alternative socially preferred according to this rule is arbitrarily close to one in sufficiently large societies.

choice rule.[4] The mechanisms we consider are detail-free in the sense that at the time she designs the mechanism, the designer does not need to know the particular features of the society, such as the number of individuals, the exact distribution of types from which individual preferences are drawn, or the importance of the choice under consideration. Hence, we interpret the proposed vote-buying mechanisms as institutions which asymptotically implement the society's choice rule, regardless of changes in distributional parameters.

Literature Review

Our question of interest has deep roots in classic mechanism design. This literature aims to design mechanisms that allow a society to choose the utilitarian optimum. All suggested mechanisms involve transfers. The VCG mechanism (Vickrey 1961, Clarke 1971 and Groves 1973) satisfies utilitarian efficiency, but is not budget-balanced and may involve substantial transfers. We would prefer a budget-balanced mechanism that requires minimal transfers, so that budget-constrained agents can signal their true preference intensity. The mechanisms by Arrow (1979) and AGV (D'Aspremont and Gerard-Varet 1979) are budget-balanced, require minimal transfers when the society is large, and attain utilitarian efficiency by requiring each agent to pay the expected externality of her choices.

A demanding feature of the AGV mechanism is that to compute the expected externality, the designer must know population parameters such the distribution from which individual preferences are drawn. We would prefer mechanisms that can be run even if the designer does not know this distribution. Fortunately, the dependence of the AGV mechanism on population parameters becomes less of a problem as the size of the society increases. Actually, if the distribution of preferences does not favor any alternative, as society becomes large, the expected externality attains a quadratic functional form.

This asymptotic behavior of the AGV mechanism suggests that in large societies, detail-

[4] We derive our main result under the assumption –standard in the literature– that agents are risk neutral. If agents' utility over wealth is instead concave, rich agents acquire more votes, so if the distribution of preferences over the social choice is wealth-specific, the preferences of the poor end up under-represented. Optimality can be restored by allowing for richer mechanisms that condition the cost of voting on individual wealth (we discuss this extension in Section 4).

free vote-buying mechanisms with a quadratic cost function $c(x) = \alpha x^2$ may inherit the efficiency properties of the AGV mechanism, while relaxing the information requirements on the planner. Indeed, Lalley and Weyl (2016) show that a vote buying mechanism with a quadratic cost function (i.e. "quadratic voting") approximates the utilitarian optimum in all equilibria in large societies, while Goeree and Zhang (2017) complement the theoretical arguments with experimental evidence that vote-buying mechanisms with quadratic cost are relevant in settings of applied interest. Several papers followed studying additional aspects of quadratic voting, such as agenda-setting (Patty and Penn 2017), heuristic behavior (Lalley and Weyl 2018), and turnout (Kaplov and Kominers 2017).

Like this literature, we study detail-free vote-buying mechanisms in the context of binary collective choice problems. Unlike it, we look beyond quadratic voting: we consider a whole class of vote-buying mechanisms and we show that they implement a large –but neatly packaged– class of normative choice rules axiomatized by Roberts (1980) and Moulin (1988).[5] Related approaches to gauge intensity of preferences through voting involve majority voting with heterogenous turnout costs, storable votes, or vote trading in a competitive market for votes. Voluntary majority voting with costly turnout (Börgers 2004; Krishna and Morgan 2015) and the possibility to store votes for future use (Casella 2005) are superior to compulsory majority voting from a utilitarian perspective. Whereas, a competitive equilibrium in a decentralized market for votes is very similar to our special case with parameter $\rho = \infty$: the cost of votes is linear, and the agent who cares most about the decision buys most votes (Dekel, Jackson and Wolinski 2008; Casella, Llorente-Saguer and Palfrey 2012).

We follow traditional Bayesian implementation (Jackson 1991) to assume that citizens share a common prior about the society they live in, but we also assume, as in robust implementation (Bergemann and Morris 2013), that the planner does not know this prior. The planner's goal, and ours, is to design a mechanism such that given the planner's choice

[5] While our results generalize the finding of Lalley and Weyl (2016) that all equilibria of quadratic voting lead to utilitarian efficiency, the two models are not nested: to obtain simpler and shorter proofs, we make assumptions on the payoff function that are substantially similar, but technically distinct.

rule, for any realization of individual preferences, in any equilibrium of the mechanism, and in any society with any common prior, the equilibrium outcome coincides with the desired social choice.

2 The Formal Framework

Summary. A set of agents must make a binary social choice. The decision is made via a vote-buying mechanism: agents purchase votes, and the alternative with the most votes is chosen. We characterize the set of social choice correspondences that are asymptotically implementable by these vote-buying mechanisms.

Social choice problem. A society N^n of size $n \in \mathbb{N}\backslash\{1\}$ must make a binary choice over $\{A, B\}$. Let the social decision $d \in \{A, B\}$ denote the alternative chosen.

Individual preferences. Each agent $i \in N^n$ has preferences over real wealth and over the social decision, and also over lotteries over wealth profiles paired with a social decision. Under standard conditions (detailed in the working paper version Eguia and Xefteris 2018b), each agent $i's$ preference relation is representable by a quasilinear expected utility function that depends only on agent $i's$ valuation of the alternatives, on the social decision, and on the net transfer of wealth received by the agent. For ease of exposition, here we work directly with the quasilinear utility representation.[6]

Agent $i's$ valuation of alternative A, denoted $\gamma\theta_i$, is the amount of real wealth that i would be willing to trade in order to assure that $d = A$, instead of letting d be randomly drawn. Parameter $\gamma \in \mathbb{R}_{++}$ is the importance of the social decision, and $\theta_i \in [-1, 1]$ as the attitude of agent i; agents with a negative attitude prefer B to A, and those with a positive attitude prefer A to B. We refer to $\gamma\theta_{N^n} \equiv \gamma(\theta_1, ..., \theta_n)$ as a valuation profile of A, or simply "valuation profile," and to $-\gamma\theta_{N^n}$ as the valuation profile of B. Let θ_{-i} denote

[6] The key conditions are separability over wealth and the social decision, and risk neutrality. It is standard in the literature to implicitly assume that preferences satisfy these conditions, and to treat the quasilinear utility function as a primitive (see, for instance, Krishna and Morgan 2015 or Lalley and Weyl 2016).

$(\theta_1, ..., \theta_{i-1}, \theta_{i+1}, ..., \theta_n)$.

Let F be a continuously differentiable cumulative distribution function over $[-1, 1]$ with strictly positive density f over its domain, and no mass at any point. Let $\mathcal{F}$ be the set of all cumulative distributions. Let $\bar{\theta}$ be a random variable with cumulative distribution F. We assume that each attitude θ_i is an independent draw of $\bar{\theta}$. Let $\bar{\theta}_{N^n}$ denote the random vector composed of n independent draws of $\bar{\theta}$, so the profile of attitudes θ_{N^n} is a realization of $\bar{\theta}_{N^n}$.

Vote-buying mechanisms. A vote-buying mechanism is defined by a cost function $c : \mathbb{R} \longrightarrow \mathbb{R}_+$. The mechanism invites each agent $i \in N^n$ to choose any action $a_i \in \mathbb{R}$. For any $a \in \mathbb{R}$, and any agent $i \in N^n$, if agent i chooses action $a_i = x$, then i pays a cost $c(x)$. All payments are redistributed equally among all other agents, so given a vector of actions $a_{N^n} \in \mathbb{R}^n$, each agent $i \in N^n$ obtains a net nominal wealth transfer $-c(a_i) + \sum\limits_{j \in N^n \backslash \{i\}} \frac{c(a_j)}{n-1}$. Since agents care about real, not nominal, wealth, their incentives are affected by the price index in society. However, we show in the working paper version (Eguia and Xefteris 2018b) that our results hold for any price index; therefore, for ease of presentation, we fix the price index to one and thereafter omit the distinction between nominal and real wealth.

Let $\mathcal{C}$ denote the set of all possible vote-buying mechanisms (all cost functions from $\mathbb{R}$ to $\mathbb{R}_+$). A perfect execution of a mechanism $c \in \mathcal{C}$ would entail society choosing $d = A$ if $\sum\limits_{j \in N^n} a_j > 0$ and $d = B$ if $\sum\limits_{j \in N^n} a_j < 0$. However, we assume that the execution of any mechanism entails some element of uncertainty, so that the mapping from actions to outcomes is stochastic: while the probability that $d = A$ is increasing in $\sum\limits_{j \in N^n} a_j$, it is not a step function.

Formally, we assume that there exists an outcome function $G : \mathbb{R} \longrightarrow [0, 1]$ such that for any $n \in \mathbb{N} \backslash \{1\}$ and any $a_{N^n} \in \mathbb{R}^n$, the probability that $d = A$ is $G\left(\sum\limits_{j \in N^n} a_j\right)$. Let $\mathcal{G}$ be the class of strictly increasing, twice continuously differentiable functions from $\mathbb{R} \longrightarrow [0, 1]$ such that for any $\tilde{G} \in \mathcal{G}$ with density $\tilde{g}$ and derivative of the density $\tilde{g}'$:

i) $\tilde{G}(x) - \frac{1}{2} = \frac{1}{2} - \tilde{G}(-x)$ for any $x \in \mathbb{R}_{++}$;

ii) $\lim\limits_{x \longrightarrow -\infty} \tilde{G}(x) = 0$ and $\lim\limits_{x \longrightarrow -\infty} \tilde{g}(x) = 0$;

iii) $\exists \hat{\varepsilon} \in \mathbb{R}_{++}$ such that $\lim_{x \to \infty} \frac{\tilde{g}'(x+\varepsilon)}{\tilde{g}(x)} \in \mathbb{R}\ \forall \varepsilon \in (-\hat{\varepsilon}, \hat{\varepsilon})$.

Condition (i) is neutrality. Condition (ii) is a responsiveness condition: if the vote margin is sufficiently large, the outcome is the one with the vote advantage with probability arbitrarily close to one. Condition iii) requires the tails of the density not to drop to zero too steeply.[7] The set $\mathcal{G}$ contains, among others, all Logistic and Student-t distributions.

We assume that $G \in \mathcal{G}$, but G is not known to the mechanism designer, and hence we will propose mechanisms whose results are robust for any $G \in \mathcal{G}$, including those that are arbitrarily close to a step function with discontinuity at zero, as in Figure 1.

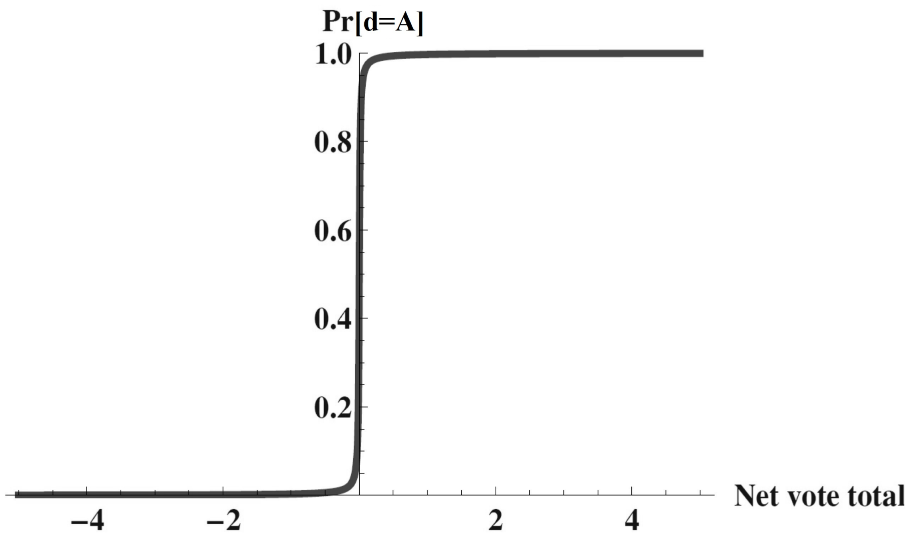

Figure 1: An outcome function G.

This –minimally- stochastic element of the outcome as a function of the equilibrium strategies can be interpreted literally as a probabilistic outcome function given the vote tally. Alternatively, with a deterministic outcome function (the alternative with a greater tallied vote total is chosen with certainty), we can interpret G to capture some aggregate noise in agents' behavior, or in the tallying and recording of the votes cast so that a number of votes is assigned stochastically in addition to those cast by agents. In any of these cases, the objective function of a voter is identical, and hence the equilibrium behavior is identical as

[7] These assumptions make it easier to derive equilibrium properties that are known to hold but harder to prove in alternative frameworks (e.g. Lalley and Weyl 2016). They do not affect the main characterization result.

well. Notice that G can be arbitrarily close to the deterministic outcome function and hence the stochastic element can be arbitrarily small.

Admissible vote-buying mechanisms. We specify the set of admissible vote-buying mechanisms $\mathcal{C}_A \subset \mathcal{C}$. Let $\hat{C} \subset \mathcal{C}$ be the set of continuously differentiable non-negative functions defined over $\mathbb{R}$ that are twice continuously differentiable over $\mathbb{R}\backslash\{0\}$. For any $c \in \hat{C}$, define $\kappa(c) \equiv \lim_{a \longrightarrow 0} \frac{ac'(a)}{c(a)}$ as the limit of the elasticity of c at zero (if it exists). Let $\mathcal{C}_A \equiv \{ c \in \hat{C} : c(0) = 0,\ c'(0) = 0,\ \kappa(c) \in (1, \infty),\ c'(a) > 0$ for any $a \in \mathbb{R}_{++},\ \lim_{a \longrightarrow \infty} c(a) = \infty$, and $c(a) = c(-a)$ for any $a \in \mathbb{R}\}$. The intuition on $\mathcal{C}_A$ is that, in addition to continuity and differentiability, an admissible cost functions has the following properties:

i) abstention (acquiring no votes) is free;

ii) to encourage positive participation, the marginal cost of votes at zero is zero, so for any strictly positive willingness to pay per vote, some strictly positive quantity of votes can be acquired at that price;

iii) but the elasticity of the cost function near zero is greater than one (so c is strictly convex) near zero, and thus the marginal cost of votes becomes positive immediately;

iv) and while elsewhere the cost function need not be convex, this marginal cost is always positive for all positive quantities;

v) and very high quantities of votes are prohibitively expensive; and

vi) neutrality: votes for A cost the same as votes against A.

Note that $\mathcal{C}_P \subset \mathcal{C}_A$, i.e. all power functions with exponent greater than one are admissible vote-buying mechanisms.

Strategies. Each agent i in society N^n with size $n \in \mathbb{N}\backslash\{1\}$, facing a social choice problem of importance $\gamma \in \mathbb{R}_{++}$ to be decided according to mechanism $c \in C$ under uncertainty $G \in \mathcal{G}$, and taking into account that the ex-ante distribution of attitudes toward the decision is given by distribution $F \in \mathcal{F}$, chooses an action $a_i \in \mathbb{R}$ as a function of the realization $\theta_i \in [-1, 1]$ of her own attitude toward the decision. We assume actions are taken simultaneously, that the tuple (n, F, γ, c, G) is common knowledge, and that each attitude θ_i is private information to agent i. Therefore, for any given tuple (n, F, γ, c, G), a pure strategy is a mapping $s : [-1, 1] \longrightarrow \mathbb{R}$. Let S be the set of all feasible pure strategies. For each $s \in S$ and each $\theta \in [-1, 1]$, let $s(\theta) \in \mathbb{R}$ be the action taken given θ according to strategy s, always given (n, F, γ, c, G). For each $s \in S$, for each $n \in \mathbb{N}\backslash\{1\}$, and for each $i \in N^n$, let $s_i = s$ denote that agent i chooses strategy s. We say that a strategy s is monotone if $\frac{\partial s}{\partial \theta} \geq 0$.

