Implementation of Efficient Investments in Mechanism Design

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Abstract

This paper studies the question of when we can eliminate investment inefficiency in a general mechanism design model with transferable utility. We show that when agents make investments only before participating in the mechanism, inefficient investment equilibria cannot be ruled out whenever an allocatively efficient social choice function is implemented. We then allow agents to make investments before *and after* participating in the mechanism. When *ex post* investments are possible and an allocatively constrained-efficient social choice function is implemented, efficient investments can be fully implemented in perfect Bayesian Nash equilibria if and only if the social choice function is *commitment-proof* (a weaker requirement than strategy-proofness). Commitment-proofness ensures the efficiency of investments by suppressing the agents' incentives to make costly *ex ante* investments which may work as a commitment device. Our result implies that in the provision of public goods, implementation of efficient investments and efficient allocations is possible even given a budget-balance requirement.

Keywords: investment efficiency, full implementation, mechanism design, commitment, *ex post* investment

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1 Introduction

Can an auction, like the spectrum auction, be designed to induce efficient investments as well as efficient allocations? A standard assumption in the mechanism design literature is that the values that the participants get out of the possible outcomes are exogenously given. In many real-life applications however, there are opportunities to invest in the values of the outcomes outside of the mechanism. In the spectrum auction, telecom companies make investments in new technologies or build base stations in anticipation of winning the spectrum licenses. In a procurement auction, participating firms make efforts to reduce the cost of production in preparation for bidding (Tan, 1992; Bag, 1997; Arozamena and Cantillon, 2004). Moreover, the firms in these auctions not only make *ex ante* investments but also make further investments if they win the auction (Piccione and Tan, 1996). These investments endogenously form the valuations of the allocations that are determined by the auction. At the same time, the incentives of both *ex ante* and *ex post* investments are affected by the structure of the allocation mechanism. Therefore, to seek an efficient mechanism, we should take account of the efficiency of the outside investments it induces, in addition to its standard efficiency within the mechanism.

The goal of this paper is to analyze when we can fully implement efficient investments, i.e., under what mechanisms *every* equilibrium of the investment game will be efficient.[1] To do this, we consider a general mechanism design model with transferable utility. This includes several important applications such as auctions, matching with transfers and the provision of public goods. The valuation functions of agents at the market clearing stage are endogenously determined. We examine the following two environments: (i) agents make investments only before the mechanism, and (ii) they make investments before and after the mechanism. In either environment, we analyze the implementability of full efficiency, which requires that given that an allocatively efficient social choice function is implemented, every equilibrium of the investment game should maximize the total expected utility of agents inclusive of the cost of investments. In particular, we characterize the social choice functions for which efficient investments are implementable in every equilibrium. The main results are summarized as follows: first, with only *ex ante* investments, we show that efficient investments are not implementable for any allocatively efficient social choice function (Theorem 1). Next, allowing for *ex post* investments, we show that a new concept of *commitment-proofness* is sufficient and necessary for implementing efficient investments when an allocatively effi-

[1]When we simply say "implementation" in this paper, this refers to full implementation. See Definition 3 and 6 for the mathematical expressions.

cient social choice function is implemented (Theorem 2).

Furthermore, as a variant of the main model, we consider the provision of public goods with budget balance. In this environment, we show that there exists a commitment-proof, allocatively efficient and budget-balanced social choice function (Proposition 1). This implies that even with budget-balance requirement, it is always possible to implement efficient investments and efficient allocations at the same time.

The investigation of full implementation advances the traditional question asked in the literature: under what mechanisms does there exist an efficient pre-mechanism investment equilibrium? Rogerson (1992) initiated this field by showing that when agents make investments prior to the mechanism, there is a socially efficient Nash equilibrium investment profile for any strategy-proof and allocatively efficient mechanisms. Hatfield, Kojima and Kominers (2015) complemented Rogerson (1992)'s findings to show that strategy-proofness is also necessary for the existence of an efficient investment equilibrium when the mechanism is allocatively efficient. In the context of information acquisition (Milgrom, 1981; Obara, 2008), Bergemann and Välimäki (2002) indicate the link between *ex ante* efficiency and strategy-proofness; the VCG mechanism ensures *ex ante* efficiency under private values. Overall, in order to induce efficient *ex ante* investment incentives, strategy-proofness is essential because the privately optimal investment choice always becomes socially optimal given other agents' investment choices.

With only *ex ante* investments, however, there may exist another inefficient equilibrium even under strategy-proof mechanisms. Many authors in the literature pointed out this problem in a particular example, but they have not developed a general result.[2] The multiplicity of equilibria is not a trivial problem because an inefficient equilibrium may not be eliminated by employing stronger equilibrium concepts such as trempling-hand perfection. Consider an example where telecom firms are competing for a spectrum license, and suppose they know the competitors' cost functions for investments. When investments are observable, the *ex ante* investment may work as a commitment device even for a firm whose investment is more costly than other firms. If it is the only firm that makes an investment, at the market clearing stage, the value of the license can be higher than the values for any other firms because the cost of investment has been sunk. Therefore, there is an equilibrium at which the firm makes a lot of costly *ex ante* investments and deters its competitors from investing. This role of *ex ante* investment has also been studied as an entry-deterring behavior for an in-

[2]For example, see Example 4 of Hatfield, Kojima and Kominers (2015). This motivated the spectrum auction example which will be introduced in the next section.

cumbent firm in an oligopolistic market (Spence, 1977, 1979; Salop 1979; Dixit, 1980). This intuition is generalized by our first result; when agents invest only before the mechanism, inefficient investment equilibria cannot be ruled out whenever an allocatively efficient social choice function is implemented (Theorem 1).

In order to eliminate such investment inefficiency while securing allocative efficiency, we consider a setting where agents can invest before *and after* participating in the mechanism. In many applications, agents make further investments after the market clearing stage to maximize the value of the outcome realized in the mechanism. In the context of bidding for government contracts, firms invest in cost reduction once they are selected by the government to perform the task (McAfee and McMillan, 1986; Laffont and Tirole, 1986, 1987). For simplicity, we model investments as an explicit choice of valuation functions. *Ex ante* and *ex post* investments are modeled in the following way. First, agents choose their own valuation functions over the outcomes prior to the mechanism. The cost of each valuation function is determined by each agent's cost type. Each agent knows her own cost type, but does not know the realization of the cost types of other agents. These *ex ante* investments are irreversible, but after participating in the mechanism, agents may make further investments by revising their valuations to more costly ones. Note that equilibrium *ex post* investments are always socially optimal given the outcome of the mechanism as we assume no externality of investments. Therefore, if agents could not make any *ex ante* investments, the problem of implementing efficient investments falls within the scope of the classical mechanism design theory. However, this is not the case when *ex ante* investments are possible. Our main theorem characterizes allocatively efficient social choice functions for which investment efficiency is guaranteed in every equilibrium; given that an allocatively constrained-efficient social choice function is implemented, commitment-proofness of the social choice function is sufficient and necessary for implementing efficient investments in any perfect Bayesian Nash equilibrium (Theorem 2).

We introduce a novel concept called *commitment-proofness* which is illustrated in the following (hypothetical) scenario. Suppose that a participant in a mechanism makes a contract with a third party, in which the agent pays some amount to the third party before the mechanism, and then the third party returns some or all of the payment to the agent contingent on the outcome of the mechanism. Since this contract manipulates the value of each outcome (based on the amount of money returned to the agent), it allows the agent to commit to behaving as a different type in the mechanism. Commitment-proofness of a social choice function requires that no agent be able to benefit from making such a commitment.[3] This is a natural requirement since the third party would always be (weakly) better off from entering this contract. The concept thus precludes an important class of *ex ante* commitments which can potentially be made in a wide range of environments.

Then, how does the possibility of *ex post* investment help us obtain a positive result together with commitment-proofness? First, as we discussed above, investment efficiency is achieved by any allocatively efficient mechanism if no agent makes *ex ante* investments. Therefore, we need to find out under what conditions no agent will have the incentive to make positive *ex ante* investments for any cost type.[4] Consider a firm whose investment is more costly than other firms in the spectrum auction explained above. Suppose that no other firms make any *ex ante* investments. The values of the spectrum license for these firms would be low if there were no *ex post* investment opportunities. But now the value for each firm should be equal to the maximum net profit from the license inclusive of the cost of investment because any firm would make the optimal investment *ex post* if it wins the auction. Thus, in order for the firm with costly investment to win, it needs to beat its competitors who value the license more than the costly firm's potential profit from the license. To completely suppress the incentive of this firm to win out by investing *ex ante*, there must be a sufficient amount of payment for the license. Commitment-proofness of social choice functions characterizes such transfer payments that are sufficient and necessary for suppressing the incentives to invest *ex ante* in a general environment. In this way, the information of firms' cost types are revealed by the presence of *ex post* investment, and commitment-proofness eliminates the incentives for making *ex ante* investment which works as a commitment device.

In our model, the difficulty of implementing efficient investments stems from the combination of the following assumptions: (i) investments are not verifiable, (ii) investments are irreversible, and (iii) the agents' cost types are not known to the mechanism designer. First, if investments were verifiable to a third party, they could just be part of the outcome of mechanisms and the standard implementation theory applies. However, investment behaviors are usually difficult to describe; they are multi dimensional and they involve the expenditure of time and effort as well as the expenditure of money (Hart, 1995). These noncontractible investments have also been a central concern in the hold-up problems (Klein, Crawford, and Alchian, 1978; Williamson, 1979, 1983; Hart and Moore, 1988). Second, if

[3]As we will show in Section 4, this property is weaker than the well-known strategy-proofness condition.

[4]In the main model, we introduce a (slight) time discounting between two investment stages so that given that the allocation rule is efficient, investment efficiency is achieved only when no agents make costly *ex ante* investments.

investments were reversible, the efficiency of allocations would not be affected by the choice of *ex ante* investments. Therefore, we could apply mechanisms proposed by the standard implementation theory and (virtually) implement efficient allocations. Finally, it is obvious that investments would be efficient if the mechanism designer knew the agents' cost types and specified the first-best allocation because investments do not have any externalities.

Unlike related papers that analyze specific mechanisms such as the first-price auction and the second-price auction (Tan, 1992; Piccione and Tan, 1996; Stegeman, 1996; Bag, 1997; Arozamena and Cantillon, 2004), we consider the entire space of social choice functions. Also, we focus on the equilibrium analysis of the investment game outside of the mechanism. That is, the analysis of the game within the mechanism to implement a social choice function is set apart from the discussion. This is because we know that a large class of social choice functions are implementable both under complete and incomplete information. For example, any social choice function can be implemented by an extensive form mechanism in subgame-perfect equilibria under quasi-linear utility and complete information environments (Moore and Repullo, 1988; Maskin and Tirole, 1999). For incomplete information cases, it is known that a large class of social choice functions are virtually implementable by a static mechanism (Abreu and Matsushima, 1992). Therefore, most of the social choice rules considered here can be (virtually) implemented by some mechanism. Hence, our theorem gives a general guideline to distinguish whether an allocatively efficient mechanism, which may have not been analyzed well, implements efficient investments. In order to detect whether a specific mechanism (which has a non-truth-telling equilibrium) implements efficient investments from our results, we need one more step to check if it implements a commitment-proof and allocatively efficient social choice function.

There is also large literature on investment incentives before competition or two-sided matching (Gul, 2001; Cole, Mailath and Postlewaite, 2001a, 2001b; Felli and Roberts, 2002; De Meza and Lockwood, 2010; Mailath, Postlewaite and Samuelson, 2013; Nöldeke and Samuelson, 2015). Although these papers have a common interest with ours, there are two major differences in the modeling choices. First, they often assume that the investments of the two sides of agents have externalilties. Therefore, it is difficult to eliminate inefficient investment equilibria in their framework due to coordination failure. Moreover, they often consider situations where trade takes place in the market clearing stage. In such contexts, it is not plausible to consider the possibility of *ex post* investments. In short, our positive result may not be directly applied to their models because of these differences in the assumptions.

The rest of the paper is organized as follows. In Section 2, we explain a numerical example of the spectrum auction to provide intuition for the results. Section 3 introduces the formal model and defines implementability of efficient investments. In Section 4, commitment-proofness is introduced, and the impossibility results without *ex post* investments and the possibility results with *ex post* investments are presented. Provision of public goods is discussed as an application of our model in Section 5. Section 6 concludes. All proofs are in the appendix.

2 Example: Spectrum Auction

Before introducing the general model, we provide intuition for our main theorems (Theorem 1 and 2) using a simple example of an auction. Consider a situation where two firms, A and B, are competing for a single spectrum license. The spectrum license is sold in the English auction, in which the price rises continuously from zero and each firm can drop out of the bidding. (We also consider another mechanism in the last part of the section.) The potential value of the spectrum license is in $[0, 10]$. Each firm $i =$ A, B makes investments to increase its own value a^i of the license outside the auction mechanism. Here, we model the investment behavior as the explicit choice of a value from the interval $[0, 10]$.[5] In order to realize $a^{\mathrm{A}}, a^{\mathrm{B}} \in [0, 10]$, each firm incurs the cost of investment which is represented by the following cost functions:

$$c^{\mathrm{A}}(a^{\mathrm{A}}) = \frac{1}{6}(a^{\mathrm{A}})^2,$$
$$c^{\mathrm{B}}(a^{\mathrm{B}}) = \frac{1}{4}(a^{\mathrm{B}})^2.$$

For simplicity, we assume that there is only one cost type for each agent. We alo assume that cost functions are common knowledge between firms and investments are observable (but not verifiable). Therefore, the information is complete between firms in the games which will be defined below.[6] The mechanism designer does not observe either their investments or cost types.

First, consider efficient investments and allocation which maximize the sum of each firm's profit from the license inclusive of the cost of investments (i.e., the social welfare). If firm A obtains the license, the optimal investment would be

$$\arg\max_{a^{\mathrm{A}} \in [0,10]} \left\{ -\frac{1}{6}(a^{\mathrm{A}})^2 + a^{\mathrm{A}} \right\} = 3.$$

[5]This means that we are assuming no externality for investments.

[6]In the general model, the complete information assumption of the cost types will be relaxed. But we still assume that investments are observable among agents.

The maximum net profit for firm A in this case is

$$\max_{a^{\mathrm{A}}\in[0,10]} \left\{ -\frac{1}{6}(a^{\mathrm{A}})^2 + a^{\mathrm{A}} \right\} = \frac{3}{2}.$$

Similarly, for firm B, the optimal investment would be

$$\arg\max_{a^{\mathrm{B}}\in[0,10]} \left\{ -\frac{1}{4}(a^{\mathrm{B}})^2 + a^{\mathrm{B}} \right\} = 2.$$

The maximum net profit for firm A in this case is

$$\max_{a^{\mathrm{B}}\in[0,10]} \left\{ -\frac{1}{4}(a^{\mathrm{B}})^2 + a^{\mathrm{B}} \right\} = 1.$$

Since there is a single license, it is clear that only one of the firms should make a positive investment to achieve investment efficiency. Therefore, the unique profile of efficient investments is $(a^{*\mathrm{A}}, a^{*\mathrm{B}}) = (3, 0)$ and we should allocate the license to firm A. The maximum social welfare is $\frac{3}{2}$.

Now we define the investment stage as a game between these two firms, and examine whether every equilibrium of the investment game achieves efficiency. The following two settings are considered: [1] firms make investments only before the mechanism, and [2] they make investments before and after the mechanism. We analyze the English auction in both cases, and also analyze another mechanism in the second setting. We consider trembling-hand perfect equilibrium (in the agent-normal form) in this section to exclude unintuitive equilibria of the English auction.[7]

[1] Investments only before the English auction.

