Homework 3

Due on 10/03

Problem 1. Exercise 13 on page 51 of the book.

Problem 2. Exercise 17 on page 51 of the book.

Problem 3. Exercise 22 on page 51 of the book.

Problem 4. Let X and Y be smooth vector fields on a smooth manifold M, and let X_t and Y_t denote the flow of X and Y for $t \in (-\epsilon, \epsilon)$ and $\epsilon > 0$ sufficiently small. Show that for $f \in C^\infty(M)$ and $p \in M$ we have the following equality

$$\lim_{s,t \to 0} \frac{f(Y_{-s}(X_{-t}(Y_s(X_t(p))))) - f(p)}{st} = [X,Y]_p f \in \mathbb{R}.$$

Problem 5. Let $\mathbb{C}P^n$ be the complex projective n-space and $\gamma_n \to \mathbb{C}P^n$ be the tautological line bundle. Prove that there is a short exact sequences of complex vector bundles

$$0 \to \mathbb{C}P^n \times \mathbb{C} \xrightarrow{i} (n+1)\gamma_n^* \xrightarrow{q} T\mathbb{C}P^n \to 0,$$

where $(n+1)\gamma_n^* := \underbrace{\gamma_n^* \oplus \cdots \oplus \gamma_n^*}_{(n+1)\text{-copies}}$ and γ_n^* is the dual line bundle. Exactness here means that i and q are injective and surjective vector bundle homomorphisms respectively such that $\mathrm{Im}(i) = \mathrm{Ker}(q) = q^{-1}(\underline{0})$ where $\underline{0}$ is the zero section in $T\mathbb{C}P^n$.
This short exact sequence is sometimes called the *Euler exact sequence*.