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Exchange-rate dynamics under stochastic regime shifts

A unified approach

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Simple techniques of regulated Brownian motion are used to analyze the behavior of the exchange rate when official policy reaction functions are subject to future stochastic changes. We examine exchange-rate dynamics in cases where the authorities promise (i) to confine a floating rate within a predetermined range, (ii) to peg the currency once it reaches a predetermined future level, and (iii) to unify a system of dual exchange rates. Similarities among these and several related examples of regime switching are stressed. We also discuss how stochastic regime changes can affect some standard statistical tests of hypotheses about exchange rates.

1. Introduction

The typical forward-looking variable in an economist's model is driven by a forcing process the form of which is fixed for all time. Yet, in the real world there are many examples in which the forcing process is subject to change once a certain event occurs. When variables such as interest rates, current accounts, inflation, or exchange rates reach certain values, authorities may not only change their policies – they also may change their policy reaction *functions*.

A number of models examine the behavior of forward-looking variables when an otherwise passive policy-maker intervenes to keep the variables from moving out of a predetermined range. In this spirit, Bentolila and

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Bertola (1990) study hiring and firing costs in the labor market; Krugman (1991), followed by Miller and Weller (1988), studies exchange-rate target zones; and Dixit (1989a), Dumas (1989a), Krugman (1988), and McDonald and Siegel (1986) study the allocation of capital. In all of these models an authority takes actions that keep a relevant forward-looking variable within a desired range or band. These models also share a simple and intuitive analytical approach, namely, the technique of regulated Brownian motion.

A related literature studies the effects on forward-looking variables of once-and-for-all changes in regime. Flood and Garber (1983) analyze a case inspired by Britain's 1925 return to the gold standard, in which the exchange rate floats freely until it reaches a preannounced value, and is then permanently pegged by the authorities. Because rational investors anticipate the transition to the peg, the dynamics of the exchange rate can differ from those under a permanent float. Another example of a permanent regime change, one of perennial importance in countries with exchange controls, is the unification of a system of dual exchange rates.

In this paper we apply techniques of regulated Brownian motion to clarify the relationship between the newer target-zone results such as Krugman's (1991), the process-switching model of Flood and Garber (1983), and models of exchange-rate unification. We derive intuitive closed-form solutions for these (and related) problems. An advantage of having such solutions in hand is that they allow a precise description and calibration of statistical estimation problems caused by possible future regime shifts. We give an example of this type of empirical application below.

The paper is organized as follows. In section 2 we lay out a general framework for analyzing exchange-rate models with regime shifts. Section 3 contains the solutions to several specific examples of importance in the literature. Section 4 discusses some implications of stochastic regime shifts for econometric work on asset pricing. Section 5 concludes.[1]

2. The model

The analysis works with a two-country monetary model of the exchange rate, in which the (log) spot exchange rate at time $t, x(t)$, is defined as the price of foreign currency in terms of domestic currency. In equilibrium $x(t)$ is the sum of a scalar index of macroeconomic fundamentals, $k(t)$, plus a speculative term proportional to the expected percentage change in the exchange rate:

$$x(t)=k(t)+\alpha\,\mathrm{E}(\mathrm{d}x(t)\,|\,\phi(t))/\mathrm{d}t, \tag{1}$$

[1] Many closely related papers appeared after the research reported herein was completed. Rather than attempting to cite them all, we direct readers to the references contained in Krugman and Miller (1991).

where the parameter α can be interpreted as the semi-elasticity of money demand with respect to the nominal interest rate; $\mathrm{E}(\cdot|\cdot)$ is the conditional expectations operator; and $\phi(t)$ is the time-t information set, which includes the current value of fundamentals, $k(t)$, as well as any explicit or implicit restrictions the authorities have placed on the future evolution of fundamentals. For example, the authorities may have announced that they will keep the exchange rate from moving outside certain limits, or that they will fix the exchange rate once it reaches a certain level. Such information about future policies would be incorporated into $\phi(t)$. Included among the fundamental factors that raise $k(t)$ is a variable measuring the domestic money supply relative to the foreign money supply; that variable is assumed to be controlled directly by the national monetary authorities. Included as well are other, exogenous determinants of exchange rates that the authorities cannot influence.[2]

Exchange-market intervention by monetary authorities may change the stochastic process governing (relative) money-supply growth.[3] This in turn will alter the process driving the fundamentals, $k(t)$. We assume that under a free float the authorities refrain from intervening to offset shocks to fundamentals. The evolution of fundamentals under a free float is described by the process:

$$\mathrm{d}k(t)=\eta\,\mathrm{d}t+\sigma\,\mathrm{d}z(t), \tag{2}$$

where η is the (constant) predictable change in k, $\mathrm{d}z$ is a standard Wiener process, and σ^2 is the (constant) variance per unit of time in the growth of k. All of our results, however, can be derived using more complex forcing processes.[4] Because the authorities can control k through intervention, k need not follow (2) under regimes other than a free float.

If speculative bubbles are ruled out, the rational-expectations assumption leads to a unique equilibrium exchange-rate path that satisfies (1). This path has the integral representation:

[2]On the monetary model of exchange rates see, for example, Frenkel and Johnson (1978) or Flood and Garber (1983). We follow Flood and Garber (1983) in adopting a continuous-time stochastic model, which allows a neater characterization of solutions than comparable discrete-time formulations. No essentials of the results, however, depend on the continuous-time assumption. By restricting $k(t)$ to be a function of money supplies and exogenous variables, we exclude sticky-price models such as the one analyzed numerically by Miller and Weller (1988). There, price levels, which are functions of past exchange rates, are among the fundamentals, and price stickiness induces mean reversion in fundamentals.

[3]It is clear from the model that it does not matter whether it is the home or foreign government that manages the exchange rate (or a committee representing both governments).

[4]Froot and Obstfeld (1991) show how the exogenous mean-reverting process,

$$\mathrm{d}k(t)=(\eta-\theta k(t))\,\mathrm{d}t+\sigma\,\mathrm{d}z,$$

can be handled in examples like those discussed below.

$$x(t)=\alpha^{-1}\int_t^{\infty} e^{(t-s)/\alpha}\,E(k(s)\,|\,\phi(t))\,ds, \tag{3}$$

a representation valid under *any* policy regime or sequence of policy regimes. In words, (3) equates the current exchange rate to the present discounted value of expected future fundamentals (the discount rate is $1/\alpha$). Below, the equilibrium exchange rate defined by the present-value formula (3) is called the *saddlepath* exchange rate.

We assume that the saddlepath solution for the exchange rate can be written as a twice continuously differentiable function of a single variable, the current fundamental:

$$x(t)=S(k(t)). \tag{4}$$

The assumption that the saddlepath exchange rate is expressible as a function of k alone is reasonable given (2) and the types of regime shift we will consider. Continuity of the function $S(k)$ is also plausible, since it is necessary to rule out excess profit opportunities. The assumed additional smoothness of $S(k)$ does not reduce significantly the generality of the analysis, and allows crucial simplification of the developments that follow.

The precise form of the function $S(k)$ depends, of course, on which regime shifts (if any) the market believes to be possible. A familiar special case is the one in which the authorities are committed to a *permanent* exchange-rate float, so that fundamentals are expected to follow process (2) forever. In this case, the conditional expectations in (3) are easy to evaluate, since they depend exclusively on current fundamentals, and not on possible future regime shifts. The saddlepath exchange rate, $S(k(t))$, for a free float is

$$\begin{aligned} x(t)&=\alpha^{-1}\int_t^{\infty} e^{(t-s)/\alpha}\,E(k(s)\,|\,k(t))\,ds \\ &=\alpha^{-1}\int_t^{\infty} e^{(t-s)/\alpha}(k(t)+(s-t)\eta)\,ds=k(t)+\alpha\eta. \end{aligned} \tag{5}$$

If, however, there is a chance that the authorities will depart from a free float in the future, fundamentals may not always follow (2), and the equilibrium exchange rate may not follow (5), even while (2) remains in effect. In such cases direct computation of the sequence of conditional expectations in the present-value formula (3) that defines $S(k)$ is likely to be burdensome. We therefore follow an alternative, two-step approach to determine $S(k)$ when a regime switch from (2) to some other process is possible. First, we characterize the family of functions of the form $x=G(k)$ that satisfy the differential equation (1) so long as fundamentals evolve according to (2). Second, we find the member of this family that satisfies boundary conditions appropriate to the stochastic regime switch under consideration. As we argue in detail below, this last function is the saddlepath solution, $S(k)$.