In this case, we model the *ex ante* investment stage as a simultaneous move game where each firm chooses its own valuation.[8] The timeline of the investment and the auction is as follows:

1. Each firm $i = \mathrm{A}, \mathrm{B}$ chooses its own valuation a^i from $[0, 10]$ simultaneously. The cost of investment $c^i(a^i)$ is paid.

2. They participate in the English auction given the valuations $(a^{\mathrm{A}}, a^{\mathrm{B}})$.

[7] In the next section, we employ perfect Bayesian Nash equilibria for the analysis of the investment game.

[8] My main results do not heavily rely on the simultaneity of investments. For example, the inefficient equilibrium in the first setting is also achieved when firm B moves first. In addition, the efficiency result in the second setting under the English auction is robust to the sequential moves of firms because firm B would not want to invest whatever the sequence of the move is.

First, consider the English auction stage. The unique trembling-hand perfect equilibrium is that each firm drops out when its value is reached.[9] Since the valuations of the license for firms are $(a^{\mathrm{A}}, a^{\mathrm{B}})$, firm $i \in \{A, B\}$ whose valuation is higher than the other, i.e., $a^i \geq a^j$ where $j \neq i$, wins the license and pays a^j in the unique equilibrium. Therefore, given the equilibrium of the English auction, for any choice of investments $(a^{\mathrm{A}}, a^{\mathrm{B}}) \in [0, 10]^2$, the net profit of firm $i = \mathrm{A}, \mathrm{B}$ is written as

$$-c^i(a^i) + (a^i - a^j)\mathbb{1}_{\{a^i \geq a^j\}}$$

where j is the other firm.[10]

Next, analyze the equilibrium of the investment stage. First, it is easy to see that the socially efficient investments $(a^{*\mathrm{A}}, a^{*\mathrm{B}}) = (3, 0)$ are achieved in equilibrium. Consider another investment profile $(a^{\mathrm{A}}, a^{\mathrm{B}}) = (0, 2)$ where firm A makes no investment and firm B chooses 2 *ex ante*. Consider firm A's incentive given $a^{\mathrm{B}} = 2$. If firm A wins the auction, the payment in the English auction would be 2, which exceeds the maximum net profit of $\frac{3}{2}$ for firm A;

$$-\frac{1}{6}(a^{\mathrm{A}})^2 + (a^{\mathrm{A}} - 2)\mathbb{1}_{\{a^{\mathrm{A}} \geq 2\}} \leq \frac{3}{2} - 2 < 0$$

for any $a^A \in [0, 10]$. Thus, firm A does not have the incentive to win the auction by making a positive investment. For firm B, it is clear that choosing 2 is optimal given that firm A does not make any investments because B will obtain the license in the auction. Therefore, this profile $(a^{\mathrm{A}}, a^{\mathrm{B}}) = (0, 2)$ is an equilibrium of the *ex ante* investment game. However, this is not an efficient investment profile because it gives less social welfare than $(a^{*\mathrm{A}}, a^{*\mathrm{B}}) = (3, 0)$. Thus, we can conclude that there is a socially inefficient trembling-hand perfect equilibrium. □

This is an example where the English auction failed to fully implement efficient investments. Unfortunately, we show that not only the English auction but any other mechanism

[9] Under complete information, there are other subgame-perfect equilibria. For example, a firm whose valuation is lower than the other drops out at price zero in a subgame-perfect equilibrium because dropping out at any low price is indifferent for the losing firm with complete information. However, it is not a trembling-hand perfect equilibrium because dropping out at its own valuation is strictly better when every action of the other firm is taken with a positive probability.

[10] $\mathbb{1}$ is an indicator function. For any proposition p, $\mathbb{1}_{\{p\}}$ is defined by

$$\mathbb{1} = \begin{cases} 1 & \text{if } p \text{ is true,} \\ 0 & \text{otherwise.} \end{cases}$$

fails to implement efficient investments in the general model when there are no *ex post* investment opportunities and the allocation is selected efficiently (Theorem 1). Next, let's consider what will happen with *ex post* investments when the same English auction is used.

[2-1] Investments before and after the English auction.

When *ex post* investments are possible, another investment stage for revising their own valuations is added after the mechanism. The timeline of the investment and the auction in this case is:

1. Each firm $i = \mathrm{A}, \mathrm{B}$ chooses its own valuation a^i from $[0, 10]$ simultaneously. The cost of investment $c^i(a^i)$ is paid.

2. They participate in the English auction.

3. Each firm $i = \mathrm{A}, \mathrm{B}$ again chooses its own valuation $\bar{a}^i$ from $[a^i, 10]$. The cost of additional investment $c^i(\bar{a}^i) - c^i(a^i)$ is paid.

As we discuss in the next section, we assume the irreversibility of investments; $\bar{a}^i$ can be only chosen from $[a^i, 10]$. Also, the cost function is assumed to be unchanged over time so that for a fixed total amount $\bar{a}^i$, the total cost of investment is $c^i(\bar{a}^i)$ and choosing any *ex ante* investments $a^i \in [0, \bar{a}^i]$ is indifferent if the allocation is fixed. However, since we consider an auction mechanism to determine the allocation, *ex ante* choices matter as they change the outcome of the auction. The net profit of firm $i = \mathrm{A}, \mathrm{B}$ is written as

$$-c^i(a^i) + (\bar{a}^i - p)\mathbb{1}_{\{i \text{ wins the auction}\}} - (c^i(\bar{a}^i) - c^i(a^i))$$

where p is the payment in the auction, whose equilibrium value will be computed below.

Although the investment game is different from the first setting, efficient investments and allocation are unchanged; firm A should obtain the license and it makes investments $(a^{*\mathrm{A}}, \bar{a}^{*\mathrm{A}}) \in [0, 10]^2$ such that $a^{*\mathrm{A}} \leq \bar{a}^{*\mathrm{A}} = 3$. Firm B should not make any investment, i.e., $(a^{*\mathrm{B}}, \bar{a}^{*\mathrm{B}}) = (0, 0)$.

The equilibrium is solved by backward induction. Consider firm A's optimal strategy in the *ex post* investment stage. Given any *ex ante* valuation choice $a^{\mathrm{A}} \in [0, 10]$, the profit from the license in the last stage is

$$\bar{a}^{\mathrm{A}} - \left(c^{\mathrm{A}}(\bar{a}^{\mathrm{A}}) - c^{\mathrm{A}}(a^{\mathrm{A}})\right).$$

Thus, it makes further investment only when it obtains the license and a^{A} is less than 3. The optimal *ex post* investment strategy given a^{A} is

$$\bar{a}^{\mathrm{A}} = \begin{cases} \max\{3, a^{\mathrm{A}}\} & \text{if firm A obtains the license,} \\ a^{\mathrm{A}} & \text{otherwise.} \end{cases}$$

Similarly, firm B's optimal *ex post* investment strategy given a^{B} is

$$\bar{a}^{\mathrm{B}} = \begin{cases} \max\{2, a^{\mathrm{B}}\} & \text{if firm B obtains the license,} \\ a^{\mathrm{B}} & \text{otherwise.} \end{cases}$$

Next, analyze the English auction. Again, in the unique trembling-hand perfect equilibrium, the firm with the higher willingness to pay should win and it pays the other firm's valuation. Let $b^i(a^i)$ be the value of the license in the auction stage when firm i chooses a^i *ex ante*. The following two things should be noted in calculating it; (i) $b^i(a^i)$ takes account of the optimal strategy in the *ex post* stage, and (ii) the cost of *ex ante* investment is sunk. For each $a^{\mathrm{A}} \in [0, 10]$, it is

$$b^{\mathrm{A}}(a^{\mathrm{A}}) = \max_{\bar{a}^{\mathrm{A}} \in [a^{\mathrm{A}}, 10]} \left\{ \bar{a}^{\mathrm{A}} - \left(c^{\mathrm{A}}(\bar{a}^{\mathrm{A}}) - c^{\mathrm{A}}(a^{\mathrm{A}}) \right) \right\} = \begin{cases} \frac{3}{2} + \frac{1}{6}(a^{\mathrm{A}})^2 & \text{if } a^{\mathrm{A}} \in [0, 3) \text{ and} \\ a^{\mathrm{A}} & \text{if } a^{\mathrm{A}} \in [3, 10], \end{cases}$$

and for each $a^{\mathrm{B}} \in [0, 10]$,

$$b^{\mathrm{B}}(a^{\mathrm{B}}) = \max_{\bar{a}^{\mathrm{B}} \in [a^{\mathrm{B}}, 10]} \left\{ \bar{a}^{\mathrm{B}} - \left(c^{\mathrm{B}}(\bar{a}^{\mathrm{B}}) - c^{\mathrm{B}}(a^{\mathrm{B}}) \right) \right\} = \begin{cases} 1 + \frac{1}{4}(a^{\mathrm{B}})^2 & \text{if } a^{\mathrm{B}} \in [0, 2) \text{ and} \\ a^{\mathrm{B}} & \text{if } a^{\mathrm{B}} \in [2, 10]. \end{cases}$$

Intuitively, when firm i's initial investment a^i is more than the optimal value 3, $b^i(a^i)$ is equal to a^i as there is no further investment. If a^i is less than the optimal value 3, $b^i(a^i)$ is increasing in a^i exactly by the amount of $c^i(a^i)$ because more *ex ante* investment means less cost of additional investment when the license is awarded to the firm. Under the unique equilibrium of the English auction, if firm A wins the license, the payment will be $b^{\mathrm{B}}(a^{\mathrm{B}})$ and vice versa.

Given these equilibrium strategies, we can analyze the first investment stage. Consider firm B's incentive. If it wins the license in the English auction, the payment is at least $\frac{3}{2}$ because $b^{\mathrm{A}}(a^{\mathrm{A}}) \geq \frac{3}{2}$ holds for any $a^{\mathrm{A}} \in [0, 10]$. However, since the maximum net profit from the spectrum license is 1 for firm B, it does not have the incentive to win by choosing $a^{\mathrm{B}} > \frac{3}{2}$;

$$\max\{2, a^{\mathrm{B}}\} - b^{\mathrm{A}}(a^{\mathrm{A}}) - \frac{1}{4}\left(\max\{2, a^{\mathrm{B}}\}\right)^2 \leq 1 - \frac{3}{2} < 0.$$

Therefore, firm B refrains from making investments in equilibrium, and chooses $a^{*\mathrm{B}} = 0$. Since firm A always wins the auction with the payment $b^{\mathrm{B}}(0) = 1$, it is indifferent to choose any investments $(a^{*\mathrm{A}}, \bar{a}^{*\mathrm{A}})$ such that $a^{*\mathrm{A}} \leq \bar{a}^{*\mathrm{A}} = 3$.[11] Therefore, investment efficiency is achieved in any trembling-hand perfect equilibrium. □

Now allowing for *ex post* investments, any trembling-hand perfect equilibrium achieves investment efficiency in the English auction. Why did this become possible? Intuitively, with only *ex ante* investments, if firm A has not made any investment, it will drop out at price zero in the English auction and firm B will choose an investment $a^{\mathrm{B}} = 2$ to maximize its profit. Furthermore, firm A will optimally choose not to make any investment given $a^{\mathrm{B}} = 2$ because firm B will stay too long in the English auction for firm A to make a profit from any positive investment. On the other hand, with *ex post* investments, firm A will stay in the English auction until the price reaches $\frac{3}{2}$ because firm A can make a profit when firm B drops out before $\frac{3}{2}$. Now, since firm B's payment exceeds $\frac{3}{2}$ if it wins the auction, it cannot make a profit from any positive investment.

However, when we consider other mechanisms, allowing *ex post* investment does not always solve the problem. More importantly, this is not because the mechanism fails to allocate the license efficiently, but because an inefficient investment equilibrium exists even though the mechanism always selects an efficient allocation (according to the valuations in the auction stage).

To introduce such an example of a mechanism, we review the literature of (subgame-perfect) implementation. A seminal paper by Moore and Repullo (1988) showed that under complete information and quasi-linear utility functions, any social choice function is subgame-perfect implementable. This implies that by their mechanism, we can implement an efficient allocation rule with any transfer rule. Consider here one such mechanism: a Moore-Repullo mechanism which always chooses an efficient allocation according to $(b^{\mathrm{A}}(a^{\mathrm{A}}), b^{\mathrm{B}}(a^{\mathrm{B}}))$ and does not impose any transfers.[12]

[2-2] Investments before and after the efficient Moore-Repullo mechanism with no transfers.

The timeline of the investment game is the same as in the previous case [2-1]. The English

[11] When there is a strict time discounting as we consider in the general model, the unique optimal investment is $(a^{*\mathrm{A}}, \bar{a}^{*\mathrm{A}}) = (0, 3)$.

[12] In some countries such as Japan, spectrum licenses are still allocated to firms for free once they are screened by the government. Although this process is not a mechanism, if the government correctly observes the valuations $(b^{\mathrm{A}}(a^{\mathrm{A}}), b^{\mathrm{B}}(a^{\mathrm{B}}))$, it is exactly the social choice function implemented by this Moore-Repullo mechanism.

auction is replaced by the following mechanism.

Stage 1:

1-1. Firm A announces its own valuation $\bar{b}^{\mathrm{A}}$.

1-2. Firm B decides whether to challenge firm A's announcement $\bar{b}^{\mathrm{A}}$.

If firm B does not challenge it, go to stage 2.

If firm B challenges, firm A pays 20 to the mechanism designer. Firm B receives 20 if the challenge is successful, but pays 20 to the mechanism designer if it is a failure. Whether it is a success or a failure is determined by the following game: The license is sold in the second-price auction. Firm B chooses some $\bar{b}^{\mathrm{B}}$ to submit to the auction and a positive value $\eta > 0$, and asks firm A to choose one of them:

(i) submitting any value,

(ii) submitting $\bar{b}^{\mathrm{A}}$ and receiving an additional transfer η.

The challenge is successful only if firm A picks (i). Stop.

Stage 2: Same as stage 1, but the roles of A and B are switched.

Stage 3: If there are no challenges in stage 1 and 2, the license is given for free to firm i such that $\bar{b}^i \geq \bar{b}^j$ where j is the other firm.

Given the optimal strategies in the *ex post* investment stage, for any profile of *ex ante* investments $(a^{\mathrm{A}}, a^{\mathrm{B}})$, it is shown that the unique subgame-perfect (and also trembling-hand perfect) equilibrium of this mechanism is such that each firm $i =$ A, B announces its true valuation $b^i(a^i)$, and no firm challenges the other firm's claim (Moore and Repullo, 1988). The intuitive reason is that in the challenge phase, the other firm j can choose some $\bar{b}^j$ and $\eta > 0$ so that the challenge is successful (firm i optimally chooses (i)) whenever the announcement $\bar{b}^i$ of firm i is different from $b^i(a^i)$. Also, the other firm's challenge would never be successful when the announcement is truthful since (ii) is always chosen by a truthful firm. Therefore, the allocation is always determined efficiently and no transfer is imposed in equilibrium.

