

Strategic asset allocation in a continuous-time VAR model

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This paper derives an approximate solution to a continuous-time intertemporal portfolio and consumption choice problem. The problem is the continuous-time equivalent of the discrete-time problem studied by Campbell and Viceira (1999), in which the expected excess return on a risky asset follows an AR(1) process, while the riskless interest rate is constant. The paper also shows how to obtain continuous-time parameters that are consistent with discrete-time econometric estimates. The continuous-time solution is the limit of that of Campbell and Viceira and has the property that conservative long-term investors have a large positive intertemporal hedging demand for stocks.

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1 Introduction

This paper studies the impact of predictable variation in stock returns on intertemporal optimal portfolio choice and consumption. We consider a model in which an infinitely lived investor with utility defined over consumption makes portfolio and consumption decisions continuously.

Two assets are available, a riskless asset with a constant interest rate, and a risky asset (“stocks”) whose expected return is time-varying. The realized return on stocks and the state variable driving changes in expected stock returns follow a joint homoskedastic, continuous-time vector autoregressive process (VAR). Thus the Sharpe ratio of the risky asset is linear in the state variable. An important characteristic of our model is that the instantaneous correlation between stock returns and expected returns need not be perfect; in other words, markets need not be complete.

To separate the effects of risk aversion from the effects of the investor’s willingness to substitute consumption intertemporally, we assume that the investor has recursive preferences. Recursive utility is a generalization of power utility that allows both the coefficient of relative risk aversion and the elasticity of intertemporal substitution in consumption to be constant free parameters. We adopt Duffie and Epstein’s (1992a, 1992b) parameterization of recursive preferences in continuous time.

A discrete-time version of this model has been previously studied by Campbell and

Viceira (1999), who derive an approximate analytical solution for the optimal portfolio rule, and show that this rule is linear in the state variable. Because Campbell and Viceira work in discrete time, no exact portfolio solutions are available in their model except in the trivial case of unit risk aversion, which implies myopic portfolio choice. By working in continuous time we show that this model has an exact analytical solution when the elasticity of intertemporal substitution of consumption equals one, for any value of the coefficient of relative risk aversion. For elasticities of intertemporal substitution different from one, our solution is still approximate. The solution is intuitive, and is the limit, as the frequency with which the investor can rebalance increases, of the discrete-time solution in Campbell and Viceira.¹ However the continuous-time solution is likely to be more appealing and intuitive to finance theorists who are accustomed to working in continuous time. The solution presented here extends the continuous-time results of Kim and Omberg (1996), who consider a finite-horizon model with consumption only at a single terminal date, and of Wachter (2002), who assumes that innovations to future expected stock returns are perfectly correlated with unexpected returns.

Campbell and Viceira (1999) calibrate their model to U.S. stock market data for the postwar period, and find that intertemporal hedging motives greatly increase the average

¹Campbell and Viceira (1999) claim that their solution becomes exact in the limit of continuous time when the elasticity of intertemporal substitution equals one. They base this claim on the fact that they use an approximation to the investor's intertemporal budget constraint which becomes exact as the time interval of their model shrinks.

demand for stocks by investors whose relative risk aversion coefficients exceed one. We show that our continuous-time VAR model has a unique correspondence with their discrete-time VAR model. This allows us to use their discrete-time estimates, appropriately time-disaggregated, to calibrate our own model. The calibration results show that our model exhibits similar properties.

The structure of the paper is as follows. Section 2 describes the investment opportunity set, and show how to time-aggregate our continuous-time VAR model for realized and expected stock returns. Section 3 solves and calibrates the intertemporal consumption and portfolio choice problem. Section 4 concludes.

