Homework 7

Due on 11/17

Problem 1. Let M be a topological space.

(1) Prove that the sheafification of the constant presheaf is the locally constant sheaf. (Be aware: it is sometimes also called the constant sheaf, for instance, in Warner's book)

(2) Give an example of your favorite locally constant sheaf and explain.

Problem 2. Given a morphism of sheaves of $\mathbb{K}$-modules $\varphi\colon \mathscr{F} \to \mathscr{G}$, prove the following:

(1) For all $p \in M$, we have that $\mathscr{K}er(\varphi)_p \cong \ker(\varphi_p)$ and $\mathscr{I}m(\varphi)_p \cong \operatorname{Im}(\varphi_p)$, where φ_p is the map induced on stalks.

(2) The morphism φ is injective (respectively, surjective) if and only if the induced maps on stalks φ_p are injective (respectively, surjective) for all $p \in M$.

(3) Show that a sequence of morphisms of sheaves

$$\cdots \to \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^{i} \xrightarrow{\varphi^{i}} \mathscr{F}^{i+1} \xrightarrow{\varphi^{i+1}} \cdots$$

is exact if and only if we have an exact sequence of $\mathbb{K}$-modules for all $p \in M$

$$\cdots \to \mathscr{F}^{i-1}_p \xrightarrow{\varphi^{i-1}_p} \mathscr{F}^{i}_p \xrightarrow{\varphi^{i}_p} \mathscr{F}^{i+1}_p \xrightarrow{\varphi^{i+1}_p} \cdots .$$

(4) The morphism φ is surjective if and only if the following condition holds: for every open set $U \subset M$ and for every $s \in \mathscr{G}(U)$, there exists covering $\{U_i\}_{i\in I}$ of U and elements $t_i \in \mathscr{F}(U_i)$ such that $\varphi(t_i) = \rho_{U_i,U}(s) = s|_{U_i}$. (For simplicity, one can use the notation $s|_U := \rho_{U,V}(s)$ for $s \in \mathscr{F}(V)$ if it causes no confusions).

(5) Give your favorite example of a surjective morphism of sheaves $\varphi\colon \mathscr{F} \to \mathscr{G}$ and an open set $U \subset M$ such that $\varphi(U)\colon \mathscr{F}(U) \to \mathscr{G}(U)$ is NOT surjective.

Problem 3. Let M be a smooth manifold and Ω^k_M be the sheaf of smooth differential k-forms on M for any fixed $k \geq 0$. Show that q-th Čech cohomology satisfies $\check{H}^q(M, \Omega^k_M) = 0$ for $q \geq 1$.

Problem 4.[Computing Čech cohomology using good covers for smooth manifolds]

Let M be a smooth manifold of dimension n. We say an open cover $\underline{U} = \{U_\alpha\}_{\alpha\in I}$ is a good cover if all nonempty finite intersections $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ are diffeomorphic to $\mathbb{R}^n$ for $p \geq 0$. A smooth manifold is called of finite type if it admits a finite good cover. We will assume the follow fact: Every smooth manifold has a good cover. In particular, a compact smooth manifold has a finite good cover. [Proof (sketch): Equip M with a Riemannian metric. A nontrivial theorem in Riemannian geometry says that every point in a Riemannian manifold has a *geodesically convex neighborhood*. The intersections of two geodesically convex neighborhoods is also geodesically convex and geodesically convex neighborhoods are diffeomorphic to $\mathbb{R}^n$.]

Let I be the set of all open covers of M, this set is directed under the relation $<$ given by refinement of open covers, that is to say, we have $(I, <)$ satisfying

- (reflexivity) $\underline{U} < \underline{U}$ for all $\underline{U} \in I$.
- (transitivity) If $\underline{U} < \underline{V}$ and $\underline{V} < \underline{W}$, then $\underline{U} < \underline{W}$.
- (upper bound) For any $\underline{U}, \underline{V} \in I$, there is an element $\underline{W} \in I$ such that $\underline{U} < \underline{W}$ and $\underline{V} < \underline{W}$.

A subset $J \subset I$ is called cofinal if for every $\underline{U} \in I$ there is a $\underline{V} \in J$ such that $\underline{U} < \underline{V}$. Think about why the subset J consisting of good covers is cofinal in the set I of all covers of a manifold M.

(1) Given a cofinal sequence of good covers $\{\underline{U_i}\}_{i\in\mathbb{Z}}$ such that $\cdots < \underline{U_1} < \underline{U_2} < \cdots$. Show that there is an isomorphism between direct limits of $\mathbb{K}$-modules

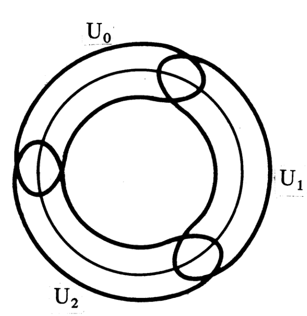
$$\check{H}(M, \mathscr{F}) := \lim_{\underline{U}\in I} \check{H}(\underline{U}, \mathscr{F}) \cong \lim_{i\to\infty} \check{H}(\underline{U_i}, \mathscr{F}).$$

[This implies that Čech cohomology of M with values in a presheaf $\mathscr{F}$ can be computed by $\check{H}(\underline{U}, \mathscr{F})$ for a sufficiently fine good cover $\underline{U}$.]

(2) Compute $\check{H}^*(\underline{U}, \mathscr{F})$ with respect to the good cover $\underline{U} = \{U_0, U_1, U_2\}$ on S^1 in the figure below. The presheaf $\mathscr{F}$ is defined by $\mathscr{F}(U_i) = \mathbb{Z}$ for $i = 0, 1, 2$ and the restriction maps are defined by

$$\rho_{U_{01},U_0} = \rho_{U_{01},U_1} = id,$$
$$\rho_{U_{12},U_1} = \rho_{U_{12},U_2} = id,$$
$$\rho_{U_{02},U_2} = -id, \quad \rho_{U_{02},U_0} = id,$$

where $U_{ij} = U_i \cap U_j$.



Problem "Just for fun".[Monodromy representation of a locally constant sheaf]

Read Chapter 2 section 13 (mainly page 141-147) of Bott and Tu "Differential forms in algebraic topology". (Be aware: his notation for restriction morphism is ρ_U^V if $U \subset V$)

Answer the question that when is a locally constant sheaf constant? (Give the statement and no proofs are required, but you are free to provide further detailed explanations).