Consider firm B's incentive in the first investment stage. Now firm B has the incentive to invest more than firm A as long as A's investment is socially efficient, i.e., $a^{\mathrm{A}} \leq 3$. This is because the price of the license is zero in the mechanism and firm B would still earn a positive profit by winning the auction: for some $a^{\mathrm{B}} \in (3, 4)$,

$$\max\{2, a^{\mathrm{B}}\} - 0 - \frac{1}{4}\big(\max\{2, a^{\mathrm{B}}\}\big)^2 > 0.$$

Actually, there is a mixed strategy equilibrium in which $a^{\mathrm{B}} > 0$ occurs with a positive probability. Thus, efficient investments are not implemented by this allocatively efficient Moore-Repullo mechanism with no transfers. □

In the English auction with *ex post* investments, firm B could not make a profit by investing $a^{\mathrm{B}} = 2$ because the price of the license was greater than $\frac{3}{2}$. However, in this zero-payment mechanism, $a^{\mathrm{B}} = 2$ remains profitable because firm B does not pay anything in the auction. This shows that the range of the price of the license is critical for inducing the right incentive for firm B. Suppose that the allocation is always efficiently determined, and that firm A does not make any *ex ante* investment, i.e., $a^{\mathrm{A}} = 0$. Then, firm B would lose the auction when choosing $a^{\mathrm{B}} = 0$, but would win the auction if it chooses $a^{\mathrm{B}} = 2$. In order to prevent firm B from choosing 2, the price p of the license when firms choose $(a^{\mathrm{A}}, a^{\mathrm{B}}) = (0, 2)$ *ex ante* should satisfy

$$0 \geq b^{\mathrm{B}}(2) - c^{\mathrm{B}}(2) - p \;\Leftrightarrow\; p \geq 1.$$

Obviously, the English auction satisfied this condition, but the Moore-Repullo mechanism with no transfers violated it. This idea of disincentivizing *ex ante* investment with a right transfer rule can be applied to more general environments. Our main contribution is to discover a property of a social choice function, which we call *commitment-proofness*, in the general model and to show that it is sufficient and necessary for implementing efficient investments.

3 General Model

There is a finite set I of agents and a finite set Ω of alternatives. A valuation function of agent $i \in I$ is $v^i : \Omega \to \mathbb{R}$. The valuation function is endogenously determined by each agent's investment decision as described below. The set of possible valuation functions is $V^i \subseteq \mathbb{R}^{\Omega}$. Assume that V^i is a compact set. Denote the profile of the sets of valuations by $V \equiv \times_{i \in I} V^i$. We assume that investments are not verifiable to a third party. Therefore, a mechanism chooses an alternative and transfers, but does not choose agents' investment behaviors. We discuss the relationship between social choice rules and mechanisms later in this section.

Each agent makes an investment decision to determine her own valuation over alternatives. The investment is modeled as an explicit choice of a valuation function with the cost of investment determined by a cost function $c^i : V^i \times \Theta^i \to C^i \subseteq \mathbb{R}_+$ where Θ^i is a finite set of cost types of agent i. Each agent i knows her own cost type $\theta^i \in \Theta^i$, but may be unsure about $\theta^{-i} \equiv (\theta^j)_{j\in I\setminus\{i\}}$. There is a common prior distribution on $\Theta \equiv \times_{i\in I}\Theta^i$, denoted p. Conditional on knowing her own cost type θ^i, agent i's posterior distribution over $\Theta^{-i} \equiv \times_{j\in I\setminus\{i\}}\Theta^j$ is denoted $p(\cdot|\theta^i)$. $p(\cdot|\theta^i)$ is computed by Bayes rule whenever θ^i occurs with a positive probability, i.e., $\sum_{\theta^{-i}\in\Theta^{-i}} p(\theta^i, \theta^{-i}) > 0$.[13] Assume that C^i is a compact set and $0 \in C^i$. Denote the profile of the sets of possible costs by $C \equiv \times_{i\in I}C^i$. Without loss of generality, the cost of investment is assumed to be non-negative, and we also assume that for each $\theta^i \in \Theta^i$, there is $v^i \in V^i$ such that $c^i(v^i, \theta^i) = 0$. There are two investment stages; before and after participating in the mechanism. We model each of the investment stages as a simultaneous move game by all agents. Assume that the investment is irreversible; if agent i with cost type θ^i chooses $v^i \in V^i$ before the mechanism, she can only choose a valuation function from the set $\{\bar{v}^i \in V^i | c^i(\bar{v}^i, \theta^i) \geq c^i(v^i, \theta^i)\}$ in the second investment stage.[14] To clarify, the timeline of the investment game induced by a mechanism is:

0. Each agent i observes her own cost type $\theta^i \in \Theta^i$.

1. Each agent makes a prior investment by choosing a valuation function $v^i \in V^i$ simultaneously.

2. Agents participate in a mechanism.

3. After the mechanism is run, each agent can make an additional investment, i.e., each agent chooses a valuation function from $\{\bar{v}^i \in V^i | c^i(\bar{v}^i, \theta^i) \geq c^i(v^i, \theta^i)\}$.

We assume that chosen valuation functions are observable among agents (but not verifiable). Also, assume that cost functions are common knowledge among agents and the mechanism designer. However, each agent only knows her own cost type and the distribution of other agents' cost types. The mechanism designer does not know the realized cost type vector θ or the common prior distribution p. The investment game is an incomplete information game if p is a non-degenerate distribution. We allow for the complete information case where $p(\theta) = 1$ for some $\theta \in \Theta$. Throughout the analyses in this paper, we fix the set I of agents, the set Ω of alternatives, the set Θ of cost types and the common prior distribution p.

The *ex ante* utility function of an agent has the following three components: the valuation functions she chooses in the first and the second investment stages, the cost function and a

[13] If $\sum_{\theta^{-i}\in\Theta^{-i}} p(\theta^i, \theta^{-i}) = 0$, we assign any arbitrary posterior distribution $p(\cdot|\theta^i)$.

[14] The essential assumption is actually that the cost of *ex ante* investment is sunk, rather than the (physical) irreversibility of an investment itself. However, we maintain the assumption of irreversibility since it keeps the analysis simple and easy to understand.

discount factor. Let $\beta \in (0,1]$ be a discount factor which discounts the utility realized in the second stage and later.[15] For an alternative $\omega \in \Omega$, a transfer vector $t \equiv (t^i)_{i\in I} \in \mathbb{R}^I$ and an investment schedule $(v^i, \bar{v}^i) \in (V^i)^2$ where v^i is the valuation function chosen before the mechanism and $\bar{v}^i$ is the final valuation function, the *ex ante* utility of agent i with cost type θ^i is defined by

$$-c^i(v^i,\theta^i) + \beta\Big[\bar{v}^i(\omega) - t^i - \big(c^i(\bar{v}^i,\theta^i) - c^i(v^i,\theta^i)\big)\Big].^{16} \tag{1}$$

In the first stage, only the cost $c^i(v^i,\theta^i)$ of *ex ante* investment is paid. In the second stage, the outcome (ω, t) of the mechanism is evaluated by the final valuation function $\bar{v}^i$. And in the last stage, the additional cost $c^i(\bar{v}^i,\theta^i) - c^i(v^i,\theta^i) \geq 0$ of revising the valuation function is paid. Throughout the paper, we consider this quasi-linear utility function, i.e., utility to be perfectly transferable.

When agents face the mechanism in the second stage, the cost of investment made in the first stage is already sunk. Moreover, in any equilibrium, an alternative $\omega \in \Omega$ is evaluated by a valuation function which is the optimal choice of the *ex post* investment. Therefore, we can define the valuations of agents at the time of the mechanism as follows using the notation b^{c^i,θ^i,v^i} for any cost function $c^i : V^i \times \Theta^i \to C^i$, cost type $\theta^i \in \Theta^i$ and the prior investment $v^i \in V^i$.

Definition 1. *The valuation function $b^{c^i,\theta^i,v^i} : \Omega \to \mathbb{R}$ at the time of the mechanism* given a cost function $c^i : V^i \times \Theta^i \to C^i$, a cost type $\theta^i \in \Theta^i$ and a valuation function $v^i \in V^i$ is defined by

$$b^{c^i,\theta^i,v^i}(\omega) = \max_{\bar{v}^i \in \{\tilde{v}^i \in V^i \mid c^i(\tilde{v}^i,\theta^i) \geq c^i(v^i,\theta^i)\}} \Big\{\bar{v}^i(\omega) - c^i(\bar{v}^i,\theta^i)\Big\} + c^i(v^i,\theta^i)$$

for each $\omega \in \Omega$. Let $b^{c,\theta,v} \equiv (b^{c^i,\theta^i,v^i})_{i\in I}$.

The equation is taken from the second term of equation (1), and takes account of each agent's optimal *ex post* investment choice given the cost type. Given a prior investment $v^i \in V^i$ and an alternative $\omega \in \Omega$, the optimal choice of the *ex post* investment should be

[15] There is no time discounting between the mechanism stage and the *ex post* investment stage, but this is without loss of generality.

[16] Here, we assume that the same cost function is used for both investment stages. Some of the main results, however, still hold when the cost functions differ across time. For example, the sufficiency part of our possibility theorem (Theorem 2) holds as long as the *ex post* cost function is weakly lower than the *ex ante* cost function.

$\bar{v}^i \in V^i$ which maximizes the net value $\bar{v}^i(\omega) - c^i(\bar{v}^i, \theta^i)$ among the set of feasible valuation functions, which is $\{\tilde{v}^i \in V^i | c^i(\tilde{v}^i, \theta^i) \geq c^i(v^i, \theta^i)\}$.[17]

A social choice function $h : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ is defined as a mapping from the potential set $\mathbb{R}^{\Omega \times I}$ of valuation functions at the time of the mechanism to the set Ω of alternatives and the set $\mathbb{R}^I$ of transfer vectors. A social choice function $h \equiv (h_\omega, h_t)$ has the following two components; $h_\omega : \mathbb{R}^{\Omega \times I} \to \Omega$ is called an allocation rule and $h_t : \mathbb{R}^{\Omega \times I} \to \mathbb{R}^I$ is called a transfer rule. The transfer rule for each agent is denoted by $h_t^i : \mathbb{R}^{\Omega \times I} \to \mathbb{R}$ and $h_t(b) = (h_t^i(b))_{i \in I}$ holds for any $b \in \mathbb{R}^{\Omega \times I}$. Note that the domain $\mathbb{R}^{\Omega \times I}$ of social choice functions is not restricted by V, but defined to include any potential valuation functions at the time of the mechanism. Therefore, a social choice function is defined only for a tuple (I, Ω). As we see below, when we define the implementability of efficient investments given a social choice function, we consider any possible set $V \subseteq \mathbb{R}^{\Omega \times I}$ of valuation functions and a profile of cost functions $c : V \times \Theta \to C$.

We are interested in whether efficient investments are fully implementable in perfect Bayesian Nash equilibria when an allocatively efficient social choice function is implemented. In this paper, we focus on the analysis of an investment game induced by a social choice function, and do not explicitly consider mechanisms to implement the social choice function. Although we do not discuss whether a specific social choice function is implementable, the literature has shown several positive results under both complete and incomplete information. For example, Moore and Repullo (1988) showed that any social choice function is subgame-perfect implementable by their extensive form mechanism under transferable utility and complete information environments.[18] Their extensive form mechanism only works under complete information, but even under incomplete information, Abreu and Matsushima (1992) showed that a large class of social choice functions are virtually implementable. Therefore, we take these positive theorems as given, and simply consider the entire space of social choice functions in this paper. We leave the equilibrium analysis within a mechanism outside the scope of the paper, and concentrate on finding out the properties of social choice functions

[17]If the cost of *ex ante* investments is refundable, the valuation function at the time of the mechanism only shifts by a constant for any choice of *ex ante* investment (since the first term of $b^{c^i, \theta^i, v^i}(\omega)$ would then be fixed). This means that concepts such as allocative efficiency (defined shortly) are not essentially affected by the *ex ante* investment behaviors. Therefore, we focus on the non-trivial cases where *ex ante* investment is irreversible.

[18]To make use of the Moore-Repullo mechanism, the utility of agents must be uniformly bounded. Thus, the amount of penalty used in this mechanism needs to depend on (V, C), but it can be appropriately chosen in this setting because V and C are both bounded.

which enable us to implement efficient investments.

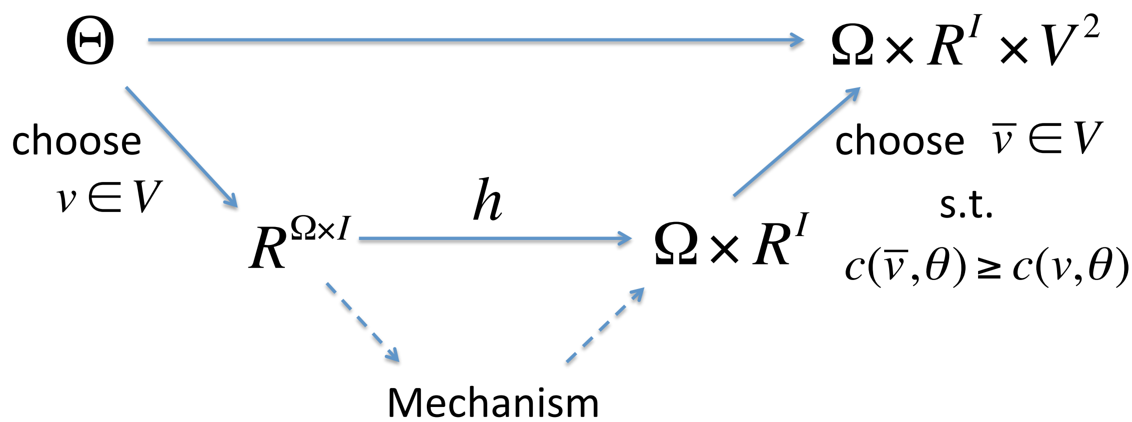

Figure 1. The structure of a social choice function and the investment game.

To introduce the implementability of efficient investments, we first define a perfect Bayesian Nash equilibrium of an investment game induced by a given social choice function. For the set of strategies in the first investment stage, we denote the set of all mappings from Θ^i to V^i by Σ^i. For the set of strategies in the last investment stage, we denote the set of all mappings from $V^i \times \Omega \times \Theta^i$ to V^i as $\mathcal{M}^i$. Let $\Sigma \equiv \times_{i\in I}\Sigma^i$ and $\mathcal{M} \equiv \times_{i \in I}\mathcal{M}^i$.

Definition 2. For any $V \subseteq \mathbb{R}^{\Omega\times I}$ and any profile of cost functions $c : V \times \Theta \to C$, a profile of investment strategies $(\sigma^*, \mu^*) \in \Sigma \times \mathcal{M}$ is a *perfect Bayesian Nash equilibrium (PBNE) of the investment game* given a social choice function $h : \mathbb{R}^{\Omega\times I} \to \Omega \times \mathbb{R}^I$ and a discount factor $\beta \in (0, 1]$ if for each $i \in I$ and $\theta^i \in \Theta^i$,

1. $\mu^{*i}(v^i, \omega, \theta^i) \in \underset{\bar{v}^i\in\{\tilde{v}^i\in V^i|c^i(\tilde{v}^i,\theta^i)\geq c^i(v^i,\theta^i)\}}{\arg\max} \Big\{\bar{v}^i(\omega) - c^i(\bar{v}^i, \theta^i)\Big\}$

   for any $v^i \in V^i$ and $\omega \in \Omega$, and

2. $$\sigma^{*i}(\theta^i) \in \underset{v^i\in V^i}{\arg\max} \Big\{ - c^i(v^i, \theta^i) + \beta \textstyle\sum_{\theta^{-i}\in\Theta^{-i}} p(\theta^{-i}|\theta^i) \Big[\mu^{*i}(v^i, h_\omega(b^{c^i,\theta^i,v^i}, b^{-i}), \theta^i)(h_\omega(b^{c^i,\theta^i,v^i}, b^{-i})) - h^i_t(b^{c^i,\theta^i,v^i}, b^{-i}) - c^i(\mu^{*i}(v^i, h_\omega(b^{c^i,\theta^i,v^i}, b^{-i}), \theta^i), \theta^i) + c^i(v^i, \theta^i)\Big]\Big\}$$

   where $b^{-i} \equiv b^{c^{-i},\theta^{-i},\sigma^{*-i}(\theta^{-i})}$

hold.