Step one of the procedure outlined above – finding the general solution $x=G(k)$ – uses Itô's lemma and eq. (1) to express expected depreciation while (2) holds as[5]

$$\mathrm{E}(\mathrm{d}x|\phi)/\mathrm{d}t=\mathrm{E}(\mathrm{d}G(k)|\phi)/\mathrm{d}t=\eta G'(k)+\frac{\sigma^2}{2}G''(k), \tag{6}$$

where we have assumed $G(k)$ is twice continuously differentiable. Combining (1) and (6) yields a second-order differential equation that the exchange rate in (1) and (3) must satisfy:

$$G(k)=k+\alpha\eta G'(k)+\frac{\alpha\sigma^2}{2}G''(k). \tag{7}$$

The general solution to (7) is

$$G(k)=k+\alpha\eta+A_1\,\mathrm{e}^{\lambda_1 k}+A_2\,\mathrm{e}^{\lambda_2 k}, \tag{8}$$

where $\lambda_1>0$ and $\lambda_2<0$ are the roots to the quadratic equation in λ,

$$\lambda^2\alpha\sigma^2/2+\lambda\alpha\eta-1=0,$$

and A_1 and A_2 are constants of integration. Eq. (8) forms the basis of our analysis below: as just discussed, a single member of the family defined by (8)

[5]Where it does not create confusion, we drop the time-dependence notation. It is worth noting that while we refer to $G(k)$ as a 'general' solution, it is general only if attention is restricted to solutions that depend on current fundamentals alone. In fact, (1) has even more general solutions, for example solutions that are functions not only of current fundamentals, but also of variables extraneous to the model. Such solutions are not considered here, but their exclusion is not restrictive given the economic problems we are considering.

will turn out to be equivalent to the present-value formula for x in (3). This is just the function $S(k)$.[6]

There are two parts to the general solution (8), one linear, the other nonlinear, in k. The linear part, $k+\alpha\eta$, would be the standard linear saddlepath solution if no change in the fundamentals process (2) was possible, so that a free float was permanently in effect [see eq. (5)].

Although the nonlinear terms in (8) would represent deviations from the saddlepath under a permanent free float, we do not want to throw them away in solving for the saddlepath exchange rate under a free float that could terminate.[7] When there is some possibility of regime change, fundamentals may *not* remain permanently a random walk with trend, and the present-value formula (3) therefore need *not* equal the simple linear expression (5). Under a possible regime change, the saddlepath value of the exchange rate prior to the switch will generally depend on the nonlinear terms in (8). Just which initial conditions A_1 and A_2 are appropriate depends on the boundary conditions associated with the regime switch, conditions to be determined in step two of the two-step solution procedure outlined above.

Before proceeding to this second step in the next section, however, it is useful to inspect graphically the paths given by (8). Fig. 1 shows these paths in the symmetric case $A_1=-A_2$, for $\eta>0$. The line FF is the linear solution (5), which corresponds to the case $A_1=A_2=0$. FF, once again, is the saddlepath under a permanent free-float regime. All the nonlinear paths in the figure are nonsaddlepath solutions to (1) under a permanent free float. These alternative solutions are supported by their different curvatures, which translate the expected growth rate of fundamentals into different expected rates of exchange-rate change. (The effect of nonlinearity on expectations reflects Jensen's inequality). The apparent asymmetry in the paths is due to the positive trend in the growth of fundamentals, η.

3. Examples

This section carries out the second step of the solution method outlined in

[6]Note that all of our subsequent analysis extends immediately to a class of fundamentals processes somewhat broader than (2). Let $f(k)$ be any continuous function of fundamentals and suppose (1) is replaced by

$$x(t)=f(k(t))+\alpha\,\mathrm{E}(\mathrm{d}x(t)\,|\,\phi(t))/\mathrm{d}t, \tag{1'}$$

so that the time-t fundamental is $f(k(t))$ rather than $k(t)$ itself. The general solution to (1') is

$$G(k(t))=\int_t^\infty \mathrm{e}^{(t-s)/\alpha}\,\mathrm{E}(f(k(s))\,|\,k(t))\,\mathrm{d}s+A_1\,\mathrm{e}^{\lambda_1 k(t)}+A_2\,\mathrm{e}^{\lambda_2 k(t)}, \tag{8'}$$

where the integral is just the free-float saddlepath. The results developed below can be applied directly to show how various regime shifts induce departures from free-float exchange rates. Of course, the integral in (8') usually will be difficult to evaluate.

[7]Froot and Obstfeld (1990) show why nonlinear terms such as those in (8) would describe a rational speculative bubble if policy regime (2) were permanent.

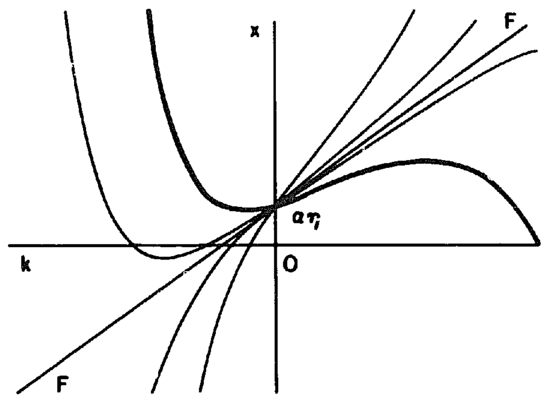

Fig. 1

section 2. The discussion takes up sequentially the boundary conditions implied by several possible regime-switching scenarios. In terms of the mathematics, all that is involved is the appropriate choice of the two arbitrary constants in (8), A_1 and A_2. A single unifying principle leads to solutions for all of the problems considered.

Before solving the model under alternative policy scenarios, we note that policy intentions regarding exchange rates can be conveyed to the market in several ways, not all of which lead to a determinate equilibrium exchange rate, or to a unique rule for managing the fundamentals. For example, the announcement, 'We will let the exchange rate float freely until it reaches $\bar{x}$, and then peg it', leaves the market with too little information about future fundamentals to set a unique rate prior to pegging: various equilibria can be supported by suitable accommodating policies. Because announcing exchange-rate objectives without specifying the accompanying policies is not generally enough to determine a unique equilibrium, our examples always make explicit the policies on *fundamentals* through which the authorities manage exchange rates.

3.1. Exchange-rate target zones

Suppose the authorities want to keep the exchange rate from penetrating the lower and upper levels, $\underline{x}$ and $\bar{x}$. When the exchange rate reaches one of these boundaries, the authorities alter fundamentals so as to keep x from moving outside of its range. However, they do not prevent a movement of x back into the interior of the range. Exchange-rate behavior within such a target zone was first studied by Krugman (1991).

How do the authorities defend the target zone? One way to think of this process is to imagine that they place lower and upper limits, $\underline{k}$ and $\bar{k}$, on the *fundamentals*. If the fundamentals are prevented from moving outside the range $[\underline{k}, \bar{k}]$, and if (as will turn out to be true below) $S(k)$ is monotonically increasing in k, the exchange rate will be confined between the lower and upper values $S(\underline{k})$ and $S(\bar{k})$. [As usual, $S(k)$ is the saddlepath value of the exchange rate within the target zone.] We will show below that by choosing $[\underline{k}, \bar{k}]$ appropriately, the desired target exchange-rate zone, $\underline{x}=S(\underline{k})$, $\bar{x}=S(\bar{k})$, can be enforced. Since the exchange rate is free to move back within the zone after it has touched one of its edges, the bounds $[\underline{k}, \bar{k}]$ are essentially *reflecting* barriers on the fundamentals process.

