

Quantitative Methods in Economics

Residual regression

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Roadmap, Part I

1. Linear predictors and least squares regression
2. Conditional expectations
3. Some functional forms for linear regression
4. **Regression with controls and residual regression**
5. Panel data and generalized least squares

Takeaways for these slides

- ▶ Regress outcome on K variables, we are only interested in X_K
- ▶ Get the same result by first regressing X_K on the other regressors, then the outcome on the residual.
- ▶ Omitted variables: How dropping one regressor changes the other coefficients

Residual regression

- ▶ Consider regression with K regressors:

$$E^*(Y | 1, X_1, \dots, X_K) = \beta_0 + \beta_1 X_1 + \dots + \beta_K X_K$$

- ▶ Regress X_K on the other X s:

$$E^*(X_K | 1, X_1, \dots, X_{K-1}) = \gamma_0 + \gamma_1 X_1 + \dots + \gamma_{K-1} X_{K-1}$$

- ▶ Define residual regressor:

$$\tilde{X}_K = X_K - E^*(X_K | 1, X_1, \dots, X_{K-1})$$

- ▶ Claim:

$$\beta_K = E(Y \tilde{X}_K) / E(\tilde{X}_K^2).$$

Questions for you

- ▶ Interpret this last equation.
- ▶ Prove it.

Solution:

- ▶ Rewrite X_K in terms of linear predictor and residual:

$$\begin{aligned} E^*(Y|1, X_1, \dots, X_K) &= \beta_0 + \beta_1 X_1 + \dots + \beta_{K-1} X_{K-1} \\ &\quad + \beta_K (\gamma_0 + \gamma_1 X_1 + \dots + \gamma_{K-1} X_{K-1} + \tilde{X}_K) \\ &= \tilde{\beta}_0 + \tilde{\beta}_1 X_1 + \dots + \tilde{\beta}_{K-1} X_{K-1} + \beta_K \tilde{X}_K, \end{aligned}$$

with

$$\tilde{\beta}_j = \beta_j + \beta_K \gamma_j \quad (j = 0, 1, \dots, K-1).$$

- ▶ Use orthogonality of \tilde{X}_K and the other X s:

$$\begin{aligned} 0 &= \langle Y - \tilde{\beta}_0 - \tilde{\beta}_1 X_1 - \dots - \tilde{\beta}_{K-1} X_{K-1} - \beta_K \tilde{X}_K, \tilde{X}_K \rangle \\ &= \langle Y, \tilde{X}_K \rangle - \beta_K \langle \tilde{X}_K, \tilde{X}_K \rangle. \end{aligned}$$

- ▶ Thus:

$$\beta_K = \langle Y, \tilde{X}_K \rangle / \langle \tilde{X}_K, \tilde{X}_K \rangle = E(Y \tilde{X}_K) / E(\tilde{X}_K^2).$$

Sample Counterpart

- ▶ Data

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad x_j = \begin{pmatrix} x_{1j} \\ \vdots \\ x_{nj} \end{pmatrix} \quad (j = 0, 1, \dots, K).$$

- ▶ Linear predictor and residual regressor:

$$\hat{y}_i | 1, x_1, \dots, x_K = b_0 + b_1 x_{i1} + \dots + b_K x_{iK}.$$

$$\tilde{x}_{iK} = x_{iK} - (\hat{x}_{iK} | 1, x_1, \dots, x_{K-1}).$$

- ▶ Claim (as before):

$$b_K = \frac{1}{n} \sum_{i=1}^n y_i \tilde{x}_{iK} \Bigg/ \frac{1}{n} \sum_{i=1}^n \tilde{x}_{iK}^2$$

Omitted variables

- ▶ Long regression:

$$E^*(Y | 1, X_1, \dots, X_K) = \beta_0 + \beta_1 X_1 + \dots + \beta_K X_K$$

- ▶ Short regression:

$$E^*(Y | 1, X_1, \dots, X_{K-1}) = \alpha_0 + \alpha_1 X_1 + \dots + \alpha_{K-1} X_{K-1}$$

- ▶ Auxiliary regression:

$$E^*(X_K | 1, X_1, \dots, X_{K-1}) = \gamma_0 + \gamma_1 X_1 + \dots + \gamma_{K-1} X_{K-1}$$

- ▶ Claim:

$$\alpha_j = \beta_j + \beta_K \gamma_j \quad (j = 0, 1, \dots, K-1)$$

Questions for you

- ▶ Interpret this last equation.
- ▶ Prove it.

Sample Counterpart

- ▶ Long regression:

$$\hat{y}_i | 1, x_1, \dots, x_K = b_0 + b_1 x_{i1} + \dots + b_K x_{iK} \quad (\text{Long})$$

- ▶ Short regression:

$$\hat{y}_i | 1, x_1, \dots, x_{K-1} = a_0 + a_1 x_{i1} + \dots + a_{K-1} x_{i,K-1} \quad (\text{Short})$$

- ▶ Auxiliary regression:

$$\hat{x}_{iK} | 1, x_1, \dots, x_{K-1} = c_0 + c_1 x_{i1} + \dots + c_{K-1} x_{i,K-1} \quad (\text{Aux})$$

- ▶ Claim:

$$a_j = b_j + b_K c_j \quad (j = 0, 1, \dots, K-1)$$

Matrix version

- ▶ Linear predictor:

$$X = \begin{pmatrix} X_0 \\ X_1 \\ \vdots \\ X_K \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_K \end{pmatrix}, \quad E^*(Y|X) = \sum_{j=0}^K X_j \beta_j = X' \beta.$$

- ▶ Orthogonality conditions:

$$E[X_j(Y - X'\beta)] = 0 \quad (j = 0, 1, \dots, K),$$

$$E[X(Y - X'\beta)] = 0,$$

$$E(XY) - E(XX')\beta = 0$$

- ▶ Solution:

$$\beta = [E(XX')]^{-1} E(XY).$$

Sample Counterpart

- ▶ Data:

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad x_j = \begin{pmatrix} x_{1j} \\ \vdots \\ x_{nj} \end{pmatrix} \quad (j = 0, 1, \dots, K), \quad b = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_K \end{pmatrix},$$

$$x = (x_0 \quad x_1 \quad \dots \quad x_K) = \begin{pmatrix} x_{10} & x_{11} & \dots & x_{1K} \\ \vdots & \vdots & & \vdots \\ x_{n0} & x_{n1} & \dots & x_{nK} \end{pmatrix}$$

- ▶ Linear predictor:

$$\hat{y} = x_0 b_0 + x_1 b_1 + \dots + x_K b_K = xb.$$

- ▶ Orthogonality conditions:

$$\sum_{i=1}^n x_{ij}(y_i - \hat{y}_i) = x'_j(y - xb) = 0 \quad (j = 0, 1, \dots, K),$$

$$\begin{pmatrix} x'_0 \\ x'_1 \\ \vdots \\ x'_K \end{pmatrix} (y - xb) = x'(y - xb) = 0,$$

$$x'y - x'xb = 0,$$

- ▶ Solution:

$$b = (x'x)^{-1}x'y.$$