The first condition of a PBNE is the optimality in the *ex post* investment stage. Since the investment does not have an externality, this is simply an individual maximization problem. Therefore, it is defined for each information set in the last stage, which is characterized by the choice of the first stage investment, the realized alternative and the cost type of the agent. The second condition requires that σ^* forms a Bayesian Nash equilibrium of the first stage investment game, given the optimal *ex post* investment strategy μ^* and the social choice function h.

Full implementation of efficient investments requires that any PBNE of the investment game should be socially efficient. More precisely, given that a social choice function h is implemented, efficient investments are said to be implementable in PBNE if for any profile of the sets of valuations and cost functions, any PBNE of the investment game given h and a discount factor β maximizes the sum of expected utility of agents inclusive of the cost of investments given h and β.

Definition 3. Given a social choice function $h : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ and a discount factor $\beta \in (0, 1]$, *efficient investments are implementable in perfect Bayesian Nash equilibria* if for any $V \subseteq \mathbb{R}^{\Omega \times I}$ and any profile of cost functions $c : V \times \Theta \to C$, any perfect Bayesian Nash equilibrium $(\sigma^*, \mu^*) \in \Sigma \times \mathcal{M}$ satisfies the following equation:

$$(\sigma^*, \mu^*) \in \arg\max_{(\sigma,\mu)\in\Sigma\times\mathcal{M}} \sum_{\theta\in\Theta} p(\theta) \sum_{i\in I} \Big\{ - c^i(\sigma^i(\theta^i), \theta^i)$$
$$+ \beta\Big[\mu^i(\sigma^i(\theta^i), h_\omega(b^{c,\theta,\sigma(\theta)}), \theta^i)(h_\omega(b^{c,\theta,\sigma(\theta)})) - c^i(\mu^i(\sigma^i(\theta^i), h_\omega(b^{c,\theta,\sigma(\theta)}), \theta^i), \theta^i) + c^i(\sigma^i(\theta^i), \theta^i)\Big]\Big\}.$$

Next, we define the properties of social choice functions. There are two versions of allocative efficiency. The first definition of allocative efficiency is standard; the allocation rule chooses an alternative to maximize the sum of the valuation of agents. A social choice function $h : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ is allocatively efficient if for any $b \in \mathbb{R}^{\Omega \times I}$,

$$h_\omega(b) \in \arg\max_{\omega\in\Omega} \sum_{i\in I} b^i(\omega).$$

Our main theorem (Theorem 2) holds for a weaker notion of allocative efficiency, which is called allocative constrained-efficiency. This guarantees allocative efficiency within a certain subset of alternatives.

Definition 4. A social choice function $h : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ is *allocatively constrained-efficient for $\Omega' \subseteq \Omega$ with $\Omega' \neq \emptyset$* if for any $b \in \mathbb{R}^{\Omega \times I}$, the allocation rule satisfies

$$h_\omega(b) \in \arg\max_{\omega\in\Omega'} \sum_{i\in I} b^i(\omega).$$

Note that Ω' in the definition above can be a singleton set. Thus a constant social choice function $\bar{h} : \mathbb{R}^{\Omega\times I} \to \Omega\times\mathbb{R}^I$ such that $\bar{h}_\omega(b) = \bar{\omega} \in \Omega$ for any $b \in \mathbb{R}^{\Omega\times I}$ also satisfies allocative constrained-efficiency for $\Omega' \equiv \{\bar{\omega}\}$. We also say that an allocation rule $h_\omega : \mathbb{R}^{\Omega\times I} \to \Omega$ is allocatively (constrained-) efficient if a social choice function $h \equiv (h_\omega, h_t)$ is allocatively (constrained-) efficient.

As mentioned in the introduction, a new concept called commitment-proofness plays a crucial role in our possibility theorem (Theorem 2). Since it will need a careful explanation, we will defer the definition of commitment-proofness to subsection 4.2 where we begin to discuss the possibility of implementing efficient investments.

4 Implementation of Efficient Investments

4.1 Impossibility without Ex Post Investments

In the literature, it is often assumed that investments are made only before the mechanism. In such a situation, Rogerson (1992) and Hatfield, Kojima and Kominers (2015) showed that we can find an efficient equilibrium of the investment game given allocatively efficient and strategy-proof social choice functions. But at the same time, another inefficient equilibrium exists in many examples. This is due to the fact that the *ex ante* investment stage incentivizes some agents to make more investments than at the efficient level and generates a multiplicity of equilibria. To see if this observation can be generalized, we consider the implementability of efficient investments without the post-mechanism investments in our model. For this purpose, we need to redefine the implementability of efficient investments for this environment accordingly.

When *ex post* investments are not allowed, the investment game induced by a social choice function is a one-shot game which takes place before the mechanism. Thus, the equilibrium concept we employ in the investment game reduces to a Bayesian Nash equilibrium in this case.

Definition 5. For any $V \subseteq \mathbb{R}^{\Omega\times I}$ and any profile of cost functions $c : V \times \Theta \to C$, a profile of investment strategies $\sigma^* \in \Sigma$ is a *Bayesian Nash equilibrium of the ex ante investment game* given a social choice function $h : \mathbb{R}^{\Omega\times I} \to \Omega \times \mathbb{R}^I$ and a discount factor $\beta \in (0, 1]$ if for each $i \in I$ and $\theta^i \in \Theta^i$,

$$\sigma^{*i}(\theta^i) \in \arg\max_{v^i \in V^i} \Big\{ -c^i(v^i, \theta^i) + \beta \sum_{\theta^{-i}\in\Theta^{-i}} p(\theta^{-i}|\theta^i) \Big[v^i(h_\omega(v^i, \sigma^{*-i}(\theta^{-i}))) - h^i_t(v^i, \sigma^{*-i}(\theta^{-i})) \Big] \Big\}$$

holds.

Implementability of efficient investments is redefined in the following way. In this environment, investment efficiency requires that the total expected utility of agents be maximized given that agents cannot revise their original choices of valuation functions after the mechanism.

Definition 6. Given a social choice function $h : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ and a discount factor $\beta \in (0, 1]$, *efficient ex ante investments are Bayesian Nash implementable* if for any $V \subseteq \mathbb{R}^{\Omega \times I}$ and any profile of cost functions $c : V \times \Theta \to C$, any Bayesian Nash equilibrium $\sigma^* \in \Sigma$ satisfies the following equation:

$$\sigma^* \in \arg\max_{\sigma \in \Sigma} \sum_{\theta \in \Theta} p(\theta) \sum_{i \in I} \Big\{ - c^i(\sigma^i(\theta^i), \theta^i) + \beta \sigma^i(\theta^i)(h_\omega(\sigma(\theta))) \Big\}.$$

The question is whether efficient *ex ante* investments are Bayesian Nash implementable given certain social choice functions. Unfortunately, the result is negative when we require allocative efficiency; for any allocatively efficient social choice function, there is a profile of the sets of valuations and cost functions under which there exists an inefficient equilibrium of the *ex ante* investment game.

Theorem 1. *Suppose $|I| \geq 2$ and $|\Omega| \geq 2$. Given any allocatively efficient social choice function $h : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ and any discount factor $\beta \in (0, 1]$, there exists $V \subseteq \mathbb{R}^{\Omega \times I}$ and a profile of cost functions $c : V \times \Theta \to C$ such that an inefficient Bayesian Nash equilibrium of the ex ante investment game exists, which means that efficient ex ante investments are not Bayesian Nash implementable.*

We show Theorem 1 by considering the following two cases: when the social choice function h is strategy-proof and when it is not. Here strategy-proofness plays a key role because *ex post* investments are not allowed and hence the model has the same structure as those considered by Rogerson (1992) and Hatfield, Kojima and Kominers (2015). Therefore, there exists an efficient Bayesian Nash equilibrium of the *ex ante* investment game if h is strategy-proof, and there may not if it is not strategy-proof. In both cases, we construct an example where the cost functions are constant across any type profile $\theta \in \Theta$ of agents, so that the investment game is under complete information.

When h is not strategy-proof, the logic follows Theorem 1 and 2 of Hatfield, Kojima and Kominers (2015) who show that for an allocatively efficient social choice function h, if agent i's *ex ante* choice of a valuation that maximizes her own utility always maximizes the social welfare given other agents' valuations, then h must be strategy-proof for i. Therefore, when it is not strategy-proof, we can construct a profile of cost functions under which, given other agents' valuations, the privately optimal *ex ante* investment choice for agent i does not achieve investment efficiency.[19]

On the other hand, for any strategy-proof social choice function, the logic of the English auction example in the previous section applies. Thus, we can always construct a case where an inefficient investment equilibrium exists in addition to the efficient one. This is because the *ex ante* investment stage gives commitment power to more than one agents although their cost functions are different. Once some agent makes a large investment, then other more efficient agents may refrain from making investments as it is costly to compete with them in the mechanism. Hence, the mechanism allows them to achieve a socially inefficient outcome in equilibrium. In the next subsection, we introduce *commitment-proofness* to eliminate such incentives when further investments are possible after the mechanism.

4.2 Commitment-proofness

The previous subsection demonstrated that inefficient equilibria cannot be ruled out if there are no *ex post* investment opportunities. In this paper, we seek the possibility of implementation by allowing the *ex post* investment opportunities. When investments are possible both *ex ante* and *ex post*, there are two opposing forces which influence the implementability of efficient investments. The *ex post* investment stage helps to achieve it by allowing agents to reflect the information of their cost types onto the valuations at the time of the mechanism. As we saw in Theorem 1 however, the *ex ante* investment stage does the opposite by preventing us from extracting the information of their cost types. Which of these two forces dominates the other depends on the characteristics of the social choice function to be implemented. To answer this question, we introduce a new concept of a social choice function called *commitment-proofness*.

Definition 7. A social choice function $h : \mathbb{R}^{\Omega\times I} \to \Omega \times \mathbb{R}^I$ is *commitment-proof* if for any $i \in I$, $b \in \mathbb{R}^{\Omega\times I}$, $\tilde{b}^i \in \mathbb{R}^\Omega$ and $x \geq 0$ such that $\tilde{b}^i(\omega) \leq b^i(\omega) + x$ for all $\omega \in \Omega$,

$$\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h^i_t(\tilde{b}^i, b^{-i}) - x \leq b^i(h_\omega(b)) - h^i_t(b). \tag{2}$$

[19]Note that the construction of cost functions is slightly different from Hatfield, Kojima and Kominers (2015) because the cost of investment in our model is non-negative whereas it is not assumed as such in their paper.

The concept of commitment-proofness involves a manipulation of an agent's true valuation through a certain commitment behavior given that the social choice function is implemented. Conceptually, this is distinct from a misreport of valuations when the social choice function is regarded as a direct mechanism, but there is indeed a close relationship with the strategy-proofness condition. This point will be demonstrated shortly. In equation (2), the non-negative value x can be interpreted as the cost of commitment. Consider a situation where each agent can make a contract with a third party that the agent will pay x to the third party in advance, and x or less will be paid back to the agent depending on the alternative chosen by the social choice function (the payback can be negative). We argue that the supposition of this situation (or other situations which bring about the same effect) is not demanding because the third party wouldn't lose anything from this contract. By making this agreement, the agent can commit to having a different valuation in the mechanism because the value from each realization of $\omega \in \Omega$ has been manipulated even though the genuine value of ω is unchanged. When agent i's original valuation function is b^i, her new valuation function given this contract will be $\tilde{b}^i$ which satisfies $\tilde{b}^i(\omega) \leq b^i(\omega) + x$ for all $\omega \in \Omega$. Equation (2) requires that no agent be able to benefit from such a commitment under h.

The following example gives a numerical illustration of a commitment $(\tilde{b}^i, x)$ for a given b^i, and shows how we detect commitment-proofness of social choice functions.

Example 1. Consider an auction with a single item and two bidders. Let $I = \{i, j\}$ and $\Omega = \{\omega^i, \omega^j\}$ where ω^i and ω^j each represent the alternatives where i and j obtain the item respectively. Suppose that the original valuation function (at the time of the mechanism) of agent i is $b^i : \Omega \to \mathbb{R}$ such that

$$b^i(\omega^i) = 10,$$
$$b^i(\omega^j) = 0.$$

Consider $x = 5$ and another valuation function (at the time of the mechanism) $\tilde{b}^i : \Omega \to \mathbb{R}$ such that

$$\tilde{b}^i(\omega^i) = 15,$$
$$\tilde{b}^i(\omega^j) = 0.$$

These x and $\tilde{b}^i$ satisfy the condition that $\tilde{b}^i(\omega) \leq b^i(\omega) + x$ for all $\omega \in \Omega$. Thus, $(\tilde{b}^i, x)$ is one of the commitments given b^i.

Suppose agent j's valuation function is fixed to $b^j : \Omega \to \mathbb{R}$ such that

$$b^j(\omega^i) = 0,$$
$$b^j(\omega^j) = 11.$$

Consider the following two social choice functions:[20]

1. The second-price auction h^{SPA} which gives the item to who values it most and has the winner pay the other agent's value, and

2. The half-price auction h^{half} which gives the item to who values it most and has the winner pay the half of her own value.

We examine whether the equation (2) holds for the example of valuation functions $b^i, \tilde{b}^i$ and b^j above.

[1] Under h^{SPA}, the RHS of equation (2) is 0 because agent i loses the auction. On the LHS, i wins the auction when her true valuation is $\tilde{b}^i$, and the utility from the auction is $\tilde{b}^i(h^{SPA}_\omega(\tilde{b}^i, b^j)) - h^{SPA,i}_t(\tilde{b}^i, b^j) = 15 - 11 = 4$. However, including the cost of commitment $x = 5$, we have

$$\tilde{b}^i(h^{SPA}_\omega(\tilde{b}^i, b^j)) - h^{SPA,i}_t(\tilde{b}^i, b^j) - x = -1 < 0 = b^i(h^{SPA}_\omega(b)) - h^{SPA,i}_t(b).$$

Thus, equation (2) holds for this example of valuation functions.[21]

[2] Under h^{half}, the RHS of equation (2) is again 0 for the same reason. On the LHS, i wins the auction when her true valuation is $\tilde{b}^i$, and the utility from the auction is $\tilde{b}^i(h^{half}_\omega(\tilde{b}^i, b^j)) - h^{half,i}_t(\tilde{b}^i, b^j) = 15 - 7.5 = 7.5$. Then, even with the cost of commitment $x = 5$, we have

$$\tilde{b}^i(h^{half}_\omega(\tilde{b}^i, b^j)) - h^{half,i}_t(\tilde{b}^i, b^j) - x = 2.5 > 0 = b^i(h^{half}_\omega(b)) - h^{half,i}_t(b).$$

Therefore, we know that the half-price auction h^{half} is not commitment-proof. □

Commitment-proofness is defined as a property of a social choice function and is not directly related to the structure of the investment game. Our main theorem establishes a strong connection between this concept and the implementability of efficient investments;

[20] The social choice function should be defined for $\mathbb{R}^{\Omega \times I}$ in general in this paper, but for this example we only consider the following domains, $B^i = \{a\mathbb{1}_{\{\omega=\omega^i\}} | a \in \mathbb{R}_+\}$ and $B^j = \{a\mathbb{1}_{\{\omega=\omega^j\}} | a \in \mathbb{R}_+\}$, to simplify the exposition of auction rules.

[21] Indeed, it is shown that this holds for any other valuation functions concerned in the definition of commitment-proofness, and that the second-price auction is commitment-proof.

commitment-proofness is sufficient and necessary for implementing efficient investments in PBNE. Intuitively, for any sets of valuation functions and cost functions, it will be shown that the cost of any costly *ex ante* investment corresponds to the cost of commitment (x) in the definition of commitment-proofness. Thus, no agent has the incentive to make a costly investment before the mechanism is run, and investment efficiency is achieved. As we will see in more detail in the next subsection, commitment-proofness works as a dividing ridge for understanding the interaction of two investment stages; (only) when commitment-proof social choice functions are implemented, the role of the *ex post* investment stage outweighs that of the *ex ante* investment stage.

