Homework 1

Due on 09/12

Problem 1. Exercise 2 on Page 50 of the book.

Problem 2.

(a) Prove that the n-sphere $S^n = \{x_0^2 + x_1^2 + \cdot + x_n^2 = 1\} \subset \mathbb{R}^{n+1}$ is a n-dimensional smooth manifold by equipping it with a smooth structure generated by two charts. [Hint: use stereographic projections]

(b) Show that $\mathbb{C}P^n$ admits a structure of a *complex* n-manifold, that is, it can be covered by charts whose overlap maps, $\psi_\alpha \circ \psi_\beta^{-1}$, are holomorphic maps between open subsets of $\mathbb{C}^n$.

Problem 3. [Smooth structure on the quotient space] Suppose a group G acts *properly discontinuously* on a smooth n-manifold $\tilde{M}$ by diffeomorphisms. (This means for each point $p \in \tilde{M}$ there exists a neighborhood U such that $\forall g \in G$ and $g \neq id \Longrightarrow gU \cap U = \emptyset$).

(a) Show that the quotient topological space $M = \tilde{M}/G$ admits a smooth structure

$$\mathcal{F} := \{(\pi(U), \phi \circ \pi^{-1}|_U) \mid (U, \phi) \in \mathcal{F}_{\tilde{M}}, \pi|_U \text{ is injective}\},$$

where $\pi \colon \tilde{M} \to M$ is the projection map and $\mathcal{F}_{\tilde{M}}$ defines a smooth structure on $\tilde{M}$.

(b) Show that this smooth structure is the unique one on the quotient space M satisfying the following
• the projection map π is a *local diffeomorphism* ($\forall p \in \tilde{M}$ and $\pi(p) \in M$, $\exists U_p, V_{\pi_p}$ such that $\pi|_{U_p} \colon U_p \to V_{\pi_p}$ is a diffeomorphism).
• If N is another smooth manifold, a continuous map $f \colon M \to N$ is smooth if and only if the map $f \circ \pi \colon \tilde{M} \to N$ is smooth.

Problem 4. Exercise 6 on Page 50 of the textbook, which asks you to prove that a bijective immersion is a diffeomorphism.

Problem 5. Given a smooth map $\psi \colon M \to N$. The differential of ψ at $m \in M$, $d\psi|_m \colon T_m M \to T_{\psi(m)} N$, is defined by

$$d\psi|_m v(f) = v(f \circ \psi) \in \mathbb{R}, \text{ for all } v \in T_m M, f \in \tilde{F}_{\psi(m)},$$

where $\tilde{F}_{\psi(m)}$ denotes the germs at $\psi(m)$. Show that $d\psi|_m v$ is a well-defined element of $T_{\psi(m)} N$ for all $v \in T_m M$, i.e., it defines a linear derivation of $\tilde{F}_{\psi(m)}$.