2 Investment Opportunity Set

2.1 A continuous-time VAR

We start by assuming that there are two assets available to the investor, a riskless asset with instantaneous return

$$\frac{dB_t}{B_t} = rdt, \tag{1}$$

and a risky asset (“stocks”) whose instantaneous return is given by

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_S d\tilde{Z}_S. \tag{2}$$

where

$$d\mu_t = \kappa(\theta - \mu_t)dt + \sigma_\mu d\tilde{Z}_\mu. \quad (3)$$

The shocks to dS_t/S_t and μ_t are given by $d\tilde{Z}_S = dZ_S$ and $d\tilde{Z}_\mu = \rho dZ_S + \sqrt{1 - \rho^2} dZ_\mu$, where $dZ_{S,t}$ and $dZ_{\mu,t}$ are independent Wiener processes.

Equations (2)–(3) imply that the instantaneous return on stocks (dS_t/S_t) follows a diffusion process whose drift (or instantaneous expected return) μ_t is mean-reverting and instantaneously correlated with the instantaneous return itself, with correlation coefficient equal to ρ . These equations define in fact a continuous-time vector autoregressive (VAR) process for the instantaneous return on stocks and its expectation. For convenience, we work with instantaneous log returns, and rewrite the system as

$$\begin{aligned} \begin{bmatrix} d(\log S_t + \frac{1}{2}\sigma_S^2 t - \theta t) \\ d(\mu_t - \theta) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & -\kappa \end{bmatrix} \begin{bmatrix} \log S_t + \frac{1}{2}\sigma_S^2 t - \theta t \\ \mu_t - \theta \end{bmatrix} dt \\ &+ \begin{bmatrix} \sigma_S & 0 \\ \sigma_\mu \rho & \sigma_\mu \sqrt{1 - \rho^2} \end{bmatrix} \begin{bmatrix} dZ_{S,t} \\ dZ_{\mu,t} \end{bmatrix}, \end{aligned} \quad (4)$$

where we have applied Itô's Lemma to equation (2) to obtain the process for the instantaneous log return. We show in Section 2.3 that the system (4) is the continuous-time counterpart of the discrete-time VAR(1) process in Campbell and Viceira (1999).

2.2 Time-aggregation of the continuous-time VAR

We can write the continuous VAR model (4) in compact form as

$$dy_t = Ay_t dt + CdZ_t, \quad (5)$$

where the definition of the variables and coefficient matrices is obvious. Note that the instantaneous variance of dy is given by CC' :

$$\text{Var}(dy) = CC' = \begin{bmatrix} \sigma_S^2 & \rho\sigma_S\sigma_\mu \\ \rho\sigma_S\sigma_\mu & \sigma_\mu^2 \end{bmatrix}.$$

Bergstrom (1984) and Campbell and Kyle (1993) show how to derive the discrete-time process implied by a continuous-time VAR when we take point observations of the continuous time process at evenly spaced points $\{t_0, t_1, \dots, t_n, t_{n+1}, \dots\}$, with $\Delta t = t_n - t_{n-1}$. Direct application of their results shows that the process y in (5) has the following discrete-time VAR(1) representation:

$$y_{t_n+\Delta t}^p = \exp\{\Delta t A\} y_{t_n}^p + u_{t_n+\Delta t}^p, \quad (6)$$

where

$$u_{t_n+\Delta t}^p = \int_{\tau=0}^{\Delta t} \exp\{(\Delta t - \tau) A\} CdZ_{t_n+\tau}, \quad (7)$$

and

$$\exp\{A\} = I + \sum_{r=1}^{\infty} \frac{A^r}{r!}. \quad (8)$$

We prove in Appendix A that $\exp\{sA\}$ is equal to

$$\exp(As) = \begin{bmatrix} 1 & \frac{1}{\kappa}(1 - e^{-\kappa s}) \\ 0 & e^{-\kappa s} \end{bmatrix}. \quad (9)$$

