Homework 1 Solution

Due on 09/12

Problem 1. If $\mathcal{F} = \{(\mathbb{R}, id\colon \mathbb{R} \to \mathbb{R})\}$ and $\mathcal{F}' = \{(\mathbb{R}, f\colon \mathbb{R} \to \mathbb{R}, t \mapsto t^3)\}$. Show that as collection of charts for $\mathbb{R}$, we have $\mathcal{F} \neq \mathcal{F}'$.

Proof. We first prove that $\mathcal{F} \neq \mathcal{F}'$ as collections of charts. Since we know that $id \in \mathcal{F}$, it is sufficient to show $id \notin \mathcal{F}'$. Suppose that $id\colon \mathbb{R} \to \mathbb{R}$ is another chart in $\mathcal{F}'$. Consider the overlap map

$$id \circ f^{-1}\colon f(\mathbb{R} \cap \mathbb{R}) = \mathbb{R} \to id(\mathbb{R} \cap \mathbb{R}) = \mathbb{R},$$

one observes that this function $id \circ f^{-1}(t) = t^{\frac{1}{3}}$ is not smooth at $t = 0$. Hence we conclude that $id\colon \mathbb{R} \to \mathbb{R}$ is NOT a smooth chart for $\mathbb{R}$ when equipping with the smooth structure $\mathcal{F}'$. Now we define the map $h(t) = t^{\frac{1}{3}}\colon (\mathbb{R}, \mathcal{F}) \to (\mathbb{R}, \mathcal{F}')$ and we will show that h is a diffeomorphism. It is sufficient to check that

$$f \circ h \circ id^{-1} = f \circ h = f(t^{\frac{1}{3}}) = t \text{ and } id \circ h^{-1} \circ f^{-1}(t) = h^{-1}(t^3) = t$$

are both smooth. This completes the proof. □

Remark 0.1. We observe that in the above proof, it suffices to check smoothness of $f \circ h \circ id^{-1}$ and $id^{-1} \circ h \circ f$, because for any other charts $\phi\colon U \to \mathbb{R}$ in $\mathcal{F}$ and $\psi\colon V \to R$ in $\mathcal{F}'$, we know that $id \circ \phi^{-1}$, $\phi \circ id^{-1}$ and $f \circ \psi^{-1}$, $\psi \circ f^{-1}$ are smooth by maximality. This implies that, for instance,

$$(\psi \circ f^{-1}) \circ (f \circ h \circ id^{-1}) \circ (id \circ \phi^{-1}) = \psi \circ h \circ \phi^{-1}$$

is smooth because composition of smooth functions is smooth.

Problem 2. We will assume that S^n and $\mathbb{C}P^n$ are n-dimensional and $2n$-dimensional topological manifolds and describe their smooth and complex structures respectively.

Proof. (a) Let $(x_0, x_1, \cdots, x_n)$ be the coordinates on $\mathbb{R}^{n+1}$ and set the coordinates of "north pole" and "south pole" to be

$$N = \{(0, 0, \cdots, 0, 1)\} \text{ and } S = \{(0, \cdots, 0, -1)\}$$

Then the smooth structure is defined by $\mathcal{F} = \{(S^n - \{N\}, p_N), (S^n - \{S\}, p_S)\}$, where p_N and p_S are stereographic projection maps to N and S. For instance, the map p_N sends $p_0 \in S^n - \{N\}$ to the intersection of $\mathbb{R}^n = \{x_{n+1} = 0\} \subset \mathbb{R}^{n+1}$ with the line defined by N and p_0. Explicitly, the stereographic projection maps are given by

$$p_N \colon S^n \to \mathbb{R}^n, (x_1, \cdots, x_{n+1}) \mapsto (\frac{x_1}{1 - x_{n+1}}, \frac{x_2}{1 - x_{n+1}}, \cdots, \frac{x_n}{1 - x_{n+1}}),$$
$$p_S \colon S^n \to \mathbb{R}^n, (x_1, \cdots, x_{n+1}) \mapsto (\frac{x_1}{1 + x_{n+1}}, \frac{x_2}{1 + x_{n+1}}, \cdots, \frac{x_n}{1 + x_{n+1}}).$$