More precisely, one can think of the authorities as defending the target zone through *infinitesimal* interventions that alter k only when process (2) has brought k to $\underline{k}$ or to $\bar{k}$. In other words, intervention at the margin is just sufficient to prevent fundamentals from falling below the lower bound $\underline{k}$ or rising above the upper bound $\bar{k}$; but no intervention occurs when k is strictly within those limits. We emphasize that the infinitesimal character of the marginal interventions implies that k can never take a discrete jump.[8]

The foregoing intervention policy can be formalized as follows (see the appendix for details). Define $K(t)$ to be the *unregulated* fundamentals at time t, that is, the value of fundamentals $k(t)$ that would prevail under the counterfactual assumption that intervention never occurs. By (2), $K(t)$ follows:

$$\mathrm{d}K(t)=\eta\,\mathrm{d}t+\sigma\,\mathrm{d}z(t). \tag{9}$$

Let $L(t)$ be the integral of all intervention purchases of foreign exchange up to t at the lower bound $\underline{k}$ (these are infinitesimal increases in k); and define $U(t)$ similarly with respect to $\bar{k}$ [so that $U(t)$ is an integral of infinitesimal decreases in k, which are intervention sales]. Then the fundamentals variable that enters (1) is given by $k(t)=K(t)+L(t)-U(t)$, and by (9), this variable follows the process

$$\mathrm{d}k(t)=\eta\,\mathrm{d}t+\sigma\,\mathrm{d}z(t)+\mathrm{d}L(t)-\mathrm{d}U(t), \tag{10}$$

where $\mathrm{d}L(t)$ and $\mathrm{d}U(t)$ are the smallest interventions that confine $k(t)$ to

[8]Flood and Garber (1989) have examined an alternative formulation of target zones in which a noninfinitesimal jump in k does occur at the margins. In that setting, the saddlepath relation $x=S(k)$ turns out to be nonmonotonic over $[\underline{k}, \bar{k}]$. Specifically, x is below its maximal value for k near $\bar{k}$, and above its minimal value for k near $\underline{k}$, a reflection of the discrete changes in k expected to occur at these boundaries. Although Flood and Garber do not emphasize this aspect of their results, their examples illustrate nicely the point made at the start of this section, namely that the same target zone for the exchange rate can be supported by many different specifications of policy. In general, one cannot pin down the equilibrium relation between fundamentals and the exchange rate merely by assuming limits on the range of possible exchange-rate values allowed by the authorities.

$[\underline{k}, \bar{k}]$. These interventions are zero except at $\underline{k}$ and $\bar{k}$, so $\mathrm{d}k(t) = \mathrm{d}K(t)$ for $k(t) \in (\underline{k}, \bar{k})$.

It is assumed from now on that the chosen target zone, along with the intervention policies used to defend it, are fully and permanently credible. Such credibility is feasible if the central banks that maintain the zone are willing to adjust domestic credit appropriately.

To determine exchange-rate behavior within a credible target zone, we solve for the exchange-rate path that satisfies (1) given that k evolves according to (10). The solution is a special case of (3):

$$x(t) = S(k(t)) = \alpha^{-1} \int_t^{\infty} \mathrm{e}^{(t-s)/\alpha} \, \mathrm{E}(k(s) \,|\, k(t), k(s) \in [\underline{k}_r, \bar{k}_r]) \, \mathrm{d}s, \tag{11}$$

where the 'r' subscript indicates that the barriers on fundamentals are reflecting. As noted in section 2, direct evaluation of the conditional expectation in (11) is much more difficult than in the case of a permanent free float [eq. (5)]: under (10), the saddlepath exchange rate, $S(k)$, will no longer be a purely linear function of k.

We have already taken the first step in finding $S(k)$ by deriving the general nonlinear solution $x = G(k)$ given by (8). Some member of this family of solutions must characterize exchange-rate behavior when k is *in the interior of* $[\underline{k}, \bar{k}]$, where (1) and (2) simultaneously hold. But eq. (8) remains relevant *at the boundary* of this interval as well, that is, at the barriers $k = \underline{k}$ and $k = \bar{k}$. The reason is the continuity of $S(k)$, a property that precludes excess anticipated profit opportunities at $\underline{k}$ or $\bar{k}$. Because $S(k)$ is continuous on the entire interval $[\underline{k}, \bar{k}]$, it cannot coincide with a function of the form $G(k)$ on the interior of that interval unless it coincides with the same function at the edges.

All that remains, then, is to determine the boundary conditions on $G(k)$ implied by the reflecting barriers. These conditions deliver unique values for the undetermined coefficients A_1 and A_2 in (8), and therefore tie down uniquely the member of the class $G(k)$ that coincides with $S(k)$ when k lies between the reflecting barriers.

The appropriate boundary conditions on $G(k)$ are the value-matching conditions suggested by Krugman (1991):

$$G'(\underline{k}) = 0 \tag{12}$$

and

$$G'(\bar{k}) = 0. \tag{13}$$

A formal proof that these conditions are necessary is given in the appendix. However, the following three-step argument clarifies the intuition behind the proof.[9]

(i) Consider, for example, condition (12). Let $G(k)$ be a 'candidate' saddlepath solution; because the saddlepath $S(k)$ satisfies (7) everywhere on $[\underline{k}, \bar{k}]$, and in particular at $k=\underline{k}$, (1) and (6) imply that

$$\mathrm{E}(\mathrm{d}G(\underline{k})\,|\,\phi)=\alpha^{-1}(G(\underline{k})-\underline{k})\,\mathrm{d}t=\left(\eta G'(\underline{k})+\frac{\sigma^2}{2}G''(\underline{k})\right)\mathrm{d}t$$

if $G(k)=S(k)$. In words, the point $(\underline{k}, G(\underline{k}))$ lies on a (nonsaddlepath) solution to (1) for the free-float case, in which the fundamentals follow (2).

(ii) But $(\underline{k}, G(\underline{k}))$ is also an equilibrium point under the target zone, so (1) holds there when $x=G(k)$ on $[\underline{k}, \bar{k}]$ even though the fundamentals are generated in this case by a process different from (2). At $k=\underline{k}$, investors now have a one-sided bet on fundamentals: they know that because k follows (10), k can only rise from $\underline{k}$, and not fall. Under this condition, however, Itô's lemma, combined with (1), implies that

$$\begin{aligned}\mathrm{E}(\mathrm{d}G(\underline{k})\,|\,\phi)&=\alpha^{-1}(G(\underline{k})-\underline{k})\,\mathrm{d}t\\&=\left(\eta G'(\underline{k})+\frac{\sigma^2}{2}G''(\underline{k})\right)\mathrm{d}t+G'(\underline{k})\,\mathrm{d}L,\end{aligned}$$

where we have used (10) combined with the fact that $\mathrm{d}U=0$ at $k=\underline{k}$. (See the appendix for the derivation.) In words, there is a positive expected intervention purchase $\mathrm{d}L$ when $k=\underline{k}$, and this raises the expected change in $G(k)$ in proportion to $G'(\underline{k})$.

(iii) The equations displayed in the previous two paragraphs, however, are mutually contradictory unless $G'(\underline{k})=0$. Thus, the point $(\underline{k}, G(k))$ can lie on a solution to (1) valid under both a free-float and a target zone only if (12) holds. A similar argument completes the proof by establishing that the general solution $G(k)$ corresponding to $S(k)$ satisfies (13).