Thus we can write (6) in matrix form as:

$$\begin{aligned} & \begin{bmatrix} \log S_{t_n+\Delta t} + \frac{\sigma_S^2}{2}(t_n + \Delta t) - \theta(t_n + \Delta t) \\ \mu_{t_n+\Delta t} - \theta \end{bmatrix} \\ &= \begin{bmatrix} 1 & \frac{1}{\kappa}(1 - e^{-\kappa\Delta t}) \\ 0 & e^{-\kappa\Delta t} \end{bmatrix} \begin{bmatrix} \log S_{t_n} + \frac{\sigma_S^2}{2}t_n - \theta t_n \\ \mu_{t_n} - \theta \end{bmatrix} + u_{t_n+\Delta t}, \end{aligned} \quad (10)$$

where

$$u_{t_n+\Delta t} = \begin{bmatrix} u_{S,t_n+\Delta t} \\ u_{\mu,t_n+\Delta t} \end{bmatrix} = \int_0^{\Delta t} \begin{bmatrix} 1 & \frac{1}{\kappa}(1 - e^{-\kappa(\Delta t-\tau)}) \\ 0 & e^{-\kappa(\Delta t-\tau)} \end{bmatrix} \begin{bmatrix} \sigma_S & 0 \\ \rho\sigma_\mu & \sigma_\mu\sqrt{1-\rho^2} \end{bmatrix} dZ_{t_n+\tau}. \quad (11)$$

>From equation (11), it follows that the variance-covariance matrix of the innovations $u_{t_n+\Delta t}$ in the discrete-time representation of the continuous-time VAR is given by

$$\begin{aligned} \text{Var}(u_{t_n+\Delta t}^p) &= \int_{\tau=0}^{\Delta t} \exp\{(\Delta t - \tau)A\} CC' \exp\{(\Delta t - \tau)A'\} d\tau \\ &= \int_0^{\Delta t} \begin{bmatrix} B_{11} & B_{12} \\ B_{12} & B_{22} \end{bmatrix} d\tau, \end{aligned} \quad (12)$$

where

$$\begin{aligned}
B_{11} &= \sigma_S^2 + \frac{2\rho\sigma_S\sigma_\mu}{\kappa} (1 - e^{-\kappa(\Delta t - \tau)}) + \frac{\sigma_\mu^2}{\kappa^2} (1 - e^{-\kappa(\Delta t - \tau)})^2, \\
B_{12} &= \rho\sigma_S\sigma_\mu e^{-\kappa(\Delta t - \tau)} + \frac{\sigma_\mu^2}{\kappa} (e^{-\kappa(\Delta t - \tau)} - e^{-2\kappa(\Delta t - \tau)}), \\
B_{22} &= \sigma_\mu^2 e^{-2\kappa(\Delta t - \tau)}.
\end{aligned}$$

Therefore, given values for the parameters of the continuous-time process (4), we can easily aggregate to any frequency Δt , by using (10) and (12). The discrete-time representation is especially useful in recovering the parameters of the continuous-time VAR (4) from estimates of the equivalent discrete-time VAR (10). We do this in the next section.

2.3 Recovering continuous-time parameters from a discrete-time VAR

In their analysis of optimal consumption and portfolio choice with time-varying expected returns, Campbell and Viceira (1999, 2000) assume that the log excess returns on stocks is described by the following discrete-time VAR(1):

$$\begin{bmatrix} \Delta \log S_{t_n + \Delta t} - r_f \\ x_{t_n + \Delta t} \end{bmatrix} = \begin{bmatrix} 0 \\ (1 - \phi)\mu \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & \phi \end{bmatrix} \begin{bmatrix} \Delta \log S_{t_n} - r_f \\ x_{t_n} \end{bmatrix} + \begin{bmatrix} \varepsilon_{t_n + \Delta t} \\ \eta_{t_n + \Delta t} \end{bmatrix}, \quad (13)$$

where r_f is the return (assumed constant) on a T-bill with maturity Δt , and $x_{t_n + \Delta t}$ is the conditional expected log excess return on stocks.