Their inverses can be computed to be

$$p_N^{-1} \colon \mathbb{R}^n \to S^n, (y_1, \cdots, y_n) \mapsto (\frac{2y_1}{y_1^2 + \cdots + y_n^2 + 1}, \cdots, \frac{2y_n}{y_1^2 + \cdots + y_n^2 + 1}, \frac{y_1^2 + \cdots + y_n^2 - 1}{y_1^2 + \cdots + y_n^2 + 1}),$$
$$p_S^{-1} \colon \mathbb{R}^n \to S^n, (y_1, \cdots, y_n) \mapsto (\frac{2y_1}{1 + y_1^2 + \cdots + y_n^2}, \cdots, \frac{2y_n}{1 + y_1^2 + \cdots + y_n^2}, \frac{1 - y_1^2 - \cdots - y_n^2}{1 + y_1^2 + \cdots + y_n^2}).$$

Now if suffices to check yourself that on the overlap $S^n - \{N, S\}$, the transition functions

$$p_S \circ p_N^{-1} \text{ and } p_N \circ p_S^{-1}$$

are smooth functions (we omit the computations here). This completes the proof.

(b) Let $(z_0, z_1, \cdots, z_n)$ be the complex coordinates on $\mathbb{C}^{n+1}$. The complex projective space is the quotient of $\mathbb{C}^{n+1} - \{0\}$ by the diagonal action by $\mathbb{C}^*$

$$\lambda \cdot (z_0, z_1, \cdots, z_n) \mapsto (\lambda z_0, \lambda z_1, \cdots, \lambda z_n), \quad \lambda \in \mathbb{C}^*,$$

this means that we declare $(z_0, z_1, \cdots, z_n)$ and $(\lambda z_0, \lambda z_1, \cdots, \lambda z_n)$ to be equivalent whenever $\lambda \in \mathbb{C}^*$. Denote the equivalence class by $[z_0 \colon z_1 \colon \cdots \colon z_n]$. This defines so-called *homogeneous coordinates* on $\mathbb{C}P^n$. Set $U_i = \{z_i \neq 0\}$ for $i = 0, 1, \cdots, n$. We define the i-th chart to be

$$\phi_i \colon U_i \to \mathbb{C}^n, [z_0 \colon z_1 \colon \cdots \colon z_n] \mapsto (\frac{z_0}{z_i}, \frac{z_1}{z_i}, \cdots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \cdots, \frac{z_n}{z_i}).$$

Now on the overlap $U_i \cap U_j$ for $i \neq j$, say $i < j$, then we check that

$$\begin{aligned} & \phi_i \circ \phi_j^{-1}(w_0, \cdots, w_{j-1}, w_{j+1}, \cdots, w_n) \\ = \quad & \phi_i[w_0 \colon \cdots \colon w_{j-1} \colon 1 \colon w_{j+1} \colon \cdots \colon w_n] \\ = \quad & (\frac{w_0}{w_i}, \frac{w_1}{w_i}, \cdots, \frac{w_{i-1}}{w_i}, \frac{w_{i+1}}{w_i}, \cdots, \frac{w_{j-1}}{w_i}, \frac{1}{w_i}, \frac{w_{j+1}}{w_i}, \cdots, \frac{w_n}{w_i}). \end{aligned}$$

which is a holomorphic functions on $U_i \cap U_j = \{z_i \neq 0 \text{ and } z_j \neq 0\}$. Similarly for $\phi_j \circ \phi_i^{-1}$. This completes the proof. □

Problem 3.

