Final Exam – 140A-1: Geometric Analysis

Due on 12/22

Part I: Complete 5 problems from Problems 1-6.

Problem 1. Let $f\colon \mathbb{R}P^3 \to T^3 := (\mathbb{R}/2\pi\mathbb{Z})^3$ be a smooth map. Show that f cannot be an immersion.

Problem 2. Let M be a smooth manifold of dimension n. Suppose α is a closed nowhere-zero $(n-1)$-form on M. Show that for every $p \in M$ there exists a neighborhood U of p and a smooth chart $\varphi = (x_1, \cdots, x_n)\colon U \to \mathbb{R}^n$ near p such that in this chart we have

$$\alpha|_U = dx_2 \wedge dx_3 \wedge \cdots \wedge dx_n.$$

Problem 3. Let U and V be smooth vector fields on $\mathbb{R}^3$ defined by

$$U(x,y,z) = \frac{\partial}{\partial x} \text{ and } V(x,y,z) = F(x,y,z)\frac{\partial}{\partial y} + G(x,y,z)\frac{\partial}{\partial z},$$

where $F(x,y,z)$ and $G(x,y,z)$ are smooth functions on $\mathbb{R}^3$. Show that there exists a proper foliation of $\mathbb{R}^3$ by 2-dimensional embedded submanifolds such that the vector fields U and V everywhere span the tangent spaces of these submanifolds if and only if

$$F(x,y,z) = f(y,z)e^{h(x,y,z)} \text{ and } G(x,y,z) = g(y,z)e^{h(x,y,z)}$$

for some $f, g \in C^\infty(\mathbb{R}^2)$ and $h \in C^\infty(\mathbb{R}^3)$ such that (f,g) does not vanish on $\mathbb{R}^2$.

Problem 4. Let M and N be connected smooth manifolds and $\pi_1\colon M \times N \to M$ and $\pi_2\colon M \times N \to N$ the projection maps onto M and N respectively. Show directly from the definitions that the homomorphism defined by

$$\Psi\colon H^1_{\text{deR}}(M) \oplus H^1_{\text{deR}}(N) \to H^1_{\text{deR}}(M \times N), \quad ([\alpha],[\beta]) \mapsto [\pi_1^*\alpha + \pi_2^*\beta]$$

is well-defined and an isomorphism.

Problem 5. [Poincare duality for de Rham cohomology with compact support]

Let M be an oriented manifold of dimension n and possibly non-compact. Let $\Omega_c^*(M)$ denote the cochain complexes of smooth differential forms with compact support on M such that

$$\Omega_c^p(M) = \{\omega \in \Omega^p(M) \mid \operatorname{supp}(\omega) \subset M \text{ is compact}\}, \forall p \geq 1.$$

We denote the cohomology of this cochain complex under the exterior derivative d by the de Rham cohomology of compact support $H^*_{\text{deR,c}}(M)$ for M.

(a) Show that for each $0 \leq p \leq n$ the pairing

$$H^p_{\mathrm{deR}}(M) \otimes H^{n-p}_{\mathrm{deR,c}}(M) \to \mathbb{R}, \quad [\alpha] \otimes [\beta] \to \int_M \alpha \wedge \beta$$

is well-defined.

(b) Show that the above pairing is non-degenerate if $M = \mathbb{R}^n$.

(c) Suppose that M has a good cover $\underline{U} = \{U_\alpha\}_{\alpha \in I}$ (i.e., the intersections $U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_k}$ is either empty or diffeomorphic to $\mathbb{R}^n$ for all k). Show that the pairing is nondegenerate for M.

Problem 6. [Gluing of sheaves] Let X be a topological space and $\underline{U} = \{U_\alpha\}_{\alpha \in I}$ be an open cover of X, and suppose we are given for each α a sheaf $\mathscr{F}_\alpha$ of abelian groups on U_α, and for each α, β we are given a sheaf isomorphism $\varphi_{\alpha\beta} \colon \mathscr{F}_\beta|_{U_\alpha \cap U_\beta} \to \mathscr{F}_\alpha|_{U_\alpha \cap U_\beta}$ such that

(1) For each α, we have $\varphi_{\alpha\alpha} = id$,

(2) (Cocycle condition) For each $\alpha, \beta, \gamma \in I$, we have $\varphi_{\beta\gamma}\varphi_{\gamma\alpha} = \varphi_{\beta\alpha}$ on all the triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$.

Show that there exists a unique sheaf $\mathscr{F}$ on X together with isomorphisms $\psi_\alpha \colon \mathscr{F}|_{U_\alpha} \xrightarrow{\cong} \mathscr{F}_\alpha$ such that for all $\alpha, \beta \in I$, we have that $\psi_\beta = \varphi_{\beta\alpha} \circ \psi_\alpha$ on all double overlaps $U_\alpha \cap U_\beta$. Where is the cocycle condition used in your proof?

Part II: Complete Problems 7-9 as homework for Hodge theory.

Problem 7. [Harmonic forms on the torus] Let $\mathbb{R}^n$ be the Euclidean space equipped with the standard Riemannian metric g on $\mathbb{R}^n$ given by $g_p(\frac{\partial}{\partial x_i}|_p, \frac{\partial}{\partial x_j}|_p) = \delta_{ij}$ where $\frac{\partial}{\partial x_1}|_p, \cdots \frac{\partial}{\partial x_n}|_p$ are the standard basis of $T_p\mathbb{R}^n \cong \mathbb{R}^n$ and for all $p \in \mathbb{R}^n$.