Utilities. Given a society N^n with $(n, \gamma, F, G) \in \mathbb{N}\backslash\{1\} \times \mathbb{R}_{++} \times \mathcal{F} \times \mathcal{G}$ and given a mechanism $c \in C$, for any agent $i \in N^n$, we can compute the expected utility of agent i as a function of her attitude θ_i, her strategy s_i and the strategy profile of every other player s_{-i}. Let $EU_i : [-1, 1] \times S^n \longrightarrow \mathbb{R}$ denote the expected utility of agent i. Then, for any $\theta_i \in [-1, 1]$ and $s_{N^n} \in S^n$, $EU_i[\theta_i, s_{N^n}]$ is equal to the expected utility from the social decision plus the expected wealth transfer. For any $s_{N^n} \in S^n$ and any $\theta_i \in [-1, 1]$, let $\bar{d}(s_{N^n}, \theta_i, \bar{\theta}_{-i})$ denote the social decision given that agents play the strategy profile s_{N^n}, and agent i has attitude θ_i. Note that $\bar{d}(s_{N^n}, \theta_i, \bar{\theta}_{-i})$ is a random variable that depends on the realization of the attitude profile θ_{-i}, and on the realization of the outcome given $G\left(\sum\limits_{k=1}^{n} s_k(\theta_k)\right)$. Then $EU_i[\theta_i, s_{N^n}]$ is equal to

$$\gamma\theta_i \left(\Pr[\bar{d}(s_{N^n}, \theta_i, \bar{\theta}_{-i}){=}A]\text{-} \Pr[\bar{d}(s_{N^n}, \theta_i, \bar{\theta}_{N^n}){=}B]\right) \text{-}c(s_i(\theta_i)) + \frac{1}{n\text{-}1} \sum_{j \in N^n\backslash\{i\}} \int_{-1}^{1} f(x)c(s_j(x))dx, \tag{1}$$

where

$$\Pr[\bar{d}(s_{N^n}, \theta_i, \bar{\theta}_{-i}) = A] = \int\limits_{\theta_{-i} \in [-1,1]^{n-1}} \left(\prod_{j \in N^n\backslash\{i\}} f(\theta_j)\right) G\left(s_i(\theta_i) + \sum_{j \in N^n\backslash\{i\}} s_j(\theta_j)\right) d\theta_{-i},$$

and $\Pr[\bar{d}(s_{N^n}, \theta_i, \bar{\theta}_{-i}) = B] = 1 - \Pr[\bar{d}(s_{N^n}, \theta_i, \bar{\theta}_{-i}) = A]$.

Game. For each tuple $(n, \gamma, F, c, G) \in \mathbb{N}\backslash\{1\} \times \mathbb{R}_{++} \times \mathcal{F} \times C \times \mathcal{G}$, let $\Gamma^{(n,\gamma,F,c,G)}$ denote the game played by the n players in society N^n, with strategy set S for each agent, and expected utility given by EU_i in Expression (1) for each $n \in \mathbb{N}\backslash\{1\}$ and each $i \in N^n$.

Equilibrium. For any tuple $(n, \gamma, F, c, G) \in \mathbb{N}\backslash\{1\} \times \mathbb{R}_{++} \times \mathcal{F} \times C \times \mathcal{G}$, let $BNE^{(n,\gamma,F,c,G)} \subseteq S^n$ denote the set of pure Bayes Nash Equilibria of game $\Gamma^{(n,\gamma,F,c,G)}$. We are interested in the subset of symmetric pure BNE, in which each player plays the same pure, monotone strategy. Let $E^{(n,\gamma,F,c,G)} \subseteq S$ denote the set of pure and monotone strategies that constitute a symmetric Bayes Nash equilibrium of game $\Gamma^{(n,\gamma,F,c,G)}$. Hereafter, an "equilibrium" is always a strategy $s \in E^{(n,\gamma,F,c,G)}$.

Sequence of societies. We consider a sequence of societies $\{N^n\}_{n=2}^{\infty}$. We will establish results for sufficiently large societies. Note that aside from size $n \in \mathbb{N}\backslash\{1\}$, (γ, F, G) are the characteristics that identify a particular social choice problem. These characteristics are common knowledge among members of the society, but they are unobserved by the mechanism designer, who only knows that $\gamma \in \mathbb{R}_{++}$, $F \in \mathcal{F}$ and $G \in \mathcal{G}$. The problem we address is to design a mechanism that has desirable properties for any $(\gamma, F, G) \in \mathbb{R}_{++} \times \mathcal{F} \times \mathcal{G}$, for any sufficiently large n.

Social preferences. For each $n \in \mathbb{N}\backslash\{1\}$, let R^n denote a complete and transitive relation over $\mathbb{R}^n$, interpreted as a preference over valuation profiles: for any $\gamma \in \mathbb{R}_{++}$ and for any $\theta_{N^n}, \tilde{\theta}_{N^n} \in [-1, 1]^n$, we interpret $(\gamma\theta_{N^n})\, R^n(\gamma\tilde{\theta}_{N^n})$ to mean that according to preference R^n, valuation profile $\gamma\theta_{N^n}$ is preferable to valuation profile $\gamma\tilde{\theta}_{N^n}$. We can interpret this preference as a preference held by the mechanism designer, or as an abstract preference relation over valuation profiles. Let $R \equiv \{R^n\}_{n=2}^{\infty}$ denote an infinite sequence of such preferences over valuation profiles. For each $n \in \mathbb{N}\backslash\{1\}$, define as well the strict preference P^n by $\gamma\theta_{N^n} P^n(\gamma\tilde{\theta}_{N^n}) \iff \neg(\gamma\tilde{\theta}_{N^n})R^n\,(\gamma\theta_{N^n})$, where $\neg$ denotes the negation of a logical statement.

A sequence R of preferences over valuation profiles determines a social preference over $\{A, B\}$ as a function of n, γ and θ_{N^n}.

Definition 1 *For any* $(n, \gamma, \theta_{N^n}) \in \mathbb{N}\backslash\{1\} \times \mathbb{R}_{++} \times \mathbb{R}^n_{++}$, *and any preference over valuation* *profiles R^n, alternative A is* ***socially weakly preferred*** *to B if and only if $(\gamma\theta_{N^n})\, R^n\, (-\gamma\theta_{N^n})$, and is socially strictly preferred if $(\gamma\theta_{N^n})\, P^n\, (-\gamma\theta_{N^n})$.*

Alternative B is socially weakly [strictly] preferred given R^n if A is not socially strictly [weakly] preferred.

Welfare representation. If the preference relation over valuation profiles R^n is continuous, then it can be represented by a continuous function (Debreu 1954). We refer to this utility representation as a "welfare" function.[8] We say that a welfare function $W : \mathbb{R}_{++} \times \bigcup_{n=2}^{\infty} [-1,1]^n \longrightarrow \mathbb{R}$ represents a sequence $\{R^n\}_{n=1}^{\infty}$ if for any $n \in \mathbb{N}$, for any $\gamma \in \mathbb{R}_+$, and for any $\theta_{N^n}, \tilde{\theta}_{N^n} \in [-1,1]^n$, $W(\gamma, \theta_{N^n}) \geq W(\gamma, \tilde{\theta}_{N^n})$ if and only if $\gamma\theta_{N^n} R^n \gamma\tilde{\theta}_{N^n}$.

Let $sgn : \mathbb{R} \longrightarrow \{-1, 0, 1\}$ be the sign function, defined by $sgn(x) = -1$ if $x < 0$, $sgn(x) = 0$ and $sgn(x) = 1$ if $x > 0$. For each $\rho \in \mathbb{R}_{++}$, define the Bergson welfare function W_ρ (Burk 1936) by

$$W_\rho(\gamma, \theta_{N^n}) \equiv \sum_{i \in N^n} sgn(\theta_i) |\gamma\theta_i|^\rho,$$

and let R_ρ^n denote the preference relation over valuation profiles in $\mathbb{R}^n$ represented by W_ρ. We refer to the set $\bigcup_{\rho \in \mathbb{R}_{++}} \{R_\rho^n\}$ as the set of Bergson preference relations. Bergson preference relations are the only ones that satisfy the following collection of axioms: continuity, anonymity, neutrality, monotonicity, separability, and scale invariance.[9] These axioms, together with a particular value $\rho \in \mathbb{R}_{++}$, uniquely identifies a particular preference relation R_ρ^n over $\mathbb{R}^n$. Parameter ρ measures how much the preference over valuation profiles responds to intensity of individual preferences over alternatives. Each value $\rho \in \mathbb{R}_{++}$ can be interpreted as a distinct normative axiom on preferences over valuations, in addition to the collection above.

[8] Given that this function represents preferences on the social choice only, our approach is in line with the standard "micro" version of welfarism. See Moulin (2004) for a discussion of the advantages of microwelfarism over macrowelfarism.

[9] See the working paper version (Eguia and Xefteris 2018b) for a definition of the axioms, and Eguia and Xefteris (2018a) for a more detailed explanation. The original axiomatization is due to Roberts (1980) and Moulin (1988).

Social Choice correspondences. For any $n \in \mathbb{N}$, a social choice correspondence $SC^n : \mathbb{R}_{++} \times [-1,1]^n \rightrightarrows \{A, B\}$ maps a pair (γ, θ_{N^n}) into the subset of normatively desirable social decisions $SC(\gamma, \theta_{N^n})$. Let $SC \equiv \{SC^n\}_{n=1}^{\infty}$ denote a sequence of social choice correspondences. For each $\rho \in \mathbb{R}_{++}$, and for each $n \in \mathbb{N}$, define the Bergson choice correspondence SC_ρ^n by

$$SC_\rho^n(\gamma, \theta_{N^n}) \equiv \begin{cases} B \text{ if } \sum\limits_{i \in N^n} sgn(\theta_i)|\gamma\theta_i|^\rho < 0 \\ \{A, B\} \text{ if } \sum\limits_{i \in N^n} sgn(\theta_i)|\gamma\theta_i|^\rho = 0 \\ A \text{ if } \sum\limits_{i \in N^n} sgn(\theta_i)|\gamma\theta_i|^\rho > 0. \end{cases}$$

Note that SC_ρ^n is the social choice correspondence that chooses the alternative(s) that are socially preferred given the Bergson preference over valuation profiles R_ρ^n (which is represented by the Bergson welfare function W_ρ). Define the sequence of Bergson social choice correspondences $SC_\rho \equiv \{SC_\rho^n\}_{n=2}^{\infty}$.

Asymptotically equivalent Social Choice correspondences. We say that two sequences of social choice correspondences SC and $\widetilde{SC}$ are asymptotically equivalent if the probability that they select the same outcome converges to one, as $n \longrightarrow \infty$. We say a property holds generically if it holds in an open dense subset of the set under consideration. To formally define generic asymptotic equivalence of SC and $\widetilde{SC}$ over $\mathcal{F}$, we need to define more structure on $\mathcal{F}$.

Let $C[-1,1]$ denote the set of all continuous functions over $[-1,1]$ and let d_∞ be the sup-metric over $C[-1,1]$, so that for any $\varphi, \hat{\varphi} \in C[-1,1]$, $d_\infty(\varphi, \hat{\varphi}) \equiv \sup\limits_{\theta \in [-1,1]} \{|\varphi(\theta) - \hat{\varphi}(\theta)|\}$. We consider the metric space $(\mathcal{F}, d_{\infty,\infty})$ with distance function $d_{\infty,\infty} : \mathcal{F} \times \mathcal{F} \longrightarrow \mathbb{R}_+$ defined by $d_{\infty,\infty}(F, \hat{F}) \equiv d_\infty(F, \hat{F}) + d_\infty(f, \hat{f})$.[10] A subset $\mathcal{F}^D \subset \mathcal{F}$ is dense in $\mathcal{F}$ if the closure of $\mathcal{F}^D$ is equal to $\mathcal{F}$ (so any cumulative distribution $F \in \mathcal{F} \backslash \mathcal{F}^D$ is the limit of a sequence of distributions in $\mathcal{F}^D$). We can now precisely define the desired asymptotic equivalence notion.

[10] This is the standard distance to metricize the set of continuously differentiable functions (Ok 2007, Chapter C, Example 2[3]).

Definition 2 *For any $F \in \mathcal{F}$, two sequences of social choice correspondences SC and $\widetilde{SC}$ are* ***asymptotically equivalent*** *with respect to F if $\lim_{n \longrightarrow \infty} \Pr\left[SC(\gamma, \bar{\theta}_{N^n}) \neq \widetilde{SC}(\gamma, \bar{\theta}_{N^n})\right] = 0$.*

We say that SC and $\widetilde{SC}$ are generically asymptotically equivalent if they are asymptotically equivalent for any F in an open dense set $\mathcal{F}^D \subseteq \mathcal{F}$.

For ease of exposition, and since all our results are asymptotic, we refer to generically asymptotically equivalent sequences as "generically equivalent."

Implementability. We say that a vote-buying mechanism c asymptotically implements a sequence of social choice correspondences SC over a given subdomain $\hat{\mathcal{F}} \subseteq \mathcal{F}$ of possible distribution functions from which attitudes are drawn if two conditions hold: i) an equilibrium exists for any large society; and ii) in equilibrium, the probability that the social decision coincides with the alternative chosen by SC converges to one. For any subclass of vote-buying mechanisms $C \subseteq \mathcal{C}$, we say that a sequence $SC \in \mathcal{SC}$ is implementable by C over $\hat{\mathcal{F}}$ if there exists $c \in C$ that implements SC over $\hat{\mathcal{F}}$.

For any $F \in \mathcal{F}$ and any $n \in \mathbb{N}\backslash\{1\}$, let $\bar{d}_F^n(s, \bar{\theta}_{N^n})$ be the social decision considered as a random variable that depends on the realization of the attitude profile θ_{N^n} and on the realization of the outcome given $G(\sum_{i=1}^{n} s_i(\theta_i))$, given that $s_i = s$ for each $i \in N^n$. The formal definition of implementation is then as follows.

Definition 3 *For any $\hat{\mathcal{F}} \subseteq \mathcal{F}$, a vote-buying mechanism $c \in \mathcal{C}$* ***asymptotically implements*** *a sequence of social choice correspondences SC over $\hat{\mathcal{F}}$ if for any $(\gamma, F, G) \in \mathbb{R}_{++} \times \hat{\mathcal{F}} \times \mathcal{G}$,*

i) there is $\hat{n} \in \mathbb{N}$ such that for any $n \geq \hat{n}$, the set of equilibria $E^{(n,\gamma F,c,G)}$ is non empty, and

ii) for any $\varepsilon \in (0,1)$ and for any sequence of equilibria $\{s^t\}_{t=\hat{n}}^{\infty}$, there exists $n_{\varepsilon,\gamma,F,G} \in \mathbb{N}$ such that for any $n > n_{\varepsilon,\gamma,F,G}$, $\Pr[\bar{d}_F^n(s^n, \bar{\theta}_{N^n}) = SC^n(\gamma, \bar{\theta}_{N^n})] > 1 - \varepsilon$.