Using (8), we can write the conditions (12) and (13) as

$$1+A_1\lambda_1\,\mathrm{e}^{\lambda_1 \underline{k}}+A_2\lambda_2\,\mathrm{e}^{\lambda_2 \underline{k}}=0 \tag{14}$$

[9]Conditions (12) and (13) are sometimes called 'smooth-pasting' conditions in the literature on target zones; but as Dumas (1989b) points out, the term is usually applied in the context of intertemporal maximization problems involving the costly regulation of state variables that follow Brownian motion. [See also Harrison (1985) and Dixit (1989b) on optimal regulation of Brownian motion.] Since (12) and (13) are entirely due to asset-price continuity (as the following proof demonstrates), we refer to them instead as value-matching conditions.

and

$$1+A_1\lambda_1\,\mathrm{e}^{\lambda_1\bar{k}}+A_2\lambda_2\,\mathrm{e}^{\lambda_2\bar{k}}=0. \tag{15}$$

Eqs. (14) and (15) yield the following solution for the saddlepath value of the exchange rate [formula (11)] under the target zone:

$$\begin{aligned} x&=S(k)\\ &=k+\alpha\eta+\left(\frac{\lambda_2\,\mathrm{e}^{\lambda_2\bar{k}+\lambda_1 k}-\lambda_2\,\mathrm{e}^{\lambda_2\underline{k}+\lambda_1 k}+\lambda_1\,\mathrm{e}^{\lambda_1\underline{k}+\lambda_2 k}-\lambda_1\,\mathrm{e}^{\lambda_1\bar{k}+\lambda_2 k}}{\lambda_1\lambda_2\,\mathrm{e}^{\lambda_2\underline{k}+\lambda_1\bar{k}}-\lambda_1\lambda_2\,\mathrm{e}^{\lambda_2\bar{k}+\lambda_1\underline{k}}}\right). \end{aligned} \tag{16}$$

If we let the lower barrier, $\underline{k}$, go to minus infinity, (16) simplifies to

$$x=k+\alpha\eta-\lambda_1^{-1}\,\mathrm{e}^{\lambda_1(k-\bar{k})}. \tag{17}$$

If in addition we let the upper barrier, $\bar{k}$, go to infinity, (17) becomes the linear saddlepath in (5):

$$x=k+\alpha\eta. \tag{18}$$

Only when both boundaries are infinitely distant is the exchange rate a linear function of fundamentals.

Notice that the saddlepath solution given in (16) is of the form initially hypothesized: it is a function of the current state k and the two barriers. It is also straightforward to verify that $S(k)$ is monotonically increasing over its domain of definition, as claimed earlier. Eq. (16) therefore implies that confining fundamentals to the zone $[\underline{k},\bar{k}]$, where the boundaries are reflecting barriers, restricts exchange rates to the target zone $[\underline{x},\bar{x}]=[S(\underline{k}),S(\bar{k})]$.

Eq. (16) generalizes the solution found by Krugman (1991) for the case $\eta=0$. Krugman, however, assumes that the authorities announce directly the exchange-rate band $[\underline{x},\bar{x}]$, and that they keep the exchange rate within these limits with infinitesimal interventions that occur only when x equals $\underline{x}$ or $\bar{x}$. His solution procedure and ours are equivalent because any reflecting exchange-rate zone determines a unique fundamentals zone, while any reflecting fundamentals zone determines a unique exchange-rate zone. In effect, Krugman assumes that the exchange-rate zone $[\underline{x},\bar{x}]$ determines a corresponding fundamentals zone $[\underline{k},\bar{k}]$; he then uses (14) and (15) (as we do) to solve for A_1 and A_2 in terms of $\underline{k}$ and $\bar{k}$, and then solves for $\underline{k}$ and $\bar{k}$ in terms of $\underline{x}$ and $\bar{x}$ using (16). Under a policy of infinitesimal interventions at the margins, any target zone for exchange rates is supported by confining fundamentals to a uniquely-determined range. The resulting equilibrium relationship between fundamentals and the exchange rate is uniquely determined by (12) and (13).

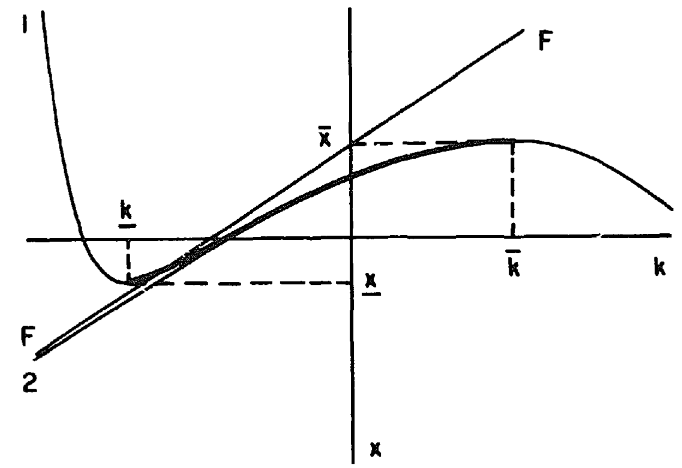

Fig. 2

A policy of intervening infinitesimally when the *exchange rate* reaches limiting values is easily translated into the language we used above to describe formally the regulation of fundamentals [see eq. (10)]. For this purpose it is easiest to assume that the authorities announce they will buy any quantity of foreign exchange the market desires to sell when $x=\underline{x}$, and sell any quantity the market desires to buy when $x=\bar{x}$. At any time t, the processes $L(t)$ and $U(t)$ are equal, respectively, to the cumulated intervention purchases of foreign currency at $\underline{x}$ and sales at $\bar{x}$.[10]

Fig. 2 illustrates two possible exchange-rate paths described by (16). The paths share a common upper barrier, $\bar{k}$, but differ with respect to the lower barrier. Path 1 in the figure, an 'S-shaped' curve, shows the behavior of the exchange rate when there are finite reflecting barriers at $\underline{k}$ and $\bar{k}$. This path has several noteworthy features. First, its shape reflects the influence of expected policy changes at the barriers. In the neighborhood of $\bar{k}$, for example, the exchange rate is below FF, the saddlepath under a hypothetical free float. This bending below FF near $\bar{k}$ reflects a lower expected increase in x compared with a situation without boundaries; and that expectation, in turn, reflects the local concavity of the saddlepath. Second, the equilibrium solution behaves much like the free-float path when the exchange rate is within the band but not close to either boundary. A wider band would leave the equilibrium solution closer to FF for a greater range of fundamentals. Path 2, for example, shows the case in which the lower boundary is infinitely distant [eq. (17)]. This graphical intuition is made precise in (18), which shows that the equilibrium solution converges to the saddlepath when both

[10]This equivalence shows, incidentally, that the authorities can operate a reflecting barrier on fundamentals without being able to observe fundamentals directly; it is sufficient to observe the exchange rate and maintain fixed buying and selling rates for foreign currency.

barriers are infinitely distant. For a narrow band, however, the free-float solution FF will usually not be a good approximation to the true equilibrium path.

3.2. Stochastic exchange-rate pegging

The target-zone system is a regime combining aspects of both floating and fixed exchange rates: the exchange rate is free to move within a band, but imperfectly flexible at the edges of the band. We now turn to a *permanent* change of regime, a stochastically-determined transition from freely floating to rigidly fixed rates. Flood and Garber (1983) originally studied this problem, but were unable to calculate a closed-form solution for the exchange rate. A solution is given below. To derive it, we initially set up the problem in a general manner that has no exact counterpart in reality, but that serves to clarify the solutions of more realistic special cases.[11]

Suppose now that authorities wish to let the exchange rate float until it reaches a lower or an upper level, $\underline{x}$ or $\bar{x}$, at which time they plan to fix x permanently. To keep the spot rate fixed at one of these levels, the authorities must hold the fundamentals constant at $\underline{k}=S^{-1}(\underline{x})$ or $\bar{k}=S^{-1}(\bar{x})$, respectively. This class of problems contains the one posed by Flood and Garber (1983), who consider the behavior of a floating exchange rate when the authorities plan to switch to a fixed-rate regime at a single, predetermined level of the exchange rate, $\bar{x}$.