We now show that there is pathwise convergence between the continuous-time VAR given in (4) and the discrete-time VAR given in (13). This implies that no other continuous-time process can generate (13) when aggregated at any time interval—although there might be other continuous-time processes whose moments would match the moments generated by (13) at specific time intervals. To see this, note that we can rewrite the discrete-time aggregation of y in (10) as follows:

$$\begin{aligned} \begin{bmatrix} \Delta \log S_{t_n+\Delta t} - r\Delta t \\ \mu_{t_n+\Delta t} \end{bmatrix} &= \begin{bmatrix} \left(\theta - \frac{\sigma_S^2}{2} - r\right) \Delta t - \frac{1}{\kappa} (1 - e^{-\kappa\Delta t}) \theta \\ (1 - e^{-\kappa\Delta t}) \theta \end{bmatrix} \\ &+ \begin{bmatrix} 1 & \frac{1}{\kappa} (1 - e^{-\kappa\Delta t}) \\ 0 & e^{-\kappa\Delta t} \end{bmatrix} \begin{bmatrix} \log S_{t_n} + \frac{\sigma_S^2}{2} t_n - \theta t_n \\ \mu_{t_n} \end{bmatrix} + \begin{bmatrix} u_{S,t_n+\Delta t} \\ u_{\mu,t_n+\Delta t} \end{bmatrix}. \end{aligned} \quad (14)$$

Using the following linear transformation for the process μ_t ,

$$v_t = \left(\theta - \frac{\sigma_S^2}{2} - r\right) \Delta t - \frac{1}{\kappa} (1 - e^{-\kappa\Delta t}) \theta + \frac{1}{\kappa} (1 - e^{-\kappa\Delta t}) \mu_t,$$

we can further rewrite (14) in the same form as (13):

$$\begin{aligned} \begin{bmatrix} \Delta \log S_{t_n+\Delta t} - r\Delta t \\ v_{t_n+\Delta t} \end{bmatrix} &= \begin{bmatrix} 0 \\ (1 - e^{-\kappa\Delta t}) \left(\theta - \frac{\sigma_S^2}{2} - r\right) \Delta t \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 1 \\ 0 & e^{-\kappa\Delta t} \end{bmatrix} \begin{bmatrix} \Delta \log S_{t_n+\Delta t} - r\Delta t \\ v_{t_n} \end{bmatrix} \\ &+ \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\kappa} (1 - e^{-\kappa\Delta t}) \end{bmatrix} \begin{bmatrix} u_{S,t_n+\Delta t} \\ u_{\mu,t_n+\Delta t} \end{bmatrix}. \end{aligned} \quad (15)$$

A simple comparison of the coefficients in (13) and (15) gives us a system of equations that relate the parameters of the discrete-time VAR process in Campbell and Viceira (1999) to the parameters of our continuous-time VAR process. For the intercept and slope parameters we have the following equivalence relations:

$$r_f = r\Delta t, \quad (16)$$

$$\mu = \left(\theta - \frac{\sigma_S^2}{2} - r \right) \Delta t, \quad (17)$$

$$\phi = e^{-\kappa\Delta t}. \quad (18)$$

Solving the integral (12) and matching parameters, we obtain the following equivalence relations for the variance and covariance parameters:

$$\begin{aligned} \text{Var}_{t_n} (\eta_{t_n+\Delta t}) &= \frac{1}{\kappa^2} (1 - e^{-\kappa\Delta t})^2 \text{Var}_{t_n} (u_{\mu,t_n+\Delta t}) \\ &= \frac{\sigma_\mu^2}{2\kappa^3} (1 - e^{-\kappa\Delta t})^2 (1 - e^{-2\kappa\Delta t}), \end{aligned} \quad (19)$$

$$\begin{aligned} \text{Cov}_{t_n} (\varepsilon_{t_n+\Delta t}, \eta_{t_n+\Delta t}) &= \frac{1}{\kappa} (1 - e^{-\kappa\Delta t}) \text{Cov}_t (u_{S,t_n+\Delta t}, u_{\mu,t_n+\Delta t}) \\ &= \frac{\rho\sigma_S\sigma_\mu}{\kappa^2} (1 - e^{-\kappa\Delta t})^2 + \frac{\sigma_\mu^2}{\kappa^3} (1 - e^{-\kappa\Delta t})^2 \\ &\quad - \frac{\sigma_\mu^2}{2\kappa^3} (1 - e^{-2\kappa\Delta t}) (1 - e^{-\kappa\Delta t}), \end{aligned} \quad (20)$$