As mentioned above, commitment-proofness has an interesting relationship with the more well-known strategy-proofness; any strategy-proof social choice function is commitment-proof. To see this, we first define strategy-proofness. A social choice function $h : \mathbb{R}^{\Omega\times I} \to \Omega \times \mathbb{R}^I$ is strategy-proof if for any $i \in I$, $b \in \mathbb{R}^{\Omega\times I}$ and $\tilde{b}^i \in \mathbb{R}^\Omega$,

$$b^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i}) \le b^i(h_\omega(b)) - h_t^i(b)$$

Showing that commitment-proofness is implied by strategy-proofness is straightforward: for any $i \in I$, $b \in \mathbb{R}^{\Omega\times I}$, $\tilde{b}^i \in \mathbb{R}^\Omega$ and $x \ge 0$ such that $\tilde{b}^i(\omega) \le b^i(\omega) + x$ for all $\omega \in \Omega$,

$$\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i}) - x \le b^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i}) \le b^i(h_\omega(b)) - h_t^i(b),$$

where the first inequality follows from the definition of $\tilde{b}^i$, and the second inequality holds from the strategy-proofness of h. Commitment-proofness concerns behaviors to manipulate the agents' true types outside the mechanism, rather than their misreports in the mechanism. Nonetheless, the fact that commitment-proofness is weaker than strategy-proofness implies that the consequence of commitments considered in this definition is translated into a type of misreports when the social choice function is regarded as a direct mechanism.

From this relationship, we know that the VCG auction, which is known to be strategy-proof, satisfies commitment-proofness. The VCG social choice function h^{VCG} is defined as follows: for any $b \in \mathbb{R}^{\Omega\times I}$,

$$\begin{aligned} h_\omega^{VCG}(b) &\in \arg\max_{\omega\in\Omega} \sum_{i\in I} b^i(\omega), \\ h_t^{VCG,i}(b) &= \max_{\omega\in\Omega} \sum_{j\in I\setminus\{i\}} b^j(\omega) - \sum_{j\in I\setminus\{i\}} b^j(h_\omega^{VCG}(b)) \text{ for any } i \in I. \end{aligned}$$

The second-price auction is a special case of the VCG auction, so it is also commitment-proof.

Since commitment-proofness is weaker than strategy-proofness, there exists a non-strategy-proof social choice function which is commitment-proof. Consider a class of social choice functions $h^\alpha : \mathbb{R}^{\Omega\times I} \to \Omega \times \mathbb{R}^I$ parameterized by $\alpha \in [0,1)$ such that the alternative is efficiently chosen and the payment is a convex combination of the VCG payment and each agent's own valuation from the alternative itself: for any $b \in \mathbb{R}^{\Omega\times I}$,

$$h^\alpha_\omega(b) \in \arg\max_{\omega\in\Omega} \sum_{i\in I} b^i(\omega),$$
$$h^{\alpha,i}_t(b) = \alpha\Big\{\max_{\omega\in\Omega} \sum_{j\in I\setminus\{i\}} b^j(\omega) - \sum_{j\in I\setminus\{i\}} b^j(h^\alpha_\omega(b))\Big\} + (1-\alpha)b^i(h^\alpha_\omega(b)) \text{ for any } i \in I$$

for some $\alpha \in [0,1)$. This h^α is not strategy-proof because for some valuations of other agents, an agent will be strictly better off by decreasing her report of valuation without changing the alternative chosen by h^α. However, this is shown to be commitment-proof. The first part of the payment is exactly the VCG payment, and we know that the VCG social choice function satisfies equation (2). Regarding the second part of the payment, it is easy to see that for any $i \in I$, $b \in \mathbb{R}^{\Omega\times I}$, $\tilde{b}^i \in \mathbb{R}^\Omega$ and $x \geq 0$ such that $\tilde{b}^i(\omega) \leq b^i(\omega) + x$ for all $\omega \in \Omega$,

$$\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h^i_t(\tilde{b}^i, b^{-i}) - x = -x \leq 0 = b^i(h_\omega(b)) - h^i_t(b)$$

holds. Therefore, equation (2) is satisfied when the transfer rule is a convex combination of these two, and hence h^α is commitment-proof.

4.3 Possibility with Ex Ante and Ex Post Investments

Now we formally present the possibility theorem in our original model. In what follows, we demonstrate how commitment-proofness makes it possible to implement efficient investments when *ex post* investments are allowed.

First, for the purpose of the main theorem, we prove the following lemma.

Lemma 1. *For any agent $i \in I$, $V^i \subseteq \mathbb{R}^{\Omega\times I}$ and a cost function $c^i : V^i \times \Theta^i \to C^i$,*

$$c^i(v^i, \theta^i) \geq \max_{\omega\in\Omega} \Big\{ b^{c^i,\theta^i,v^i}(\omega) - b^{c^i,\theta^i,v^{0i}}(\omega) \Big\}$$

holds for any $\theta^i \in \Theta^i$, $v^i \in V^i$, and $v^{0i} \in V^i$ such that $c^i(v^{0i}) = 0$.

Proof: From the definition of the valuation at the time of the mechanism,

$$\begin{aligned} b^{c^i,\theta^i,v^{0i}}(\omega) &= \max_{\bar{v}^i\in V^i} \Big\{ \bar{v}^i(\omega) - c^i(\bar{v}^i, \theta^i) \Big\} \\ &\geq \max_{\bar{v}^i\in\{\tilde{v}^i\in V^i | c^i(\tilde{v}^i,\theta^i)\geq c^i(v^i,\theta^i)\}} \Big\{ \bar{v}^i(\omega) - c^i(\bar{v}^i, \theta^i) \Big\} \\ &= b^{c^i,\theta^i,v^i}(\omega) - c^i(v^i, \theta^i) \end{aligned}$$

holds for any $\omega \in \Omega$. Thus, we have $c^i(v^i, \theta^i) \geq \max_{\omega \in \Omega} \left\{ b^{c^i,\theta^i,v^i}(\omega) - b^{c^i,\theta^i,v^{0i}}(\omega) \right\}$. $\square$

This lemma shows that the cost of changing the original valuation $b^{c^i,\theta^i,v^{0i}}$ with least costly *ex ante* investment v^{0i} to another valuation b^{c^i,θ^i,v^i} with some *ex ante* investment v^i is at least as large as the maximum element of the difference between $b^{c^i,\theta^i,v^{0i}}$ and b^{c^i,θ^i,v^i}. This is useful when we connect the definition of commitment-proofness to the structure of the investment game in the following theorem.

The next result is the main theorem of this paper which identifies when efficient investments are implementable; for allocatively constrained-efficient social choice functions, commitment-proofness is sufficient and necessary for implementing efficient investments in PBNE for any discount factor $\beta \in (0, 1)$.

Theorem 2. *Consider any I, Ω and any social choice function $h : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ which is allocatively constrained-efficient for some $\Omega' \subseteq \Omega$ with $\Omega' \neq \emptyset$. Given the social choice function h, efficient investments are implementable in PBNE for any discount factor $\beta \in (0, 1)$ if and only if h is commitment-proof.*

The proof consists of the following two parts; (i) commitment-proofness of h as sufficient for implementing efficient investments, and (ii) it also being necessary. First, we characterize the set of PBNE when h is commitment-proof. We show that under commitment-proof social choice functions, no agent has the incentive to make a costly investment *ex ante* for any cost type. This is because the cost of any (costly) investment corresponds to x in the definition of commitment-proofness as shown in Lemma 1, and every agent i prefers to have the valuation $b^{c^i,\theta^i,v^{0i}}$ with least costly *ex ante* investment at the mechanism stage. And we show that any such PBNE maximizes the expected social welfare when the social choice function h is allocatively constrained-efficient. For the necessity part, we show that if h is not commitment-proof for agent i, there is a set of valuations and a profile of cost functions under which agent i has the incentive to make a costly investment *ex ante*, which is socially inefficient. Therefore, we conclude that only under commitment-proof social choice functions, the incentive for making a commitment through *ex ante* investment is completely suppressed by the presence of the *ex post* investment stage, and efficient investments are implemented.

Regarding the two distinct features of our main result that (i) inefficient investment equilibria are eliminated when (ii) post-mechanism investments are allowed, Piccione and Tan (1996) provided a closely related result in the literature. They analyze a procurement auction in which firms make R&D investments prior to the auction and the firm that wins the procurement contract exerts an additional effort to reduce costs. One of the main results of their paper is that the full-information solution (in which investments and alternative are efficient) can be uniquely implemented by the first-price and second-price auctions when the R&D technology exhibits decreasing returns to scale. Although the model is similar to ours, the focus of their theorem is different. Their result determines the structure of cost functions which enable unique implementation under those two common auction rules. On the other hand, we characterize the set of social choice functions for which efficient investments are implementable. Also, our cost functions allow any arbitrary heterogeneity among agents, which is not allowed in Piccione and Tan (1996), but we assume a certain relationship between *ex ante* and *ex post* cost functions. (See footnote 16.) Since we do not analyze the equilibrium of specific mechanisms such as the first-price auction, it would be an interesting direction to analyze such mechanisms and see how the result relates to Piccione and Tan (1996).

In the rest of the section, we provide two examples to show the importance of (i) β being strictly less than one and (ii) the allocative constrained-efficiency of h in Theorem 2.

First, a strict time discounting plays an important role. Although commitment-proofness implies implementability of efficient investments for any β which is arbitrarily close to one, it does not when β is exactly one. Intuitively, this is because when β is one, there are cases where the choice between investing *ex ante* and *ex post* is indifferent and there exists an equilibrium in which more than one agents chooses costly *ex ante* investments, which is socially inefficient. We provide an example where given $\beta = 1$ and a VCG social choice function (see subsection 4.2 for the definition), which is allocatively efficient and strategy-proof, efficient investments are not implementable in PBNE.

Observation 1. *Suppose* $|I| \geq 2$, $|\Omega| \geq 2$. *Given a VCG social choice function* $h^{VCG} : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ *and* $\beta = 1$, *efficient investments are not implementable in perfect Bayesian Nash equilibria.*

Example 2. Let $\{i, j\} \subseteq I$ and $\{\omega_1, \omega_2\} \subseteq \Omega$. Consider the following sets of valuations:

$$\begin{aligned} V^i &= \{b^i, \tilde{b}^i\}, \\ V^j &= \{b^j, \tilde{b}^j\}, \\ V^k &= \{0\} \text{ for any } k \in I \setminus \{i, j\} \end{aligned}$$

where

$$b^i(\omega_1) = b^j(\omega_1) = 5,\ b^i(\omega_2) = b^j(\omega_2) = 4,\ b^i(\omega) = b^j(\omega) = 0 \text{ for any } \omega \in \Omega \setminus \{\omega_1, \omega_2\}$$
$$\tilde{b}^i(\omega_1) = \tilde{b}^j(\omega_1) = 0,\ \tilde{b}^i(\omega_2) = \tilde{b}^j(\omega_2) = 6,\ \tilde{b}^i(\omega) = \tilde{b}^j(\omega) = 0 \text{ for any } \omega \in \Omega \setminus \{\omega_1, \omega_2\}.$$

Consider the following cost functions: for any $\theta \in \Theta$ with $p(\theta) > 0$,

$$\begin{aligned} &c^i(b^i, \theta^i) = c^j(b^j, \theta^j) = 0, \\ &c^i(\tilde{b}^i, \theta^i) = c^j(\tilde{b}^j, \theta^j) = 2, \\ &c^k(0, \theta^k) = 0 \text{ for any } k \in I \setminus \{i, j\}. \end{aligned}$$

Since the only choice of valuation is 0 for any $k \in I \setminus \{i, j\}$, we can ignore these agents. Given a VCG social choice function h^{VCG}, the most efficient investment schedules of agents i and j is $((b^i, b^j), (b^i, b^j))$. This is because it achieves the maximum social welfare $\beta(5+5) = \beta 10 = 10$ as h^{VCG} chooses ω_1 for (b^i, b^j), and the cost of (b^i, b^j) is zero for any $\theta \in \Theta$ which occurs with a positive probability.

Next, consider an investment strategy $(\tilde{\sigma}^l, \mu^l) \in \Sigma^l \times \mathcal{M}^l$ for each agent $l = i, j$ where $\tilde{\sigma}^l(\theta^l) = \tilde{b}^l$ and μ^l is the optimal *ex post* investment strategy. First, because $c^l(\tilde{b}^l, \theta^l) > c^l(b^l, \theta^l)$ for each agent $l = i, j$,

$$\mu^l(\tilde{b}^l, \omega, \theta^l) = \tilde{b}^l$$

holds for any $\omega \in \Omega$ and $\theta^l \in \Theta^l$. Thus, for the *ex ante* investment strategy $\tilde{\sigma}^l(\theta^l) = \tilde{b}^l$, the valuation at the time of the mechanism is $\tilde{b}^l$.

Suppose that agent j takes this investment strategy $(\tilde{\sigma}^j, \mu^j) \in \Sigma^j \times \mathcal{M}^j$, and consider agent i's incentive. When she chooses b^i in the first stage, since $b^i(\omega) \geq \tilde{b}^i(\omega) - c^i(\tilde{b}^i)$ holds for any $\omega \in \Omega$, the valuation at the time of the mechanism is

$$b^{c^i, b^i}(\omega) = \max_{\bar{v}^i \in \{b^i, \tilde{b}^i\}} \left\{ \bar{v}^i(\omega) - c^i(\bar{v}^i) \right\} = b^i(\omega)$$

for each $\omega \in \Omega$. In this case, the outcome of the social choice function should be

$$\begin{aligned} &h_\omega^{VCG}(b^i, \tilde{b}^j, 0) = \omega_2, \text{ and} \\ &h_t^{VCG,i}(b^i, \tilde{b}^j, 0) = 0. \end{aligned}$$

The total utility of agent i would be $4\beta = 4$. On the other hand, when she chooses $\tilde{b}^j$ in the first stage, the outcome of the social choice function will be

$$\begin{aligned} &h_\omega^{VCG}(\tilde{b}^i, \tilde{b}^j, 0) = \omega_2, \text{ and} \\ &h_t^{VCG,i}(\tilde{b}^i, \tilde{b}^j, 0) = 0. \end{aligned}$$

The total utility of agent i would be $6\beta - 2 = 4$. Since these choices are indifferent, choosing $\tilde{b}^i$ in the first stage can be a best response for agent i. Therefore, the same logic applies to agent j, and $\{(\tilde{\sigma}^l, \mu^l) \in \Sigma^l \times \mathcal{M}^l\}_{l=i,j}$ constitutes a PBNE of the investment game. But

$\{(\tilde{\sigma}^l, \mu^l) \in \Sigma^l \times \mathcal{M}^l\}_{l=i,j}$ gives the social welfare of 8, which is less than that of $\{(\sigma^l, \mu^l) \in \Sigma^l \times \mathcal{M}^l\}_{l=i,j}$ such that $\sigma^l(\theta^l) = b^l$ for any $\theta^l \in \Theta^l$, which gives the social welfare of 10. Thus, efficient investments are not implementable in PBNE given h^{VCG} and $\beta = 1$. $\square$

As a second observation, the sufficiency of commitment-proofness in Theorem 2 no longer holds if the social choice function is not allocatively constrained-efficient for any $\Omega' \subseteq \Omega$. The next example demonstrates that efficient investments are not implementable given a strategy-proof (and hence, commitment-proof) social choice function which is not allocatively constrained-efficient.

