Solution 4

Due on 10/19

Problem 1.

Proof. For (a), Given a point $p = (x, y)$ in $\mathbb{T}^2$ and let U be any open set that contains p. We want to show that $\mathrm{Im}(\phi) \cap U \neq \emptyset$, where ϕ is the smooth map

$$\phi \colon \mathbb{R} \to S^1 \times S^1, t \mapsto (e^{2\pi i t}, e^{2\pi i \alpha t}) \text{ for } \alpha \in \mathbb{R} - \mathbb{Q}.$$

This is equivalent to the fact that there exists $t_0 \in \mathbb{R}$ such that $x = e^{2\pi i t_0}$ and $\{e^{2\pi i \alpha(t_0+k)}\}_{k\in\mathbb{Z}}$ intersect arbitrary open set of y in S^1, that is, the set $\{e^{2\pi i \alpha(t_0+k)}\}_{k\in\mathbb{Z}}$ is dense in S^1. To see this, it suffices to prove the following Lemma.

Lemma 0.1. *Let $x \in \mathbb{R}$ and we denote by $[x]$ the decimal part of the real number x. Then we have that*

$$A = \{[nx] \mid n \in \mathbb{Z} \text{ and } x \in \mathbb{R} - \mathbb{Q}\}$$

is dense in $[0, 1)$.

Proof of Lemma (based on Te Cao's solution): First, one notices that A is an infinite set. This is because if $n, m \in \mathbb{Z}$ and $n \neq m$ then we have that $[nx] \neq [mx]$. Otherwise, $nx - mx \in \mathbb{Z}$ which contradicts to the fact that $x \in \mathbb{R} - \mathbb{Q}$. Using that fact that A is infinite, we will show that given any $\epsilon \in (0, 1)$, there exits $n_\epsilon \in \mathbb{Z}$ such that $[n_\epsilon \alpha] \in [0, \epsilon)$ as follows. Given any $k \in \mathbb{N}^*$, we can subdivide the interval equally $[0, 1]$ into k pieces given by

$$[0, \frac{1}{k}) \cup [\frac{1}{k}, \frac{2}{k}) \cup \cdots \cup [\frac{k-1}{k}, 1).$$

Since the cardinality of A is infinite, there must exist two distinct elements $m_k, n_k \in \mathbb{Z}$ (depending on k) such that $[n_k x]$ and $[m_k x]$ lie in the same interval, that is, we have that $[(n_k - m_k)x] \in (0, \frac{1}{k})$. Given any $\epsilon \in (0, 1)$, we can take $n_\epsilon = n_k - m_k$ for $k \in \mathbb{N}^*$ and $\frac{1}{k} \leq \epsilon$. Now for any $p \in [0, 1)$ and an arbitrary open neighbourhood U of p, we can find an interval $I \subset U$ of length ϵ_0 centered at p for some $\epsilon_0 \in (0, 1)$. Then we have that $[k \cdot n_{\frac{\epsilon_0}{2}} \alpha] \in A$. This shows that A is dense in $[0, 1)$.

For (b): For a fixed $\beta \in S^1$, we first show that $f_\beta(t) = (\beta e^{2\pi i t}, e^{2\pi i \alpha t})$ is an injective immersion. Suppose $f_\beta(t_1) = f_\beta(t_2)$, we have $e^{2\pi i t_1} = e^{2\pi i t_2}$ and $e^{2\pi i \alpha t_1} = e^{2\pi i \alpha t_2}$, which implies that $t_1 - t_2 \in \mathbb{Z}$ and $\alpha t_1 - \alpha t_2 \in \mathbb{Z}$. Since $\alpha \in \mathbb{R} - \mathbb{Q}$, we conclude that $t_1 = t_2$. We also have that f_β is an immersion because the differential

$$(df_\beta)_t \colon T_t\mathbb{R} \to T_{f_\beta(t)}\mathbb{T}^2, \quad (df_\beta)_t = [2\pi i \beta e^{2\pi i t}, 2\pi i e^{2\pi i \alpha t}]^T$$

