Solution 3

Due on 10/03

Problem 1.

(a) We first check that for $p \in M$, the tangent vector defined by $[X,Y]_p$ gives a derivation on the space of germs of smooth function at $p \in M$. (We will write f for the equivalence class of its germ $\underline{f}_p$ below.)

For $f, g \in C^\infty(M)$, we have that

$$\begin{aligned}
[X,Y]_p(fg) &= X_p(Y(fg)) - Y_p(X(fg)) = X_p(fY(g)+gY(f)) - Y_p(fX(g)+gX(f)) \\
&= \big(f(p)X_p(Y(g)) + g(p)X_p(Y(f)) + Y_p(g)X_p(f) + Y_p(f)X_p(g)\big) \\
&- \big((f(p)Y_p(X(g)) + g(p)Y_p(X(f)) + X_p(g)Y_p(f) + X_p(f)Y_p(g)\big) \\
&= f(p)[X,Y]_p(g) + g(p)[X,Y]_p(f).
\end{aligned}$$

Let U be a neighborhood of $p \in M$ such that $f|_U = g|_U$, then we have $X(f)|_U = (Xg)|_U$ and $Y(f)|_U = Y(g)|_U$ as smooth functions defined on U. Now we know that X_p and Y_p depends only on the germs of function at $p \in U$, one concludes that

$$[X,Y]_p f = [X,Y]_p g.$$

By Proposition 1.43 we know that $X(f)$, $Y(f)$, $X(Y(f))$ and $Y(X(f))$ are smooth functions, so one concludes that $[X,Y](f) = X(Y(f)) - Y(X(f))$ is a smooth function and hence $[X,Y]$ defines a smooth vector field.

(b) By (a), it suffices to check for $f, g \in C^\infty(M)$ we have that

$$\begin{aligned}
[fX, gY](h) &= fX(gY(h)) - gY(fX(h)) \\
&= fX(g)Y(h) + fg(X(Y(h)) - gY(f)X(h) - gfY(X(h)) \\
&= fg[X,Y](h) + fX(g)Y(h) - g(Yf)X(h).
\end{aligned}$$

(c) Similarly, we have $[X,Y](f) = X(Y(f)) - Y(X(f)) = -[Y,X](f)$.

(d) For $f \in C^\infty(M)$, one has

$$\begin{aligned}
[[X,Y],Z](f) &= [X,Y](Zf) - Z([X,Y](f)) \\
&= (X(Y(Zf)) - Y(X(Z(f)))) - (Z(X(Y(f) - Y(Xf)) \\
&= X(Y(Zf)) - Y(X(Zf)) - Z(X(Yf)) + Z(Y(X(f)).
\end{aligned}$$

We can Cyclicly permute X, Y and Z in the above equation

$$[[Y,Z],X]f = Y(Z(X(f)) - Z(Y(X(f)) - X(Y(Z(f)) + X(Z(Yf))$$
$$[[Z,X],Y]f = Z(X(Y(f)) - X(Z(Y(f)) - Y(Z(X(f)) + Y(X(Zf))$$

and they sum up to zero.

Problem 2. Let M be a smooth manifold and $\gamma\colon (a,b) \to M$ be any maximal integral curve of a given vector field $X \in \Gamma(M, TM)$. We want to show that $(a,b) = \mathbb{R}$.

One can take a sequence $t_n \in (a,b)$ converging to the value $b \in \mathbb{R}$. We know M is compact, so there is a subsequence of $\gamma(t_n)$ converges to some point $p \in M$. By part (2) of Theorem 1.48, one has the local flow of X near p is defined

$$(-\epsilon, \epsilon) \times U \to M, \quad (t,q) \mapsto \phi_t(q), \quad \epsilon \in (0, |a|).$$

