Homework 2

Due on 09/19

Problem 1. Suppose M is a compact (nonempty) manifold of dimension n and $f\colon M \to \mathbb{R}^n$ is a smooth map. Show that f is not an immersion.

Proof. Given a smooth map $f\colon M \to \mathbb{R}^n$. Let $\pi_1\colon \mathbb{R}^n \to \mathbb{R}$ be the projection onto the first coordinate. Consider the composition

$$\pi_1 \circ f\colon M \to \mathbb{R}.$$

Since M is compact, the image is a compact subset of $\mathbb{R}$ and the function $\pi \circ f$ has to reaches its maximum at some point $p \in M$. For some chart $\phi\colon U \to \mathbb{R}^n$ at its maximum $p \in M$, we have,

$$d(\pi \circ f)|_p = \sum_{i=1}^{n} \frac{\partial(\pi_1 \circ f \circ \phi^{-1})}{\partial x_i} dx_i = 0.$$

By the chain rule, one has

$$d\pi_1|_{f(p)} \circ df|_p = d(\pi \circ f)|_p = 0.$$

Hence the differential $df|_p\colon T_pM \to T_{f(p)}\mathbb{R}^n \cong \mathbb{R}^n$ is not surjective (as $d\pi|_{f(p)}$ is surjective but $d(\pi \circ f)|_p$ is not). Since the dimension of T_pM is $n \geq 1$, it follows that $df|_p$ is not injective at $p \in M$ and f is not an immersion (at p).

□

Problem 2.

(a) For what values of $t \in \mathbb{R}$, is the following subspace $\{(x_1, \cdots, x_{n+1}) \,|\, x_1^2 + \cdots + x_{n+1}^2 = t\}$ a smooth embedded submanifold of $\mathbb{R}^{n+1}$?

(b) For such values of t, determine the diffeomorphism type of the submanifolds.

Proof. For (a): Let $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ be a smooth map defined by

$$f(x_1, \cdots, x_{n+1}) = x_1^2 + \cdots + x_{n+1}^2.$$

We compute the differential of this map at $\mathbf{x} = (x_1, \cdots, x_{n+1})$,

$$d_{\mathbf{x}}f\colon T_{\mathbf{x}}\mathbb{R}^{n+1} \to T_{f(\mathbf{x})}\mathbb{R} \cong \mathbb{R}, \quad d_{\mathbf{x}}f = 2x_1 dx_1|_{\mathbf{x}} + \cdots + 2x_{n+1}dx_{n+1}|_{\mathbf{x}}.$$

Now we have $df|_{\mathbf{x}} = 0 \iff x_1 = \cdots = x_{n+1} = 0$. Hence $df|_{\mathbf{x}}$ is surjective unless $\mathbf{x} = (0, \cdots, 0)$ or equivalently $f(\mathbf{x}) = 0$. Now by Theorem 1.38, we have if $t \neq 0$, then the preimage

$$S(t) := \{\mathbf{x} \in \mathbb{R}^{n+1} | x_1^2 + \cdots x_{n+1}^2 = t\}$$

is a smooth embedded submanifold of $\mathbb{R}^{n+1}$ of codimension 1.

One notices that if $t = 0$, the preimage $f^{-1}(0) = \{(0, \cdots, 0)\}$ is a smooth zero dimensional submanifold of $\mathbb{R}^{n+1}$ of the wrong codimension, so the Implicit Function Theorem is violated. Also, in general the preimage at a non-transverse is NOT a submanifold as in this case.

For (b): By the Implicit Function Theorem (embedded version), the preimage of an embedded submanifold $t \neq 0$ in $\mathbb{R}$ has a unique smooth structure that makes $S(t)$ an embedded submanifold of $\mathbb{R}^{n+1}$. Which standard smooth manifold is this?

Let $S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} | x_1^2 + \cdots x_{n+1}^2 = 1\}$ be the standard unit n-sphere. For every $t \neq 0$, one shows that the multiplication maps

$$\cdot\sqrt{t}\colon S^n \to S(t) \text{ and } \cdot\frac{1}{\sqrt{t}}\colon S(t) \to S^n$$

are smooth maps and inverses to each other. (We omit the computations in local coordinates using stereographic projections). Hence for $t \neq 0$, the preimage has the diffeomorphism type of the unit n-sphere and for $t = 0$, the preimage has the diffeomorphism of a point $\mathbf{x} = (0, \cdots, 0) \in \mathbb{R}^{n+1}$. $\square$

Problem 3.