is nonsingular (i.e. has full rank) for all $t \in \mathbb{R}$. To see $\mathbb{T}^2 = \bigcup_{\beta \in S^1} \mathrm{Im}(f_\beta)$, given any point $p = (x, y)$ in $\mathbb{T}^2$, there exist a unique β such that $(x, y) = (\beta e^{2\pi i t}, e^{2\pi i \alpha t})$. In fact, we can solve $\beta = x \cdot (e^{2\pi i t})^{-1}$ for the unique $t \in S^1$ satisfying $y = e^{2\pi i \alpha t}$. However, this foliation is NOT proper because for each $\beta \in S^1$, the image of f_β is NOT homeomorphic to $\mathbb{R}$ by part (a) (since $\overline{\mathrm{Im}(\phi)} = \mathbb{T}^2$). Hence f_β is not an embedding. Furthermore, the leaves f_β of this foliation intersect each other and hence they do NOT partition $\mathbb{T}^2$. ∎

Problem 2.

Proof. For (a), we compute $i_X\eta$ as follows.

$$\begin{aligned} i_X\eta &= i_X(dx)\wedge dy\wedge dz - dx\wedge i_X(dy)\wedge dz + dx\wedge dy\wedge i_X(dz) \\ &= xdy\wedge dz + ydz\wedge dx + zdx\wedge dz = \omega \end{aligned}$$

For (b), **there is a typo in the original homework problem**: The spherical coordinates (ρ, ϕ, θ) in $\mathbb{R}^3$ is given by

$$(x, y, z) = (\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi)).$$

We express ω in spherical coordinates as follows. The differentials satisfying

$$\begin{aligned} dx &= \sin(\phi)\cos(\theta)d\rho + \rho\cos(\phi)\cos(\theta)d\phi - \rho\sin(\phi)\sin(\theta)d\theta, \\ dy &= \sin(\phi)\sin(\theta)d\rho + \rho\cos(\phi)\sin(\theta)d\phi + \rho\sin(\phi)\cos(\theta)d\theta, \\ dx &= \cos(\phi)d\rho - \rho\sin(\phi)d\phi. \end{aligned}$$

Using the fact that $dx\wedge dx = 0$ and $dx\wedge dy = -dy\wedge dx$, we obtain

$$\begin{aligned} dy\wedge dz &= -\rho\sin(\theta)d\rho\wedge d\phi - \rho\sin(\phi)\cos(\phi)\cos(\theta)d\rho\wedge d\theta + \rho^2\sin^2(\phi)\cos(\theta)d\phi\wedge d\theta, \\ dz\wedge dx &= \rho\cos(\theta)d\rho\wedge d\phi - \rho\sin(\phi)\cos(\phi)\sin(\theta)d\rho\wedge d\theta + \rho^2\sin^2(\phi)\sin(\theta)d\phi\wedge d\theta, \\ dx\wedge dy &= \rho\sin^2(\phi)d\rho\wedge d\theta + \rho^2\cos(\phi)\sin(\phi)d\phi\wedge d\theta. \end{aligned}$$

$\Longrightarrow$

$$\begin{aligned} &xdy\wedge dz + ydz\wedge dx + xdx\wedge dy \\ &= \big(-\rho^2\sin(\phi)\cos(\theta)\sin(\theta) + \rho^2\sin(\phi)\sin(\theta)\cos(\theta)\big)d\rho\wedge d\phi \\ &+\big(-\rho^2\sin^2(\phi)\cos^2(\theta)\cos(\phi) - \rho^2\sin^2(\phi)\sin^2(\theta)\cos(\phi) + \rho^2\cos(\phi)\sin^2(\phi)\big)d\rho\wedge d\theta \\ &+\big(\rho^3\sin^3(\phi)\cos^2(\theta) + \rho^3\sin^3(\phi)\sin^2(\theta) + \rho^3\cos^2(\phi)\sin(\phi)\big)d\phi\wedge d\theta \\ &= \rho^3\sin(\phi)d\phi\wedge d\theta. \end{aligned}$$