Now we choose t_n in this sequence such that $b - t_n < \epsilon$ and $\gamma(t_n) \in U$. Let $\beta\colon (-\epsilon, \epsilon) \to M$ be the unique integral curve of X such that $\beta(0) = \gamma(t_n)$. Then we can define another integral curve as the composite of $\gamma(t)$ and $\beta(t)$ by

$$\alpha\colon (a, t_n + \epsilon) \to M, \quad \alpha(t) = \begin{cases} \gamma(t), \text{ if } t \in (a,b) \\ \beta(t - t_n), \text{ if } t \in (t_n - \epsilon, t_n + \epsilon). \end{cases} \tag{1}$$

By definition, one checks that $\tilde{\gamma}(t) = \gamma(t + t_n)$ is an integral curve of X for $t \in (-\epsilon, b - t_n)$ and hence $\tilde{\gamma} = \beta$ on $(-\epsilon, b - t_n)$ by uniqueness of integral curves. This implies that $\alpha\colon (a, t_n + \epsilon) \to M$ is a well-defined integral curve of X. However we have that $t_n + \epsilon > b$ and $\alpha\big|_{(a,b)} = \gamma$, this will contradicts the fact that γ is the maximal integral curve of X *unless* $b = \infty$. We can prove that $a = -\infty$ similarly by applying this argument to the vector field $-X$. This shows that $(a,b) = \mathbb{R}$ for any vector field X defined on a compact manifold M.

Problem 3. Given an integral curve $\gamma\colon (a,b) \subset \mathbb{R} \to M$ such that $\gamma'(t_0) = 0$ for some $t_0 \in (a,b)$ and set $p := \gamma(t_0)$, we have that

$$X(p) = X(\gamma(t_0)) = \gamma'(t_0) = 0.$$

Let $\tilde{\gamma}\colon (a,b) \subset\to M$ be the constant curve defined by $\tilde{\gamma}(t) = p$ for all $t \in (a,b)$, we have $\tilde{\gamma}$ also satisfies

$$\tilde{\gamma}'(t) = X(p) = X(\tilde{\gamma}(t)), \quad \forall t \in (a,b).$$

By uniqueness Theorem for first-order ODE with initial conditions we have $\gamma = \tilde{\gamma}$ on (a,b). Hence γ is the constant map.

Problem 4. Let M be a smooth manifold M and $X, Y \in \Gamma(M, TM))$ and X_t and Y_t denote the flow of X and Y for $t \in (-\epsilon, \epsilon)$. Show that for $f \in C^\infty(M)$ and $p \in M$, we have

$$\lim_{s,t\to 0} \frac{f(Y_{-s}(X_{-t}(Y_s(X_t(p))))) - f(p)}{st} = [X,Y]_p f \in \mathbb{R}.$$

Remark: This problems says that the "rate of change of" the "difference between $Y_s \circ X_t(p)$ and $X_t \circ Y_s(p)$" for $p \in M$ is measured by the Lie bracket.

Proof. For a fixed $p \in M$, by part (3) of Theorem 1.48, we know that there exist $\epsilon > 0$ and an open neighborhood U of p such that the flow of X is defined on

$$(-\epsilon, \epsilon) \times U \to M, \quad (t, p) \mapsto X_t(p).$$

Similarly, by shrinking $(-\epsilon, \epsilon)$ and U if necessary (we will still denote them as $\epsilon > 0$ and U), we can conclude that there is a well-defined smooth map

$$(-\epsilon, \epsilon)^4 \times U, \quad (a, b, c, d, x) \mapsto Y_a \circ X_b \circ Y_c \circ X_d(p).$$

Given a smooth function $f \in C^\infty(M)$ near p, we will consider the composition

$$F\colon (-\epsilon, \epsilon)^4 \subset \mathbb{R}^4 \to \mathbb{R}, \quad (a, b, c, d) \mapsto f\big(Y_a \circ X_b \circ Y_c \circ X_d(p)\big),$$
$$\text{and } G\colon (-\epsilon, \epsilon)^2 \subset \mathbb{R}^2 \to \mathbb{R}, \quad G(s,t) = F(-s, -t, s, t).$$