$$\begin{aligned} \text{Var}_{t_n} (\varepsilon_{t_n+\Delta t}) &= \text{Var}_{t_n} (u_{S,t_n+\Delta t}) \\ &= \left(\sigma_S^2 + \frac{2\rho\sigma_S\sigma_\mu}{\kappa} + \frac{\sigma_\mu^2}{\kappa^2} \right) \Delta t - \frac{2\rho\sigma_S\sigma_\mu}{\kappa^2} (1 - e^{-\kappa\Delta t}) \\ &\quad - \frac{2\sigma_\mu^2}{\kappa^3} (1 - e^{-\kappa\Delta t}) + \frac{\sigma_\mu^2}{2\kappa^3} (1 - e^{-2\kappa\Delta t}). \end{aligned} \quad (21)$$

The nonlinear system of equations (16)–(21) has a unique solution. That is, there is a

unique correspondence between the set of parameters that define the continuous time model (4), and the set of parameters that define the discrete-time model (13). To see this, note that equations (16) and (18) uniquely relate r_f and r , and ϕ and κ respectively. Given κ , equation (19) uniquely relates $\text{Var}_{t_n}(\eta_{t_n+\Delta t})$ and σ_μ^2 ; given κ and σ_μ^2 , equation (20) uniquely relates $\text{Cov}_{t_n}(\varepsilon_{t_n+\Delta t}, \eta_{t_n+\Delta t})$ and $\sigma_{\mu S} \equiv \rho\sigma_S\sigma_\mu$; and given κ , σ_μ^2 and $\sigma_{\mu S}$, equation (21) uniquely relates $\text{Var}_{t_n}(\varepsilon_{t_n+\Delta t})$ and σ_S^2 . Finally, given σ_S^2 and r , equation (17) uniquely relates θ and μ .

Campbell and Viceira (2000) report estimates of the VAR(1) given in (13) based on US quarterly data for the period 1947.Q1-1995.Q4. Table 1 shows the value of the parameters of the continuous-time equivalent VAR implied by their estimates.²

3 Intertemporal Portfolio Choice

In this section we solve the intertemporal consumption and portfolio choice problem of an investor who faces the investment opportunity set described in Section 2. To this end we use the approximation techniques described in Chacko and Viceira (1999) and Campbell and Viceira (2002), and show that the solution is invariant to the choice of discrete-time or

²There is an estimation error in Campbell and Viceira (1999) that results in an underestimation of the degree of predictability in stock returns in their paper. Campbell and Viceira (2000) report correct estimates, and calibration results based on the corrected estimates.

continuous-time approximations.

3.1 Assumptions on investment opportunities and preferences

We consider an investor who has only two assets available for investment, a riskless short-term bond and stocks, and no labor income. Return dynamics are given by (1) and the bivariate system (2)–(3). These assumptions on investment opportunities imply that the wealth dynamics for the investor are given by

$$dW_t = rW_t dt + \alpha_t W_t [(\mu_t - r) dt + \sigma_S dZ_S] - C_t dt, \quad (22)$$

where α_t is the fraction of wealth invested in stocks, and C_t denotes consumption.

We assume that the investor has recursive preferences over consumption. We use Duffie and Epstein's (1992a, b) continuous-time parameterization:

$$J_t = \int_t^\infty f(C_s, J_s) ds,$$

where $f(C_s, J_s)$ is a normalized aggregator of current consumption and continuation utility that takes the form

$$f(C, J) = \frac{\beta}{1 - \frac{1}{\psi}} (1 - \gamma) J \left[\left(\frac{C}{((1 - \gamma) J)^{\frac{1}{1-\gamma}}} \right)^{(1 - \frac{1}{\psi})} - 1 \right]. \quad (23)$$

Here $\beta > 0$ is the rate of time preference, $\gamma > 0$ is the coefficient of relative risk aversion, and $\psi > 0$ is the elasticity of intertemporal substitution. The Duffie-Epstein continuous-time

preference specification is normalized differently from the Epstein-Zin (1989, 1991) discrete-time specification—in particular, its value function that is homogeneous of degree $(1 - \gamma)$ in wealth, while the Epstein-Zin value function is linearly homogeneous in wealth—but it generates equivalent decision rules.