For any subset of vote-buying mechanisms $C \subseteq \mathcal{C}$, we say that a sequence of social choice correspondences SC is asymptotically implementable by C over $\hat{\mathcal{F}}$ if there exists a mechanism $c \in C$ that asymptotically implements SC over $\hat{\mathcal{F}}$.

Since our implementation results are always asymptotic, if a mechanism c asymptotically implements SC, then we say simply that c "implements SC."

This implementation notion requires that, if the society is sufficiently large, the outcome in every equilibrium of the game induced by the mechanism must be the outcome desired by the social choice rule with probability arbitrarily close to one, for any distribution parameters. Depending on the domain of distributions $\hat{\mathcal{F}}$ under consideration, such robustness across societies may not be attainable. We then seek, as a second best, a mechanism that works for most societies in the domain under consideration.

We define generic asymptotic implementability accordingly.

Definition 4 *A vote-buying mechanism $c \in \mathcal{C}$* ***asymptotically implements*** *a sequence of social choice correspondences SC* ***generically*** *if there exists an open $\mathcal{F}^D$ dense in $\mathcal{F}$ such that c implements SC over $\mathcal{F}^D$.*

For any $C \subseteq \mathcal{C}$, we say that a sequence of social choice correspondences SC is generically asymptotically implementable by $C \subseteq \mathcal{C}$ if there exists a mechanism $c \in C$ that generically asymptotically implements SC.

If a mechanism c asymptotically implements a sequence of social choice correspondences SC generically, we say simply that c "implements SC generically."

3 Results

We can now state our main result: a complete characterization of the class of sequences of social choice correspondences that are generically implementable by each admissible vote-buying mechanism. We show that the set of social choice correspondences implemented by any given admissible vote-buying mechanism c is entirely determined by the mechanism's limit elasticity $\kappa(c)$.

Theorem 1 *A sequence of social choice correspondences SC is generically implementable by the class of admissible vote-buying mechanisms if and only if SC is generically equivalent to a sequence of Bergson correspondences SC_ρ for some $\rho \in \mathbb{R}_{++}$.*

Further, each admissible vote-buying mechanism c generically implements any sequence of social choice correspondences that is generically equivalent to the sequence of Bergson correspondences $SC_{\frac{1}{\kappa(c)-1}}$.

That is, only sequences of Bergson choice correspondences and those generically equivalent to them, are generically implementable by admissible vote-buying mechanisms, and each admissible vote-buying mechanism c with limit elasticity $\kappa(c)$ implements the Bergson sequence with importance of intensity of individual preferences ρ equal to $\frac{1}{\kappa(c)-1}$.

Equivalently, each Bergson sequence SC_ρ –and any other sequence generically equivalent to it- are generically implemented by any admissible vote-buying mechanism c with limit elasticity $\kappa(c) = \frac{1+\rho}{\rho}$.

A corollary follows: for any $k \in (1, \infty)$, since $\kappa(c) = k$ for any power function $c(a) = |a|^k$, the set of sequences of choice correspondences generically implementable by any given admissible vote-buying mechanism c with $\kappa(c) = k$ is exactly the same as the set of sequences of choice correspondences implementable by vote-buying mechanisms $c(a) = |a|^k$. Therefore, the set of all sequences of social choice correspondences generically implementable by the class of admissible vote-buying mechanisms is the same as the set generically implementable by the subclass of power function vote-buying mechanisms.

Corollary 1 *If SC is generically implemented by $c \in \mathcal{C}_A$, then SC is generically implemented by $\hat{c}(a) = |a|^{\kappa(c)}$.*

Hence, the class of sequences of correspondences implementable by admissible vote-buying mechanisms include only Bergson sequences of correspondences, and sequences asymptotically equivalent to them. That's all.

Goeree and Zhang (2017) and Lalley and Weyl (2018) provide a heuristic intuition for the special case of implementing utilitarianism with a quadratic power mechanism: if agents

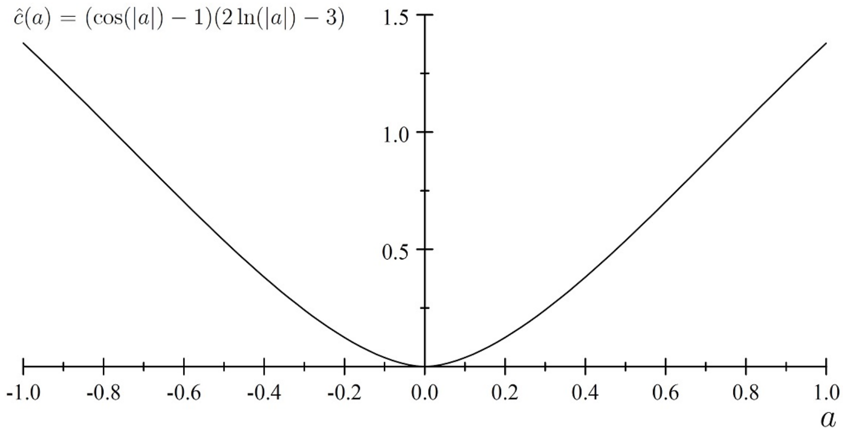

Figure 2: A non-polynomial mechanism $\hat{c}$ that implements utilitarianism.

assume that their marginal benefit of acquiring votes is constant in the quantity of votes acquired, then agents infer that their marginal benefit of acquiring votes is linear in their attitude. Given a mechanism $c(a)$ with derivative $c'(a)$ that is linear in a, agents equate perceived marginal benefit and marginal cost by acquiring votes in proportion to their attitude, which leads to utilitarian efficiency.

This heuristic intuition is useful as far as other power cost mechanisms are concerned, but beyond these functions (or those that scale like them), it does not generalize well: what matters for asymptotic implementation is the limit elasticity $\kappa(c)$, and not the shape of the derivative $c'(a)$. Consider, for example, a non-power mechanism such as $\hat{c} \in C$ depicted in Figure 2, and defined by $\hat{c}(a) = (\cos(|a|) - 1)(2\ln(|a|) - 3)$ for any $a \in [-1, 1]$ (and with $\hat{c}$ increasing arbitrarily for higher quantities).

Notice that $\hat{c}(a)$ and $c(a) = |a|^2$ are generically unequal. In fact, $\lim_{a \longrightarrow 0^+} \frac{\hat{c}(a)}{c(a)} = \lim_{a \longrightarrow 0^+} \frac{\hat{c}'(a)}{c'(a)} = +\infty$, ($c$ converges to zero arbitrarily faster than $\hat{c}$). The marginal cost $\hat{c}'(a)$ is a (cumbersome) trigonometric function, suggesting that if the heuristic intuition based on the marginal cost were correct, perhaps mechanism $\hat{c}$ would implement a social choice correspondence that maximized some trigonometric welfare function. But this is not the case: It is easy to check that $\kappa(\hat{c}) = 2$, so $\hat{c}$ implements utilitarianism as well. Put differently: quadratic voting implements utilitarianism not because its marginal cost is linear, but rather, because its limit elasticity at zero is 2, and any other mechanism with limit elasticity of 2 also implements utilitarianism.[11]

We next sketch the most relevant steps of the proof, relegating the formal details, and all lemmata, to the appendix.

For any admissible vote-buying mechanism, and for any society size, the game satisfies Reny's (2011) existence conditions, so an equilibrium exists (Lemma 1).

In any sequence of equilibria, individual vote acquisitions converge to zero: $\lim_{n\to\infty} s^n(\theta) = 0$ (Lemma 3). As individual acquisitions converge to zero, the ratio of the marginal costs corresponding, for instance, to two distinct types of alternative A supporters, must converge to the ratio of the attitudes of these types (Lemma 5). That is, for every $(\theta, \hat{\theta}) \in (0,1]^2$, we get:

$$\lim_{n\to\infty} \frac{c'(s^n(\theta))}{c'(s^n(\hat{\theta}))} = \frac{\theta}{\hat{\theta}} \Rightarrow \lim_{n\to\infty} \ln \frac{c'(s^n(\theta))}{c'(s^n(\hat{\theta}))} = \ln \frac{\theta}{\hat{\theta}}.$$

Moreover, the function $J : \mathbb{R}^2_{++} \to \mathbb{R}$ given by:

$$J(x,y) = \begin{cases} \frac{yc''(y)}{c'(y)} & \text{if } x = y \\ \frac{\ln \frac{c'(x)}{c'(y)}}{\ln \frac{x}{y}} & \text{if } x \neq y, \end{cases}$$

converges to $\kappa(c) - 1$ as $(x,y) \to (0,0)$ (Lemma 9). Hence,

$$\lim_{n\to\infty} \frac{\ln \frac{c'(s^n(\theta))}{c'(s^n(\hat{\theta}))}}{\ln \frac{s^n(\theta)}{s^n(\hat{\theta})}} = \kappa(c) - 1 \Longrightarrow \lim_{n\to\infty} \ln \left(\frac{c'(s^n(\theta))}{c'(s^n(\hat{\theta}))} \right) = \lim_{n\to\infty} \ln \left(\frac{s^n(\theta)}{s^n(\hat{\theta})} \right)^{\kappa(c)-1},$$

[11] Computational analysis shows that convergence toward implementing the desired optimum is often fast. For instance, if G is the CDF of a Logistic distribution with mean zero and variance one, and F assigns probability $\frac{3}{4}$ to valuations about -1 and probability $\frac{1}{4}$ to valuations about 10, then the probability of implementing the utilitarian optimum is greater than 99.3% for any size $n \geq 1,000$ and greater than 99.9% for any $n \geq 10,000$ if society uses a vote-buying mechanism with a quadratic cost function, and greater than 97.9% for any $n \geq 1,000$ and greater than 99.5% for any $n \geq 10,000$ if society uses the trigonometric cost function $\hat{c}$; whereas, it is less than 0.01% for any $n \geq 1,000$ under simple majority voting.

and thus substituting the left hand side according to $\lim_{n\to\infty} \ln \frac{c'(s^n(\theta))}{c'(s^n(\hat{\theta}))} = \ln \frac{\theta}{\hat{\theta}}$, we get

$$\ln \frac{\theta}{\hat{\theta}} = \lim_{n\to\infty} \ln \left(\frac{s^n(\theta)}{s^n(\hat{\theta})} \right)^{\kappa(c)-1} \Rightarrow \lim_{n\to\infty} \frac{s^n(\theta)}{s^n(\hat{\theta})} = \left(\frac{\theta}{\hat{\theta}} \right)^{\frac{1}{\kappa(c)-1}}.$$

That is, the equilibrium vote acquisitions become proportional to the ratio of the attitudes raised to a power that depends on the limit cost elasticity; and this leads to the implementation of the Bergson choice correspondence $SC_{\frac{1}{\kappa(c)-1}}$, which is SC_ρ if $\kappa(c) = \frac{1+\rho}{\rho}$. Notice, that the above argument does not depend on the nature of the outcome function or on other particular assumptions that we made. As long as the equilibrium ratio of marginal costs converges to its intuitive level (i.e. the ratio of valuations), vote acquisitions become proportional to power functions of the valuations and, hence, the implemented social choice correspondences cannot substantially differ to the Berson ones.

4 Discussion

We characterize the set of rules that are generically implementable by admissible vote-buying mechanisms in sufficiently large societies: a sequence of choice rules is generically implementable if and only if it asymptotically follows a Bergson rule. In particular, any admissible mechanism c with limit cost elasticity $\kappa(c) \in (1, \infty)$ generically implements the Bergson rule with parameter $\rho = \frac{1}{\kappa(c)-1}$.[12]

Utilitarianism is the Bergson rule with $\rho = 1$, so it is implemented by any mechanism with limit elasticity $\kappa(c) = 2$, such as a quadratic cost function. Majority rule is equivalent to the limit $\rho = 0$: as the limit cost elasticity $\kappa(c)$ (defined as marginal cost over average cost) at zero diverges to ∞, the marginal cost of votes becomes arbitrarily larger than the average cost, so everyone converges toward acquiring the same amount of votes.[13] A decentralized,

[12] Note that admissible vote-buying mechanisms are "bounded" in the sense of Jackson (1992), but they are not "strategically simple" in the sense of Börgers and Li (2017). Nor are they robust to coalitional deviations (Bierbrauer and Hellwig 2016).

[13] For instance, if $c(a) = |a|^\infty$, any quantity of votes smaller than one is free, while any quantity of votes

competitive market for votes, similar to the ones proposed for instance by Dekel, Jackson and Wolinski (2008) and Casella, Llorente-Saguer and Palfrey (2012), implements the opposite extreme, $\rho = \infty$: as the limit cost elasticity converges to 1, the marginal cost of votes becomes identical to the average cost –as in a competitive market– and the agent or agents with most intense preferences purchase most votes and determine the social decision.

Casella, Llorente-Saguer and Palfrey (2012) interpret the outcome with a market for votes as a utilitarian welfare loss. We interpret the finding differently: the outcome is optimal if the society aims to choose according to the wishes of whoever has the most intense preference. If that's the goal, a centralized market for votes with linear pricing such as ours, or a decentralized one like Casella, Llorente-Saguer and Palfrey's (2012), are optimal. If that is not society's normative goal, then we should not price votes linearly. Rather, we should choose the pricing scheme that implements society's normative goal.

Finally, we address a substantive concern: wealth inequality. A common criticism of vote-buying mechanisms is that in practice they would favor the rich, effectively disenfranchising the poor. In our theory, as in previous theories of vote-buying mechanisms, agents are risk neutral and preferences over wealth are separable and individual transfers converge to zero, so there are no wealth or budget effects: agents' actions are independent of their wealth. Concerns about the effects of wealth inequality arise if agents are risk averse and the distribution of preferences over the social choice depends on wealth. If so, wealthier agents acquire more votes, their preferences are overweighed, and the optimality of the mechanisms is lost: the axiom of anonymity is violated.

Fortunately, if individual wealth is observable and contractible (as it should be in a company, via payslips, and in a state, via tax reports), then we can restore optimality by using mechanisms such that the cost function conditions on individual wealth. By compensating the lower marginal utility of wealth of rich agents with an individualized higher monetary cost per vote for these agents, a wealth-dependent vote-buying mechanism induces all agents

above one is infinitely expensive, leading all players to acquire exactly one vote.

to condition their vote acquisitions exclusively on their intensity of preferences over the social decision, and not on their wealth.

Appendix (for online publication)

In this Appendix, we prove our results. The proof is long. It proceed in nine steps.

One - We note existence of an equilibrium for any parameter tuple (Lemma 1).

Two - We prove that net vote acquisitions for A are strictly increasing in attitude θ (Lemma 2), and we use this result to write the first order condition of the individual optimization problem (Equation (3)).

Three - We prove that equilibrium vote acquisitions converge to zero (Lemma 3 establishes the result for most attitudes; later Lemma 8 extends this result to all attitudes).

Four - We prove that the ratio of marginal costs converges to the ratio of attitudes (Lemma 5).

Five - We prove that the marginal benefit of acquiring votes converges to zero (Lemma 7), and use this result to prove that the third and fourth steps extend to all attitudes (Lemma 8, Corollary 2).