In order to avoid potential multiple equilibria, we assume that the authorities inform investors that fundamentals will follow (2) until k reaches $\underline{k}$ or $\bar{k}$. At that time the authorities will fix k, thereby fixing the exchange rate at $\underline{x}=S(\underline{k})$ or $\bar{x}=S(\bar{k})$, respectively. Thus, k is not allowed to jump discontinuously at the moment the transition between regimes is made.[12]

Given the boundaries, $\underline{k}$ and $\bar{k}$, the saddlepath solution is

$$x(t)=S(k(t))=\alpha^{-1}\int_t^{\infty} e^{(t-s)/\alpha}\,\mathrm{E}(k(s)\,|\,k(t),k(s)\in[\underline{k}_a,\bar{k}_a])\,\mathrm{d}s, \tag{19}$$

where the 'a' subscript denotes that the barriers on fundamentals are now

[11]See Froot and Obstfeld (1991) for a more extensive discussion of stochastic exchange-rate pegging. Flood and Garber applied a first-stopping-time methodology to the regime-switching problem, but only recently has Smith (1991) produced a closed-form solution using that approach. The solution presented below naturally agrees with the one found by Smith. Smith and Smith (1990) present an interesting empirical analysis of the 1925 British return to the gold standard that illustrates the usefulness of closed-form solutions of the type derived by Smith (1991) and by us.

[12]For a detailed examination of multiple solutions involving discrete jumps in k, see Froot and Obstfeld (1991). The problem is briefly identified by Obstfeld and Stockman (1985, section 2.3).

absorbing barriers. As before, direct evaluation of (19) is cumbersome. The methods used above apply directly, however, and lead to a simple answer.

The first step once again is to examine the value of the exchange rate at the boundaries. Fortunately, the boundary values of the integral (19) are easy to evaluate. They are:

$$S(\underline{k})=\alpha^{-1}\int_t^\infty \mathrm{e}^{(t-s)/\alpha}\,\mathrm{E}(k(s)\,|\,k(t)=\underline{k}_{\mathrm{a}})\,\mathrm{d}s$$

$$=\alpha^{-1}\int_t^\infty \mathrm{e}^{(t-s)/\alpha}\underline{k}\,\mathrm{d}s=\underline{k}=\underline{x}, \tag{20}$$

$$S(\bar{k})=\alpha^{-1}\int_t^\infty \mathrm{e}^{(t-s)/\alpha}\,\mathrm{E}(k(s)\,|\,k(t)=\bar{k}_{\mathrm{a}})\,\mathrm{d}s$$

$$=\alpha^{-1}\int_t^\infty \mathrm{e}^{(t-s)/\alpha}\bar{k}\,\mathrm{d}s=\bar{k}=\bar{x}. \tag{21}$$

In words, once fundamentals are fixed permanently, the exchange rate is just the value of current fundamentals, either $\underline{k}$ or $\bar{k}$.

At the boundaries, (8) and either (20) or (21) must hold. Together they imply:

$$\alpha\eta+A_1\,\mathrm{e}^{\lambda_1\underline{k}}+A_2\,\mathrm{e}^{\lambda_2\underline{k}}=0, \tag{22}$$

$$\alpha\eta+A_1\,\mathrm{e}^{\lambda_1\bar{k}}+A_2\,\mathrm{e}^{\lambda_2\bar{k}}=0. \tag{23}$$

These two equations lead to a unique solution for the two constants in (8), and to the following saddlepath solutions for (19):

$$x=S(k)=k+\alpha\eta\left(1+\frac{\mathrm{e}^{\lambda_2\bar{k}+\lambda_1 k}-\mathrm{e}^{\lambda_2\underline{k}+\lambda_1 k}+\mathrm{e}^{\lambda_1\underline{k}+\lambda_2 k}-\mathrm{e}^{\lambda_1\bar{k}+\lambda_2 k}}{\mathrm{e}^{\lambda_2\underline{k}+\lambda_1\bar{k}}-\mathrm{e}^{\lambda_2\bar{k}+\lambda_1\underline{k}}}\right). \tag{24}$$

If we let the lower bound, $\underline{k}$, go to minus infinity, (24) simplifies to

$$x=k+\alpha\eta(1-\mathrm{e}^{\lambda_1(k-\bar{k})}). \tag{25}$$

If, in addition, the upper bound, $\bar{k}$, goes to infinity, we again get the linear solution in (5).

Fig. 3 illustrates eq. (24). It shows two exchange-rate paths that share the

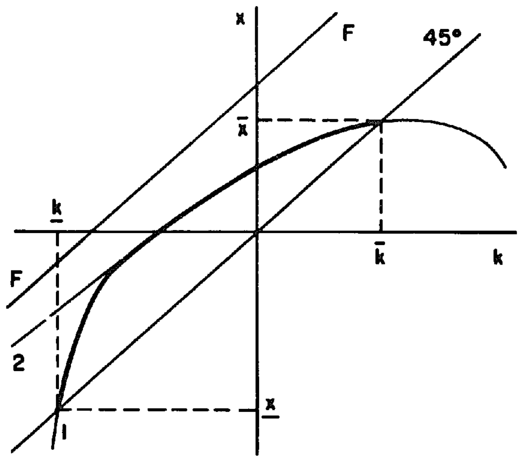

Fig. 3

same upper bound, but that have different lower bounds. Path 1 shows the behavior of x when the absorbing barriers are the points $\underline{k}$ and $\bar{k}$ in the figure. Path 2 is drawn to correspond to the extreme case in (25) where the lower bound is at minus infinity. It is clear from (20) and (21) that the exchange rate must lie on the 45° line through the origin at both absorbing barriers. When both boundaries are infinitely distant, the saddlepath is just the free-float saddlepath, FF. Notice also that if there is no trend growth in fundamentals, $\eta=0$, *all* solutions correspond to the 45° line (which then coincides with FF), regardless of the boundary values. This result seems paradoxical at first glance, but it is actually quite natural. At the absorbing boundaries of the interval $[\underline{k}, \bar{k}]$, just as in its interior, the distribution of possible movements in k is symmetric upward and downward. (This is not the case when the barriers on k are reflecting.)

Why does a nonzero η give a rise to a curved saddlepath solution in fig. 3? The saddlepath exchange rate is the present discounted value of fundamentals, and the evolution of fundamentals depends in part on their deterministic trend growth rate, η. Suppose that $\eta>0$ (the case the figure assumes). As k approaches either $\bar{k}$ or $\underline{k}$, the probability that the exchange rate will still be floating on any given future date declines; and since η is set permanently to zero at the moment of pegging, the expected rate of monetary growth on any future date also declines as either absorbing barrier is approached. As a result, there is a progressive currency appreciation relative to FF as k moves toward one of the barriers. For $\eta<0$, FF would lie below the 45° line and the saddlepath solution would be the mirror image of the one in fig. 4. When $\eta=0$ the bending effects are absent because absorption of k has no effect on the expected change in fundamentals (which remains zero). The saddlepath lies between FF and the 45° line, which coincide when $\eta=0$.

Eq. (25) is significant because it is the closed-form solution for the problem raised by Flood and Garber (1983). Flood and Garber try to solve directly the integral representation for the exchange rate under a single absorbing barrier at $\bar{k}$:

$$x(t)=\alpha^{-1}\int_t^{\infty} e^{(t-s)/\alpha}\,\mathrm{E}(k(s)\,|\,k(t), k(s)\leq \bar{k}_a)\,\mathrm{d}s. \tag{26}$$

With a single absorbing barrier at $\bar{k}$, we know from (8) and (21) that the exchange rate must satisfy

$$x=k+\alpha\eta(1-e^{\lambda_1(k-\bar{k})})+A_2(e^{\lambda_2 k}-e^{(\lambda_2-\lambda_1)\bar{k}+\lambda_1 k}), \tag{27}$$

where A_2 is an arbitrary constant to be determined. Clearly, the unique choice of A_2 that makes (27) equal to the integral (26) is zero. Note first that as k becomes infinitely small, the presence of the barrier $\bar{k}$ has a negligible effect on the conditional expectation of future levels of k in (26). For such small k, the exchange rate should therefore be approximately linear in the fundamentals [as in the no-boundary solution, eq. (5)]. Next, note that (27) becomes linear in k as $k\rightarrow-\infty$ if and only if $A_2=0$. But setting $A_2=0$ just gives solution (25), which was found by letting the lower bound on k become unboundedly negative.