Proof. (a) We will prove that $M = \tilde{M}/G$ admits a smooth structure

$$\mathcal{F} := \{(\pi(U), \phi \circ (\pi|_U)^{-1}) \mid (U, \phi) \in \mathcal{F}_{\tilde{M}}, \pi|_U \text{ is injective}\}$$

by checking the axioms $(S1)$ and $(S2)$ in the definitions of smooth manifolds:

(S1): Since G acts properly discontinuously on $\tilde{M}$, then we know that π is a covering map. Hence if U is open in $\tilde{M}$, then $\pi(U)$ is open in M. Also, for all $p \in \tilde{M}$ there exists a chart $(U, \phi) \in \mathcal{F}_{\tilde{M}}$ near p such that $\pi|_U$ is injective by the covering property. Hence if $\bigcup U_i = \tilde{M}$ and $\pi|_{U_i}$ is injective for all i, then $\bigcup \pi(U_i) = M$.

(S2) For any two charts $(\pi(U), \phi \circ (\pi|_U)^{-1})$ and $(\pi(V), \psi \circ (\pi|_V)^{-1})$ in $\mathcal{F}$, we will prove that the composition

$$\phi_{UV} := \big(\phi \circ (\pi|_U)^{-1}\big) \circ \big(\pi|_V \circ \psi^{-1}\big) \colon \psi\big((\pi|_V)^{-1}(\pi(U))\big) \to \phi\big((\pi|_U)^{-1}(\pi(V))\big)$$

is smooth by showing $(\pi|_U)^{-1} \circ \pi|_V$ is a diffeomorphism. This is due to the following Lemma.

Lemma 0.2. *If U and V are open sets in $\tilde{M}$ and $\pi|_U$ and $\pi|_V$ are injective, then the composition*

$$\pi_{UV} := (\pi|_U)^{-1} \circ \pi|_V \colon (\pi|_V)^{-1}(\pi(U)) \to (\pi|_U)^{-1}(\pi(V))$$

is a diffeomorphism

- First, the map π_{UV} is a homoemorphism as $\pi|_U \colon U \to \pi(U)$ and $\pi|_V \colon V \to \pi(V)$ are. Hence it suffices to show that for any point $p \in (\pi|_V)^{-1}(\pi(U))$ there is a neighborhood $V_p \subset (\pi|_V)^{-1}(\pi(U))$ such that $\pi_{UV}|_{V_p}$ is a diffeomorphism.
- Because G acts properly discontinuously, we have that for any $p \in \pi^{-1}|_V(\pi(U))$ and $q = \pi_{UV}(p)$, there exists a unique $g \in G$ such that $q = g \cdot p$ as $\pi \colon \tilde{M} \to M$ is a covering map.
- One can verify that in this neighborhood $V_p \subset (\pi|_V)^{-1}(\pi(U))$ of p, we have $\pi_{UV}|_{V_p} = g|_{V_p}$. Now because by definition $g \in G$ acts by diffeomorphism, we know $g|_{V_p}$ is a diffeomorphism and so is π_{UV}.

□

Proof. (b) In fact, we will show that this smooth structure is the unique one on M satisfying *either of the following two conditions*:
(i) the projection map π is a *local diffeomorphism* ($\forall p \in \tilde{M}$ and $\pi(p) \in M$, $\exists U_p, V_{\pi_p}$ such that $\pi|_{U_p} \colon U_p \to V_{\pi_p}$ is a diffeomorphism).
(ii) If N is another smooth manifold, a continuous map $f \colon M \to N$ is smooth if and only if the map $f \circ \pi \colon \tilde{M} \to N$ is smooth.

For (i): ($\Longrightarrow$) With respect to this smooth structure, we have that the transition functions $\phi_{UV} := \phi \circ \pi_{UV} \circ \psi^{-1}$ is a diffeomorphism whenever $(\pi(U), \phi \circ (\pi|_U)^{-1}), (\pi(V), \psi \circ (\pi|_V)^{-1}) \in \mathcal{F}$.
($\Longleftarrow$) Suppose that $\overline{\mathcal{F}}$ is another smooth structure on M such that π is a local diffeomorphism. We will show that $\mathcal{F} \subset \overline{\mathcal{F}}$. as follows. For $(U, \phi), (V, \psi) \in \mathcal{F}_{\tilde{M}}$ and $\pi|_U$ and $\pi|_V$ are injective, we have the composition

$$\phi \circ (\pi|_U)^{-1} \circ (\psi \circ (\pi|_V)^{-1})^{-1} = \phi \circ (\pi|_U)^{-1} \circ \pi|_V \circ \psi^{-1}$$

is a diffeomorphism because $(\pi|_U)^{-1} \circ \pi|_V$ is a diffeomorphism. This implies that $\mathcal{F} \subset \overline{\mathcal{F}}$ then by maximality condition we conclude that $\mathcal{F} = \overline{\mathcal{F}}$.