Six - We prove that the ratio of vote acquisitions converges to a power function of the ratio of attitudes; first we prove it piecewise (Lemma 10) and then over the whole domain (Lemma 11).

Seven - After two technical lemmas (Lemma 12 and Lemma 13) we establish a sufficient condition for a sequence of social choice correspondences to be implementable over a subset of distribution functions that is open and dense over the set over all cumulative distribution functions (Proposition 1).

Eight - We find a necessary condition for such implementation (Proposition 2).

Nine - We show that the necessary condition is sufficient for generic implementability, establishing our main result (Theorem 1).

Lemma 1 *For any tuple* $(n,\gamma,F,c,G)\in\mathbb{N}\backslash\{1\}\times\mathbb{R}_{++}\times\mathcal{F}\times\mathcal{C}_A\times\mathcal{G}$, *an equilibrium of game* $\Gamma^{(n,\gamma,F,c,G)}$ *exists.*

Proof. For any tuple $(n,\gamma,F,c,G)\in\mathbb{N}\backslash\{1\}\times\mathbb{R}_{++}\times\mathcal{F}\times\mathcal{C}_A\times\mathcal{G}$, define S^γ as the set of all functions with domain $[-1,1]$ and codomain $[-c^{-1}(2\gamma),c^{-1}(2y)]$, and let $\Gamma_R^{(n,\gamma,F,c,G)}$ denote the restricted game played by the n players in society N^n with strategy set S^γ and the same payoff functions as in the unrestricted game $\Gamma^{(n,\gamma,F,c,G)}$. Game $\Gamma_R^{(n,\gamma,F,c,G)}$ satisfies the nine conditions for existence of a symmetric, pure monotone Bayes-Nash equilibrium (in our jargon, an "equilibrium") in Theorem 4.5 in Reny (2011).[14] For any $i\in N^n$ and for any $\theta_i\in[-1,1]$, any action $a_i\notin[-c^{-1}(2\gamma),c^{-1}(2y)]$ is dominated by $a_i=0$. Hence the equilibrium of game $\Gamma_R^{(n,\gamma,F,c,G)}$ is also an equilibrium of game $\Gamma^{(n,\gamma,F,c,G)}$. ■

Lemma 2 *For any* $(n,\gamma,F,c,G)\in\mathbb{N}\backslash\{1\}\times\mathbb{R}_{++}\times\mathcal{F}\times\mathcal{C}_A\times\mathcal{G}$, *for any* $s^n\in E^{(n,\gamma,F,c,G)}$, $s^n:[0,1]\longrightarrow\mathbb{R}$ *is strictly increasing.*

Proof. Fix $(\gamma,F,c,G)\in\mathbb{R}_{++}\times\mathcal{F}\times\mathcal{C}_A\times\mathcal{G}$. Recall $X\equiv[-c^{-1}(2\gamma),c^{-1}(2\gamma)]$, and for any $n\in\mathbb{N}\backslash\{1\}$ and any $x\in(n-1)X$, define $\varphi^n(x)\equiv\Pr\left[\sum_{k\in N^n\backslash\{i\}}s^n(\bar{\theta}_k)=x\right]$, and define $h^n:(n-1)X\longrightarrow\mathbb{R}_+$ as the probability density of H^n such that

$$\sum_{x\in(n-1)X}\varphi^n(x)+\int_{x\in(n-1)X}h^n(x)dx=1.$$

Then, given any equilibrium $s^n\in E^{(n,\gamma,F,c,G)}$, the optimization problem of agent $i\in N^n$ with

[14] We discuss these conditions and explain why they hold in the working paper version (Eguia and Xefteris 2018).

attitude $\theta_i \in [-1,1]$ is

$$\max_{a_i \in X} \gamma\theta_i \left(\sum_{x\in(n-1)X} \varphi^n(x)G(x+a_i) + \int_{x\in(n-1)X} h^n(x)G(x+a_i)dx \right)$$
$$-\gamma\theta_i \left(\sum_{x\in(n-1)X} \varphi^n(x)(1-G(x+a_i)) + \int_{x\in(n-1)X} h^n(x)(1-G(x+a_i))dx \right) - c(a_i),$$

or equivalently

$$\max_{a_i \in X} \gamma\theta_i \left(\sum_{x\in(n-1)X} \varphi^n(x)(2G(x+a_i)-1) + \int_{x\in(n-1)X} h^n(x)(2G(x+a_i)-1)dx \right) - c(a_i).$$

Since G is continuously differentiable and the constraint $a_i \in X$ is not binding, we obtain a solution by the First Order Condition

$$2\gamma\theta_i \left(\sum_{x\in(n-1)X} \varphi^n(x)g(x+a_i) + \int_{x\in(n-1)X} h^n(x)g(x+a_i)dx \right) = c'(a_i). \tag{2}$$

Note that since g is strictly positive in $\mathbb{R}$, and $\sum\limits_{x\in(n-1)X} \varphi^n(x) + \int\limits_{x\in(n-1)X} h^n(x)dx = 1$, it follows that the summation within the parenthesis on the left-hand side of Equation (2) is strictly positive for any $a_i \in X$, and thus the left hand side is overall strictly increasing in θ_i. Assume $a_i = a \in X$ is a solution to the First Order Condition (2) for agent i with attitude θ_i, and for an arbitrary agent $j \in N^n \backslash \{i\}$, assume $\theta_j \neq \theta_i$; without loss of generality assume $\theta_j > \theta_i$. Then, the left hand side of Equation (2) has a lower value than the left hand side of the analogous First Order Condition to the optimization problem of agent j. Hence, $a_j = a$ cannot solve $j's$ first order condition, so it must be $s^n(\theta_j) \neq s^n(\theta_j)$ and thus for any $\theta, \theta' \in [-1,1]$ such that $\theta \neq \theta'$ we obtain $s^n(\theta) \neq s^n(\theta')$, which, since s^n is weakly increasing, implies s^n is strictly increasing. ■

As an immediate corollary to Lemma 2, H^n does not have a mass point, so for each $n \in N\backslash\{1\}$, we can define the probability density function $h : (n-1)X \longrightarrow \mathbb{R}_+$ so that $\int\limits_{-(n-1)X}^{x} h^n(t)dt = H^n(t)$.

Given any equilibrium $s^n \in E^{(n,\gamma,F,c,G)}$, the first order condition for the optimization problem of player $i \in N^n$ with attitude $\theta_i \in [-1,1]$ can be simplified to:

$$2\gamma\theta_i \int\limits_{x\in(n-1)X} h^n(x)g(x+a_i)dx = c'(a_i). \tag{3}$$

Lemma 3 establishes that vote acquisitions converge to zero. We use the Berry-Esseen theorem (Berry 1941; Esseen 1942).

Lemma 3 *For any tuple $(\gamma, F, c, G) \in \mathbb{R}_{++} \times \mathcal{F} \times \mathcal{C}_A \times \mathcal{G}$, and any sequence $\{s^n\}_{n=1}^{\infty}$ such that $s^n \in E^{(n,\gamma,F,c,G)}$ for each $n \in \mathbb{N}\backslash\{1\}$, $\lim\limits_{n\to\infty} s^n(\theta) = 0$ for each $\theta \in (-1,1)$.*

Proof. Proof by contradiction. For any tuple $(\gamma, F, c, G) \in \mathbb{R}_{++} \times \mathcal{F} \times \mathcal{C}_A \times \mathcal{G}$, assume that $\{s^n\}_{n=2}^{\infty}$ is a sequence of monotone, symmetric, pure equilibrium strategies of game $\Gamma^{(n,\gamma,F,c,G)}$, and assume (absurd) that there exists $\theta' \in (-1,1)$ such that $\lim\limits_{n\to\infty} s^n(\theta') \neq 0$. Then there exist a $\delta \in \mathbb{R}_{++}$ and an infinite subsequence $\{s^{n(\tau)}\}_{\tau=1}^{\infty}$ of $\{s^n\}_{n=2}^{\infty}$ with $n : \mathbb{N}\backslash\{1\} \longrightarrow \mathbb{N}$ strictly increasing, such that $|s^{n(\tau)}(\theta')| \geq \delta$ for every $\tau \in \mathbb{N}$. Note $n(\tau)$ is the size of the society in the $\tau - th$ element of the subsequence. By monotonicity of $s^{n(\tau)}(\theta)$ with respect to $\theta \in [-1,1]$ for each $\tau \in \mathbb{N}$, it follows that if $\theta' \in (-1,0)$, then $s^{n(\tau)}(\theta) \leq -\delta$ for any $\theta \in [-1,\theta']$ and for any $\tau \in \mathbb{N}$, and if $\theta' \in (0,1)$, then $s^{n(\tau)}(\theta) \geq \delta$ for any $\theta \in [\theta',1]$.

For each $n \in \mathbb{N}\backslash\{1\}$, and for each $k \in \{1,...,n\}$, let $E[s^n(\bar{\theta})]$ denote the expectation of the random variable $s^n(\bar{\theta}_k)$, where we drop the subindex k because the expectation does not depend on k. For each $n \in N\backslash\{1\}$ and for each $k \in \{1,..,n\}$, define as well the independent, identically distributed random variables

$$q^n(\bar{\theta}_k) \equiv s^n(\bar{\theta}_k) - E[s^n(\bar{\theta})] \text{ and } q^n(\bar{\theta}) \equiv s^n(\bar{\theta}) - E[s^n(\bar{\theta})];$$

let $E[q^n(\bar{\theta})]$ and $Var[q^n(\bar{\theta})]$ denote their expectation and variance, which do not depend on k. Note that for each $n \in \mathbb{N}\backslash\{1\}$, and for each $k \in \{1,..,n\}$, $E[q^n(\bar{\theta})] = 0$. Since $|s^{n(\tau)}(\theta)| \geq \delta$ for every $\tau \in \mathbb{N}$ either for any $\theta \in [\theta', 1]$ or for any for any $\theta \in [-1, \theta']$, there exists $\hat{\delta} \in \mathbb{R}_{++}$ such that $Var[q^{n(\tau)}(\bar{\theta})] > \hat{\delta}$ for any $\tau \in \mathbb{N}\backslash\{1\}$. Note $Var[q^{n(\tau)}(\bar{\theta})] \equiv E\left[\left(q^{n(\tau)}(\bar{\theta})\right)^2\right] - \left(E[q^{n(\tau)}(\bar{\theta})]\right)^2 = E\left[\left(q^{n(\tau)}(\bar{\theta})\right)^2\right]$, so $E\left[\left|q^{n(\tau)}(\bar{\theta})\right|^2\right] > \hat{\delta}$, which implies $E\left[\left|q^{n(\tau)}(\bar{\theta})\right|\right] > 0$ and $E\left[\left|q^{n(\tau)}(\bar{\theta})\right|^3\right] > 0$. Since $E\left[\left|q^{n(\tau)}(\bar{\theta}_k)\right|\right] = E\left[\left|q^{n(\tau)}(\bar{\theta})\right|\right]$ for any $k \in \{1, ..., n(\tau)\}$, for any $\tau \in \mathbb{N}$, let $E\left[\left|q^{n(\tau)}(\bar{\theta})\right|^2\right]$ and $E\left[\left|q^{n(\tau)}(\bar{\theta})\right|^3\right]$ respectively denote $Var[q^{n(\tau)}(\bar{\theta}_k)]$ and $E\left[\left|q^{n(\tau)}(\bar{\theta}_k)\right|^3\right]$ for any $k \in \{1, ..., n(\tau)\}$, for any $\tau \in \mathbb{N}$.

For each $\tau \in \mathbb{N}$, define $V^\tau(\bar{\theta}_{N^{n(\tau)}\backslash\{i\}})$ as the cumulative distribution of the random variable $\frac{\sum_{k \in N^{n(\tau)}\backslash\{i\}} q^{n(\tau)}(\bar{\theta}_k)}{\sqrt{n(\tau)-1}\sqrt{E[\left(q^{n(\tau)}(\bar{\theta})\right)^2]}}$. By the Berry-Esseen theorem (Berry 1941; Esseen 1942), there exists a $\kappa \in \mathbb{R}_{++}$ such that for any $\tau \in \mathbb{N}$ and any $x \in \mathbb{R}$,

$$|V^\tau(x) - N[0,1](x)| \leq \frac{\kappa E[\left|q^{n(\tau)}(\bar{\theta})\right|^3]}{\left(\sqrt{n(\tau)-1}\right)\left(E[\left(q^{n(\tau)}(\bar{\theta})\right)^2]\right)^{\frac{3}{2}}}.$$

For each $\tau \in \mathbb{N}$, define $\hat{H}^\tau(\bar{\theta}_{N^{n(\tau)}\backslash\{i\}})$ as the cumulative distribution of the random variable $\sum_{k \in N^{n(\tau)}\backslash\{i\}} q^{n(\tau)}(\bar{\theta}_k)$, and let $\hat{h}^\tau(\bar{\theta}_{N^{n(\tau)}\backslash\{i\}})$ be its density function. For any $z \in \mathbb{R}_{++}$ and any $x \in \mathbb{R}$, let $N[0,z](x)$ denote value at x of the cumulative distribution of a normal distribution with mean zero and variance z. Then,

$$\left|\hat{H}^\tau(x) - N[0, E[\left(q^{n(\tau)}(\bar{\theta})\right)^2]\,(n(\tau)-1)](x)\right| < \frac{\kappa E[\left|q^{n(\tau)}(\bar{\theta})\right|^3]}{\left(\sqrt{n(\tau)-1}\right)\hat{\delta}^{\frac{3}{2}}}, \tag{4}$$

Since $\{s^n(\bar{\theta})\}_{n=1}^\infty$ is bounded for any $n \in \mathbb{N}\backslash\{1\}$, both $\{E[s^{n(\tau)}(\bar{\theta})]\}_{n=1}^\infty$ and $\{q^{n(\tau)}(\bar{\theta})\}_{n=1}^\infty$ are bounded as well for any $\tau \in \mathbb{N}$, and hence $\{E[\left|q^{n(\tau)}(\bar{\theta})\right|^3]\}_{\tau=1}^\infty$ is bounded, and the right hand side of Inequality (4) converges to zero as τ diverges to infinity. Thus, the random variable $\sum_{k \in N^{n(\tau)}\backslash\{i\}} q_k^{n(\tau)}(\bar{\theta}) = \sum_{k \in N^{n(\tau)}\backslash\{i\}} \left(s_k^{n(\tau)}(\bar{\theta}) - E[s^{n(\tau)}(\bar{\theta})]\right)$ with cumulative distribution $\hat{H}^\tau(x)$ converges as $\tau \longrightarrow \infty$ to a mean zero Normal distribution with variance $E[(q^{n(\tau)}(\bar{\theta}))^2](n(\tau)-1)$. Since $E[(q^{n(\tau)}(\bar{\theta}))^2] \geq \hat{\delta}$ for any $\tau \in \mathbb{N}$, it follows that $E[(q^{n(\tau)}(\bar{\theta}))^2](n(\tau)-1)$ diverges to infinity as $\tau \longrightarrow \infty$. Therefore,

$$\lim_{\tau\longrightarrow\infty}\left(\hat{H}^{\tau}(x)-\hat{H}^{\tau}(-x)\right)=0 \text{ for any } x\in\mathbb{R}_{++}. \tag{5}$$