3.3. Unification of a dual exchange-rate system

A final application of our approach is set in an economy the government of which maintains separate exchange rates for current- and capital-account transactions. Dual or parallel exchange-rate schemes are sometimes adopted when authorities wish to insulate cross-border trade flows from the factors that influence portfolio demands for currencies. Often the arrangements are temporary. Here we examine the effects of expected unification on a dual-rate system in which both rates float freely but clear separate markets.[13]

The model's equilibrium conditions turn out to be described by two differential equations rather than one, an equation for each exchange rate. The path of the commercial rate clears the economy's current-account balance, while the path of the financial rate depends on the domestic interest rate and the premium of the financial rate over the commercial rate. Because the two rates are simultaneously determined, the solution technique for the model is based on a multivariate extension of the methods used above.[14]

[13] Flood and Marion (1983) develop a model quite similar to the one introduced below, simpler in some respects but more realistic in others. There are several types of dual exchange-rate regime, and most often the commercial rate is fixed. For a general survey, see Dornbusch (1986).

[14] Klein (1990) applies a similar multivariate approach in a different setting.

An analysis of dual rates requires some changes in the model used so far, so we give a relatively detailed description of the economic setting now assumed. As usual, all variables other than interest rates are natural logarithms. We consider a small economy facing given and constant values of the foreign price level, p^*, and nominal interest rate, i^*. Along with domestic money, residents of the economy hold both domestic- and foreign-currency bonds, which pay interest at the rates $i(t)$ and i^*, respectively. Two key institutional restrictions support the dual exchange-rate system. First, the stock of foreign-currency bonds is frozen (through prohibitions on asset trade with foreigners) at some fixed level. Second, all foreign exchange receipts on current account must either be sold to domestic importers at a market-clearing price, or spent immediately on imports.

The price of foreign assets in terms of home currency is x^{f}, the financial exchange rate. All current-account transactions – including, importantly, the repatriation of interest on foreign-currency assets – take place at the commercial exchange rate, x^{c}. Under free capital mobility the financial and commercial markets would be unified, with $x^{\mathrm{c}}=x^{\mathrm{f}}=x$.

Because domestic residents may trade foreign assets among themselves, an interest parity condition links the returns on domestic bonds and domestically-held foreign bonds. This condition is written

$$i(t)=i^*-i^*(x^{\mathrm{f}}(t)-x^{\mathrm{c}}(t))+\mathrm{E}(\mathrm{d}x^{\mathrm{f}}(t)\,|\,\phi(t))/\mathrm{d}t. \tag{28}$$

Condition (28) differs from what would apply under a unified exchange market, because it recognizes that the difference $x^{\mathrm{f}}(t)-x^{\mathrm{c}}(t)$ acts as a tax on foreign interest. If no change in $x^{\mathrm{f}}(t)$ is expected, (28) implies that financial foreign exchange sells at a premium relative to commercial foreign exchange when $i(t)<i^*$, and at a discount in the opposite case.[15]

Goods prices are linked internationally through an assumption of strict purchasing power parity. Since it is the commercial exchange rate at which goods are traded, that condition states that the domestic price level $p(t)$ is given by

$$p(t)=x^{\mathrm{c}}(t)+p^*. \tag{29}$$

[15]For a detailed discussion, see Flood and Marion (1983). The commercial rate would not enter eq. (28) if foreign interest were repatriated at the financial exchange rate; but in this case that rate would be indeterminate in our model [see Dornbusch's chapter in Frenkel and Johnson (1978) for a similar case]. Determinacy could be restored, as in Cumby (1984) and Lizondo (1987), by allowing money demand to depend on total domestic wealth in addition to the other factors listed below.

The price level therefore moves one-for-one with the commercial exchange rate.

Money-market equilibrium determines the domestic interest rate. With the notation reflecting the assumptions that the money supply, m, and output, y, are fixed, the equilibrium condition is

$$m-p(t)=\psi y-\alpha i(t). \tag{30}$$

Combining (30) with (28) and (29) leads to the differential equation governing the financial exchange rate:

$$x^{\mathrm{f}}(t)=\kappa+(1-1/(\alpha i^*))x^{\mathrm{c}}(t)+(1/i^*)\,\mathrm{E}(\mathrm{d}x^{\mathrm{f}}(t)\,|\,\phi(t))/\mathrm{d}t, \tag{31}$$

where $\kappa=1+(1/\alpha i^*)(m-p^*-\psi y)$. For simplicity, $\kappa=0$ is assumed.

A current-account balance condition leads to the model's second differential equation, which governs the commercial rate. The current account is a decreasing function of domestic real money balances, an increasing function of the expected real interest rate, and a decreasing function of a random trade-balance shock, $\tilde{k}(t)$, which can be thought of as a shock to public spending.[16] Under capital controls and a floating commercial rate, and in the absence of direct government borrowing abroad, there is no channel through which the economy will change its net stock of foreign claims. The implied requirement of a balanced current account is written

$$\xi(i(t)-\mathrm{E}(\mathrm{d}p(t)\,|\,\phi(t))/\mathrm{d}t)-\gamma(m-p(t))-\tilde{k}(t)=0. \tag{32}$$

Let $\chi=\xi/((\xi/\alpha)+\gamma)$, $\kappa'=(m-p^*)-(\chi\psi/\alpha)y$, and $k(t)=(\chi/\xi)\tilde{k}(t)$. After substitution using (29) and (30), (32) can be rewritten as

$$x^{\mathrm{c}}(t)=\kappa'+k(t)+\chi\,\mathrm{E}(\mathrm{d}x^{\mathrm{c}}(t)\,|\,\phi(t))/\mathrm{d}t. \tag{33}$$

This is the second differential equation describing equilibrium asset prices. (We will again simplify and assume $\kappa'=0$.)

The disturbance $k(t)$ in (33) is approximated by the stochastic process (2), but with η set at zero. Since (33) is of the same form as eq. (1), the general solution for the commercial exchange rate is

$$x^{\mathrm{c}}=k+A_1\,\mathrm{e}^{\rho k}+A_2\,\mathrm{e}^{-\rho k}, \tag{34}$$

where A_1 and A_2 are arbitrary constants and $\rho=(2/\chi\sigma^2)^{1/2}$.

[16] Real balances affect the current account by reducing saving (a wealth effect). An increase in the expected real interest rate is assumed to raise saving.

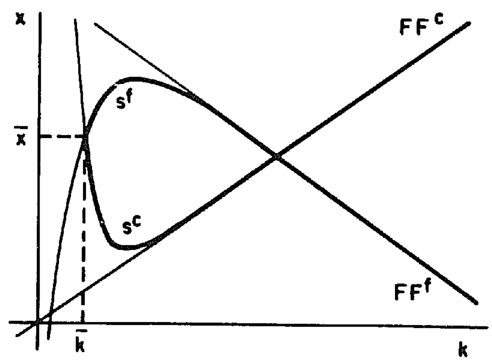

Fig. 4

Finding the solution to the financial exchange rate equation, (31), is not so straightforward, because its forcing variable is the endogenous price $x^c(t)$. It is easy to check, however, that the following general expression for the financial rate satisfies (31):

$$x^f = \left(\frac{\alpha i^* - 1}{\alpha}\right)\left(\frac{k}{i^*} - \left(\frac{\chi}{1-\chi i^*}\right)(A_1 e^{\rho k} + A_2 e^{-\rho k})\right)$$

$$+ A_3 e^{\delta k} + A_4 e^{-\delta k}, \tag{35}$$

where A_3 and A_4 are arbitrary constants and $\delta = (2i^*/\sigma^2)^{1/2}$.

The straight lines in fig. 4 show how the two floating exchange rates depend on the fundamental $k(t)$ when no regime change is possible. The upward-sloping one, FF^c, graphs the linear part of (34). It shows that a rise in k (an exogenous incipient deterioration in the current account) causes a rise in x^c which restores external balance by lowering real monetary wealth and raising the real interest rate. The downward-sloping line, FF^f, graphs the linear part of (35), which is decreasing in k if the interest elasticity of money demand at $i = i^*, \alpha i^*$, is less than one. By raising x^c, a rise in k raises the nominal domestic interest rate by dx^c/α. Since the domestic–foreign interest differential equals $i^*(x^c - x^f)$, x^f must fall absolutely to maintain interest parity when $\alpha i^* < 1$.