Now the left hand side of the equation becomes

$$\lim_{s,t\to 0} \frac{f(Y_{-s}(X_{-t}(Y_s(X_t(p))))) - f(p)}{st} = \lim_{s,t\to 0} \frac{G(s,t) - G(0,0)}{st},$$

Since we know that $X_0 = Y_0 = id_M, X_{-t} \circ X_t = id_{\mathcal{D}(X)}$ and $Y_{-s} \circ Y_s = id_{\mathcal{D}(Y)}$, this implies that $G(s, 0) \equiv G(0, t) \equiv 0$ and $G(0, 0) = 0$. Now one has

$$\begin{aligned}
\lim_{s,t\to 0} \frac{G(s,t) - G(0,0)}{st} &= \lim_{s,t\to 0} \frac{G(s,t) - G(s,0) + G(s,0) - G(0,0)}{st} = \lim_{s,t\to 0} \frac{G(s,t) - G(s,0)}{st} \\
&= \lim_{s\to 0} \Big(\lim_{t\to 0} \frac{G(s,t) - G(s,0)}{t} \Big) \cdot \frac{1}{s} = \frac{\partial^2 G}{\partial_s \partial_t}\bigg|_{(0,0)},
\end{aligned}$$

where we have use the fact that G is a smooth function on $(-\epsilon, \epsilon)^2$ to conclude the above limits exists and are well-defined. Now we write this as

$$\frac{\partial^2 G}{\partial_s \partial_t}\bigg|_{(0,0)} = \frac{\partial^2 F}{\partial_a \partial_b}\bigg|_{(0,0,0,0)} - \frac{\partial^2 F}{\partial_a \partial_d}\bigg|_{(0,0,0,0)} - \frac{\partial^2 F}{\partial_b \partial_c}\bigg|_{(0,0,0,0)} + \frac{\partial^2 F}{\partial_c \partial_d}\bigg|_{(0,0,0,0)}.$$

We compute the first term as follows

$$\begin{aligned}
\frac{\partial^2 F}{\partial a \partial b}\bigg|_{(0,0,0,0)} &= \frac{\partial}{\partial b}\Big(\frac{\partial}{\partial a} f(Y_a \circ X_b(p)))\big|_{a=0}\Big)\bigg|_{b=0} = \frac{\partial}{\partial b}\Big(d_{X_b(p)} f\big(\frac{\partial}{\partial a} Y_a(X_b(p))\big)\big|_{a=0}\Big)\bigg|_{b=0} \\
&= \frac{\partial}{\partial b}(d_{X_a(p)} f(Y|_{X_a(p)}))\big|_{b=0} = \frac{\partial}{\partial b}\big((Yf)(X_b(p))\big)\bigg|_{b=0} = X_p(Yf).
\end{aligned}$$

For the other terms, we claim that similar calculations give the following equations

$$\left.\frac{\partial^2 F}{\partial a \partial d}\right|_{(0,0,0,0)} = X_p(Yf), \quad \left.\frac{\partial^2 F}{\partial b \partial c}\right|_{(0,0,0,0)} = Y_p(Xf), \text{ and } \left.\frac{\partial^2 F}{\partial c \partial d}\right|_{(0,0,0,0)} = X_p(Yf).$$

This completes the proof since $[X, Y]_p(f) = X_p(Yf) - Y_p(Xf)$ by definition for $p \in M$ and $f \in C^\infty(M)$. $\square$

Problem 5. Let $\mathbb{C}P^n$ be the complex projective n-space and $\gamma_n \to \mathbb{C}P^n$ be the tautological line bundle. Prove that there is a short exact sequences of complex vector bundles

$$0 \to \mathbb{C}P^n \times \mathbb{C} \xrightarrow{i} (n+1)\gamma_n^* \xrightarrow{q} T\mathbb{C}P^n \to 0.$$

We give two solutions to this problem as follows. I encourage you to read Solution 2.