Observation 2. *Suppose $|I| \geq 2$ and $|\Omega| \geq 2$. There is a strategy-proof social choice function $h : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ which is not allocatively constrained-efficient for any $\Omega' \subseteq \Omega$ such that efficient investments are not implementable in perfect Bayesian Nash equilibria given h and some $\beta \in (0, 1)$.*

Example 3. Let $\{i, j\} \subseteq I$ and $\{\omega_1, \omega_2\} \subseteq \Omega$. Consider a social choice function $h : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ such that for any $b \in \mathbb{R}^{\Omega \times I}$,

$$h_\omega(b) \in \arg\max_{\omega \in \Omega} \left\{ b^i(\omega) \right\},$$
$$h_t^k(b) = 0 \text{ for any } k \in I.$$

This means that the best alternative for agent i is always chosen and no transfer is made under h. This h is strategy-proof because i does not have the incentive to manipulate her type and j's report does not affect the outcome. It is clear that h is not allocatively constrained-efficient because other agents' valuations are not taken into account. Consider the following sets of valuations:

$$V^i = \{b^i, \tilde{b}^i\},$$
$$V^j = \{b^j\},$$
$$V^k = \{0\} \text{ for any } k \in I \setminus \{i, j\}$$

where

$$b^i(\omega_1) = 5,\ b^i(\omega_2) = 4,\ b^i(\omega) = 0 \text{ for any } \omega \in \Omega \setminus \{\omega_1, \omega_2\}$$
$$\tilde{b}^i(\omega_1) = 5,\ \tilde{b}^i(\omega_2) = 6,\ \tilde{b}^i(\omega) = 0 \text{ for any } \omega \in \Omega \setminus \{\omega_1, \omega_2\}$$
$$b^j(\omega_1) = 0,\ b^j(\omega_2) = 3,\ b^j(\omega) = 0 \text{ for any } \omega \in \Omega \setminus \{\omega_1, \omega_2\}.$$

Also consider the following cost functions: for any $\theta \in \Theta$ with $p(\theta) > 0$,

$$\begin{aligned} &c^i(b^i, \theta^i) = 0,\ c^i(\tilde{b}^i, \theta^i) = 3, \\ &c^j(b^j, \theta^j) = 0, \\ &c^k(0, \theta^k) = 0 \text{ for any } k \in I \setminus \{i, j\}. \end{aligned}$$

Since the only choice of valuation is 0 for any $k \in I \setminus \{i, j\}$, we can ignore these agents. For j, the only choice of valuation is b^j.

Consider the optimal choice for agent i in the second investment stage for any $\theta^i \in \Theta^i$ which occurs with a positive probability. If i chooses b^i before the mechanism, since $b^i(\omega) > \tilde{b}^i(\omega) - c^i(\tilde{b}^i, \theta^i)$ holds for any $\omega \in \Omega$, her optimal valuation after the mechanism is b^i. If i chooses $\tilde{b}^i$ before the mechanism, then the only valuation she can choose after the mechanism is $\tilde{b}^i$ because $c^i(\tilde{b}^i, \theta^i) > c^i(b^i, \theta^i)$. In either case, when the same valuation is taken *ex ante* and *ex post*, the valuation at the time of the mechanism is also that valuation. To summarize, agent i's optimal *ex post* investment strategy and the valuation at the time of the mechanism is as follows:

Ex Ante Valuation	Valuation at the Mechanism	Optimal *Ex Post* Valuation
b^i	b^i	ω_1: b^i ω_2: b^i
$\tilde{b}^i$	$\tilde{b}^i$	ω_1: $\tilde{b}^i$ ω_2: $\tilde{b}^i$

Thus, we can compare two investment choices b^i and $\tilde{b}^i$ of agent i in the first stage to analyze the investment efficiency and the equilibrium.

First, we show that $\tilde{b}^i$ gives higher social welfare than b^i for sufficiently large $\beta \in (0, 1)$. Given j's valuation b^j, the social welfare when i chooses $\tilde{b}^i$ is

$$-3 + \beta(6 + 3) = 9\beta - 3.$$

The social welfare when i chooses b^i is

$$0 + \beta(5 + 0) = 5\beta.$$

Since the former is larger for $\beta > \frac{3}{4}$, choosing $\tilde{b}^i$ is socially efficient, and choosing b^i is not for such β.

Next, consider the incentive of agent i. Given j's valuation b^j, compare the utility of i when she chooses $\tilde{b}^i$ and b^i in the first stage. When i chooses $\tilde{b}^i$, her utility is $6\beta - 3$ whereas it is 5β when i chooses b^i. Since

$$6\beta - 3 < 5\beta \text{ for any } \beta \in (0, 1),$$

agent i chooses b^i in a PBNE. Thus, agent i chooses b^i in a PBNE of the investment game, but it does not maximize the social welfare for $\beta > \frac{3}{4}$. Therefore, efficient investments are not implementable in PBNE given h and such β. □

5 Provision of Public Goods

In this section, we consider a variant of the original problem; providing public goods through the finances of agents. The provision of public goods is represented by a choice of an alternative $\omega \in \Omega$ in our model. We still assume perfectly transferable utility and allow for transfers $(t^i)_{i\in I}$ in the mechanism. The only difference from the original model is that we require a budget balance for social choice functions, i.e., the sum of the transfers must be equal to zero.

Definition 8. A social choice function h is *budget-balanced* if

$$\sum_{i\in I} h_t^i(b) = 0$$

for any $b \in \mathbb{R}^{\Omega\times I}$.

Budget balance is considered to be part of allocative efficiency because the transfer collected by the mechanism designer is regarded as the loss of welfare in this problem. In this environment, it is known that there is no social choice function that is strategy-proof, allocatively efficient and budget-balanced (Green and Laffont, 1977; Hölmstrom, 1979; Walker, 1980). Therefore, when there is only an *ex ante* investment stage, it is impossible to even ensure the existence of efficient investment equilibria if we require budget balance and allocatively efficiency of the social choice function (Hatfield, Kojima and Kominers, 2015). However, we can show that commitment-proofness is compatible with these two properties; there is a social choice function which is commitment-proof, allocatively efficient and budget-balanced.

Proposition 1. *For any I, Ω and an efficient allocation rule $h_\omega : \mathbb{R}^{\Omega\times I} \to \Omega$, there exists a transfer rule $h_t : \mathbb{R}^{\Omega\times I} \to \mathbb{R}^I$ with which $h = (h_\omega, h_t)$ is commitment-proof and budget-balanced.*

Proposition 1 is shown by proposing a specific transfer rule h_t: for any agent $i \in I$, h_t^i is defined by

$$h_t^i(b) = b^i(h_\omega(b)) - \frac{1}{n}\sum_{i\in I} b^i(h_\omega(b)).$$

By this transfer rule, the maximized social welfare is equally divided to all agents. Consider the definition of commitment-proofness. Under this transfer rule, the value $\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h^i_t(\tilde{b}^i, b^{-i})$ from the social choice function h under type $\tilde{b}^i$ increases from the original value $b^i(h_\omega(b)) - h^i_t(b)$ under type b^i by only $\frac{1}{n}$ of the increment of the social welfare. On the other hand, since x satisfies $x \geq \max_{\omega \in \Omega}\{\tilde{b}^i(\omega) - b^i(\omega)\}$, x should be larger than the increment of social welfare. Therefore, the equation of commitment-proofness is satisfied under this transfer rule. It is easy to see that this h is not strategy-proof because agents have the incentive to underreport their valuations to reduce the payment.

By the result of Theorem 2, we obtain the following corollary; with the *ex post* investments, budget balance does not preclude the implementation of efficient investments.

Corollary 1. *For any I and Ω, there exists an allocatively efficient and budget-balanced social choice function $h : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ such that efficient investments are implementable in perfect Bayesian Nash equilibria given h and any discount factor $\beta \in (0, 1)$.*

6 Concluding Remarks

Our main result shows that allowing for *ex post* investments, commitment-proofness is equivalent to the implementability of efficient investments for allocatively efficient social choice functions. This has the following two implications. First, whenever it is possible, the mechanism should be run sufficiently before the actual production or consumption is carried out. This allows agents to reflect the information of their cost types onto their valuations at the mechanism stage through the optimal behavior in the *ex post* investment stage. Otherwise, according to Theorem 1, we cannot eliminate the possibility of inefficient equilibria. Second, commitment-proofness of the mechanism is essential. This ensures that no agent has the incentive to commit to having a different valuation in the mechanism by making prior investments. Moreover, this is not a restrictive concept since it is much weaker than the strategy-proofness condition.

In this paper, we allow for incomplete information about the cost types of other agents, but we assume that every agent knows her own cost function and it is unchanged over time. This assumption allows us to characterize the set of PBNE, in which no agent makes a costly *ex ante* investment. Although out result still holds for some systematic changes of cost functions after the mechanism (see footnote 16), we do not know what will happen if the *ex post* cost function is uncertain *ex ante*. When agents are unsure about thier own *ex post* cost functions, they may need to make some investments *ex ante* to improve their own *ex post* cost functions. This uncertain investment model will be related to Piccione and Tan (1996) and other papers on information acquisition (Bergemann and Välimäki, 2002; Obara, 2008). Under this uncertain investment setting, we hope to obtain conditions on social choice functions or cost structures which make the implementation of investment efficiency possible.

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Appendix A Proofs of Main Results

A.1 Proof of Theorem 1

Consider any I and Ω with $|I| \geq 2$ and $|\Omega| \geq 2$. Consider any arbitrary allocatively efficient social choice function $h : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$. We will examine two cases where h is not strategy-proof and h is strategy-proof. In the former case, we construct the sets of valuations and cost functions under which an inefficient Bayesian Nash equilibrium exists in the investment game, exploiting the equation that strategy-proofness of h is violated. In the latter case, we show that a simple auction has multiple equilibria in the investment game and one of them is less efficient than the other for any strategy-proof h.

[1] When h is not strategy-proof. Since the social choice function h is not strategy-proof, there are $i \in I$, $v \in \mathbb{R}^{\Omega \times I}$ and $\tilde{v}^i \in \mathbb{R}^\Omega$ such that

$$v^i(h_\omega(\tilde{v}^i, v^{-i})) - h_t^i(\tilde{v}^i, v^{-i}) > v^i(h_\omega(v)) - h_t^i(v). \tag{3}$$

Consider the sets of valuations $V \subseteq \mathbb{R}^{\Omega \times I}$ such that

$$\begin{aligned} &V^i = \{v^i, \tilde{v}^i\} \text{ and} \\ &V^j = \{v^j\} \text{ for all } j \in I \setminus \{i\}. \end{aligned}$$

Consider a profile of cost functions $c : V \times \Theta \to C$ such that for any $\theta \in \Theta$ with $p(\theta) > 0$,

$$\begin{aligned} &c^i(v^i, \theta^i) = \max\left\{0, \beta\Big[v^i(h_\omega(v)) - h_t^i(v) - \big(\tilde{v}^i(h_\omega(\tilde{v}^i, v^{-i})) - h_t^i(\tilde{v}^i, v^{-i})\big)\Big]\right\}, \\ &c^i(\tilde{v}^i, \theta^i) = \max\left\{0, \beta\Big[\tilde{v}^i(h_\omega(\tilde{v}^i, v^{-i})) - h_t^i(\tilde{v}^i, v^{-i}) - \big(v^i(h_\omega(v)) - h_t^i(v)\big)\Big]\right\} \text{ and} \\ &c^j(v^j, \theta^j) = 0 \;\text{ for all } j \in I \setminus \{i\}. \end{aligned}$$

Note that

$$c^i(v^i, \theta^i) - c^i(\tilde{v}^i, \theta^i) = \beta\Big[v^i(h_\omega(v)) - h_t^i(v) - \big(\tilde{v}^i(h_\omega(\tilde{v}^i, v^{-i})) - h_t^i(\tilde{v}^i, v^{-i})\big)\Big]$$

always holds. Here, the only choice of valuations for each $j \in I \setminus \{i\}$ is v^j. Also, since the cost of investment is constant across any type $\theta \in \Theta$ with $p(\theta) > 0$, we can concentrate on the types which occur with a positive probability, and a Bayesian Nash equilibrium reduces to a Nash equilibrium in this case. Thus, we only need to analyze the choice of agent i's valuation for Nash equilibria and efficient choices.

First, consider i's incentive for choosing between v^i and $\tilde{v}^i$. For any cost type $\theta^i \in \Theta^i$ which occurs with a positive probability, the total utility from choosing v^i when the valuations of other agents are v^{-i} is

$$-c^i(v^i,\theta^i)+\beta\Big[v^i(h_\omega(v))-h_t^i(v)\Big],$$

and that from choosing $\tilde{v}^i$ is

$$-c^i(\tilde{v}^i,\theta^i)+\beta\Big[\tilde{v}^i(h_\omega(\tilde{v}^i,v^{-i}))-h_t^i(\tilde{v}^i,v^{-i})\Big].$$

The difference is

$$\begin{aligned}
& -c^i(v^i,\theta^i)+\beta\Big[v^i(h_\omega(v))-h_t^i(v)\Big]-\Big\{-c^i(\tilde{v}^i,\theta^i)+\beta\Big[\tilde{v}^i(h_\omega(\tilde{v}^i,v^{-i}))-h_t^i(\tilde{v}^i,v^{-i})\Big]\Big\}\\
= \;& \beta\Big[v^i(h_\omega(v))-h_t^i(v)-\big(\tilde{v}^i(h_\omega(\tilde{v}^i,v^{-i}))-h_t^i(\tilde{v}^i,v^{-i})\big)\Big]-\Big(c^i(v^i,\theta^i)-c^i(\tilde{v}^i,\theta^i)\Big)\\
= \;& 0.
\end{aligned}$$

Therefore, v^i and $\tilde{v}^i$ are indifferent for agent i, and both v and $(\tilde{v}^i,v^{-i})$ are Nash equilibria of the investment game.

Next, compare the social welfare between v and $(\tilde{v}^i,v^{-i})$. For v, the sum of utility of all agents is

$$\sum_{j\in I}\Big\{-c^j(v^j,\theta^j)+\beta v^j(h_\omega(v))\Big\}=-c^i(v^i,\theta^i)+\beta\sum_{j\in I}v^j(h_\omega(v)).$$

And for $(\tilde{v}^i,v^{-i})$, the sum of utility of all agents is

$$-c^i(\tilde{v}^i,\theta^i)+\beta\Big[\tilde{v}^i(h_\omega(\tilde{v}^i,v^{-i}))+\sum_{j\in I\setminus\{i\}}v^j(h_\omega(\tilde{v}^i,v^{-i}))\Big].$$

The difference of these two is:

$$\begin{aligned}
& -c^i(v^i,\theta^i)+\beta\sum_{j\in I}v^j(h_\omega(v))+c^i(\tilde{v}^i,\theta^i)-\beta\Big[\tilde{v}^i(h_\omega(\tilde{v}^i,v^{-i}))+\sum_{j\in I\setminus\{i\}}v^j(h_\omega(\tilde{v}^i,v^{-i}))\Big] && (4)\\
\geq \;& \beta\Big[\sum_{j\in I}v^j(h_\omega(\tilde{v}^i,v^{-i}))-\tilde{v}^i(h_\omega(\tilde{v}^i,v^{-i}))-\sum_{j\in I\setminus\{i\}}v^j(h_\omega(\tilde{v}^i,v^{-i}))\Big] && (5)\\
& -(c^i(v^i,\theta^i)-c^i(\tilde{v}^i,\theta^i)) && (6)\\
= \;& \beta\Big[v^i(h_\omega(\tilde{v}^i,v^{-i}))-\tilde{v}^i(h_\omega(\tilde{v}^i,v^{-i}))\Big]-(c^i(v^i,\theta^i)-c^i(\tilde{v}^i,\theta^i)) && (7)\\
> \;& \beta\Big[v^i(h_\omega(v))-h_t^i(v)+h_t^i(\tilde{v}^i,v^{-i})-\tilde{v}^i(h_\omega(\tilde{v}^i,v^{-i}))\Big]-(c^i(v^i,\theta^i)-c^i(\tilde{v}^i,\theta^i)) && (8)\\
= \;& 0, && (9)
\end{aligned}$$

in which the inequality in (5) follows from the allocative efficiency of h; the inequality in (8) follows from equation (3). Therefore, $(\tilde{v}^i,v^{-i})$ is not an efficient investment profile although it is supported by a Nash equilibrium. Hence, there is an inefficient equilibrium of the investment game, and efficient *ex ante* investments are not Bayesian Nash implementable given h.