We can now consider how currency prices respond to the prospect of a stochastically determined unification of the exchange markets. In principle, there are several systems of unified rates that could be considered. Because a single basic solution principle applies in all cases, however, we consider only the case in which the authorities announce contingencies under which they will fix both rates at a given level, $\bar{x}$.

As has been emphasized above, however, some contingency plans simply do not give the market enough information to determine unique equilibrium exchange rates. For example, an announcement that the two rates will be pegged at their common level once both reach $\bar{x}$ results in indeterminacy. In addition, the authorities must, as a general rule, announce the level of the fundamental k, $\bar{k}$ say, at which unification will occur.[17]

Fig. 4 shows how solutions to this unification problem behave. The figure is drawn on the assumption that exchange markets will be unified at a fundamental $\bar{k}$ below the current value (and at an exchange rate $\bar{x}$). Intuitively, the authorities plan to unify once the exogenous component of the current account has improved to a prespecified level, and they will do so at an exchange rate that ensures an initial current-account surplus. This setup is a variant of the absorbing-barrier problem of the previous subsection, and the exchange-rate solutions it yields are analogous. Because neither rate can take a discontinuous jump at the moment k reaches $\bar{k}$, the functions $S^c(k)$ and $S^f(k)$ must describe the commercial and financial rates prior to unification. These functions curve away from the linear loci to intersect at the unification point, but they are approximately linear when unification is distant. Intuitively, the approach of unification causes both rates to move closer to their unified level.[18]

Clearly a different choice of $\bar{k}$, with $\bar{x}$ unchanged, would imply a different equilibrium configuration. This observation shows why the equilibrium will be under-determined if an absorbing exchange rate barrier is announced without the accompanying announcement of what is effectively an absorbing barrier for the current-account fundamental.

An interesting implication of fig. 4 concerns the behavior of the two exchange rates for values of k above but close to $\bar{k}$. Because of the shapes of the curves $S^c(k)$ and $S^f(k)$ near $\bar{k}$, insignificant changes in fundamentals can have large and seemingly perverse effects on exchange rates.

4. Some empirical implications

The nonlinearities induced by stochastically-triggered policy shifts have numerous implications for empirical studies of asset pricing. In this section we explore some empirical implications of one of our exchange-rate examples, the target zone. Empirical studies of target zones have often relied

[17]An exception occurs when the current account does not depend on the real interest rate ($\xi=0$). In this case, x^c does not behave like a speculative price, so its value always lies along the linear locus FF^c, even when regime shifts are possible. An announcement that unification will occur when $x^f=x^c=\bar{x}$ therefore ties down a unique value $\bar{k}=\gamma(m-\bar{x}-p^*)$ at which the switch will occur.

[18]Explicit solutions for A_1, A_2, A_3, and A_4 are left to the interested reader.

on linear econometric methods, so we focus on the small-sample biases that such methods might entail when the true exchange-rate model is nonlinear.

The first point to be made is that the presence of permanent exchange-rate bands induces an unconditional exchange-rate distribution that is covariance stationary. This property means that many standard inference techniques retain their asymptotic justification, despite the fact that the exchange-rate process may approximate a random walk near the center of the band. We can actually be fairly precise about the unconditional distribution of x when bands are in place. Consider, for example, the case in which fundamentals follow a trendless random walk within the band ($\eta=0$). Then the unconditional distribution of k is uniform over $[\underline{k}, \bar{k}]$. Reflection at the boundaries simply nudges k back into the range $[\underline{k}, \bar{k}]$; and because there is no trend, k is equally likely to be in any two equally-sized subintervals.[19] The unconditional distribution of the exchange rate is nonstandard, but just as easily understood. By the change-of-variables formula, the probability density function for the exchange rate will turn out to be bimodal, with modes at the bands. This bimodality is a direct result of the S-shaped saddlepath: near the bands there are a greater number of (equally likely) values of k for any given interval of the exchange rate.

What do bands imply about the properties of linear econometric methods? Consider a standard test for the presence of a unit root, in which the exchange rate is regressed on its own lagged value and a constant:

$$x_{t+1}=c+\beta x_t+\varepsilon_{t+1}. \tag{36}$$

If there are no bands present, and if fundamentals evolve according to (2), then $\beta=1$ and the residual ε_{t+1} is purely random. With bands in place, however, the exchange rate is statistically stationary, so the coefficient β should converge to a value less than one. In an infinite-sized sample, the hypothesis that the exchange rate follows a random walk would be rejected with probability one. The question of practical importance concerns the amount of data needed for reasonably powerful tests.

To answer this question we performed Monte Carlo simulations, assuming that fundamentals follow the random-walk process (2) within credible reflecting barriers, so that the exchange rate is given by (16). We chose four different sets of bands on fundamentals which, measured from the initial starting point, were 6, 18, 70 and 160 (log) percent in either direction. We chose the model's free parameters as: the annual semi-elasticity of money

[19]For a formal proof, see Harrison (1985, p. 90), who also shows that the unconditional distribution of k for $\eta\neq 0$ is exponential.

Table 1

Monte Carlo simulations of the power of unit root tests against an unspecified target-zone alternative.

	Monthly data			Daily data	
Time-series sample:	100 years	25 years	10 years	10 years	1 year
Exchange-rate band width					
±2.25%	100	12.4	10.1	9.2	4.6
±13.8%	9.6	5.5	3.4	3.0	2.6
±66%	2.7	2.7	1.9	2.6	2.6
±157%	2.5	2.7	1.9	2.6	2.6

Notes: Figures reported in the first three columns are the percentage of rejections at a 2.5 percent level of significance of the hypothesis that the exchange rate follows a random walk, given that fundamentals evolve within the band according to eq. (2) in the text, and that the exchange rate evolves according to (16). Simulation parameters are reported in the text. (Percentages are measured as log differences.)

demand, $\alpha = 4$; annual standard deviation of k, $\sigma = 0.1$; and trend growth in relative fundamentals, $\eta = 0$.[20] These bands on k translate into smaller bands on the exchange rate of plus or minus 2.25, 13.8, 66 and 157 percent, respectively. The smallest of these corresponds to the bilateral band width for most currencies within the European Monetary System (EMS).

Table 1 reports rates of rejection at the 2.5 percent level of the hypothesis $\beta = 1$ in eq. (36). To construct confidence intervals we used the usual Dickey–Fuller critical values. The rejection rates are estimated using 1,000 independently-drawn simulations of the model. In an infinite sample, in the absence of bands, we would expect to reject the random walk model 2.5 percent of the time. We report results for three different sample periods using monthly data (100, 25 and 10 years), and two different sample periods using daily data (10 years and 1 year).

There are several noteworthy aspects of the estimates. First, with a target zone of plus or minus 2.25 percent and 100 years of data, we reject the random walk hypothesis in every single draw. Second, this result is quite fragile: it disappears when we increase the width of the bands, or decrease the size of the sample. The second row of the table reports the rejection rate when the bands are set at plus or minus 13.8 percent. Even with 100 years of data, the rejection rate falls from 100 to 9.6 percent. Thus, the effects of moderately-sized bands could be hard to detect econometrically even in comparatively long time-series samples. The second and third columns report

[20]These parameter values are well within the range of previously published empirical estimates. As a check on their sensibility, note that they generate an annual standard deviation for the exchange rate of about 10 percent (when the exchange rate is far from both bands). This is close to the average standard deviation of the effective value of the U.S. dollar over the 1973–1988 floating-rate period.

results for shorter samples of 25 and 10 years, respectively. The rejection rates here are uniformly low, even with very narrow bands. In 25 years of data we can detect the presence of very tight bands (2.25 percent up or down) only 12.4 percent of the time. The power of the unit root test against the target-zone alternative therefore falls rapidly with increases in zone size or with decreases in sample size. Third, by comparing the results for 10 year sample periods using daily and monthly data, we can see that sampling frequency has no effect on the results.[21]

To sum up, the nonlinearities induced by prospective policy shifts have two main empirical implications. They obviously imply that standard linear models are misspecified. Perhaps more important, however, is the implication that the possible events causing nonlinearities (such as bumping up against a boundary) may occur only infrequently in time-series samples of the usual size. In sample of the size that have been used to study the EMS, say, it may be difficult to detect evidence of the covariance stationarity of the exchange rate induced by the target zone. Indeed, the Monte Carlo results reported above suggest that the exchange rate must run into its limits many times to generate detectable evidence that those limits exist.