Solution 1: (Define explicit bundle maps in local charts). We first define the bundle maps i and q explicitly as follows. We consider $i = (f_0, f_1, \cdots f_n)\colon \mathbb{C}P^n \times \mathbb{C} \to (n+1)\gamma_n^*$ componentwise given by

$$f_j\colon \mathbb{C}P^n \times \mathbb{C} \to \gamma_n^*, \quad f_j(l,\lambda)(v_0, v_1, \cdots, v_n) = \lambda \cdot v_j, \quad \forall v = (v_0, v_1, \cdots, v_n) \in (\gamma_n)_l.$$

Locally in the chart U_i, the map i can be explicitly written as

$$i(l,\lambda) = (l, \lambda \cdot \frac{X_0}{X_i}, \cdots, \lambda \cdot \frac{X_n}{X_i}), \text{ where } l = [X_0\colon X_1\colon \cdots \colon X_n].$$

The map i is fibre-wise linear. Because if for $v \in (\gamma_n)|_l \subset \mathbb{C}^{n+1} - \{0\}$, there exists a nonzero component v_i for some i. This implies that the corresponding dual vector $f_j(l,\lambda)$ is nonzero, that is, f_i is not the zero map. This proves that i is an injective vector bundle homomorphism.

To specify the bundle map $q\colon (n+1)\gamma_n^* \to T\mathbb{C}P^n$, we define it locally on each U_i as follows. Recall that there is a local trivialization of γ_n given by $h_i^{-1}\colon U_i \times \mathbb{C} \to \gamma_n|_{U_i}$.

$$h_i^{-1}([X_0\colon X_1\colon \cdots \colon X_n], c) \mapsto ([X_0\colon X_1\colon \cdots \colon X_n], (c\frac{X_0}{X_i}, \cdots, c\frac{X_n}{X_i})).$$

We let $z_{ij} := \frac{X_j}{X_i}$ for $0 \le j \le n$ and $i \ne j$ and set $\mathbf{z}_i := (z_{i0}, z_{i1}, \cdots, z_{in})$, then we have

$$(id, \mathbf{z}_i)\colon U_i \to U_i \times \gamma_n|_{U_i}$$

defines a section of γ_n locally over U_i. Now given $(n+1)$ local sections $(f_0, \cdots, f_n)$ of γ_n^* over U_i, we define the bundle map

$$q|_{U_i}\colon (n+1)\gamma^*|_{U_i} \to T\mathbb{C}P^n|_{U_i}, \;\; q(l, f_0, f_1, \cdots, f_n) \mapsto \sum_{i \ne j} \big(f_j(l, \mathbf{z}_i(l)) - z_{ij} f_i(l, \mathbf{z}_i(l))\big)\partial_{z_{ij}}, \quad (2)$$

This map defines a surjective vector bundle homomorphism because $q|_{U_i}$ is surjective for all i by definition.

To check exactness of the short exact sequences, it suffices to check it locally in each U_i, we have $\ker(q|_{U_i}) = \mathrm{im}(i|_{U_i})$. This is true because that $q|_{U_i}(l, f_0, f_1, \cdots, f_n) = 0$ if and only if $f_j(l, \mathbf{z}_i(l)) - z_{ij} f_i(l, \mathbf{z}_i(l)) = 0$ which implies that $f_j = z_{ij} f_i$ and hence $\ker(q|_{U_i}) = \mathrm{im}(i|_{U_i})$ for each i

To see in fact q is well-defined globally, we check that the above definition of q is invariant under transition maps. For $k \ne i$, by definition we have that $z_{kl} = z_{ik}^{-1} \cdot z_{il}$, this implies that for partial derivatives, one has

$$\frac{\partial}{\partial z_{ij}} = \sum_{k \ne l} \frac{\partial z_{kl}}{\partial z_{ij}} \frac{\partial}{\partial z_{kl}} = \begin{cases} z_{ik}^{-1} \frac{\partial}{\partial z_{kj}}, & \text{if } j \ne k \\ -z_{ik}^{-2}(\frac{\partial}{\partial z_{ki}} + \sum_{l \ne i,k} z_{il} \frac{\partial}{\partial z_{kl}}), & \text{if } j = k. \end{cases}$$

