Quantitative Methods in Economics
Inference: Testing and confidence sets

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Roadmap, Part II

1. Asymptotics of least squares
2. **Inference: Testing and confidence sets**

Takeaways for these slides

- Testing: two types of error
- Neyman Pearson Lemma: How to minimize both
- Testing for normal variables
- Confidence sets
- Testing and confidence sets are closely related
- Using asymptotic normality of least squares to get confidence sets

Testing and the Neyman Pearson lemma

- testing as a decision problem
- goal: decide whether $H_0 : \theta \in \Theta_0$ is true
- decision $a \in \{0, 1\}$ (true / not true)
- statistical test is a decision function $\varphi : X \Rightarrow \{0, 1\}$
- $\varphi = 1$ corresponds to rejecting the null hypothesis
- more generally: randomized tests $\varphi : X \Rightarrow [0, 1]$
- reject H_0 with probability $\varphi(X)$
  (for technical reasons only, as we will see)

Two types of classification error

	truth	
decision a	$\theta \in \Theta_0$	$\theta \notin \Theta_0$
0	☺	Type II error
1	Type I error	☺

The power function

- suppose $X \sim f_\theta(x)$
- f: probability mass function or probability density function
- probability of rejecting H_0 given θ:
  power function

$$\beta(\theta) = E_\theta[\varphi(X)] = \int \varphi(x) f_\theta(x) dx.$$

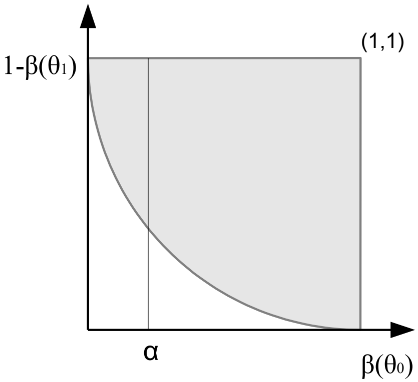
- For discrete distributions:

$$\beta(\theta) = \sum_x \varphi(x) f_\theta(x).$$

Classification errors

- suppose that θ has only two points of support, θ_0 and θ_1
- then
  1. P(Type I error) $= \beta(\theta_0)$.
  2. P(Type II error) $= 1 - \beta(\theta_1)$.
- $\beta(\theta_0)$ is called "level" or "**significance**" of the test, often denoted α.
- $\beta(\theta_1)$ is called the "**power**" of a test, and is often denoted β.
- would like to have a small α and a large β

Figure: testing as a decision problem



Suppose we want φ^* that solves

$$\max_{\varphi} \beta(\theta_1) \quad \text{s.t. } \beta(\theta_0) = \alpha$$

for a prespecified level α.

Lemma (Neyman-Pearson)

The solution to this problem is given by

$$\varphi^*(x) = \begin{cases} 1 \text{ for } f_1(x) > \lambda f_0(x) \\ \kappa \text{ for } f_1(x) = \lambda f_0(x) \\ 0 \text{ for } f_1(x) < \lambda f_0(x) \end{cases}$$

where λ and κ are chosen such that $\int \varphi^*(x) f_0(x) dx = \alpha$.

Practice problem

Try to prove this!

Hint:
our problem is to solve

$$\max_{\varphi} \int \varphi(x) f_1(x) dx$$

subject to

$$\int \varphi(x) f_0(x) dx = \alpha$$

and

$$\varphi(x) \in [0,1].$$

Recall the proposed solution,

$$\varphi^*(x) = \begin{cases} 1 \text{ for } f_1(x) > \lambda f_0(x) \\ \kappa \text{ for } f_1(x) = \lambda f_0(x) \\ 0 \text{ for } f_1(x) < \lambda f_0(x) \end{cases}$$

Proof:

- let $\varphi(x)$ be any other test of level α i.e. $\int \varphi(x) f_0(x) dx = \alpha$.
- need to show that $\int \varphi^*(x) f_1(x) dx \geq \int \varphi(x) f_1(x) dx$.
- Note that

$$\int (\varphi^*(x) - \varphi(x))(f_1(x) - \lambda f_0(x)) dx \geq 0$$

since $\varphi^*(x) = 1 \geq \varphi(x)$ for all x such that $f_1(x) - \lambda f_0(x) > 0$ and $\varphi^*(x) = 0 \leq \varphi(x)$ for all x such that $f_1(x) - \lambda f_0(x) < 0$.