5. Conclusions

This paper has shown how techniques of regulated Brownian motion can be applied to models of exchange-rate determination under a variety of possible future regime switches. The techniques used above are far simpler and more intuitive than the method of calculating exchange rates directly as expected present values of fundamentals. In this paper we restricted our attention to three examples of exchange-rate regime change, but there are clearly many other potential applications of the methodology, to exchange rates as well as to other asset prices.

All the scenarios we examined above can introduce nonlinearities into relationships between exchange rates and fundamentals, and into those between current and lagged exchange rates. These nonlinearities may be difficult to discern empirically in small samples, but will nevertheless invalidate the asymptotic distribution theory that econometric inference typically invokes. Evidence favoring particular nonlinear exchange-rate models is already accumulating, however. Engel and Hamilton (1990), for example, reject the hypothesis that the exchange rate follows a random walk

[21]We began each simulation with the exchange rate at the middle of the zone. This tends to lower the power of the tests. On the other hand, adding realistic probabilities of realignment would tend to make us reject less frequently.

against the alternative of stochastic shifts between two exchange-rate regimes. Whether stochastic regime shifts such as those studied above are responsible for these rejections can be determined only through additional empirical work.

Appendix: Formalizing the target-zone model

This appendix provides a more formal treatment of the target-zone model of subsection 3.1. The analysis draws on Harrison's (1985) discussion of the two-sided infinitesimal regulator.

According to eq. (9) in the main text, the unregulated fundamentals variable, $K(t)$, is generated by the stochastic process $\mathrm{d}K(t)=\eta\,\mathrm{d}t+\sigma\,\mathrm{d}z(t)$, where $\mathrm{d}z$ is a standard Wiener process.[22] We define the lower and upper regulators, $L(t)$ and $U(t)$, such that:

(i) $L(t)$ and $U(t)$ are increasing and continuous functions of time.
(ii) The variable $k(t)=K(t)+L(t)-U(t)$ is in $[\underline{k},\bar{k}]$, for all $t>0$.
(iii) $L(t)$ increases only when $k(t)=\underline{k}$.
(iv) $U(t)$ increases only when $k(t)=\bar{k}$.

Initially, $K(0)\in[\underline{k},\bar{k}]$ and $L(0)$, $U(0)$ are normalized to zero.

Changes in $L(t)$ and $U(t)$, denoted $\mathrm{d}L(t)$ and $\mathrm{d}U(t)$, are interpreted as policy interventions that alter the fundamentals. Thus, the fundamentals process $k(t)$ equals the unregulated process $K(t)$, adjusted for the integral of past interventions, $L(t)-U(t)=\int_0^t(\mathrm{d}L(s)-\mathrm{d}U(s))$. Changes in $L(t)$ and $U(t)$ offset changes in $K(t)$ that might otherwise take $k(t)$ below $\underline{k}$ or above $\bar{k}$. They can be interpreted as marginal official purchases and sales, respectively, of foreign currency. The continuity requirement in (i) ensures that such interventions are infinitesimal, i.e. at the smallest level that keeps fundamentals from exiting the interval $[\underline{k},\bar{k}]$. Harrison (1985, ch. 2) shows that functions L and U satisfying (i)–(iv) exist and are unique for any continuous stochastic process $\{K(t)\}$.

The goal is to find appropriate boundary conditions for the saddlepath exchange rate, $S(k(t))$, given the joint dynamics of the exchange rate, $x(t)$, and the regulated fundamentals, $k(t)$:

$$x(t)=f(k(t))+\alpha\,\mathrm{E}(\mathrm{d}x(t)\,|\,\phi(t))/\mathrm{d}t, \tag{A.1}$$

$$\mathrm{d}k(t)=\eta\,\mathrm{d}t+\sigma\,\mathrm{d}z(t)+\mathrm{d}L(t)-\mathrm{d}U(t), \tag{A.2}$$

[22] The argument presented below goes through, however, for a wider class of processes, including mean-reverting processes.

where $f(k)$ is any continuous function of the fundamentals. The key assumption we make is that $S(k)$ is a twice continuously differentiable function. Itô's lemma applied to (A.2) shows that[23]

$$S(k(t)) = S(k(0)) + \sigma \int_0^t S'(k(s))\,\mathrm{d}z(s) + \eta \int_0^t S'(k(s))\,\mathrm{d}s$$

$$+ \frac{\sigma^2}{2} \int_0^t S''(k(s))\,\mathrm{d}s + \int_0^t S'(k(s))\,\mathrm{d}L(s) - \int_0^t S'(k(s))\,\mathrm{d}U(s). \tag{A.3}$$

Since $\mathrm{d}L(s) = 0$ unless $k(s) = \underline{k}$ and $\mathrm{d}U(s) = 0$ unless $k(s) = \bar{k}$, (A.3) can be expressed in the usual shorthand notation as

$$\mathrm{d}S(k) = \sigma S'(k)\,\mathrm{d}z + \left(\eta S'(k) + \frac{\sigma^2}{2} S''(k)\right)\mathrm{d}t + S'(\underline{k})\,\mathrm{d}L - S'(\bar{k})\,\mathrm{d}U \tag{A.4}$$

[see also Harrison (1985, p. 82)].

The value-matching conditions satisfied by $S(k)$ at the boundaries of the fundamentals band can now be derived.

Theorem. In the target-zone model with fundamentals band $[\underline{k}, \bar{k}]$, *the sandlepath exchange-rate function,* $S(k)$, *satisfies the derivative conditions*

$$S'(\underline{k}) = S'(\bar{k}) = 0.$$

Proof. Write (A.4) in integral form and take its $t=0$ conditional expectation:

$$\mathrm{E}(S(k(t)) \,|\, \phi(0)) = S(k(0)) + \int_0^t \mathrm{E}\left(\eta S'(k(s)) + \frac{\sigma^2}{2} S''(k(s)) \,|\, \phi(0)\right)\mathrm{d}s$$

$$+ S'(\underline{k})\,\mathrm{E}(L(t) \,|\, \phi(0)) - S'(\bar{k})\,\mathrm{E}(U(t) \,|\, \phi(0)). \tag{A.5}$$

Because

$$\mathrm{E}(S(k(t)) \,|\, \phi(0)) = S(k(0)) + \alpha^{-1} \int_0^t \mathrm{E}(S(k(s)) - k(s) \,|\, \phi(0))\,\mathrm{d}s$$

[23] The continuous increasing functions $L(t)$ and $U(t)$ have zero quadratic variation over any finite time interval. As a result, their entire joint contribution to $S(k(t)) - S(k(0))$ is given by the last two integrals on the right-hand side of eq. (A.3) below.

by (1), however, eq. (7), which $S(k)$ must satisfy even at the boundaries of $[\underline{k}, \bar{k}]$, leads to

$$\mathrm{E}(S(k(t))|\phi(0)) = S(k(0)) + \int_0^t \mathrm{E}\left(\eta S'(k(s)) + \frac{\sigma^2}{2} S''(k(s))|\phi(0)\right) \mathrm{d}s. \tag{A.6}$$

Together (A.5) and (A.6) imply

$$S'(\underline{k})\,\mathrm{E}(L(t)|\phi(0)) - S'(\bar{k})\,\mathrm{E}(U(t)|\phi(0)) = 0. \tag{A.7}$$

But if $\underline{k}$ and $\bar{k}$ are finite, the regulators $L(t)$ and $U(t)$ can satisfy (A.7) for all initial information sets, and for all subsequent dates, only if $S'(\underline{k}) = S'(\bar{k}) = 0$.

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