Now we need to check two cases, if $j \neq i, k$, we have

$$z_{ki}^{-1} f_j(l, \mathbf{z}_k(l)) - z_{ki}^{-2} z_{kj} f_i(l, z_k(l))) z_{ik}^{-1} \frac{\partial}{\partial z_{kj}} = (f_j(l, \mathbf{z}_k(l)) - z_{ij} f_i(l, \mathbf{z}_k(l))) \frac{\partial}{\partial z_{kj}} \quad (3)$$

which is exactly the jth component in the summation $j \neq i$ in (2) when $j \neq k$. Now if $j = k$, we have

$$\begin{aligned} &\big(z_{ki}^{-1} f_k(l, \mathbf{z}_k(l)) - z_{ki}^{-2} f_i(l, \mathbf{z}_k(l))\big) \cdot \big(- z_{ik}^{-2} \frac{\partial}{\partial z_{ki}} - \sum_{j \neq i,k} \frac{z_{ij}}{z_{ik}^2} \frac{\partial}{\partial z_{kj}} \big) \\ &= \big(f_i(l, \mathbf{z}_k(l)) - z_{ki} f_k(l, \mathbf{z}_k(l))\big) \cdot \big(\frac{\partial}{\partial z_{ki}} + \sum_{j \neq i,k} z_{ij} \frac{\partial}{\partial z_{kj}} \big). \end{aligned} \quad (4)$$

Combining terms in (3) and (4), we obtain exactly definition (2) with index i replaced by k in the expression. Hence q gives rise to a well-defined bundle map.

□

Remark 0.1. *In fact, we have shown that the Euler exact sequence is a short exact sequence of holomorphic vector bundles on $\mathbb{C}P^n$ (where $\mathbb{C}P^n$ is equipped with the standard holomorphic structure defined in Homework 1 solution 2). A holomorphic vector bundle of rank k admits trivializations $\{(U_\alpha, h_\alpha)\}_{\alpha \in \mathcal{A}}$, where $h_\alpha \colon V|_{U_\alpha} \to U_\alpha \times \mathbb{C}^k$ are biholomorphisms. Equivalently, we require that the transition maps $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to GL(n, \mathbb{C})$ are holomorphic maps for all $\alpha, \beta \in \mathcal{A}$. A holomorphic vector bundle homomorphism is a holomorphic map $f \colon V \to W$ which is fibre-wise $\mathbb{C}$-linear and given local trivializations $h_i \colon V|_{U_i} \to U_i \times \mathbb{C}^k$ and $\bar{h}_j \colon W|_{U_j} \to U_j \times \mathbb{C}^l$, one has on the overlap $U_i \cap U_j$*

$$\bar{h}_j \circ h_i^{-1} \colon U_i \times \mathbb{C}^k \to U_j \times \mathbb{C}^l$$

are holomorphic maps with respect to the given complex structures.

Solution 2: (A geometric interpretation of the Euler sequence). The complex projective space $\mathbb{C}P^n$ is the quotient of the $\mathbb{C}^*$-action on $\mathbb{C}^{n+1} - \{0\}$ given by

$$p := (X_0, X_1, \cdots, X_n)) \mapsto \lambda \cdot p := (\lambda \cdot X_0, \lambda \cdot X_1, \cdots, \lambda \cdot X_n), \ \lambda \in \mathbb{C}^*.$$