For (c), the expression of $d\omega$ in Cartesian coordinates is

$$d\omega = dx\wedge dy\wedge dz + dy\wedge dz\wedge dx + dz\wedge dx\wedge dy = 3dx\wedge dy\wedge dz.$$

In spherical coordinates, we have

$$d\omega = 3\rho^2\sin(\phi)d\rho\wedge d\phi\wedge d\theta.$$

We verify that we represent the same 3-form

$$\begin{aligned} 3dx\wedge dy\wedge dz &= 3(\rho\sin^2(\phi)d\rho\wedge d\theta + \rho^2\cos(\phi)\sin(\phi)d\phi\wedge d\theta)\wedge(\cos(\phi)d\rho - \rho\sin(\phi)d\phi) \\ &= 3\rho^2\sin(\phi)(\sin^2(\phi) + \cos^2(\phi))d\rho\wedge d\phi\wedge d\theta \\ &= 3\rho^2\sin(\phi)d\rho\wedge d\theta\wedge d\theta. \end{aligned}$$

For (d), when restricting to the unit n-sphere, we have $\rho = 1$

$$(x, y, z) = (\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)).$$

On the set where the spherical coordinates are well-defined when $\phi \in (0, \pi)$, we have

$$\omega|_{S^2} = \sin(\phi)d\phi\wedge d\theta.$$

For (e), Let $N = (0, 0, 1)$ and $S = (0, 0, -1)$ be the North and South poles of S^2, on the open set $S^2 - \{N, S\}$ the volume form is nowhere zero because $\sin(\phi) \neq 0$ when $\phi \in (0, \pi)$. □

Problem 3.

Proof. Let $\alpha \in \Omega^k(M)$ be a k-form and $X_0, X_1, \cdots, X_k \in \Gamma(M, TM)$ be smooth vector field. We will verify that

$$\begin{aligned} d\alpha(X_0, X_1, \cdots, X_k) &= \sum_{i=0}^{k} (-1)^i X_i\big(\alpha(X_0, \cdots, X_{i-1}, \widehat{X}_i, X_{i+1}, \cdots, X_k)\big) \\ &+ \sum_{0\le i<j\le k} (-1)^{i+j}\alpha([X_i, X_j], X_0, X_1, \cdots, \widehat{X}_i, \cdots, \widehat{X}_j, \cdots, X_k). \end{aligned} \tag{0.1}$$

The general strategy to prove tensor identities: Verify both sides are tensorial and check the equality locally using local coordinates for (r, s)-tensors.

In our case as a $(k+1)$-form, we know that the $d\omega$ is **tensorial**, that is,

(1) For each point $p \in M$, the value of $d\omega(X_0, \cdots, X_k)$ depends only on the value of $d\omega|_p$ and $(X_i)|_p$ for $i = 0, 1, \cdots, k$.

(2) The $(k+1)$-form depends an alternating $C^\infty(M)$-multilinear functional

$$d\omega\colon \Gamma(M, TM)^{k+1} \to C^\infty(M).$$

We will prove the equation (0.1) by first verifying the *right hand side* of (0.1) is also tensorial. Hence it defines differential $(k+1)$-form. Then we check the equality in local coordinates. The first term satisfies

$$\begin{aligned} &\sum_{i=0}^{k} (-1)^i X_i\big(\alpha(fX_0, \cdots, X_{i-1}, \widehat{X}_i, X_{i+1}, \cdots, X_k)\big) \\ &= (-1)^0 (fX_0)\big(\alpha(X_1, \cdots, X_k)\big) + \sum_{i=1}^{k} (-1)^i X_i\big(f\alpha(X_0, \cdots, \widehat{X}_i, \cdots, X_k)\big) \\ &= \sum_{i=1}^{k} (-1)^i X_i(f)\alpha(X_0, \cdots, \widehat{X}_i, \cdots, X_k) + f\sum_{i=0}^{k} (-1)^i X_i(\alpha(X_0, \cdots, \widehat{X}_i, \cdots, X_k)), \end{aligned} \tag{0.2}$$

where we have used that fact that α is $C^\infty(M)$-linear in each input as a k-form in the second equality and product rule for the last equality. Similar, we check the second term on the right hand side.