[2] When h is strategy-proof. We consider a slight modification of Example 4 of Hatfield, Kojima and Kominers (2015); an auction where two agents bid for a single good. Consider any social choice function h which is allocatively efficient and strategy-proof. Suppose $\{i, j\} \subseteq I$ and $\{\omega^i, \omega^j\} \subseteq \Omega$. Since $|I|$ and $|\Omega|$ may be more than two, we choose the sets of valuation functions in the following way:

$$V^i = \{a\mathbb{1}_{\{\omega=\omega^i\}} : a \in [0, 10]\},$$
$$V^j = \{a\mathbb{1}_{\{\omega=\omega^j\}} : a \in [0, 10]\},$$
$$V^k = \{0\} \text{ for any } k \in I \setminus \{i, j\}.$$

Here ω^i and ω^j each represent the alternatives where i and j obtain the item respectively. Consider the following cost functions: for any $\theta \in \Theta$ with $p(\theta) > 0$,

$$c^i(a\mathbb{1}_{\{\omega=\omega^i\}}, \theta^i) = \frac{1}{6}\beta a^2,$$
$$c^j(a\mathbb{1}_{\{\omega=\omega^j\}}, \theta^j) = \frac{1}{4}\beta a^2,$$
$$c^k(0, \theta^k) = 0 \text{ for any } k \in I \setminus \{i, j\}.$$

Since the utility of agents other than i and j is always zero, focus on the investment choices of agents i and j. Also, since they have the same cost of investment for any cost types which occur with a positive probability, we can concentrate on such types, and a Bayesian Nash equilibrium reduces to a Nash equilibrium.

First, consider efficient investment profiles under this allocatively efficient h. It is clear that only one of agents i and j should make a positive investment. If agent i obtains the item, the optimal choice of valuation should be

$$\arg\max_{a\in[0,10]} \beta\Big\{ -\frac{1}{6}a^2 + a\Big\} = 3.$$

If agent j obtains it, the optimal choice of valuation should be

$$\arg\max_{a\in[0,10]} \beta\Big\{ -\frac{1}{4}a^2 + a\Big\} = 2.$$

The social welfare achieved by $(3\mathbb{1}_{\{\omega=\omega^i\}}, 0)$ is

$$\beta\Big\{ -\frac{3}{2} + 3\Big\} = \frac{3}{2}\beta$$

and the social welfare achieved by $(0, 2\mathbb{1}_{\{\omega=\omega^i\}})$ is

$$\beta\Big\{-1+2\Big\} = \beta.$$

Thus, $(3\mathbb{1}_{\{\omega=\omega^i\}}, 0)$ is the unique investment profile of i and j which maximizes the social welfare.

Then consider the other investment profile $(0, 2\mathbb{1}_{\{\omega=\omega^j\}})$, and show that it is a Nash equilibrium of the investment game. First, it is clear that the valuation of agent j is a best response to i's choice 0 because it maximizes the value of the item. Next, given $\bar{v}^j \equiv 2\mathbb{1}_{\{\omega=\omega^j\}}$,

$$\begin{aligned}
& \arg\max_{v^i \in V^i} \Big\{ -c^i(v^i, \theta^i) + \beta v^i(h_\omega(v^i, \bar{v}^j)) + \beta \bar{v}^j(h_\omega(v^i, \bar{v}^j)) \Big\} \\
= \; & \arg\max_{v^i \in V^i} \Big\{ -\frac{1}{\beta} c^i(v^i, \theta^i) + v^i(h_\omega(v^i, \bar{v}^j)) + \bar{v}^j(h_\omega(v^i, \bar{v}^j)) \Big\} \\
= \; & 0
\end{aligned}$$

holds. This is because given agent j's valuation $\bar{v}^j = 2\mathbb{1}_{\{\omega=\omega^j\}}$, the equation is maximized when agent j obtains the item and agent i does not make any investments (the value of the second equation becomes 2, which cannot be achieved by any positive valuation of agent i since the sum of the first two terms do not exceed $\frac{3}{2}$). Since h is allocatively efficient and strategy-proof, $h_t^i(\cdot, \bar{v}^j)$ is written as a Groves function (Green and Laffont, 1977):

$$h_t^i(v^i, \bar{v}^j) = g(\bar{v}^j) - \bar{v}^j(h_\omega(v^i, \bar{v}^j)).$$

Hence,

$$\begin{aligned}
& \arg\max_{v^i \in V^i} \Big\{ -\bar{c}^i(v^i, \theta^i) + v^i(h_\omega(v^i, \bar{v}^j)) - h_t^i(v^i, \bar{v}^j) \Big\} \\
= \; & \arg\max_{v^i \in V^i} \Big\{ -\bar{c}^i(v^i, \theta^i) + v^i(h_\omega(v^i, \bar{v}^j)) - g(\bar{v}^j) + \bar{v}^j(h_\omega(v^i, \bar{v}^j)) \Big\} \\
= \; & \arg\max_{v^i \in V^i} \Big\{ -\bar{c}^i(v^i, \theta^i) + v^i(h_\omega(v^i, \bar{v}^j)) + \bar{v}^j(h_\omega(v^i, \bar{v}^j)) \Big\}
\end{aligned}$$

should hold for any cost function $\bar{c}^i : V^i \times \Theta^i \to C^i$. Thus, we have

$$\begin{aligned}
& \arg\max_{v^i \in V^i} \Big\{ -c^i(v^i, \theta^i) + \beta v^i(h_\omega(v^i, \bar{v}^j)) - \beta h_t^i(v^i, \bar{v}^j) \Big\} \\
= \; & \arg\max_{v^i \in V^i} \Big\{ -\frac{1}{\beta} c^i(v^i, \theta^i) + v^i(h_\omega(v^i, \bar{v}^j)) - h_t^i(v^i, \bar{v}^j) \Big\} \\
= \; & \arg\max_{v^i \in V^i} \Big\{ -\frac{1}{\beta} c^i(v^i, \theta^i) + v^i(h_\omega(v^i, \bar{v}^j)) + \bar{v}^j(h_\omega(v^i, \bar{v}^j)) \Big\} \\
= \; & 0.
\end{aligned}$$

This means that 0 is the best response for agent i, and hence $(0, 2\mathbb{1}_{\{\omega=\omega^j\}})$ is a Nash equilibrium of the investment game. However, this does not achieve investment efficiency given h because it is less efficient than $(3\mathbb{1}_{\{\omega=\omega^i\}}, 0)$. Therefore, there is an inefficient equilibrium of the investment game, which means that efficient *ex ante* investments are not Bayesian Nash implementable given h.

A.2 Proof of Theorem 2

For the sufficiency of commitment-proofness, first we characterize the set of PBNE of the investment game given a commitment-proof social choice function h. Whenever h is commitment-proof, the set of PBNE is characterized by the following two properties: (i) no agent chooses a costly *ex ante* investment given her cost type, and (ii) the *ex post* investment is optimal for any information set. Next, we show that any PBNE maximizes the expected social welfare when h is allocatively constrained-efficient for some $\Omega' \subseteq \Omega$ with $\Omega' \neq \emptyset$.

For the necessity of commitment-proofness, we show that whenever h is allocatively constrained-efficient but is not commitment-proof, we can construct a profile of the sets of valuations and associated cost functions for which there exists a PBNE of the investment game that does not maximize the expected social welfare.

[1] Sufficiency of commitment-proofness. Take any $\beta \in (0, 1)$, $V \subseteq \mathbb{R}^{\Omega \times I}$ and $c : V \times \Theta \to C$, and fix them. We show that when h is commitment-proof, the set of PBNE of the investment game given h and β is characterized by $\Sigma^* \times \mathcal{M}^*$ such that

$$\Sigma^* \equiv \left\{ \sigma \in \Sigma \;\middle|\; \text{for each } i \in I, c^i(\sigma^i(\theta^i), \theta^i) = 0 \text{ for any } \theta^i \in \Theta^i \right\}, \text{ and}$$

$$\mathcal{M}^* \equiv \Big\{ \mu \in \mathcal{M} \;\Big|\; \text{for each } i \in I,$$

$$\mu^i(v^i, \omega, \theta^i) \in \underset{\bar{v}^i \in \{\tilde{v}^i \in V^i | c^i(\tilde{v}^i, \theta^i) \geq c^i(v^i, \theta^i)\}}{\arg\max} \left\{ \bar{v}^i(\omega) - c^i(\bar{v}^i, \theta^i) \right\} \text{ for any } (v^i, \omega, \theta^i) \in V^i \times \Omega \times \Theta^i, \Big\}.$$

First, by the definition of a PBNE of the investment game, it is obvious that the equilibrium *ex post* investment strategies are written as $\mathcal{M}^*$.

Next, we analyze the *ex ante* investment game given the optimal *ex post* investment strategies $\mu^* \in \mathcal{M}^*$. Take any agent $i \in I$, and consider i's incentive for the *ex ante* investment when her cost type is $\theta^i \in \Theta^i$. Consider two arbitrary *ex ante* investments with the following properties: $v^{0i} \in V^i$ such that $c^i(v^{0i}, \theta^i) = 0$, and $v^i \in V^i$ such that $c^i(v^i, \theta^i) > 0$.

We can show that for any strategies $\sigma^{-i} \in \Sigma^{-i}$ of other agents, v^{0i} gives a strictly higher expected utility than v^i for agent i. To see this, take any cost types $\theta^{-i} \in \Theta^{-i}$ of other agents and let $b^{-i} \equiv b^{c^{-i},\theta^{-i},\sigma^{-i}(\theta^{-i})}$. Agent i's *ex ante* utility from choosing v^i for $\theta^{-i} \in \Theta^{-i}$ is written as:

$$
\begin{aligned}
& -c^i(v^i,\theta^i) + \beta\Big[\mu^{*i}(v^i, h_\omega(b^{c^i,\theta^i,v^i}, b^{-i}), \theta^i)(h_\omega(b^{c^i,\theta^i,v^i}, b^{-i})) - h_t^i(b^{c^i,\theta^i,v^i}, b^{-i}) && (10)\\
& -c^i(\mu^{*i}(v^i, h_\omega(b^{c^i,\theta^i,v^i}, b^{-i}), \theta^i), \theta^i) + c^i(v^i,\theta^i)\Big] && (11)\\
= \;& \beta\Big[\mu^{*i}(v^i, h_\omega(b^{c^i,\theta^i,v^i}, b^{-i}), \theta^i)(h_\omega(b^{c^i,\theta^i,v^i}, b^{-i})) - h_t^i(b^{c^i,\theta^i,v^i}, b^{-i}) && (12)\\
& -c^i(\mu^{*i}(v^i, h_\omega(b^{c^i,\theta^i,v^i}, b^{-i}), \theta^i), \theta^i)\Big] - (1-\beta)c^i(v^i,\theta^i) && (13)\\
< \;& \beta\Big[\mu^{*i}(v^i, h_\omega(b^{c^i,\theta^i,v^i}, b^{-i}), \theta^i)(h_\omega(b^{c^i,\theta^i,v^i}, b^{-i})) - h_t^i(b^{c^i,\theta^i,v^i}, b^{-i}) && (14)\\
& -c^i(\mu^{*i}(v^i, h_\omega(b^{c^i,\theta^i,v^i}, b^{-i}), \theta^i), \theta^i)\Big] && (15)\\
= \;& \beta\Big[b^{c^i,\theta^i,v^i}(h_\omega(b^{c^i,\theta^i,v^i}, b^{-i})) - h_t^i(b^{c^i,\theta^i,v^i}, b^{-i}) - c^i(v^i,\theta^i)\Big] && (16)\\
\le \;& \beta\Big[b^{c^i,\theta^i,v^i}(h_\omega(b^{c^i,\theta^i,v^i}, b^{-i})) - h_t^i(b^{c^i,\theta^i,v^i}, b^{-i}) && (17)\\
& -\max\Big\{0, \max_{\omega\in\Omega}\big\{b^{c^i,\theta^i,v^i}(\omega) - b^{c^i,\theta^i,v^{0i}}(\omega)\big\}\Big\}\Big] && (18)\\
\le \;& \beta\Big[b^{c^i,\theta^i,v^{0i}}(h_\omega(b^{c^i,\theta^i,v^{0i}}, b^{-i})) - h_t^i(b^{c^i,\theta^i,v^{0i}}, b^{-i})\Big] && (19)\\
= \;& \beta\Big[\mu^{*i}(v^{0i}, h_\omega(b^{c^i,\theta^i,v^{0i}}, b^{-i}), \theta^i)(h_\omega(b^{c^i,\theta^i,v^{0i}}, b^{-i})) - h_t^i(b^{c^i,\theta^i,v^{0i}}, b^{-i}) && (20)\\
& -c^i(\mu^{*i}(v^{0i}, h_\omega(b^{c^i,\theta^i,v^{0i}}, b^{-i}), \theta^i))\Big], && (21)
\end{aligned}
$$

in which the last equation (20)-(21) is agent i's *ex ante* utility from choosing v^{0i} for $\theta^{-i} \in \Theta^{-i}$. The inequality in (14) holds because $c^i(v^i,\theta^i) > 0$ and $\beta < 1$; the equality in (16) follows from the definition of b^{c^i,θ^i,v^i}; the inequality in (17) follows from Lemma 1; the inequality in (19) follows from the fact that h is commitment-proof; and the equality in (20) follows from the definition of $b^{c^i,\theta^i,v^{0i}}$. Note that when there are more than one valuations $v^{0i}, \tilde{v}^{0i} \in V^i$ such that $c^i(v^{0i},\theta^i) = c^i(\tilde{v}^{0i},\theta^i) = 0$, they give exactly the same utility. Since the above inequality holds for any cost types $\theta^{-i} \in \Theta^{-i}$ of other agents, taking the expectation over Θ^{-i}, we have

$$
\begin{aligned}
v^{0i} \in \arg\max_{v^i\in V^i}\Big\{ & -c^i(v^i,\theta^i) + \beta \sum_{\theta^{-i}\in\Theta^{-i}} p(\theta^{-i}|\theta^i)\\
& \Big[\mu^{*i}(v^i, h_\omega(b^{c^i,\theta^i,v^i}, b^{-i}), \theta^i)(h_\omega(b^{c^i,\theta^i,v^i}, b^{-i})) - h_t^i(b^{c^i,\theta^i,v^i}, b^{-i})\\
& - c^i(\mu^{*i}(v^i, h_\omega(b^{c^i,\theta^i,v^i}, b^{-i}), \theta^i), \theta^i) + c^i(v^i,\theta^i)\Big]\Big\}.
\end{aligned}
$$

Thus, for any strategies $\sigma^{-i} \in \Sigma^{-i}$ of other agents, the best response for agent i with cost type θ^i is to choose a least costly investment $v^{0i} \in V^i$ such that $c^i(v^{0i},\theta^i) = 0$. As this is true for any cost type and any other agent, the set of equilibrium *ex ante* investment strategies is represented by Σ^*. Therefore, we can characterize the set of PBNE by $\Sigma^* \times \mathcal{M}^*$.