We denote the quotient map by

$$\pi \colon \mathbb{C}^{n+1} - \{0\} \to \mathbb{C}P^n.$$

Let $\widetilde{E} = \sum_{i=0}^n X_i \frac{\partial}{\partial X_i}$ be the "radial" vector field on $\mathbb{C}^{n+1}$. One notices that $\widetilde{E}$ is invariant under the $\mathbb{C}^*$-action, i.e, $\widetilde{E}(p) = \widetilde{E}(\lambda \cdot p)$ (because for $W_i = \lambda X_i$, we have $\frac{\partial}{\partial W_i} = \frac{1}{\lambda} \frac{\partial}{\partial X_i}$). This means that $\widetilde{E}$ descends to a well-defined vector field on $\mathbb{C}P^n$, called the **Euler vector field**. As a quotient space, one has the differential $d\pi_p \colon T_p(\mathbb{C}^{n+1} - \{0\}) \to T_{\pi(p)} \mathbb{C}P^n$ is surjective for all p and the tangent space of $\mathbb{C}P^n$ at $\pi(p)$ is spanned by

$$d\pi_p(\frac{\partial}{\partial X_i}|_p), \quad i = 0, 1, \cdots, n.$$

The kernel of $d\pi_p$ is isomorphic to the $\mathbb{C}$-linear span of $\widetilde{E}\big|_p = \sum_i X_i(p)\frac{\partial}{\partial X_i}\big|_p$. This is because the flow of $\widetilde{E}$ generates the $\mathbb{C}^*$-action on $\mathbb{C}^{n+1} - \{0\}$, which implies that the distribution spanned by $\widetilde{E}$ is tangent to the integral manifolds which are the orbits of the $\mathbb{C}^*$-action.

Now given any linear functional $f \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{n+1}, \mathbb{C})$, we can define a vector field on $\mathbb{C}^{n+1}$ by

$$\tilde{v}_i(p) = f(p)\frac{\partial}{\partial X_i}, \quad i = 0, 1, \cdots, n.$$

One checks that $d\pi_p(\tilde{v}_i(p)) = d\pi_{\lambda\cdot p}(\tilde{v}_i(\lambda\cdot p))$, which means that $\tilde{v}_i$ always descends to a vector field on $\mathbb{C}P^n$. The fibre of γ_n^* at $l = \pi(p) \in \mathbb{C}P^n$ consists of those linear functionals σ defined on the line $l \in \mathbb{C}P^n$. For example, for $l = [X_0\colon X_1\colon \cdots : X_n] \in \mathbb{C}P^n$, the homogeneous coordinate X_i defines a section of γ_n^* by

$$X_i(l)(v) = v_i, \quad \text{where } v = (v_0, v_1, \cdots, v_n) \in l \subset \mathbb{C}^{n+1} \text{ and } i = 0, 1, \cdots, n.$$

We can define maps between spaces of (holomorphic) sections of the above vector bundles

$$i\colon \Gamma(\mathbb{C}P^n \times \mathbb{C}) \to \Gamma((n+1)\gamma_n^*), \quad 1 \mapsto (X_0, X_1, \cdots, X_{n+1});$$
$$q\colon \Gamma((n+1)\gamma_n^*) \to \Gamma(T\mathbb{C}P^n), \quad (\sigma_0, \sigma_1, \cdots, \sigma_n) \mapsto d\pi(\sum_{i=0}^{n} \sigma_i(p)\frac{\partial}{\partial X_i}).$$

In fact, the maps i and q are homomorphism of $\mathcal{O}_{\mathbb{C}P^n}$-modules, where $\mathcal{O}_{\mathbb{C}P^n}$ denotes the ring of holomorphic functions on $\mathbb{C}P^n$. One has that i is injective by definition and q is surjective since the quotient map π is a submersion and $\ker(q) = \mathrm{Im}(i)$. In fact, we have shown that there is short exact sequence of $\mathcal{O}_{\mathbb{C}P^n}$-modules.

$$0 \to \Gamma(\mathbb{C}P^n \times \mathbb{C}) \to \Gamma((n+1)\gamma_n^*) \to \Gamma(T\mathbb{C}P^n) \to 0,$$

where $\Gamma(V)$ here denotes the space of holomorphic section of a holomorphic vector bundle V. We will see later in the class this statement is equivalent to the exactness of (holomorphic) vector bundles

$$0 \to \mathbb{C}P^n \times \mathbb{C} \to (n+1)\gamma_n^* \to T\mathbb{C}P^n \to 0.$$

□