For (ii):($\Longrightarrow$) With respect to this smooth structure, a continuous map $f \colon M \to N$ is smooth if for any local charts $(\pi(U), \phi \circ (\pi|_U)^{-1})$ on M and (V, ψ) on N we have $\psi \circ f \circ \pi|_U \circ \phi^{-1}$ is smooth, which is equivalent to the fact that $f \circ \pi$ is smooth.
($\Longleftarrow$) Now given another smooth structure $\overline{\mathcal{F}}$ on M satisfying (ii). Again by maximality condition, it suffices to show that $\mathcal{F} \subset \overline{\mathcal{F}}$. This is because, any charts $(\overline{U}, \overline{\phi}) \in \overline{\mathcal{F}}$ on M, $(\pi^{-1}|_{\overline{U}}, \phi)$ on $\tilde{M}$ and (V, ψ) on N, the fact that $\overline{F}$ satisfying (ii) implies that $\psi \circ f \circ \overline{\phi}^{-1}$ is smooth if and only if $\psi \circ f \circ \pi|_U \circ \phi^{-1}$ is smooth. Hence we can conclude that

$$\phi \circ \pi|_U^{-1} \circ \overline{\phi}^{-1} \text{ and } \overline{\phi} \circ \pi|_U \circ \phi^{-1}$$

are smooth (which implies that $\mathcal{F} \subset \overline{\mathcal{F}}$), because the compositions of smooth functions are smooth

$$\left(\psi \circ f \circ \pi|_U \circ \phi^{-1}\right) \circ \left(\phi \circ \pi|_U^{-1} \circ \overline{\phi}^{-1}\right) = \psi \circ f \circ \overline{\phi}^{-1},$$
$$\left(\psi \circ f \circ \overline{\phi}^{-1}\right) \circ \left(\overline{\phi} \circ \pi|_U \circ \phi^{-1}\right) = \psi \circ f \circ \pi|_U \circ \phi^{-1}.$$

□

Problem 4. Prove that a bijective immersion is a diffeomorphism.

Proof. Let $\dim M = m$ and $\dim N = n$. Since f is an immersion, then differential

$$df|_m \colon T_m M \to T_{f(m)} N$$

is injective for all $m \in M$. We can conclude that $n \leq k$, Suppose that $n = k$, then $df|_m$ is an isomorphism for all $m \in M$, then by the Inverse Function Theorem f is a local diffeomorphism for all $m \in M$. Now since f is also bijective, this implies that f is in fact a diffeomorphism.
Now it suffices to assume $m < n$. Let (V, ψ) be a fixed chart on N such that $\psi(V) = \mathbb{R}^n$. Then $f^{-1}(V)$ is a smooth m-manifold. We will show that in this case f is not surjective. It suffices to prove that $f^{-1}(V)$ is not all of V, or equivalently, the map $g := \psi \circ f \colon V \to \mathbb{R}$ is not surjective. Because $f^{-1}(V) \subset M$ is second-countable, one can choose a collection of charts $\{(U_i, \phi_i \colon U_i \to W_i \subset \mathbb{R}^m)\}_{i \in \mathbb{Z}}$ on $f^{-1}(V)$ such that $\bigcup_{i \in \mathbb{Z}} U_i = f^{-1}(V)$. Then we have that

$$g(f^{-1}(W)) = g(\bigcup_{i \in \mathbb{Z}} \phi_i^{-1}(W_i)) = \bigcup_{i \in \mathbb{Z}} g(\phi_i^{-1}(W_i)) \subset \mathbb{R}^n.$$