Since G is strictly increasing and neutral ($G(x)=1-G(-x)$), and $\lim_{x\longrightarrow-\infty}G(x)=0$, then for any $\varepsilon \in \left(0,\frac{1}{2}c(\delta)\right)$, there exist $\tilde{x}\in\mathbb{R}_{++}$ such that for any $x\in(-\infty,-\tilde{x}]\cup[\tilde{x},\infty)$,

$$[G(x+c^{-1}(2\gamma))-G(x)]2\gamma\theta' < \frac{1}{2}c(\delta)-\varepsilon$$

Since $|s^{n(\tau)}(\theta')|\geq\delta$ for every $\tau\in\mathbb{N}$ (first paragraph of this proof), it then follows that

$$[G(x+c^{-1}(2\gamma))-G(x)]2\gamma\theta' < \frac{1}{2}c(s^{n(\tau)}(\theta'))-\varepsilon$$

for any $x\in(-\infty,-\tilde{x}]\cup[x,\infty)$. Further, since $\left|s^{n(\tau)}(\theta')\right|\leq c^{-1}(2\gamma)$ (because $\left|s^{n(\tau)}(\theta')\right|>c^{-1}(2\gamma)$ implies that $s_i=s^{n(\tau)}$ is a strictly dominated strategy), it follows that for any $x\in(-\infty,-\tilde{x}]\cup[\tilde{x},\infty)$,

$$[G(x+s^{n(\tau)}(\theta'))-G(x)]2\gamma\theta' < \frac{1}{2}c(s^{n(\tau)}(\theta'))-\varepsilon. \tag{6}$$

For each $\tau\in\mathbb{N}$, and for any arbitrary agent $i\in N^{n(\tau)}$ with $\theta_i=\theta'$, the expected utility of playing $a_i=s^{n(\tau)}(\theta')$, minus the expected utility of playing $a_i=0$, is:

$$\begin{aligned}&2\gamma\theta'\left(\int_{-(n-1)c^{-1}(2\gamma)}^{-\tilde{x}}(G(x+s^{n(\tau)}(\theta'))-G(x))h^{\tau}(x)dx+\int_{-\tilde{x}}^{\tilde{x}}(G(x+s^{n(\tau)}(\theta'))-G(x))h^{\tau}(x)dx\right)\\&+2\gamma\theta'\int_{\tilde{x}}^{(n-1)c^{-1}(2\gamma)}(G(x+s^{n(\tau)}(\theta'))-G(x))h^{\tau}(x)dx-c(s^{n(\tau)}(\theta')),\end{aligned}$$

which is equal to

$$
\begin{aligned}
&2\gamma\theta'\int_{-(n-1)\left(c^{-1}(2\gamma)+E[s^n(\bar{\theta})]\right)}^{-\tilde{x}}(G(x+s^{n(\tau)}(\theta'))-G(x))\hat{h}^{\tau}(x)dx \\
&+2\gamma\theta'\int_{-\tilde{x}}^{\tilde{x}}(G(x+s^{n(\tau)}(\theta'))-G(x))\hat{h}^{\tau}(x)dx \\
&+2\gamma\theta'\int_{\tilde{x}}^{(n-1)\left(c^{-1}(2\gamma)-E[s^n(\bar{\theta})]\right)}(G(x+s^{n(\tau)}(\theta'))-G(x))\hat{h}^{\tau}(x)dx-c(s^{n(\tau)}(\theta')).
\end{aligned}
\tag{7}
$$

By Expression (5), $\lim_{\tau\longrightarrow\infty}\left(\hat{H}^{\tau}(-\tilde{x})-\hat{H}^{\tau}(\tilde{x})\right)=0$, and thus $\lim_{\tau\longrightarrow\infty}\hat{h}^{\tau}(x)=0$ for any $x\in(-\tilde{x},\tilde{x})$, and hence

$$
\lim_{\tau\longrightarrow\infty}2\gamma\theta'\int_{-\tilde{x}}^{\tilde{x}}(G(x+s^{n(\tau)}(\theta'))-G(x))\hat{h}^{\tau}(x)dx=0.
$$

Therefore, the limit of Expression (7) as $\tau\longrightarrow\infty$ is equal to the limit of

$$
\begin{aligned}
&2\gamma\theta'\int_{-(n-1)\left(c^{-1}(2\gamma)+E[s^n(\bar{\theta})]\right)}^{-\tilde{x}}(G(x+s^{n(\tau)}(\theta'))-G(x))\hat{h}^{\tau}(x)dx \\
&+2\gamma\theta'\int_{\tilde{x}}^{(n-1)\left(c^{-1}(2\gamma)-E[s^n(\bar{\theta})]\right)}(G(x+s^{n(\tau)}(\theta'))-G(x))\hat{h}^{\tau}(x)dx-c(s^{n(\tau)}(\theta')),
\end{aligned}
$$

which by Expression (6), is strictly smaller than

$$
\begin{aligned}
&\int_{-(n-1)\left(c^{-1}(2\gamma)+E[s^n(\bar{\theta})]\right)}^{-\tilde{x}}\left(\frac{1}{2}c(s^{n(\tau)}(\theta'))-\varepsilon\right)\hat{h}^{\tau}(x)dx \\
+\quad&\int_{\tilde{x}}^{(n-1)\left(c^{-1}(2\gamma)-E[s^n(\bar{\theta})]\right)}\left(\frac{1}{2}c(s^{n(\tau)}(\theta'))-\varepsilon\right)\hat{h}^{\tau}(x)dx-c(s^{n(\tau)}(\theta')) \\
<\quad& c(s^{n(\tau)}(\theta'))-\varepsilon-c(s^{n(\tau)}(\theta'))<-\varepsilon,
\end{aligned}
$$

so playing $a_i=0$ is strictly better, and hence $s_i=s^{n(\tau)}(\theta')$ is not a best response, so $s^{n(\tau)}$ is not an equilibrium. Thus, we reach a contradiction. Thus, there does not exist $\theta'\in(-1,1)$ such that $\lim_{n\to+\infty}s^n(\theta')\neq 0$, and it must be that $\lim_{n\to+\infty}s^n(\theta)=0$ for each $\theta\in(-1,1)$. ■

The next lemma reformulates the First Order Condition (3) into a form that proves more convenient for subsequent results. Recall we use the notation $X \equiv [-c^{-1}(2\gamma), c^{-1}(2\gamma)]$, so $(n-1)X = [-(n-1)c^{-1}(2\gamma), (n-1)c^{-1}(2\gamma)]$.

Lemma 4 *For any tuple $(\gamma, F, c, G) \in \mathbb{R}_{++} \times \mathcal{F} \times \mathcal{C}_A \times \mathcal{G}$, for any sequence $\{s^n\}_{n=1}^{\infty}$ such that $s^n \in E^{(n,\gamma,F,c,G)}$ for each $n \in \mathbb{N}\backslash\{1\}$, for any $n \in \mathbb{N}\backslash\{1\}$, and for each $\theta \in [-1,1]$, there exists $z^\theta : (n-1)X \longrightarrow [s^n(\theta), 0) \cup (0, s^n(\theta)]$ such that $sgn(z^\theta(x)) = sgn(\theta)$ for any $x \in [-(n-1)X, (n-1)X]$, and*

$$c'(s^n(\theta)) = 2\gamma\theta \left(\int\limits_{x\in(n-1)X} g(x)h^n(x)dx + s^n(\theta) \int\limits_{x\in(n-1)X} g'(x+z^\theta)h^n(x)dx \right). \tag{8}$$

Proof. For any given $n \in \mathbb{N}\backslash\{1\}$, only a compact subset of the domain of G, namely $[-nX, nX]$ is relevant, since $ns^n(\theta) \in nX$ for any θ. And G is twice continuously differentiable. Note that by the First Order Condition (3), for each $\theta \in [-1,1]$,

$$c'(s^n(\theta)) = 2\gamma\theta \int\limits_{x\in(n-1)X} g(x+s^n(\theta))h^n(x)dx.$$

We want to show that for any $x \in (n-1)X$, and any $\theta \in [0,1]$, there exists a $z^\theta(x) \in (0, s^n(\theta))$ such that

$$g(x+s^n(\theta)) = g(x) + s^n(\theta)g'(x+z^\theta(x)). \tag{9}$$

For each $x \in (n-1)X$, define $y_{\min} \equiv \arg \min\limits_{y\in[x,x+s^n(\theta)]} g'(y)$ and $y_{\max} \equiv \arg \max\limits_{y\in[x,x+s^n(\theta)]} g'(y)$. Then note

$$(s^n(\theta))g'(y_{\min}) \leq g(x+s^n(\theta)) - g(x) \leq (s^n(\theta))g'(y_{\max})$$

Since g is continuous, by the Intermediate Value Theorem, there exists some value $y(x) \in [x, x+s^n(\theta)]$ such that

$$(s^n(\theta))g'(y(x)) = g(x+s^n(\theta)) - g(x).$$

Then, define $z^{\theta}(x) \equiv y(x) - x$ and we obtain Equality (9).

An analogous argument, in this instance with $y(x) \in [x + s^n(\theta), x]$, establishes that for any $\theta \in [-1, 0]$, there exists a $z^{\theta}(x) \in [s^n(\theta), 0]$ such that Equality (9) holds. ■

The next lemma uses Lemma 4 to establish that the ratio of marginal costs of two agents converges to their ratio of attitudes.

Lemma 5 *For any tuple* $(\gamma, F, c, G) \in \mathbb{R}_{++} \times \mathcal{F} \times \mathcal{C}_A \times \mathcal{G}$, *for any sequence of equilibria* $\{s^n\}_{n=2}^{\infty}$, *for any* $\theta \in (-1, 1)$ *and for any* $\hat{\theta} \in (-1, 0) \cup (-1, 0)$,

$$\lim_{n \to \infty} \frac{c'(s^n(\theta))}{c'(s^n(\hat{\theta}))} = \frac{\theta}{\hat{\theta}}.$$

Proof. For any tuple $(\gamma, F, c, G) \in \mathbb{R}_{++} \times \mathcal{F} \times \mathcal{C}_A \times \mathcal{G}$, let $\{s^n\}_{n=2}^{\infty}$ be a sequence of equilibria, that is, $s^n \in E^{(n,\gamma,F,c,G)}$ for each $n \in \mathbb{N} \backslash \{1\}$.

From Lemma 4, for each $\theta \in [-1, 1]$,

$$c'(s^n(\theta)) = 2\gamma\theta \left(\int\limits_{x \in (n-1)X} g(x) h^n(x) dx + s^n(\theta) \int\limits_{x \in (n-1)X} g'(x + z^{\theta}(x)) h^n(x) dx \right).$$

Notice that since g is strictly positive and continuous, and g' is continuous, for any $x, y \in \mathbb{R}$, $\frac{g'(y)}{g(x)}$ is continuous, and over any closed interval of $\mathbb{R}$, it is bounded. Further, by Condition (iii) of the definition of $\mathcal{G}$, $\exists \hat{\varepsilon} \in \mathbb{R}_{++}$ such that for any $\varepsilon \in (0, \hat{\varepsilon})$,

$$\lim_{x \to -\infty} \frac{g'(x + \varepsilon)}{g(x)} \in \mathbb{R} \text{ and } \lim_{x \to \infty} \frac{g'(x + \varepsilon)}{g(x)} \in \mathbb{R}.. \tag{10}$$

Therefore, there exists $\lambda \in \mathbb{R}_{++}$ such that $\frac{g'(x+\varepsilon)}{g(x)} \in [-\lambda, \lambda]$, for any $\varepsilon \in (0, \bar{\varepsilon})$ and for any $x \in \mathbb{R}$. Equivalently,

$$-\lambda g(x) \leq g'(x + \varepsilon) \leq \lambda g(x) \; \forall \varepsilon \in (0, \bar{\varepsilon}), \; \forall x \in \mathbb{R}. \tag{11}$$

Since for any sequence $\{s^n\}_{n=1}^{\infty}$ of equilibria $\lim_{n\to\infty} s^n(\theta) = 0$ for each $\theta \in (-1,1)$ (Lemma 3), and since $z^\theta(x)$ defined in Lemma 4 satisfies $z^\theta(x) \in (0, s^n(\theta))$, it follows $\lim_{n\to\infty} z^\theta(x) = 0$ for each $\theta \in [-1,1]$ and for each $x \in (n-1)X$. Then, it follows from Expression (11), that that that there exists $\hat{n} \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ such that $n > \hat{n}$, for each $x \in (n-1)X$, for any $\theta \in (-1,0) \cup (0,1)$, and for any equilibrium strategy s^n, we have:

$$-\lambda g(x) < g'(x + z^\theta(x)) < \lambda g(x).$$

Therefore,

$$\begin{aligned} g(x) - s^n(\theta)\lambda g(x) &< g(x) + s^n(\theta) g'(x + z^\theta(x)) < g(x) + s^n(\theta)\lambda g(x); \\ [1 - s^n(\theta)\lambda] g(x)\theta h^n(x) &< (g(x) + s^n(\theta) g'(x + z^\theta(x)))\theta h^n(x) < (1 + s^n(\theta)\lambda) g(x)\theta h^n(x). \end{aligned}$$

Once again since $\lim_{n\to\infty} s^n(\theta) = 0$ for each $\theta \in (-1,1)$ (Lemma 3), there exists $\tilde{n}$ such that $1 - s^n(\theta)\lambda > 0$ for every $n > \tilde{n}$.

Then we can integrate x over $(n-1)X$ on all sides and multiply by 2γ to obtain:

$$\begin{aligned} & 2\gamma[1 - s^n(\theta)\lambda]\theta \int\limits_{x \in (n-1)X} g(x) h^n(x) dx \\ < \; & 2\gamma\theta \int\limits_{x \in (n-1)X} (g(x) + s^n(\theta) g'(x + z^\theta(x))) h^n(x) dx \\ < \; & 2\gamma(1 + s^n(\theta)\lambda)\theta \int\limits_{x \in (n-1)X} g(x) h^n(x) dx, \end{aligned}$$

and hence, substituting Equality (8), for any $\theta \in (-1,0) \cup (0,1)$,

$$c'(s^n(\theta)) \in \left(2\gamma(1 - s^n(\theta)\lambda)\theta \int\limits_{x \in (n-1)X} g(x) h^n(x) dx, \; 2\gamma(1 + s^n(\theta)\lambda)\theta \int\limits_{x \in (n-1)X} g(x) h^n(x) dx \right). \tag{12}$$

Then, for any $\theta, \hat{\theta} \in (-1,0) \cup (0,1)$,

$$\begin{aligned}\frac{c'(s^n(\theta))}{c'(s^n(\hat{\theta}))} &\in \left(\frac{(1-s^n(\theta)\lambda)\theta \int\limits_{x\in(n-1)X} g(x)h^n(x)dx}{(1+s^n(\hat{\theta})\lambda)\hat{\theta} \int\limits_{x\in(n-1)X} g(x)h^n(x)dx}, \frac{(1+s^n(\theta)\lambda)\theta \int\limits_{x\in(n-1)X} g(x)h^n(x)dx}{(1-s^n(\theta)\lambda)\hat{\theta} \int\limits_{x\in(n-1)X} g(x)h^n(x)dx}\right) \\ &= \left(\frac{(1-s^n(\theta)\lambda)\theta}{(1+s^n(\hat{\theta})\lambda)\hat{\theta}}, \frac{(1+s^n(\theta)\lambda)\theta}{(1-s^n(\theta)\lambda)\hat{\theta}}\right).\end{aligned}$$

Note that because $\lim\limits_{n\longrightarrow\infty} s^n(\tilde{\theta}) = 0$ for any $\tilde{\theta} \in (-1,0) \cup (0,1)$ (Lemma 3) and $s^n(0) = 0$ for any $n \in \mathbb{N}$, both limit points of the interval converge to $\frac{\theta}{\hat{\theta}}$ as n increases to infinity. Hence, for any $(\theta, \hat{\theta}) \in (-1,1)^2$, $\lim\limits_{n\longrightarrow\infty} \frac{c'(s^n(\theta))}{c'(s^n(\hat{\theta}))} = \frac{\theta}{\hat{\theta}}$. ■

The next lemma proves the following observation: a cost elasticity greater than one near zero implies that the cost function is convex near zero.