(a) Derive the explicit formula for Laplacian on the smooth differential forms of $\mathbb{R}^n$. Deduce from this that a differential k-form of the form

$$\alpha = \sum_{I=\{i_1 < i_2 < \cdots < i_k\}} \alpha_I dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$$

is a harmonic form if and only if its coefficients α_I are harmonic functions on $\mathbb{R}^n$.

(b) Let $T := (\mathbb{R}/2\pi\mathbb{Z})^n$ be the standard real torus of dimension n which is the quotient of $\mathbb{R}^n$ under the smooth action of $\Gamma = (2\pi\mathbb{Z})^n$ by translations. The metric on T is induced from the Euclidean metric g on $\mathbb{R}^n$. Show that the only harmonic functions on T are the constants. Deduce from this that there is a bijection between harmonic forms on T and the differential forms on $\mathbb{R}^n$ with constant coefficients which are Γ-invariant. (See solution to Problem 2(b) of homework 5 for Γ-invariant forms).

(c) Show that $H^k_{\mathrm{deR}}(T,\mathbb{R}) = \Lambda_k(H^1_{\mathrm{deR}}(T,\mathbb{R})) = \Lambda_k(\mathbb{R}^n)^*$, where $\Lambda_k(H^1_{\mathrm{deR}}(T,\mathbb{R}))$ is the kth exterior power of the real vector space $H^1_{\mathrm{deR}}(T,\mathbb{R})$.

Problem 8. [Spectral decomposition of Laplacian]
Let (M,g) be a compact oriented Riemannian manifold. We denote by

$$\Delta\colon \Omega^p(M) \to \Omega^p(M)$$

the Laplacian-Beltrami operator defined by g. Prove the following statements

(a) All eigenvalues of Δ are non-negative;

(b) Eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to the L_2-inner product;

(c) Eigenspaces of Δ are finite-dimensional;

(d) The set of eigenvalues of Δ has no limit point;

(e) Δ has infinitely many positive eigenvalues.

(Hints are given in Exercise 16 on Page 254 of Warner)

Problem 9. [Elliptic operators]
Let X be a smooth vector field on a smooth manifold M. We define

$$P\colon \Gamma(M,TM) \to \Gamma(M,TM) \text{ by } P(Y) = [X,Y].$$

(a) Show that P is a first-order differential operator.

(b) What is the symbol of this differential operator P?

(c) Under what conditions (on M and/or X) is the differential operator P elliptic ?

Bonus Problem(+20pts) [The exponential exact sequence and first Chern class of a complex line bundle]

Let M be a complex manifold and denote by $\mathscr{O}$ and $\mathscr{O}^*$ by the sheaf of holomorphic functions and sheaf of nonzero holomorphic functions on M.

(a) Prove that there is a short exact sequence of sheaves

$$0 \to \underline{\mathbb{Z}} \xrightarrow{i} \mathscr{O} \xrightarrow{\exp} \mathscr{O}^* \to 0,$$

where $i\colon \underline{\mathbb{Z}} \to \mathscr{O}$ is the sheaf homomorphism given the inclusion of locally constant integral functions into $\mathscr{O}$ and $\exp\colon \mathscr{O} \to \mathscr{O}^*$ is the sheaf homomorphism defined by $f \mapsto \exp(2\pi i f)$ for open set $U \subset M$ and $f \in \mathscr{O}(U)$. This is called the exponential exact sequence.

(b) Given the exponential exact sequence, there is a long exact sequence of Čech cohomology groups

$$\cdots \to \check{H}^1(M, \mathscr{O}) \to \check{H}^1(M, \mathscr{O}^*) \xrightarrow{\delta} \check{H}^2(M, \underline{\mathbb{Z}})) \to \check{H}^2(M, \mathscr{O}) \to \cdots .$$

By homework 8 problem 4, we know that $\check{H}^1(M, GL_1(\mathscr{O})) \cong \check{H}^1(M, \mathscr{O}^*)$ classifies the isomorphism classes of holomorphic line bundles, so each holomorphic line bundle L defines a class $[L] \in \check{H}^1(M, \mathscr{O}^*)$. The 1st Chern class $c_1(L)$ of L is sometimes defined to be the image of $[L] \in \check{H}^1(M, \mathscr{O}^*)$ under the composition

$$\check{H}^1(M, \mathscr{O}^*) \xrightarrow{\delta} \check{H}^2(M, \underline{\mathbb{Z}})) \xrightarrow{\varphi_*} \check{H}^2(M, \underline{\mathbb{R}}) \cong H^2_{\text{deR}}(M, \mathbb{R}) \xrightarrow{\cdot \otimes \mathbb{C}} H^2_{\text{deR}}(M, \mathbb{C}),$$

where δ is the connecting homomorphism and φ_* is the homomorphism induced by the sheaf homomorphism $\varphi\colon \underline{\mathbb{Z}} \to \underline{\mathbb{R}}$ and the last homomorphism is $H^2_{\text{deR}}(M, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong H^2_{\text{deR}}(M, \mathbb{C})$.

(c) Let $\gamma \to \mathbb{C}P^1$ be the tautological line bundle on $\mathbb{C}P^1$. Compute $\int_{\mathbb{C}P^1} c_1(\gamma^*)$, where $\mathbb{C}P^1$ has its canonical orientation as a complex manifold (i.e. $\Lambda_{top}(T\mathbb{C}P)$ has a canonical trivialization as a real line bundle) and $c_1(\gamma^*)$ is the 1st Chern class of the dual line bundle γ^* to the tautological line bundle.