Now by Proposition 1.35 (Slice Lemma), since $f \colon M \to N$ is an immersion, then with loss of generality we can assume the coordinate charts (U_i, ϕ_i) and (V, ψ) can be chosen such that $\psi \circ f \circ \phi_i^{-1}(x_1, \cdots, x_m) = (x_1, \cdots, x_m, C_{m+1}, \cdots, C_n)$ for some fixed constants $C_{m+1}, \cdots, C_n$ depending on $i \in \mathbb{Z}$. This implies that the image of $g(\phi_i^{-1}(W_i))$ is nowhere dense (in fact even measure zero) subset in $\mathbb{R}^n$ for each i. Now by the Baire category theorem, such countable union of nowhere dense subsets cannot be the entire $\mathbb{R}^m$, hence f is not surjective which give rise to a contradiction. □

Remark 0.3. This shows that there is no smooth surjective map $f \colon \mathbb{R} \to \mathbb{R}^n$ for $n > 1$. On the other hand, there does exist continuous surjective map $f \colon \mathbb{R} \to \mathbb{R}^n$ for $n > 1$.

Problem 5. Show that $d\psi|_m v$ is a well-defined element of $T_{\psi(m)}N$ for all $v \in T_m M$, i.e., it defines a linear derivation of $\tilde{F}_{\psi(m)}$.

Proof.

- We first check that $d\psi|_m v$ is a well-defined linear map from $\tilde{F}_{\psi(m)}$ to $\mathbb{R}$. Given U and V are open neighborhoods of $\psi(m)$ in N. Suppose there are smooth functions $f\colon U \to \mathbb{R}$ and $g\colon V \to \mathbb{R}$ and open subset $W \subset U \cap V$ such that $f|_W = g|_W$, i.e., $\underline{f}, \underline{g} \in \tilde{F}_{\psi(m)}$. We claim that $d\psi|_m v(\underline{f}) = d\psi|_m v(\underline{g})$. This is because

$$f \circ \psi^{-1} \colon \psi^{-1}(U) \to \mathbb{R} \text{ and } g \circ \psi^{-1} \colon \psi^{-1}(V) \to \mathbb{R}$$

are smooth functions near m and on $\psi^{-1}(W) \subset \psi^{-1}(U) \cap \psi^{-1}(V)$ we have that

$$(f \circ \psi)|_{\psi^{-1}(W)} = (g \circ \psi)|_{\psi^{-1}(W)} \Longrightarrow v(\underline{f \circ \psi}) = v(\underline{g \circ \psi}) \text{ for } \underline{f \circ \psi}, \underline{g \circ \psi} \in \tilde{F}_m.$$

- Then we check that $d\psi|_m v$ is a $\mathbb{R}$-linear map. If f and g are smooth function defined on neighbourhoods of $\psi(m)$ in N and $a, b \in \mathbb{R}$, for $\underline{f}, \underline{g} \in \tilde{F}_{\psi(m)}$ we have that

$$\begin{aligned} d\psi|_m v(a\underline{f} + b\underline{g}) &= d\psi|_m v(\underline{af + bg}) = v\big(\underline{(af + bg) \circ \psi}\big) \\ &= av(\underline{f \circ \psi}) + bv(\underline{g \circ \psi}) = a\{d\psi|_m v\}(\underline{f}) + b\{d\psi|_m v\}(\underline{g}). \end{aligned}$$

- Finally, we check that $d\psi|_m v$ is a derivation, that is, satisfies the Leibniz rule.

$$\begin{aligned} d\psi|_m v(\underline{f \cdot g}) &= v(\underline{(f \cdot g) \circ \psi}) = v((\underline{f \circ \psi}) \cdot (\underline{g \circ \psi})) \\ &= f \circ \psi(m) v(\underline{g \circ \psi}) + g \circ \psi(m) v(\underline{f \circ \psi}) \\ &= f(\psi(m))\{d\psi|_m v\}(\underline{g}) + g(\psi(m))\{d\psi|_m v\}(\underline{f}) \end{aligned}$$

□