Lemma 6 *For any $c \in \mathcal{C}_A$, there exists $\lambda_c \in \mathbb{R}_{++}$ such that $c''(a) \in \mathbb{R}_{++}$ for any $a \in (0, \lambda_c]$.*

Proof. By definition of $\mathcal{C}_A$, $c \in \mathcal{C}_A$ implies that $\lim\limits_{a\longrightarrow 0} \frac{ac'(a)}{c(a)} \in (1, \mathbb{R})$, $c(0) = 0$ and $\lim\limits_{a\longrightarrow 0} ac'(a) = 0$. Let $z \equiv \lim\limits_{a\longrightarrow 0} \frac{ac'(a)}{c(a)}$. Then $\lim\limits_{a\longrightarrow 0} \frac{ac'(a)}{c(a)} = \frac{0}{0}$; applying L'Hopital rule,

$$z = \lim_{a\longrightarrow 0} \frac{ac'(a)}{c(a)} = \lim_{a\longrightarrow 0} \left(1 + \frac{ac''(a)}{c'(a)}\right)$$

so

$$\lim_{a\longrightarrow 0} \frac{ac''(a)}{c'(a)} = z - 1.$$

Hence, for any $\varepsilon \in \mathbb{R}_{++}$, there exists $\lambda_\varepsilon \in \mathbb{R}_{++}$ such that for any $a \in (0, \lambda_\varepsilon]$,

$$\frac{ac''(a)}{c'(a)} \in (z - 1 - \varepsilon, z - 1 + \varepsilon). \tag{13}$$

Select $\varepsilon = \frac{z-1}{2}$, and since $z > 1$, note that $z - 1 - \varepsilon > 0$. Further, for any $a \in \left(0, \lambda_{\frac{z-1}{2}}\right]$, by assumption $c'(a) > 0$. Thus, from Expression (13), it follows $c''(a) > \frac{c'(a)}{a}(\frac{z-1}{2}) > 0$ for any $a \in \left(0, \lambda_{\frac{z-1}{2}}\right]$. ■

Next we establish that the marginal effect of acquiring votes over the outcome converges to zero (Lemma 7).

Lemma 7 *For any tuple* $(\gamma, F, c, G) \in \mathbb{R}_{++} \times \mathcal{F} \times \mathcal{C}_A \times \mathcal{G}$, *and for any sequence of equilibria* $\{s^n\}_{n=2}^{\infty}$,

$$\lim_{n \longrightarrow \infty} \int\limits_{x \in (n-1)X} g(x) h^n(x) dx = 0.$$

Proof. By Lemma 6, there exists a $\lambda \in \mathbb{R}_{++}$ such that c' is strictly increasing in $(0, \lambda]$. Therefore, c' is invertible over $(0, \lambda]$. Let $(c')^{-1}$ denote the inverse of c' over $(0, \lambda]$. Then, for any $\theta \in (-1, 1)$, from Expression (12) in the proof of Lemma 5,

$$s^n(\theta) \in \left(\begin{array}{c} (c')^{-1} \left(2\gamma(1 - s^n(\theta)\lambda)\theta \int\limits_{x \in (n-1)X} g(x) h^n(x) dx \right), \\ (c')^{-1} \left(2\gamma(1 + s^n(\theta)\lambda)\theta \int\limits_{x \in (n-1)X} g(x) h^n(x) dx \right) \end{array} \right)$$

and, since $\lim\limits_{n \longrightarrow \infty} s^n(\theta) = 0$ for any $\theta \in (-1, 1)$ (Lemma 3), it follows that

$$\lim_{n \longrightarrow \infty} (c')^{-1} \left(2\gamma(1 - s^n(\theta)\lambda)\theta \int\limits_{x \in (n-1)X} g(x) h^n(x) dx \right) = 0,$$

which, since $c'(0) = 0$ and thus $(c')^{-1}(0) = 0$, implies

$$\lim_{n \longrightarrow \infty} \left(2\gamma(1 - s^n(\theta)\lambda)\theta \int\limits_{x \in (n-1)X} g(x) h^n(x) dx \right) = 0,$$

which, for any $\theta \in (-1, 1) \backslash \{0\}$, implies $\lim\limits_{n \longrightarrow \infty} \int\limits_{x \in (n-1)X} g(x) h^n(x) dx = 0$. ■

Lemma 7 allows us to more easily strengthen Lemma 3 by showing that vote acquisitions converge to zero for every realization of attitudes, including $\theta \in \{-1, 1\}$.

Lemma 8 *For any* $(\gamma, F, c, G) \in \mathbb{R}_{++} \times \mathcal{F} \times \mathcal{C}_A \times \mathcal{G}$, *and any sequence* $\{s^n\}_{n=1}^{\infty}$ *such that* $s^n \in E^{(n,\gamma,F,c,G)}$ *for each* $n \in \mathbb{N}\backslash\{1\}$, *and for any* $\theta \in [-1,1]$, $\lim_{n\to\infty} s^n(\theta) = 0$.

Proof. Recall that $\lim_{n\to\infty} s^n(\theta) = 0$ for any $\theta \in (-1,1)$ by Lemma 3. For $\theta_i \in \{-1,1\}$, note that the First Order Condition (3) for agent i is

$$2\gamma\theta_i \int_{x\in(n-1)X} h^n(x)g(x+a_i)dx = c'(a_i),$$

By definition of $\mathcal{G}$, and since $G \in \mathcal{G}$, G is strictly increasing and continuously differentiable, thus g is continuous and strictly positive, and hence g and $\frac{g(x+a_i)}{g(x)}$ are bounded over any closed interval of $\mathbb{R}$. Further, also by definition of $\mathcal{G}$, $\exists\hat{\varepsilon} \in \mathbb{R}_{++}$ such that $\lim_{x\to-\infty} \frac{g'(x+\varepsilon)}{g(x)} \in \mathbb{R}$ and $\lim_{x\to\infty} \frac{g'(x+\varepsilon)}{g(x)} \in \mathbb{R}$ for any $\varepsilon \in [0,\hat{\varepsilon})$. In particular, for $\varepsilon = 0$, $\frac{g'(x)}{g(x)}$ is bounded over $\mathbb{R}$, and $\frac{g(x)+\int_x^{x+a_i} g'(t)dt}{g(x)} = \frac{g(x+a_i)}{g(x)}$ is bounded over $\mathbb{R}$ as well, so there exists some $K \in \mathbb{R}_{++}$ such that $g(x+a_i) \leq Kg(x)$ and

$$\int_{x\in(n-1)X} h^n(x)g(x+a_i)dx \leq K \int_{x\in(n-1)X} h^n(x)g(x+a_i)dx$$

and hence, by Lemma 7,

$$\lim_{n\longrightarrow\infty} \int_{x\in(n-1)X} h^n(x)g(x+a_i)dx = 0$$

so

$$\lim_{n\longrightarrow\infty} 2\gamma\theta_i \int_{x\in(n-1)X} h^n(x)g(x+a_i)dx = \lim_{n\longrightarrow\infty} c'(a_i) = 0,$$

so $\lim_{n\longrightarrow\infty} a_i = 0$. ■

As a corollary of Lemma 8, we can more strengthen Lemma 5 so that it holds for any pair of types. $(\theta,\hat{\theta}) \in [-1,1]^2$.

Corollary 2 *For any tuple* $(\gamma, F, c, G) \in \mathbb{R}_{++} \times \mathcal{F} \times \mathcal{C}_A \times \mathcal{G}$*, for any sequence of equilibria* $\{s^n\}_{n=2}^{\infty}$*, for any* $\theta \in [-1, 1]$ *and for any* $\hat{\theta} \in [-1, 0) \cup (0, 1]$*,*

$$\lim_{n\to\infty} \frac{c'(s^n(\theta))}{c'(s^n(\hat{\theta}))} = \frac{\theta}{\hat{\theta}}.$$

The proof follows step-by-step the proof of Lemma 5, noting, where needed, that $\lim_{n\to\infty} s^n(\theta) = 0$ for $\theta \in \{-1, 1\}$ by Lemma 8.

We next define an auxiliary function and prove a lemma related to it. Define $J : \mathbb{R}^2_{++} \longrightarrow \mathbb{R}_+$ by

$$J(x, y) = \begin{cases} \frac{yc''(y)}{c'(y)} & \text{if } x = y \\ \frac{\ln c'(x) - \ln c'(y)}{\ln x - \ln y} & \text{otherwise} \end{cases}.$$

And recall that for any $c \in \mathcal{C}_A$, $\kappa(c) \equiv \lim_{a\longrightarrow 0} \frac{ac'(a)}{c(a)}$.

Lemma 9 *Let* $\{x_n\}_{n=1}^{\infty} \in \mathbb{R}^{\infty}_{++}$ *and* $\{y_n\}_{n=1}^{\infty} \in \mathbb{R}^{\infty}_{++}$ *be two converging sequences with* $\lim_{n\longrightarrow\infty} x_n = \lim_{n\longrightarrow\infty} y_n = 0$*. Then* $\lim_{n\longrightarrow\infty} J(x_n, y_n) = \kappa(c) - 1$*.*

Proof. Note that for any $y \in \mathbb{R}_{++}$,

$$\lim_{x\longrightarrow 0} J(x, y) = \frac{\ln c'(0) - \ln c'(y)}{\ln 0 - \ln y} = \frac{-\infty}{-\infty},$$

applying L'Hopital rule,

$$\lim_{x\longrightarrow 0} J(x, y) = \lim_{x\longrightarrow 0} \frac{\frac{c''(x)}{c'(x)}}{\frac{1}{x}} = \lim_{x\longrightarrow 0} \frac{xc''(x)}{c'(x)}.$$

Notice that $\kappa(c) \equiv \lim_{x\longrightarrow 0} \frac{xc'(x)}{c(x)} = \frac{0}{0}$, so applying L'Hopital rule,

$$\kappa(c) = \lim_{x\longrightarrow 0} \frac{c'(x) + xc''(x)}{c'(x)} = 1 + \lim_{x\longrightarrow 0} \frac{xc''(x)}{c'(x)}$$

$$\kappa(c) - 1 = \lim_{x\longrightarrow 0} \frac{xc''(x)}{c'(x)}, \tag{14}$$

so $\lim\limits_{x \longrightarrow 0} J(x,y) = \kappa(c) - 1$. Note as well that, using L'Hopital rule

$$\lim_{\varepsilon \longrightarrow 0} J(x, x+\varepsilon) = \frac{-\frac{c''(x)}{c'(x)}}{-\frac{1}{x}} = \frac{xc''(x)}{c'(x)}$$

so J is continuous.

Define the function $v : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ by

$$v(x) = \begin{cases} \kappa(c) - 1 & \text{if } x = 0 \\ \frac{xc''(x)}{c'(x)} & \text{if } x \in \mathbb{R}_{++}. \end{cases}$$

By Equality (14), $\lim\limits_{x \longrightarrow 0} \frac{xc''(x)}{c'(x)} = \kappa(c) - 1$ and hence $\lim\limits_{x \longrightarrow 0} v(x) = \kappa(c) - 1$ and v is continuous.

Define the correspondence $x^+ : \mathbb{R}_+ \rightrightarrows \mathbb{R}_+$ by $x^+(w) = \arg\max\limits_{x \in [0,w]} v(x)$ for each $w \in \mathbb{R}_+$, and the correspondence $x^- : \mathbb{R}_+ \rightrightarrows \mathbb{R}_+$ by $x^-(w) = \arg\min\limits_{x \in [0,w]} v(x)$ for each $w \in \mathbb{R}_+$, and define the function $v^+ : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ by $v^+(w) = \max\limits_{x \in [0,w]} v(x)$ for each $w \in \mathbb{R}_+$ and the function $v^- : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ by $v^-(w) \equiv \min\limits_{x \in [0,w]} v(x)$ for each $w \in \mathbb{R}_+$. Since v is continuous, $x^+(w)$ and $x^-(w)$ are non-empty for each $w \in \mathbb{R}_+$, x^+ and x^- are upper hemi continuous, and v^+ and v^- are continuous (Berge's maximum theorem). Further, note that v^+ is non-decreasing and v^- is non-increasing.

Construct two sequences $\{x_t\}_{t=1}^{\infty} \in \mathbb{R}_+^{\infty}$ and $\{y_t\}_{t=1}^{\infty} \in \mathbb{R}_+^{\infty}$ such that $\lim\limits_{t \longrightarrow \infty} x_t = \lim\limits_{t \longrightarrow \infty} y_t = 0$. Then

$$\lim_{t \longrightarrow 0} \frac{x_t c''(x_t)}{c'(x_t)} = \lim_{t \longrightarrow 0} \frac{y_t c''(y_t)}{c'(y_t)} = \kappa(c) - 1.$$

Note that for any $y \in \mathbb{R}_{++}$, and for any $x \in (0,y)$, J is differentiable and

$$\begin{aligned} \frac{\partial J}{\partial x}(x,y) &= \frac{\frac{c''(x)}{c'(x)}(\ln x - \ln y) - (\ln c'(x) - \ln(c'(y))\frac{1}{x}}{(\ln x - \ln y)^2} \\ &= \frac{xc''(x)(\ln x - \ln y) - c'(x)(\ln c'(x) - \ln(c'(y))}{xc'(x)\,(\ln x - \ln y)^2}. \end{aligned}$$

Hence $\frac{\partial J}{\partial x}(x,y)=0$ if and only if

$$\begin{aligned} xc''(x)(\ln x-\ln y) &= c'(x)(\ln c'(x)-\ln(c'(y)) \\ \frac{xc''(x)}{c'(x)} &= \frac{\ln c'(x)-\ln c'(y)}{\ln x-\ln y}, \end{aligned}$$

that is, $\frac{\partial J}{\partial x}(x,y)=0$ if and only if $J(x,y)=\frac{xc''(x)}{c'(x)}$.

Since $x\in\arg\max\limits_{x\in(0,y)} J(x,y)$ implies $\frac{\partial J}{\partial x}(x,y)=0$, it follows that for any $y\in\mathbb{R}_{++}$ and any $x\in\arg\max\limits_{x\in(0,y)} J(x,y)$, $J(x,y)=v(x)$, so $J(x,y)\le v^{+}(x)$. Since v^{+} is non-decreasing, it follows $\max\limits_{x\in(0,y)} J(x,y)\le v^{+}(y)$. If $\arg\max\limits_{x\in(0,y)} J(x,y)=\varnothing$, then $\sup\limits_{x\in(0,y)} J(x,y)\in\left\{\lim\limits_{x\longrightarrow0} J(x,y),J(y,y)\right\}=\{z-1,v(y)\}\le v^{+}(y)$. So $\sup\limits_{x\in(0,y)} J(x,y)\le v^{+}(y)$ for any $y\in\mathbb{R}_{++}$. Similarly, it can be shown that $\sup\limits_{y\in(0,x)} J(x,y)\le v^{+}(x)$ for any $x\in\mathbb{R}_{++}$.