$$\begin{aligned} &\textstyle\sum_{j=1}^{k} (-1)^j \alpha([fX_0, X_j], X_1, \cdots, \widehat{X}_j, \cdots, X_k) + \sum_{1\le i<j\le k} (-1)^{i+j}\alpha([X_i, X_j], fX_0, \cdots, \widehat{X}_i, \cdots, \widehat{X}_j, \cdots, X_k) \\ &= \textstyle\sum_{j=1}^{k} (-1)^j \alpha(f\cdot [X_0, X_j] - X_i(f)X_0, X_1, \cdots, \widehat{X}_j, \cdots, X_k) + f\sum_{1\le i<j\le k} (-1)^{i+j}\alpha([X_i, X_j], X_0, \cdots, \widehat{X}_i, \cdots, \widehat{X}_j, \cdots, X_k) \\ &= \textstyle -\sum_{j=1}^{k} (-1)^j X_j(f)\alpha(X_0, \cdots, \widehat{X}_j, \cdots, X_k) + f\sum_{0\le i<j\le k} (-1)^{i+j}\alpha([X_i, X_j], X_0, \cdots, \widehat{X}_i, \cdots, \widehat{X}_j, \cdots, X_k). \end{aligned} \tag{0.3}$$

Combining equations (0.2) and (0.3), we see that the right hand side also defines a $C^\infty(M)$-linear multilinear functional on $\Gamma(M, TM)^k$ (technically one needs to check that each input is $C^\infty(M)$, the proof is completely analogous which we omit). By definition, we see that the right hand side of (0.1) is alternating. For fixed $p \in M$, we have that the right hand side of (0.1) at a point p depends only on the value of $X_i\big|_p$. Having checked both sides are tensorial, it suffices to check equality locally, i.e., we write

$$\alpha = f dx_I := f dx_{i_1} \wedge \cdots \wedge dx_{i_k}, i_1 < \cdots < i_k, \text{ and } X_j = \frac{\partial}{\partial x_{i_j}}.$$

Because that $[X_i, X_j] = 0$ in a local coordinate chart and $d\alpha = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \wedge dx_I$. Then the left hand side becomes

$$d\alpha(\frac{\partial}{\partial x_{j_0}}, \cdots \frac{\partial}{\partial x_{j_k}}) = \sum_{i=1}^{n}\sum_{l=0}^{k} (-1)^l \frac{\partial f}{\partial x_i} dx_i(\frac{\partial}{\partial x_{j_l}}) dx_I(\frac{\partial}{\partial x_{j_0}}, \cdots, \widehat{\frac{\partial}{\partial x_{j_l}}}, \cdots \frac{\partial}{\partial x_{j_k}}), \tag{0.4}$$

and the first term on the right hand side (as the second term vanishes) becomes

$$\sum_{l=0}^{k}(-1)^l \frac{\partial}{\partial x_{j_l}}(f dx_I(\frac{\partial}{\partial x_{j_0}}, \cdots, \widehat{\frac{\partial}{\partial x_{j_l}}}, \cdots, \frac{\partial}{\partial x_{j_k}})) \tag{0.5}$$

After simplications, one verifies that both (0.4) and (0.5) reduces to the expression

$$\sum_{l=0}^{k}(-1)^l (\frac{\partial f}{\partial x_{j_l}})\delta_{I,(j_0,\cdots,\widehat{j_l},\cdots,j_k)},$$

where $\delta_{I,(j_0,\cdots,\widehat{j_l},\cdots,j_k)}$ is zero unless $I = (j_0, \cdots, \widehat{j_l}, \cdots, j_k)$ as order sets. This completes the proof. □

Problem 4.