- Therefore, using $\alpha = \int \varphi(x) f_0(x) dx = \int \varphi^*(x) f_0(x) dx$,

$$\begin{aligned} & \int (\varphi^*(x) - \varphi(x))(f_1(x) - \lambda f_0(x)) dx \\ = & \int (\varphi^*(x) - \varphi(x)) f_1(x) dx \\ = & \int \varphi^*(x) f_1(x) dx - \int \varphi(x) f_1(x) dx \geq 0 \end{aligned}$$

  as required.
- proof in the discrete case: identical with all integrals replaced by summations.

Practice problem

- you observe $X \sim N(\mu, 1)$
- you know that either $\mu = 0$ or $\mu = 1$
- construct the test of largest power for $H_0 : \mu = 0$ and any level α

Composite alternatives

- This approach immediately extends to $H_0 : \mu \leq \mu_0$ against the one-sided alternative $H_1 : \mu > \mu_0$.
- For the two-side problem $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$, no most powerful test exists.
- Most common approach:

$$\varphi(X) = \mathbf{1}(|X - \mu_0| > z).$$

Practice problem

What value of z ensures

$$E_{\mu_0}[\varphi(X)] = \alpha?$$

Confidence sets

- **Confidence set** C:
  a set of θs,
  which is calculated as a function of data Y
- Confidence set C for θ **of level** α:

$$P(\theta \in C) \geq 1 - \alpha. \tag{1}$$

  for all distributions of Y (i.e., all θ).
- In this expression θ is fixed and C is random.
- Confidence set C_n for θ of **asymptotic level** α:

$$\lim_{n \to \infty} P(\theta \in C_n) \geq 1 - \alpha. \tag{2}$$

Testing and confidence sets

- Constructing a confidence set from a family of tests.
- Let $\varphi_{\theta_0}(X)$ be a (non-randomzied) test for the null $H0: \ \theta = \theta_0$ such that

$$E_{\theta_0}[\varphi_{\theta_0}(X)] \leq \alpha$$

  for all θ_0.
- Define the random set

$$C = \{\theta_0 : \ \varphi_{\theta_0}(X) = 0\}.$$

- Then

$$P(\theta \in C) \geq 1 - \alpha.$$

Practice problem

Prove this.

- Constructing a test from a confidence set.
- Let C be a confidence set of level α.
- Define
$$\varphi_{\theta_0}(X) = \mathbf{1}(\theta_0 \notin C).$$
- Then $\varphi_{\theta_0}(X)$ is a level α test for the null $H0: \ \theta = \theta_0$.

Practice problem

Prove this.

- Advantage of confidence sets: Make statement about many values θ at once,
- rather than focusing on one arbitrary θ, such as $\theta = 0$.

Practice problem

- Suppose $X \sim N(\mu, \sigma^2)$ with σ^2 known.
- Consider the two-sided test we constructed before, and use it to construct a corresponding confidence set.

Confidence sets for least squares

- Recall our asymptotic result:

$$\sqrt{n}(b-\beta) \xrightarrow{p} N(0, V)$$

where

$$V = [E(X_1 X_1')]^{-1} \operatorname{Var}(X_1 U_1)[E(X_1 X_1')]^{-1}.$$

- Let $\sigma^2 = V_{11}$.

Practice problem

Show that

$$C = [b_1 - 1.96\sigma, b_1 + 1.96\sigma]$$

is an asymptotic 95% confidence set for β_1.