(a) Show that the *special unitary group* defined by

$$SU_n = \{A \in \mathrm{Mat}_n\mathbb{C} \mid A \cdot \bar{A}^t = \mathbb{I}_n, \det(A) = 1\}$$

is a smooth compact manifold.

(b) What is its dimension?

(c) Compute the tangent space of SU_n at the $n \times n$ identity matrix $\mathbb{I}_n \in SU_n$.

Proof. For (a): We define the function

$$f\colon \mathrm{Mat}_n\mathbb{C} \to \mathrm{Mat}_n\mathbb{C}, \quad f(A) = A \cdot \bar{A}^t,$$

this is a smooth function because f is a polynomial map in the coefficients of $A \in \mathrm{Mat}_n\mathbb{C}$. Let

$$U_n = \{A \in \mathrm{Mat}_n\mathbb{C} \mid A \cdot \bar{A}^t = \mathbb{I}_n\}$$

be the unitary group. One notices that $U_n = f^{-1}(\mathbb{I}_n)$ and the range of the map is given by

$$f(\mathrm{Mat}_n\mathbb{C}) \subset \mathrm{Her}_n := \{A \in \mathrm{Mat}_n\mathbb{C} | A = \bar{A}^t\}.$$

We first prove that U_n is a smooth compact embedded submanifold of $\mathrm{Mat}_n\mathbb{C} \cong \mathbb{R}^{2n^2} (\cong \mathbb{C}^{n^2})$ by applying the Implicit Function Theorem to f and the smooth embedded submanifold $Y := \mathrm{Her}_n \subset \mathrm{Mat}_n$ (here Her_n is an embedded submanifold because it is a linear subspace of $\mathrm{Mat}_n\mathbb{C} \cong \mathbb{R}^{2n^2}$ defined by the linear equations $A = \bar{A}^t$ in the coefficients).

The differential of f at $\mathbb{I}_n$

$$d_{\mathbb{I}_n} f\colon T_{\mathbb{I}_n}\mathrm{Mat}_n \to T_{\mathbb{I}_n}\mathrm{Her}_n$$

can be computed as follows. Suppose $\gamma\colon \mathbb{R} \to \mathrm{Mat}_n$ is a smooth curve given by $\gamma(t) = \mathbb{I}_n + tB$ with $B \in T_{\mathbb{I}_n}\mathrm{Mat}_n \cong \mathrm{Mat}_n$. One has

$$\begin{aligned} d_{\mathbb{I}_n} f(\gamma'(0)) &= d_{\mathbb{I}_n} f\big(d_0\gamma(\frac{d}{dt})\big) = d_0(f \circ \gamma)(\frac{d}{dt}) = \frac{d}{dt}\big(f(\gamma(t))\big)|_{t=0} \\ &= \frac{d}{dt}(\mathbb{I}_n + tB) \cdot (\mathbb{I}_n + t\bar{B}^t)|_{t=0} = B + \bar{B}^t \in \mathrm{Her}_n \end{aligned}$$

Now $d_{\mathbb{I}_n} f$ is surjective (or transverse to $Y = \mathrm{Her}_n$) because its restriction to Hermitian matrices is surjective, i.e., one has

$$d_{\mathbb{I}_n} f\colon \mathrm{Her}_n \subset T_{\mathbb{I}_n}\mathrm{Mat}_n \to T_{\mathbb{I}_n}\mathrm{Her}_n, \quad B \mapsto B + \bar{B}^t = 2B.$$

For any other $B \in f^{-1}(\mathbb{I}_n) = U_n$, we define a right multiplication map

$$R_B\colon \mathrm{Mat}_n \to \mathrm{Mat}_n, \quad A \mapsto A \cdot B. \tag{1}$$

Then one has

$$f \circ L_B(A) = (A \cdot B) \cdot \overline{A \cdot B}^t = A \cdot B \cdot \bar{B}^t \cdot \bar{A}^t = A \cdot \bar{A}^t = f(A), \quad \forall A \in \mathrm{Mat}_n.$$