Proof. For identity (1): We check it on 0-forms first. Given $f \in C^\infty(M)$, we have $L_X(f) = X(f)$ and

$$d \circ i_X(f) + i_X \circ df = 0 + X(f),$$

where we have use the fact that $i_X(f) = 0$. This implies the Cartan formula holds for $f \in \Omega^*(M)$. Since both sides commutate with the exterior derivative d and defines derivations on $\Omega^*(M)$. We check it on k-forms ($k \geq 1$) in local coordinates. For $\alpha \in \Omega^1(M)$, it suffices to assume that $\alpha = f dx_i$, then we have the left hand side is

$$L_X(f dx_i) = f L_X(dx_i) + dx_i L_X(f) = f d L_X(x_i) + dx_i(X(f)) = f d(X(x_i)) + dx_i X(f).$$

The right hand side is

$$\begin{aligned} d i_X(f dx_i) + i_X d(f dx_i) &= d(f dx_i(X)) + i_X(df \wedge dx_i)) \\ &= d(f(X(x_i)) + df(X) dx_i + (-1) X(x_i) df \\ &= X(x_i)(df) + f d(X(x_i)) + X(f) dx_i - X(x_i))(df) \\ &= f d(X(x_i)) + X(f) dx_i = L_X(f dx_i). \end{aligned}$$

Now for $\alpha \in \Omega^k(M)$ for $k > 1$ we can assume that $\alpha = f dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ for some $i_1 < \cdots < i_k$ and since both sides are derivation of $\Omega^*(M)$, we conclude that $L_X = d i_X + i_X d$ holds on $\Omega^k(M)$ for all k.

Next, we will the proof of the identity

$$\iota_{[X,Y]} = L_X \circ i_Y - i_Y \circ L_X.$$

For 0-forms: given $f \in C^\infty(M)$, we have that

$$0 = \iota_{[X,Y]}(f) = L_X \circ i_Y(f) - i_Y \circ L_X(f) = 0 + i_Y(X(f)) = 0.$$

For 1-forms: given $\alpha \in \Omega^1(M)$, we assume that $\alpha = f dx_i$ and we have

$$\begin{aligned} (L_X \circ i_Y - i_Y \circ L_X)(f dx_i) &= L_X(f Y(x_i)) - i_Y\big(f d(X(x_i)) + dx_i X(f)\big) \\ &= X(f) Y(x_i) + f XY(x_i) - f YX(x_i) - Y(x_i) X(f) \\ &= f([X,Y](x_i)) = \iota_{[X,Y]}(f dx_i) \end{aligned}$$

Since both sides are graded derivations on $\Omega^*(M)$, we conclude that the identity holds for $\Omega^k(M)$ for all $k \geq 0$.

This last identity

$$[L_X, L_Y] := L_X \circ L_Y - L_Y \circ L_X = L_{[X,Y]}$$

impliest that **there is Lie algebra homomorphism** from the space of smooth vector fields $(\Gamma(M, TM), [\cdot,\cdot])$ equipped with Lie bracket of vector fields to the space of derivations on $A := \Omega^*(M)$ equipped with the commutator bracket $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$. (Check yourself that **commutator of a derivation is still a derivation** on A.)

We prove the last identity using the first two identities

$$\begin{aligned} L_{[X,Y]}(\alpha) &= d i_{[X,Y]}(\alpha) + i_{[X,Y]}(d\alpha) \\ &= d(L_X i_Y - i_Y L_X)(\alpha) + (L_X i_Y - i_Y L_X)(d\alpha) \\ &= (d i_X d i_Y - d i_Y d i_X - d i_Y i_X d)(\alpha) + (d i_X i_Y d + i_X d i_Y d - i_Y d i_X d)(\alpha) \\ &= (d i_X d i_Y + d i_X i_Y d + i_X d i_Y d)(\alpha) - (d i_Y d i_X + d i_Y i_X d + i_Y d i_X d)(\alpha) \\ &= L_X \circ L_Y(\alpha) - L_Y \circ L_X(\alpha). \end{aligned}$$

□

Problem 5.