Finally, we show that the expected social welfare given h is maximized under any PBNE $(\sigma^*, \mu^*) \in \Sigma^* \times \mathcal{M}^*$. For any cost type profile $\theta \in \Theta$, the social welfare given h under any investment strategies $(\sigma, \mu) \in \Sigma \times \mathcal{M}$ is written as:

$$\begin{aligned}
& \sum_{i \in I} \Big\{ - c^i(\sigma^i(\theta^i), \theta^i) + \beta \Big[\mu^i(\sigma^i(\theta^i), h_\omega(b^{c,\theta,\sigma(\theta)}), \theta^i)(h_\omega(b^{c,\theta,\sigma(\theta)})) && (22) \\
& -c^i(\mu^i(\sigma^i(\theta^i), h_\omega(b^{c,\theta,\sigma(\theta)}), \theta^i) + c^i(\sigma^i(\theta^i), \theta^i) \Big] \Big\} && (23) \\
= & \sum_{i \in I} \Big\{ \beta \Big[\mu^i(\sigma^i(\theta^i), h_\omega(b^{c,\theta,\sigma(\theta)}), \theta^i)(h_\omega(b^{c,\theta,\sigma(\theta)})) - c^i(\mu^i(\sigma^i(\theta^i), h_\omega(b^{c,\theta,\sigma(\theta)}), \theta^i), \theta^i) \Big] && (24) \\
& -(1-\beta) c^i(\sigma^i(\theta^i), \theta^i) \Big\} && (25) \\
\leq & \sum_{i \in I} \beta \Big[\mu^i(\sigma^i(\theta^i), h_\omega(b^{c,\theta,\sigma(\theta)}), \theta^i)(h_\omega(b^{c,\theta,\sigma(\theta)})) - c^i(\mu^i(\sigma^i(\theta^i), h_\omega(b^{c,\theta,\sigma(\theta)}), \theta^i), \theta^i) \Big] && (26) \\
\leq & \sum_{i \in I} \beta b^{c^i, \theta^i, \sigma^{*i}(\theta^i)}(h_\omega(b^{c,\theta,\sigma(\theta)})) && (27) \\
\leq & \sum_{i \in I} \beta b^{c^i, \theta^i, \sigma^{*i}(\theta^i)}(h_\omega(b^{c,\theta,\sigma^*(\theta)})) && (28) \\
= & \sum_{i \in I} \beta \Big[\mu^{*i}(\sigma^{*i}(\theta^i), h_\omega(b^{c,\theta,\sigma^*(\theta)}), \theta^i)(h_\omega(b^{c,\theta,\sigma^*(\theta)})) && (29) \\
& -c^i(\mu^{*i}(\sigma^{*i}(\theta^i), h_\omega(b^{c,\theta,\sigma^*(\theta)}), \theta^i), \theta^i) \Big]. && (30)
\end{aligned}$$

The last equation (29)-(30) is the social welfare given h and θ under strategies (σ^*, μ^*). The inequality in (26) holds because $c^i(\sigma^i(\theta^i), \theta^i) \geq 0$ and $\beta < 1$; the inequality in (27) follows from the definition of $b^{c^i, \theta^i, \sigma^{*i}(\theta^i)}$; the inequality in (28) follows from the fact that h is allocatively constrained-efficient; the equality of (29) follows from the definitions of $b^{c^i, \theta^i, \sigma^{*i}(\theta^i)}$ and μ^{*i}. Since this holds for any cost type profile $\theta \in \Theta$, taking the expectation over Θ, a PBNE (σ^*, μ^*) maximizes the expected social welfare.

Therefore, for any $V \subseteq \mathbb{R}^{\Omega \times I}$ and $c : V \times \Theta \to C$, any PBNE of the investment game given h and $\beta \in (0, 1)$ maximizes the expected social welfare, and hence efficient investments are implemented in PBNE.

[2] <u>Necessity of commitment-proofness.</u> Consider a social choice function h which is allocatively constrained-efficient for some $\Omega' \subseteq \Omega$ with $\Omega' \neq \emptyset$ but is not commitment-proof. We show that for some $V \subseteq \mathbb{R}^{\Omega \times I}$, $c : V \times \Theta \to C$ and $\beta \in (0, 1)$, there is a PBNE which does not maximize the expected social welfare.

First, since h is not commitment-proof, there are $i \in I$, $b \in \mathbb{R}^{\Omega \times I}$ and $\tilde{b}^i \in \mathbb{R}^\Omega$ such that

$$\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h^i_t(\tilde{b}^i, b^{-i}) - \Big(b^i(h_\omega(b)) - h^i_t(b)\Big) > \max\Big\{0, \max_{\omega\in\Omega}\big\{\tilde{b}^i(\omega) - b^i(\omega)\big\}\Big\}. \quad (31)$$

Consider the following profile of the set of valuations:

$$V^i = \{b^i, \tilde{b}^i\},$$
$$V^j = \{b^j\} \text{ for all } j \in I \setminus \{i\}.$$

Consider a profile of cost functions $c : V \times \Theta \to C$ such that for any $\theta \in \Theta$ with $p(\theta) > 0$,

$$c^i(b^i, \theta^i) = 0,$$
$$c^i(\tilde{b}^i, \theta^i) = \begin{cases} \max_{\omega\in\Omega}\Big\{\tilde{b}^i(\omega) - b^i(\omega)\Big\} & \text{if } \max_{\omega\in\Omega}\Big\{\tilde{b}^i(\omega) - b^i(\omega)\Big\} > 0, \\ \delta & \text{otherwise,} \end{cases}$$
$$c^j(b^j, \theta^j) = 0 \text{ for all } j \in I \setminus \{i\},$$

where $\delta > 0$. Any agent $j \in I \setminus \{i\}$ always chooses $b^j \in V^j$ in the investment game because there is only one choice in V^j.

First, let's find a PBNE of this investment game. Agent i has two choices b^i and $\tilde{b}^i$. Consider her optimal choice in the second investment stage. When i chooses $\tilde{b}^i$ prior to the mechanism, since $c^i(\tilde{b}^i, \theta^i) > c^i(b^i, \theta^i)$ for any cost type $\theta^i \in \Theta^i$ which occurs with a positive probability, the optimal choice of a valuation function in the *ex post* stage is $\tilde{b}^i$ for any $\omega \in \Omega$ because it is the unique choice for her. Thus, the valuation at the time of the mechanism is

$$b^{c^i,\theta^i,\tilde{b}^i}(\omega) = \Big\{\tilde{b}^i(\omega) - c^i(\tilde{b}^i, \theta^i)\Big\} + c^i(\tilde{b}^i, \theta^i) = \tilde{b}^i(\omega)$$

for each $\omega \in \Omega$. On the other hand, when i chooses b^i prior to the mechanism, in the *ex post* stage, she can still choose from $\{b^i, \tilde{b}^i\}$ because b^i is a costless valuation. However, by the construction of the cost function, we can see that

$$b^i(\omega) \geq \tilde{b}^i(\omega) - c^i(\tilde{b}^i, \theta^i)$$

for any $\omega \in \Omega$ and $\theta^i \in \Theta^i$ which occurs with a positive probability. Thus, the valuation at the time of the mechanism is

$$b^{c^i,\theta^i,b^i}(\omega) = \max_{\bar{v}^i\in\{b^i,\tilde{b}^i\}}\Big\{\bar{v}^i(\omega) - c^i(\bar{v}^i, \theta^i)\Big\} = b^i(\omega)$$

for each $\omega \in \Omega$. To summarize, for any $\theta^i \in \Theta^i$ which occurs with a positive probability, agent i's optimal investment strategy and the valuation at the time of the mechanism is as follows:

Ex Ante Valuation	Valuation at the Mechanism	Optimal *Ex Post* Valuation
b^i	b^i	for any ω: b^i (or $\tilde{b}^i$ if $b^i(\omega) = \tilde{b}^i(\omega) - c^i(\tilde{b}^i)$)
$\tilde{b}^i$	$\tilde{b}^i$	for any ω: $\tilde{b}^i$

Given this optimal strategy in the second stage, we consider the choice of agent i in the first investment stage. Other agents' choices are fixed to b^{-i}. For any $\theta^i \in \Theta^i$ which occurs with a positive probability, the utility of agent i when choosing an investment $\tilde{b}^i$ is

$$-c^i(\tilde{b}^i, \theta^i) + \beta\Big[\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i})\Big]$$

and when choosing an investment b^i, it is

$$\beta\Big[b^i(h_\omega(b)) - h_t^i(b)\Big].$$

The difference of these two is calculated as:

$$\begin{aligned}
& -c^i(\tilde{b}^i, \theta^i) + \beta\Big[\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i})\Big] - \beta\Big[b^i(h_\omega(b)) - h_t^i(b)\Big] \\
= \; & -(1-\beta)c^i(\tilde{b}^i, \theta^i) + \beta\Big[\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i}) - c^i(\tilde{b}^i, \theta^i)\Big] - \beta\Big[b^i(h_\omega(b)) - h_t^i(b)\Big] \\
= \; & -(1-\beta)c^i(\tilde{b}^i, \theta^i) \\
& +\beta\Big[\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i}) - \Big(b^i(h_\omega(b)) - h_t^i(b)\Big) - \max\Big\{\delta, \max_{\omega\in\Omega}\big\{\tilde{b}^i(\omega) - b^i(\omega)\big\}\Big\}\Big] \\
> \; & 0,
\end{aligned}$$

in which $c^i(\tilde{b}^i, \theta^i) = \max\Big\{\delta, \max_{\omega\in\Omega}\big\{\tilde{b}^i(\omega) - b^i(\omega)\big\}\Big\}$ holds for sufficiently small $\delta > 0$, and the final inequality holds from equation (31) when we take β sufficiently close to 1 and $\delta > 0$ sufficiently small. Therefore, for any $\theta^i \in \Theta^i$ which occurs with a positive probability, agent i chooses $\tilde{b}^i$ in a PBNE, i.e., there is a PBNE $(\sigma^*, \mu^*) \in \Sigma \times \mathcal{M}$ such that for any $\theta \in \Theta$ with $p(\theta) > 0$,

$$\begin{aligned}
& \sigma^{*i}(\theta^i) = \tilde{b}^i, \\
& \sigma^{*j}(\theta^j) = b^j \text{ for any } j \in I \setminus \{i\}, \\
& \mu^{*i}(b^i, \omega, \theta^i) = b^i \text{ for any } \omega \in \Omega, \\
& \mu^{*i}(\tilde{b}^i, \omega, \theta^i) = \tilde{b}^i \text{ for any } \omega \in \Omega, \text{ and} \\
& \mu^{*j}(b^j, \omega, \theta^j) = b^j \text{ for any } \omega \in \Omega.
\end{aligned}$$

However, this PBNE (σ^*, μ^*) does not maximize the expected social welfare. Consider another profile of strategies $(\sigma, \mu^*) \in \Sigma \times \mathcal{M}$ such that for any $\theta \in \Theta$ with $p(\theta) > 0$,

$$\begin{aligned}
&\sigma^i(\theta^i) = b^i, \\
&\sigma^j(\theta^j) = b^j \text{ for any } j \in I \setminus \{i\}, \\
&\mu^{*i}(b^i, \omega, \theta^i) = b^i \text{ for any } \omega \in \Omega, \\
&\mu^{*i}(\tilde{b}^i, \omega, \theta^i) = \tilde{b}^i \text{ for any } \omega \in \Omega, \text{ and} \\
&\mu^{*j}(b^j, \omega, \theta^j) = b^j \text{ for any } \omega \in \Omega.
\end{aligned}$$

The only difference between σ^* and σ is that agent i chooses $\tilde{b}^i$ under σ^{*i}, but she chooses b^i under σ^i. For any $\theta \in \Theta$ with $p(\theta) > 0$, the social welfare from (σ^*, μ^*) is written as:

$$\begin{aligned}
& -c^i(\tilde{b}^i, \theta^i) + \beta \tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) + \sum_{j \in I \setminus \{i\}} \beta \Big\{ b^j(h_\omega(\tilde{b}^i, b^{-i})) \Big\} && (32) \\
< \; & \beta \Big\{ \tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - c^i(\tilde{b}^i, \theta^i) + \sum_{j \in I \setminus \{i\}} b^j(h_\omega(\tilde{b}^i, b^{-i})) \Big\} && (33) \\
\leq \; & \beta \Big\{ b^i(h_\omega(\tilde{b}^i, b^{-i})) + \sum_{j \in I \setminus \{i\}} b^j(h_\omega(\tilde{b}^i, b^{-i})) \Big\} && (34) \\
\leq \; & \beta \sum_{j \in I} b^j(h_\omega(b)) && (35)
\end{aligned}$$

in which the last equation (35) is the social welfare from strategies (σ, μ^*). The inequality in (33) holds because $c^i(\tilde{b}^i, \theta^i) > 0$ and $\beta < 1$; the inequality in (34) holds because $b^i(\omega) \geq \tilde{b}^i(\omega) - c^i(\tilde{b}^i, \theta^i)$ for any $\omega \in \Omega$; the inequality in (35) follows from the fact that h is allocatively constrained-efficient. Therefore, there is a PBNE (σ^*, μ^*) which does not maximize the expected social welfare, and hence efficient investments are not implementable in PBNE given this h and β.

A.3 Proof of Proposition 1

For any efficient allocation rule h_ω, consider the following transfer rule h_t which divides the maximum sum of valuations equally among all agents:

$$h_t^i(b) = b^i(h_\omega(b)) - \frac{1}{n} \sum_{i \in I} b^i(h_\omega(b)). \tag{36}$$

It is clear that h is budget-balanced. It suffices to show that h is commitment-proof. Consider any $i \in I$, $b \in \mathbb{R}^{\Omega \times I}$, $\tilde{b}^i \in \mathbb{R}^\Omega$ and $x \geq 0$ such that $\tilde{b}^i(\omega) \leq b^i(\omega) + x$ for all $\omega \in \Omega$. We will show:

$$\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i}) - x \leq b^i(h_\omega(b)) - h_t^i(b)$$

for this transfer rule (24). Since $x \geq \max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\}$ holds,

$$
\begin{aligned}
& \text{(RHS) - (LHS)} \\
\geq\ & \left[b^i(h_\omega(b)) - h_t^i(b)\right] - \left[\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i})\right] + \max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\} \\
=\ & \frac{1}{n}\sum_{i\in I} b^i(h_\omega(b)) - \frac{1}{n}\left\{\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) + \sum_{j\in I\setminus\{i\}} b^j(h_\omega(\tilde{b}^i, b^{-i}))\right\} + \max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\} \\
=\ & -\frac{1}{n}\left\{\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) + \sum_{j\in I\setminus\{i\}} b^j(h_\omega(\tilde{b}^i, b^{-i})) - \sum_{i\in I} b^i(h_\omega(b))\right\} + \max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\} \\
=\ & -\frac{1}{n}\left\{\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - b^i(h_\omega(\tilde{b}^i, b^{-i})) + \sum_{i\in I} b^i(h_\omega(\tilde{b}^i, b^{-i})) - \sum_{i\in I} b^i(h_\omega(b))\right\} \\
& + \max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\} \\
\geq\ & -\frac{1}{n}\left\{\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - b^i(h_\omega(\tilde{b}^i, b^{-i}))\right\} + \max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\} \\
\geq\ & -\frac{1}{n}\max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\} + \max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\} \\
=\ & \frac{n-1}{n}\max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\} \\
\geq\ & 0.
\end{aligned}
$$

The second inequality holds from the allocative efficiency of h. Therefore, this h is commitment-proof and the proof is done.