This implies that $d_B f$ is surjective for all $B \in U_n$ because we have shown that $d_{\mathbb{I}_n} f$ is surjective and $d_{\mathbb{I}_n} f = d_{R_B(\mathbb{I}_n)} f \circ d_{\mathbb{I}_n} R_B$. Hence by the Implicit Function Theorem, the preimage U_n is embedded submanifold of Mat_n of dimension

$$\dim \mathrm{Mat}_n - \dim \mathrm{Her}_n = 2n^2 - 2 \cdot \frac{n(n-1)}{2} + n = n^2.$$

(Here, the condition $A = \bar{A}^t$ requires that the entries of A strictly above the diagonal can be chosen freely, this is $\frac{n(n-1)}{2}$ complex parameters, and we know that the eigenvalues of diagonal entries of a unitary matrix must be real, hence this adds n real parameters). Now the subspace U_n is compact because this is a closed and bounded subset of $\mathbb{R}^{2n^2}$ with the standard metric. (Again, the condition $A \cdot \bar{A}^t = \mathbb{I}_n$ implies the length of each row and column vector of A is 1.)

Next, we consider the determinant function

$$\det \colon \mathrm{Mat}_n \to \mathbb{C}, \quad A \mapsto \det(A),$$

which is smooth because it's defined by polynomial equations in the entries of A. When restricting to U_n, we have that the range of function becomes $g = \det \colon U_n \to S^1$ and $SU_n = g^{-1}(1)$. We will show that $d_A g$ is surjective for all $A \in g^{-1}(1)$. We first compute

$$d_{\mathbb{I}_n} g \colon T_{\mathbb{I}_n} U_n \to T_1 S^1 \text{ as follows.}$$

Let $\alpha \colon \mathbb{R} \to U_n \subset\!\!\to \mathrm{Mat}_n$ by a smooth curve defined by $\alpha(t) = e^{it} \cdot \mathbb{I}_n$ and one has

$$d_{\mathbb{I}_n} g(\alpha'(0)) = d_0\big(g \circ \alpha\big)(\frac{d}{dt}) = \frac{d}{dt}\big(g(\alpha(t))\big)|_{t=0} = \frac{d}{dt}\det(e^{it} \cdot \mathbb{I}_n)|_{t=0} = \frac{d}{dt}(e^{it})^n|_{t=0} = i \cdot n \in T_1 S^1.$$

Moreover, for $B \in g^{-1}(1) = SU_n$, we have that $R'_B \colon U_n \to U_n$ and one checks similarly that

$$g(R'_B(A)) = g(A), \forall A \in U_n \text{ and } d_{\mathbb{I}_n} g = d_{R'_B}(\mathbb{I}_n) g \circ d_{\mathbb{I}_n} R'_B \colon T_{\mathbb{I}_n} U_n \to T_B U_n \to T_1 S^1.$$

Because $d_{\mathbb{I}_n} g$ is surjective, so is $d_B g$ for all $B \in SU_n$ and this implies that 1 is a regular value of g and by the Implicit Function Theorem, we conclude that SU_n is an embedded submanifold of U_n of codimension 1 and it is compact because it is a closed subset of a compact manifold U_n.

For (b): We have shown in part (a) that $\dim_{\mathbb{R}} U_n = n^2$, so $\dim_{\mathbb{R}} SU^n = n^2 - 1$ since it's codimension 1.

For (c): By part (a), the tangent spaces of U_n and SU_n at $\mathbb{I}_n$ are

$$\ker(d_{\mathbb{I}_n} f) := \mathfrak{u}_n = \{A \in \mathrm{Mat}_n \mid A + \bar{A}^t = 0\},$$

$$\ker(d_{\mathbb{I}_n} f) \cap \ker(d_{\mathbb{I}_n} g) := \mathfrak{su}_n = \{A \in \mathrm{Mat}_n \mid A + \bar{A}^t = 0 \text{ and } \mathrm{Tr}(A) = 0\}.$$