Moreover, since $x\in\arg\min\limits_{x\in(0,y)} J(x,y)$ implies $\frac{\partial J}{\partial x}(x,y)=0$, it follows that for any $y\in\mathbb{R}_{++}$ and any $x\in\arg\min\limits_{x\in(0,y)} J(x,y)$, $J(x,y)=v(x)$, so $J(x,y)\ge v^{-}(x)$. Since v^{-} is non-decreasing, it follows $\max\limits_{x\in(0,y)} J(x,y)\ge v^{-}(y)$. If $\arg\min\limits_{x\in(0,y)} J(x,y)=\varnothing$, then $\inf\limits_{x\in(0,y)} J(x,y)\in\{\lim\limits_{x\longrightarrow0} J(x,y),J(y,y)\}=\{z-1,v(y)\}\ge v^{-}(y)$. So $\inf\limits_{x\in(0,y)} J(x,y)\ge v^{-}(y)$ for any $y\in\mathbb{R}_{++}$. Similarly, it can be shown that $\sup\limits_{y\in(0,x)} J(x,y)\ge v^{-}(y)$ for any $x\in\mathbb{R}_{++}$.

From all the above it follows that for any $t\in\mathbb{N}$, $J(x_t,y_t)\in[v^{-}(w_t),v^{+}(w_t)]$, where $w_t=\max\{x_t,y_t\}$. Notice that $\lim\limits_{t\longrightarrow\infty} w_t=0$, and thus $\lim\limits_{t\longrightarrow0} v^{-}(w_t)=\kappa(c)-1$ and $\lim\limits_{t\longrightarrow0} v^{+}(w_t)=z-1$, and hence $\lim\limits_{n\longrightarrow\infty} J(x_n,y_n)=\kappa(c)-1$. ■

We next establish a key intermediary result: equilibrium actions are asymptotically piece-wise linear in $(\theta)^{\rho}$.

Lemma 10 *For any tuple $(\gamma,F,c,G)\in\mathbb{R}_{++}\times\mathcal{F}\times\mathcal{C}_A\times\mathcal{G}$, for any $\{s^n\}_{n=1}^{\infty}$ such that $s^n\in E^{(n,\gamma,F,c,G)}$ for each $n\in\mathbb{N}\backslash\{1\}$, and for any $(\theta,\hat{\theta})^2\in[-1,0)^2\cup(0,1]^2$,*

$$\lim_{n\to\infty}\frac{s^n(\theta)}{s^n(\hat{\theta})}=\left(\frac{\theta}{\hat{\theta}}\right)^{\frac{1}{\kappa(c)-1}}.$$

Proof. For any $(\theta, \hat{\theta}) \in [-1,0)^2 \cup (0,1]^2$, by Lemma 5 and Corollary 2, $\lim_{n\to\infty} \frac{c'(s^n(\theta))}{c'(s^n(\hat{\theta}))} = \frac{\theta}{\hat{\theta}}$, and taking logarithms on both sides,

$$\lim_{n\to\infty} (\ln c'(s^n(\theta)) - \ln c'(s^n(\hat{\theta})) = \ln\left(\frac{\theta}{\hat{\theta}}\right). \tag{15}$$

By Lemma 9, for any $\{x_n\}_{n=1}^{\infty} \in \mathbb{R}_{++}^{\infty}$ with $\lim_{n\to\infty} x_n = 0$ and $\{y_n\}_{n=1}^{\infty} \in \mathbb{R}_{++}^{\infty}$ with $\lim_{n\to\infty} y_n = 0$,

$$\lim_{n\to\infty} \frac{\ln c'(x_n) - \ln c'(y_n)}{\ln \frac{x_n}{y_n}} = \kappa(c) - 1,$$

thus, in particular,

$$\begin{aligned} \lim_{n\to\infty} \frac{\ln c'(s^n(\theta)) - \ln c'(s^n(\hat{\theta}))}{\ln \frac{s^n(\theta)}{s^n(\hat{\theta})}} &= \kappa(c) - 1, \\ \lim_{n\to\infty} \left(\ln c'(s^n(\theta)) - \ln c'(s^n(\hat{\theta}))\right) &= \lim_{n\to\infty} \ln\left(\frac{s^n(\theta)}{s^n(\hat{\theta})}\right)^{\kappa(c)-1} \end{aligned}$$

and thus substituting the left hand side according to Equality 15, we obtain

$$\begin{aligned} \ln\frac{\theta}{\hat{\theta}} &= \lim_{n\to\infty} \ln\left(\frac{s^n(\theta)}{s^n(\hat{\theta})}\right)^{\kappa(c)-1}, \\ \lim_{n\to\infty} \frac{s^n(\theta)}{s^n(\hat{\theta})} &= \left(\frac{\theta}{\hat{\theta}}\right)^{\frac{1}{\kappa(c)-1}}. \end{aligned} \tag{16}$$

■

Further, we can strengthen this result, to obtain linearity in $(\theta)^{\rho}$.

Lemma 11 *For any tuple $(\gamma, F, c, G) \in \mathbb{R}_{++} \times \mathcal{F} \times \mathcal{C}_A \times \mathcal{G}$, for any $\{s^n\}_{n=1}^{\infty}$ such that $s^n \in E^{(n,\gamma,F,c,G)}$ for each $n \in \mathbb{N}\backslash\{1\}$, and for any $(\theta, \hat{\theta})^2 \in [-1,0)^2 \cup (0,1]^2$,*

$$\lim_{n\to\infty} \frac{s^n(\theta)}{s^n(\hat{\theta})} = sgn\left(\frac{\theta}{\hat{\theta}}\right)\left|\frac{\theta}{\hat{\theta}}\right|^{\frac{1}{\kappa(c)-1}}. \tag{17}$$

Proof. For any $(\theta,\hat{\theta}) \in [-1,0]^2 \cup [0,1]^2$, Equality (17) reduces to Equality (16), which holds by Lemma 10. We want to show that Equality (17) holds as well for any $(\theta,\hat{\theta}) \in ([-1,0] \times [0,1]) \cup ([0,1] \times [-1,0])$ (that is, if θ and $\hat{\theta}$ have different sign). For any $\theta \in [-1,0) \cup (0,1]$, by Lemma 5 and Corollary 2,

$$\lim_{n\to\infty} \frac{c'(s^n(\theta))}{c'(s^n(-\theta))} = -1.$$

Hence, for any $(\theta,\hat{\theta}) \in ([-1,0] \times [0,1]) \cup ([0,1] \times [-1,0])$,

$$\lim_{n\to\infty} \frac{c'(s^n(\theta))}{c'(s^n(\hat{\theta}))} = \lim_{n\to\infty} \frac{-c'(s^n(|\theta|))}{c'(s^n(|\hat{\theta}|))},$$

which, by Lemma 5 and Corollary 2, is equal to $-\frac{|\theta|}{|\hat{\theta}|}$. Thus,

$$-\lim_{n\to\infty} \frac{c'(s^n(\theta))}{c'(s^n(\hat{\theta}))} = \frac{|\theta|}{|\hat{\theta}|}. \tag{18}$$

Note that the left hand side of Expression (18) is equal to $\lim\limits_{n\to\infty} \frac{c'(|s^n(\theta)|)}{c'(|s^n(\hat{\theta})|)} \in \mathbb{R}_+$, so we can take logarithms on both side, and obtain

$$\lim_{n\to\infty} \left(\ln c'(|s^n(\theta)|) - \ln c'(|s^n(\hat{\theta})|)\right) = \ln\left(\frac{|\theta|}{|\hat{\theta}|}\right). \tag{19}$$

By Lemma 9, for any $\{x_n\}_{n=1}^{\infty} \in \mathbb{R}_{++}^{\infty}$ with $\lim\limits_{n\to\infty} x_n = 0$ and $\{y_n\}_{n=1}^{\infty} \in \mathbb{R}_{++}^{\infty}$ with $\lim\limits_{n\to\infty} y_n = 0$,

$$\lim_{n\to\infty} \frac{\ln c'(x_n) - \ln c'(y_n)}{\ln \frac{x_n}{y_n}} = \kappa(c) - 1,$$

thus, in particular,

$$\begin{aligned}
\lim_{n\to\infty} \frac{\ln c'(|s^n(\theta)|) - \ln c'(|s^n(\hat{\theta})|)}{\ln \frac{|s^n(\theta)|}{|s^n(\hat{\theta})|}} &= \kappa(c) - 1, \\
\lim_{n\to\infty} \left(\ln c'(|s^n(\theta)|) - \ln c'(|s^n(\hat{\theta})|)\right) &= \lim_{n\to\infty} \ln \left|\frac{s^n(\theta)}{s^n(\hat{\theta})}\right|^{\kappa(c)-1},
\end{aligned}$$

and thus substituting the left hand side according to Equality 19, we obtain

$$\begin{aligned}
\ln\left(\frac{|\theta|}{|\hat{\theta}|}\right) &= \lim_{n\to\infty} \ln\left|\frac{s^n(\theta)}{s^n(\hat{\theta})}\right|^{\kappa(c)-1}, \text{ so } \lim_{n\to\infty}\left|\frac{s^n(\theta)}{s^n(\hat{\theta})}\right| = \left|\frac{\theta}{\hat{\theta}}\right|^{\frac{1}{\kappa(c)-1}}, \text{ and} \\
\lim_{n\to\infty}\frac{s^n(\theta)}{s^n(\hat{\theta})} &= sgn\left(\frac{\theta}{\hat{\theta}}\right)\left|\frac{\theta}{\hat{\theta}}\right|^{\frac{1}{\kappa(c)-1}}.
\end{aligned}$$

■

So acquisitions of votes converge to linear in a power of valuations.

For any $F \in \mathcal{F}$, and for any function $\varphi : [-1,1] \longrightarrow \mathbb{R}$, let $E_F\left[\varphi(\bar{\theta})\right]$ denote the expectation of the random variable $\varphi(\bar{\theta})$, given that $\bar{\theta}$ is distributed according to F. If F is fixed and unambiguous, we drop the subindex. For any $\rho \in \mathbb{R}_{++}$, define $\mathcal{F}^\rho \subset \mathcal{F}$ by $\mathcal{F}^\rho \equiv \{F \in \mathcal{F} : E_F[sgn(\bar{\theta})|\bar{\theta}|^\rho] \neq 0\}$.

Lemma 12 *For any $\rho \in \mathbb{R}_{++}$, $\mathcal{F}^\rho$ is open and dense in $\mathcal{F}$.*

Proof. Consider an arbitrary $F \in \mathcal{F}^\rho$. By definition of $\mathcal{F}^\rho$, it follows from $F \in \mathcal{F}^\rho$ that $E_F[sgn(\bar{\theta})|\bar{\theta}|^\rho] \neq 0$. Without loss of generality, assume $E_F[sgn(\bar{\theta})|\bar{\theta}|^\rho] > 0$, that is,

$$\int_0^1 f(\theta)\theta^\rho d\theta - \int_{-1}^0 f(\theta)|\theta|^\rho d\theta = \kappa$$

for some $\kappa \in \mathbb{R}_{++}$. For any $\varepsilon \in \mathbb{R}_{++}$, let $N_\varepsilon(F)$ be the open $\varepsilon-$neighborhood around F, in the metric space $(\mathcal{F}, d_{\infty,\infty})$. For any $\varepsilon \in \mathbb{R}_{++}$, and for any $\hat{F} \in N_\varepsilon(F)$,

$$d_\infty(F,\hat{F}) + d_\infty(f,\hat{f}) < \varepsilon, \text{ that is,}$$

$$\sup_{\theta\in[-1,1]}\left\{\left|F(\theta)-\hat{F}(\theta)\right|\right\} + \sup_{\theta\in[-1,1]}\left\{\left|f(\theta)-\hat{f}(\theta)\right|\right\} < \varepsilon,$$

which implies

$$\sup_{\theta\in[-1,1]}\left\{\left|f(\theta)-\hat{f}(\theta)\right|\right\} < \varepsilon, \text{ and thus,}$$

$$
\begin{aligned}
\int_0^1 f(\theta)\theta^\rho d\theta - \int_{-1}^0 f(\theta)|\theta|^\rho d\theta - \left(\int_0^1 \hat{f}(\theta)\theta^\rho d\theta - \int_{-1}^0 \hat{f}(\theta)|\theta|^\rho d\theta\right) &< \varepsilon \int_{-1}^1 |\theta|^\rho d\theta \\
&= 2\varepsilon \frac{1}{\rho+1},
\end{aligned}
$$

so for $\varepsilon < \frac{\rho+1}{2}\kappa$, it follows that

$$
\int_0^1 f(\theta)\theta^\rho d\theta - \int_{-1}^0 f(\theta)|\theta|^\rho d\theta - \left(\int_0^1 \hat{f}(\theta)\theta^\rho d\theta - \int_{-1}^0 \hat{f}(\theta)|\theta|^\rho d\theta\right) < 2\varepsilon \frac{1}{\rho+1},
$$

$$
0 < \kappa - 2\varepsilon \frac{1}{\rho+1} < \left(\int_0^1 \hat{f}(\theta)\theta^\rho d\theta - \int_{-1}^0 \hat{f}(\theta)|\theta|^\rho d\theta\right),
$$

so for any $\hat{F} \in N_\varepsilon(F)$, $E_{\hat{F}}[sgn(\theta)|\theta|^\rho] \neq 0$, that is, $N_\varepsilon(F) \subset \mathcal{F}^\rho$ so $\mathcal{F}^\rho$ is open in $(\mathcal{F}, d_{\infty,\infty})$.

To show that $\mathcal{F}^\rho$ is dense in $(\mathcal{F}, d_{\infty,\infty})$, let $F \in \mathcal{F}$ be such that $E_F[sgn(\bar{\theta})|\bar{\theta}|^\rho] = 0$, and, for each $\delta \in \mathbb{R}_{++}$, take a cumulative distribution $F_\delta \in N_\delta(F)$ such that $F_\delta(\theta) < F(\theta)$ for any $\theta \in (-1, 1)$. Note that for each $\delta \in \mathbb{R}_{++}$, $E_F[sgn(\bar{\theta})|\bar{\theta}|^\rho] > 0$ and thus $F_\delta \in \mathcal{F}^\rho$, and the sequence $\{F_\delta\}$ with $\delta \longrightarrow 0$ converges to F. Hence, $\mathcal{F}^\rho$ is dense in $\mathcal{F}$. ■

We also use the following lemma by Pólya, presented as Exercise 127 in Part II, Chapter 3 of Pólya and Szegő (1978).

Lemma 13 (Pólya) *If a sequence of monotone (continuous or discontinuous) functions converges on a closed interval to a continuous function it converges uniformly.*

We can now prove a main proposition.