Proof. For (a), for each $p \in T_pM$ the symplectic forms restricts to a nondegenerate skew-symmetric bilinear form ω_p on each tangent space T_pM. Then we apply the structure Theorem for skew-symmetric bilinear form in linear algebra, which says that there exists a basis $v_1, \cdots, v_n$ for T_pM such that under this basis, the symplectic form has the standard form, i.e.,

$$\begin{pmatrix} J & 0 & \dots & 0 \\ 0 & J & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J \end{pmatrix}$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and this implies that every symplectic manifold is even dimensional. The procedure to find such a basis is the following: given a nonzero vector $v_1 \in T_pM$, as ω_p is non-degenerate, there is a nonzero vector v_2 such that $\omega_p(v_1, v_2) = 1$. Then we defined the ω-orthogonal complement to $\mathrm{Span}_{\mathbb{R}}\langle v_1, v_2\rangle$ in T_pM by

$$\{v \in T_pM \mid \omega_p(v_i, v) = 0 \text{ for } i = 1, 2\}.$$

By definition, this is a $(\dim(M) - 2)$-dimensional symplectic subspace of T_pM. Then we inductively apply the above procedure to find such a basis $v_1, v_2, \cdots, v_n$.

For (b), we define a homomorphism by

$$\psi \colon TM \to T^*M, X \mapsto i_X\omega.$$

Since the 2-form ω is smooth, so this map is smooth. Now if $v \in T_pM$, then $i_v\omega_p \in T_p^*M$, which implies that ψ commutes with the projections to the base. If $v_1, v_2, u \in T_pM$ and $a, b \in \mathbb{R}$, then we have that

$$(i_{av_1+bv_2}\omega_p)(u) = \omega_p(av_1 + bv_2, u) = a\omega_p(v_1, u) + b\omega_p(v_2, u) = a(i_{v_1}\omega_p)(u) + b(i_{v_2}\omega_p)(u),$$

which implies that ψ is linear on each fiber. So we conclude that ψ is a bundle homomorphism. By definition of ω being non-degenerate, we have that for $v \in T_pM$ and $v \neq 0$, there exists $u \in T_pM$ such that

$$(i_v\omega)(u) = \omega_p(X, Y) \neq 0 \Longrightarrow i_v\omega_p \neq 0 \in T_p^*M.$$

This non-degeneracy condition implies that the bundle homomorphism ψ is injective and therefore this is a bundle isomorphism as the rank of TM and T^*M are the same.

For (c): We have not use the fact that $d\omega = 0$ so far for part (a) and (b), in fact a nondegenerate 2-form that is not necessarily closed is called an almost symplectic form for which part (a) and (b) still apply. For a symplectic form, the fact $d\omega = 0$ implies that

$$L_{X_H}(\varphi_t^*\omega) = (i_{X_H} \circ d + d \circ i_{X_H})(\varphi_t^*\omega) = i_{X_H}\varphi_t^*d\omega + d \circ i_{X_H}\varphi_t^*\omega = 0 + d(\varphi_t^*(i_{d\phi_t(X_H)}\omega)). \tag{0.6}$$

Now since φ_t is the flow of the Hamiltonian vector field X_H, we have that

$$d_p\varphi_t(X_H) = d_p\varphi_t\Big(\frac{d}{ds}\varphi_s(p)\Big|_{s=0}\Big) = \frac{d}{ds}(\varphi_t \circ \varphi_s(p))\Big|_{s=0} = X_H(\varphi_t(p)).$$

This implies that $i_{d\phi_t(X_H)}\omega = i_{X_H}\omega = dH$ in equation (0.6). Therefore, one has

$$L_{X_H}(\varphi_t^*\omega) = d(\varphi_t^*dH) = \varphi_t^*d^2H = 0$$

. On the other hand, by the definition of Lie derivative,

$$L_{X_H}(\varphi_t^*\omega) = \lim_{s\to 0}\frac{(\varphi_{t+s}^*\omega)_p - (\varphi_t^*\omega)_p}{s} = \frac{d}{ds}\big((\varphi_{t+s}^*\omega)_p\big)\Big|_{s=0} = \frac{d}{ds}\big((\varphi_s^*\omega)_p\big)\Big|_{s=t} = 0$$

for all $t \in \mathbb{R}$ and $p \in M$. This is equivalent to the fact $\varphi_t^*\omega = \omega$, which completes the proof.

□