□

Problem 4. Suppose $f\colon X \to M$ and $g\colon Y \to M$ are smooth maps that are transverse to each other, that is,

$$T_{f(x)}M = \mathrm{Im} d_x f + \mathrm{Im} d_y g, \quad \forall (x,y) \in X \times Y \text{ such that } f(x) = g(y).$$

Show that

$$X \times_M Y := \{(x,y) \in X \times Y \mid f(x) = g(y)\}$$

is a smooth embedded submanifold of $X \times Y$ of codimension equal to the dimension of M and

$$T_{(x,y)}(X \times_M Y) = \{(v,w) \in T_x X \oplus T_y Y \mid d_x f(v) = d_y g(w)\}, \quad \forall (x,y) \in X \times_M Y.$$

Proof. We consider the function

$$h = (f,g)\colon X \times Y \to M \times M$$

and define an embedded submanifold of $M \times M$, called the diagonal submanifold,

$$\Delta = \{(x,x) \in M \times M \mid x \in M\}.$$

This is an embedded submanifold because it is the image of the diagonal map $d\colon M \to M \times M$, $x \mapsto (x,x)$, which is easily checked to be a topological embedding and an immersion, and the tangent space to the diagonal submanifold is

$$T_{(x,x)}\Delta = \{(v,v) \in T_p M \oplus T_p M\} \text{ for } p \in M.$$

To apply the Implicit Function Theorem, to $h\colon X \times Y \to M \times M$ and $Y = \Delta \subset M \times M$, we first check that transversality of f with g is equivalent to transversality of h to Δ. This is because for $(x,y) \in h^{-1}(\Delta)$ the condition

$$T_{h(x,y)}(M \times M) = \mathrm{Im}(d_{(x,y)}h) + T_{h(x,y)}\Delta$$

means that for all $v, w \in T_{f(x)}M \cong T_{g(y)}M$ there exists $u_1 \in T_x X$ and $u_2 \in T_y Y$ such that

$$v - d_x f(u_1) = w - d_y g(u_2).$$

This implies that

$$(v,w) = (d_x f(u_1), d_y g(u_2)) + (v - d_x f(u_1), w - d_y g(u_2) \in \mathrm{Im} d_{(x,y)}h + T_{h(x,y)}\Delta,$$

which is equivalently the condition for f and g to be transverse to each other.

Now by the Implicit Function Theorem that $h^{-1}(\Delta) = X \times_M Y$ is a smooth embedded submanifold of $M \times N$ of codimension being $\dim(M \times M) - \dim(\Delta) = m$. Lastly, the tangent space to $X \times_M Y$ is given by

$$T_{(x,y)}X \times_M Y \cong d_{(x,y)}h^{-1}(T_{(f(x),f(x))}\Delta) = \{(v,w) \in T_x X \oplus T_y Y \mid d_x f(v) = d_y g(w)\},$$

where we have an isomorphism since $\dim d_{(x,y)}h^{-1}(T_{(f(x),f(x))}\Delta) = \dim T_{(x,y)}X \times_M Y = \dim T_x X + \dim T_y Y - \dim M$. □

Problem 5. Prove that the tautological line bundle $\gamma_n \to \mathbb{C}P^n$ is a complex line bundle by describing its trivializations. What is its transition data? For $n \geq 1$, why is $\gamma_n \to \mathbb{C}P^n$ non-trivial? (i.e. it is not isomorphic to $\mathbb{C}P^n \times \mathbb{C} \to \mathbb{C}P^n$ as line bundles over $\mathbb{C}P^n$.)

Proof. The bundle projection map $\pi\colon \gamma_n \to \mathbb{C}P^n$ defined by $(l, v) \mapsto l$ for $l \in \mathbb{C}P^n$ and $v \in l$. On the coordinate chart $U_i = \{z_i \neq 0\}$, we define a trivialization of the tautological line bundle by

$$h_i\colon \gamma_n|_{U_i} \to U_i \times \mathbb{C}, \quad h_i(l, (c_0, \cdots c_{n+1})) = (l, c_i),$$

where $v = (c_0, \cdots c_{n+1})$ is the complex coordinate of the vector $v \in l \subset \mathbb{C}^{n+1}$. The inverse map is given by

$$h_i^{-1}\colon U_i \times \mathbb{C} \to \gamma_n|_{U_i}, \ [X_)$$

□