Proposition 1 *For any $\rho \in \mathbb{R}_{++}$, the sequence of social choice correspondences SC_ρ is implementable over $\mathcal{F}^\rho$ by any vote-buying mechanism $c \in \mathcal{C}_A$ such that $\kappa(c) = \frac{1+\rho}{\rho}$.*

Proof. Let c be any mechanism in $\mathcal{C}_A$ such that $\kappa(c) = \frac{1+\rho}{\rho}$. For any $(\gamma, F, G) \in \mathbb{R}_{++} \times \mathcal{F} \times \mathcal{G}$, let $\{s^n\}_{n=1}^\infty$ be a sequence such that $s^n \in E^{(n,\gamma,F,c,G)}$ for each $n \in \mathbb{N} \backslash \{1\}$. Then, by Lemma 11, for any $\theta \in [-1, 1]$,

$$
\lim_{n \to \infty} \frac{s^n(\theta)}{s^n(1)} = sgn\,(\theta)\,|\theta|^\rho. \tag{20}
$$

For each $n \in \mathbb{N}\backslash\{1\}$, define the function $\psi^n : [-1,1] \longrightarrow [-1,1]$ by $\psi^n(\theta) = \frac{s^n(\theta)}{s^n(1)}$. For each $n \in \mathbb{N}\backslash\{1\}$, ψ^n is a monotone function defined on a closed interval, and by Expression (20), the sequence $\{\psi^n\}_{n=1}^{\infty}$ converges pointwise to the continuous function $sgn\,(\theta)\,|\theta|^{\rho}$. It follows from Polya's lemma (Lemma 13) that $\{\psi^n\}_{n=2}^{\infty}$ converges uniformly to function $sgn\,(\theta)\,|\theta|^{\rho}$. That is, for any $\varepsilon \in \mathbb{R}_{++}$, there exists $\hat{n}(\varepsilon)$ such that for any $\theta \in [-1,1]$, and for any $n > \hat{n}(\varepsilon)$,

$$\left|\frac{s^n(\theta)}{s^n(1)} - sgn\,(\theta)\,|\theta|^{\rho}\right| < \varepsilon. \tag{21}$$

Take any $F \in \mathcal{F}^{\rho}$ such that $E_F[sgn(\bar{\theta})|\bar{\theta}|^{\rho}] > 0$, and any $\hat{\varepsilon} \in \left(0, E_F[sgn(\bar{\theta})|\bar{\theta}|^{\rho}]\right)$. By the weak law of large numbers, the random variable $\frac{1}{n}\sum\limits_{k=1}^{n} sgn\left(\bar{\theta}_k\right)\left|\bar{\theta}_k\right|^{\rho} - \hat{\varepsilon}$, where $\bar{\theta}_k$ is distributed according to F for each $k \in \{1, ..., n\}$, converges to its expectation

$$E_F[sgn(\bar{\theta})|\bar{\theta}|^{\rho}] - \hat{\varepsilon} > 0;$$

and therefore,

$$\lim_{n\longrightarrow\infty} \Pr\left[\sum_{k=1}^{n} sgn\left(\bar{\theta}_k\right)\left|\bar{\theta}_k\right|^{\rho} - \hat{\varepsilon} > 0\right] = 1. \tag{22}$$

Since, by Inequality (21), for any $n > \hat{n}(\hat{\varepsilon})$, $\frac{s^n(\theta)}{s^n(1)} > sgn\,(\theta)\,|\theta|^{\rho} - \hat{\varepsilon}$, it follows that $\Pr\left[\frac{s^n(\bar{\theta})}{s^n(1)} > sgn\left(\bar{\theta}\right)\left|\bar{\theta}\right|^{\rho} - \hat{\varepsilon}\right] = 1$ and then from Equality (22),

$$\lim_{n\longrightarrow\infty} \Pr\left[\sum_{k=1}^{n} \frac{s^n(\bar{\theta}_k)}{s^n(1)} - \hat{\varepsilon} > 0\right] = 1, \text{ and thus}$$

$$\lim_{n\longrightarrow\infty} \Pr\left[\sum_{k=1}^{n} s^n(\bar{\theta}_k) > 0\right] = \lim_{n\longrightarrow\infty} H^n(0) = 1. \tag{23}$$

Note that for any $F \in \mathcal{F}^{\rho}$ such that $E_F[sgn(\bar{\theta})|\bar{\theta}|^{\rho}] > 0$, since $SC^n_{\rho}(\gamma, \theta_{N^n}) = A$ if and only if $\sum\limits_{k=1}^{n} sgn\,(\theta_k)\,|\theta_k|^{\rho} > 0$, and since $\lim\limits_{n\longrightarrow\infty} \Pr\left[\sum\limits_{k=1}^{n} sgn\left(\bar{\theta}_k\right)\left|\bar{\theta}_k\right|^{\rho} > 0\right] = 1$ (by the weak law of large numbers), it follows that

$$\lim_{n\longrightarrow\infty} \Pr\left[SC^n_\rho(\gamma,\bar{\theta}_{N^n}) = A\right] = 1. \tag{24}$$

From Lemma 7,

$$\lim_{n\longrightarrow\infty} \int\limits_{x\in(n-1)X} g(x)h^n(x)dx = 0, \tag{25}$$

and since $g(x) > 0$ for any $x \in \mathbb{R}$, from Equality (25) we obtain that for any $\hat{x} \in \mathbb{R}_{++}$,

$$\lim_{n\longrightarrow\infty} \int_{-\hat{x}}^{\hat{x}} g(x)h^n(x)dx = 0.$$

Since g is continuous, it attains a minimum in $[-\hat{x},\hat{x}]$, and this minimum is strictly positive. Since $h^n(x) \in \mathbb{R}_+$ for any $x \in \mathbb{R}$ and for any $n \in \mathbb{N}\backslash\{1\}$, it then follows that

$$\lim_{n\longrightarrow\infty} \int_{-\hat{x}}^{\hat{x}} h^n(x)dx = 0,$$

which implies

$$\lim_{n\longrightarrow\infty} (H^n(\hat{x}) - H^n(-\hat{x})) = 0. \tag{26}$$

Note that equalities (23) and (26) together imply that

$$\lim_{n\longrightarrow\infty} \Pr\left[\sum_{k=1}^{n} s^n(\bar{\theta}_k) > \hat{x}\right] = \lim_{n\longrightarrow\infty} H^n(\hat{x}) = 1. \tag{27}$$

For any $\varepsilon_t \in \mathbb{R}_{++}$, and for any $\hat{x}_t \in \mathbb{R}_{++}$ such that $G(\hat{x}_t) > 1-\varepsilon_t$, Equality (27) implies that $\lim\limits_{n\longrightarrow\infty} \Pr[d^n_F(s,\bar{\theta}) = A] > 1-\varepsilon_t$, and thus, choosing a sequence $\{\varepsilon_t\}_{t=1}^{\infty}$ that converges to zero, $\lim\limits_{n\longrightarrow\infty} \Pr[d^n_F(s,\bar{\theta}) = A] = 1,$ and then, by Equation (24),

$$\lim_{n\longrightarrow\infty} \Pr[d^n_F(s,\bar{\theta}) = SC^n_\rho(\gamma,\bar{\theta}_{N^n})] = 1,$$

so c asymptotically implements the sequence of social choice correspondences SC_ρ over the set $\left\{F \in \mathcal{F}^\rho \text{ such that } E_F[sgn(\bar{\theta})|\bar{\theta}|^\rho] > 0\right\}$.

Similarly, for any $F \in \mathcal{F}^\rho$ such that $E_F[sgn(\bar{\theta})|\bar{\theta}|^\rho] < 0$, $\lim\limits_{n \longrightarrow \infty} \Pr\left[\sum\limits_{k=1}^{n} s^n(\bar{\theta}_k) < 0\right] = 1$ and $\lim\limits_{n \longrightarrow \infty} \Pr\left[SC_\rho^n(\gamma, \bar{\theta}_{N^n}) = B\right] = 1$, so c asymptotically implements SC_ρ over the set $\left\{F \in \mathcal{F}^\rho \text{ such that } E_F[sgn(\bar{\theta})|\bar{\theta}|^\rho] < 0\right\}$.

Hence, c asymptotically implements the sequence of social choice correspondences SC_ρ over the set of cumulative distributions $\mathcal{F}^\rho$. ■

Noting that for any $k \in (1, \infty)$, $\kappa(c) = k$ for $c(x) = |x|^k$, it follows from Proposition 1 that for any $\rho \in \mathbb{R}_{++}$, SC_ρ is implemented over $\mathcal{F}^\rho$ by the power vote-buying mechanism $c(x) = |x|^{\frac{1+\rho}{\rho}}$, or, equivalently, that for any any $k \in (1, \infty)$, the power vote-buying mechanism $c(x) = |x|^k$ implements $SC_{\frac{1}{k-1}}$.

After having detailed sufficient conditions for generic implementability in Proposition 1, we next prove that these conditions are (almost) also necessary. Let $\mathcal{SC}$ denote the set of all possible sequences of social choice correspondences.

Proposition 2 *Any $SC \in \mathcal{SC}$ that is not generically equivalent to SC_ρ for any $\rho \in \mathbb{R}_{++}$, is not implementable generically over $\mathcal{F}$ by $\mathcal{C}_A$.*

Proof. We prove the contrapositive. Assume $c \in \mathcal{C}_A$ implements SC generically. We show that there exists $\rho \in \mathbb{R}_{++}$ such that SC is generically equivalent to SC_ρ.

Recall that for any vote-buying mechanism $c \in \mathcal{C}_A$, $\kappa(c) \in (1, \infty)$. Then note that from Proposition 1, for any $\rho \in \mathbb{R}_{++}$, any vote-buying mechanism $c \in \mathcal{C}_A$ with $\kappa(c) = \frac{1+\rho}{\rho}$ implements SC_ρ over $\mathcal{F}^\rho$, so defining $z \equiv \frac{1+\rho}{\rho}$, and hence $\rho = \frac{1}{z-1}$, for any $z \in (1, \infty)$, any vote-buying mechanism $c \in \mathcal{C}_A$ with $\kappa(c) = z$ implements $SC_{\frac{1}{z-1}} = SC_\rho$ over $\mathcal{F}^\rho$. Since $\bigcup\limits_{z \in (1,\infty)} \{c \in \mathcal{C}_A : \kappa(c) = z\} = \mathcal{C}_A$, it follows that for any $c \in \mathcal{C}_A$, $\exists \rho \in \mathbb{R}_{++}$ such that c implements SC_ρ over $\mathcal{F}^\rho$ (in particular, $\rho = \frac{1}{\kappa(c)-1}$). Since $\mathcal{F}^\rho$ is open and dense in $\mathcal{F}$ (Lemma 12), it follows that for any $c \in \mathcal{C}_A$, there exists $\rho \in \mathbb{R}_{++}$, and there exists an open $\mathcal{F}^D$ dense in $\mathcal{F}$ such that c implements SC_ρ over $\mathcal{F}^D$, so for any $F \in \mathcal{F}^D$

$\lim_{n \longrightarrow \infty} \Pr\left[\bar{d}_F^n(s^n, \bar{\theta}_{N^n}) = SC_\rho(\gamma, \bar{\theta}_{N^n})\right] = 1.$

But since c is posited to also implement SC, there exists an open $\mathcal{F}^{D\prime}$ dense in $\mathcal{F}$ such that c implements SC_ρ over $\mathcal{F}^{D'}$, so for any $F \in \mathcal{F}^{D'}$, $\lim_{n \longrightarrow \infty} \Pr\left[\bar{d}_F(s^n, \bar{\theta}_{N^n}) = SC(\gamma, \bar{\theta}_{N^n})\right] = 1.$

It follows that for any $F \in \mathcal{F}^{D'} \cap \mathcal{F}^D$, $\lim_{n \longrightarrow \infty} \Pr\left[SC(\gamma, \bar{\theta}_{N^n}) \neq SC_\rho(\gamma, \bar{\theta}_{N^n})\right] = 0.$

Since the intersection of two open dense sets is dense (an implication of Baire's [2] Category Theorem), it follows that $\mathcal{F}^{D'} \cap \mathcal{F}^D$ is itself an open dense set in $\mathcal{F}$, so SC is generically equivalent to SC_ρ. ■

Proposition 1 and 2 together lead to our main result, the characterization of generically implementable sequences of social choice correspondences in Theorem 1. We restate the theorem more formally.

Theorem 1. *Any $SC \in SC$ is generically implementable by $\mathcal{C}_A$ if and only if there exists $\rho \in \mathbb{R}_{++}$ such that SC and SC_ρ are generically equivalent. Further, any $c \in \mathcal{C}_A$ generically implements $SC \in SC$ if and only if SC is generically equivalent to $SC_{\frac{1}{\kappa(c)-1}}$.*

Proof of Theorem 1. By Proposition 1, for any $\rho \in \mathbb{R}_{++}$, any vote-buying mechanism $c \in \mathcal{C}_A$ such that $\lim_{x \longrightarrow 0^+} \frac{xc'(x)}{c(x)} = \frac{1+\rho}{\rho}$ implements SC_ρ over $\mathcal{F}^\rho$, and $\mathcal{F}^\rho$ is an open dense subset of $\mathcal{F}$ (Lemma 12). Hence, c implements SC_ρ generically.

For any $\rho \in \mathbb{R}_{++}$ and for any $SC \in \mathcal{SC}$ that is generically equivalent to SC_ρ, there exists an open dense set $\mathcal{F}^D \subseteq \mathcal{F}$ such that for any $F \in \mathcal{F}^D$,

$$\lim_{n \longrightarrow \infty} \Pr\left[SC(\gamma, \bar{\theta}_{N^n}) \neq SC_\rho(\gamma, \bar{\theta}_{N^n})\right] = 0.$$

Since SC and SC_ρ are generically equivalent over $\mathcal{F}^\rho \cap \mathcal{F}^D$, from

$$\lim_{n \longrightarrow \infty} \Pr\left[SC(\gamma, \bar{\theta}_{N^n}) \neq SC_\rho(\gamma, \bar{\theta}_{N^n})\right] = 0 \text{ for any } F \in \mathcal{F}^\rho \cap \mathcal{F}^D, \text{ and}$$

$$\lim_{n \longrightarrow \infty} \Pr\left[\bar{d}_F^n(s^n, \bar{\theta}_{N^n}) \neq SC_\rho(\gamma, \bar{\theta}_{N^n})\right] = 0 \text{ for any } F \in \mathcal{F}^\rho \cap \mathcal{F}^D,$$

it follows that

$$\lim_{n\longrightarrow\infty} \Pr\left[\bar{d}_F^n(s^n, \bar{\theta}_{N^n}) \neq SC(\gamma, \bar{\theta}_{N^n})\right] = 0 \text{ for any } F \in \mathcal{F}^\rho \cap \mathcal{F}^D.$$

Since $\mathcal{F}^\rho$ is open and dense in $\mathcal{F}$ (Lemma 12), and since the intersection of two open dense sets is open dense (an implication of the Category Theorem by Baire (1899)), it follows that $\mathcal{F}^\rho \cap \mathcal{F}^D$ is itself an open dense set in $\mathcal{F}$, and thus c implements SC generically.

For any $SC \in \mathcal{SC}$ that is not generically equivalent to SC_ρ for any $\rho \in \mathbb{R}_{++}$, SC is not implementable generically by $\mathcal{C}_A$ over $\mathcal{F}$, by Proposition 2. ■

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