There are two interesting special cases of the normalized aggregator (23): $\psi = 1/\gamma$ and $\psi = 1$. The case $\psi = 1/\gamma$ is interesting because in that case the normalized aggregator (23) reduces to the standard, additive power utility function—from which log utility obtains by setting $\gamma = 1$. In the second special case, the aggregator $f(C_s, J_s)$ takes the following form as $\psi \rightarrow 1$:

$$f(C_s, J_s) = \beta(1 - \gamma) J \left[\log(C) - \frac{1}{1 - \gamma} \log((1 - \gamma) J) \right]. \quad (24)$$

The case $\psi = 1$ is important because it allows an exact solution to our dynamic optimization problem for investors who are more risk averse than an investor with unit coefficient of relative risk aversion. We now explore this solution, as well as an approximate solution for investors with $\psi \neq 1$ in the next section.

3.2 Bellman equation

Duffie and Epstein (1992a, b) show that the standard Bellman principle of optimality applies to recursive utility. The Bellman equation for this problem is

$$0 = \sup_{\{\alpha_t, C_t\}} \left\{ f(C_t, J_t) + J_W [W_t(r + \alpha_t(\mu_t - r)) - C_t] + J_{\mu} \kappa(\theta - \mu_t) + \frac{1}{2} J_{WW} W_t^2 \alpha_t^2 \sigma_S^2 + J_{W\mu} W_t \alpha_t \rho \sigma_S \sigma_{\mu} + \frac{1}{2} J_{\mu\mu} \sigma_{\mu}^2 \right\}, \quad (25)$$

where $f(C_t, J_t)$ is given in (23) when $\psi \neq 1$, or (24) when $\psi = 1$. J_x denotes the partial derivative of J with respect to x , except J_t , which denotes the value of J at time t .

The first order condition for consumption is given by

$$C_t = J_W^{-\psi} [(1 - \gamma) J]^{\frac{1-\gamma\psi}{1-\gamma}} \beta^{\psi}, \quad (26)$$

which reduces to $C_t = (J/J_W)(1 - \gamma)\beta$ when $\psi = 1$.

The first order condition for portfolio choice is given by

$$\alpha_t = \frac{-J_W}{W_t J_{WW}} \left(\frac{\mu_t - r}{\sigma_S^2} \right) - \frac{J_{W\mu}}{W_t J_{WW}} \left(\frac{\rho \sigma_{\mu}}{\sigma_S} \right). \quad (27)$$

Substitution of the first order conditions (26) and (27) into the Bellman equation (25) results in the following partial differential equation for the value function J :

$$0 = f \left(J_W^{-\psi} \{(1 - \gamma) J\}^{\frac{1-\gamma\psi}{1-\gamma}} \beta^{\psi}, J_t \right) - J_W \left\{ J_W^{-\psi} [(1 - \gamma) J]^{\frac{1-\gamma\psi}{1-\gamma}} \beta^{\psi} \right\} + J_W W_t r + J_{\mu} \kappa(\theta - \mu_t) + \frac{1}{2} J_{\mu\mu} \sigma_{\mu}^2 - \frac{1}{2} \left\{ \frac{J_W^2}{J_{WW}} \frac{(\mu_t - r)^2}{\sigma_S^2} + 2 \frac{J_W J_{W\mu} \rho \sigma_{\mu} (\mu_t - r)}{J_{WW} \sigma_S} + \frac{J_{W\mu}^2}{J_{WW}} \rho^2 \sigma_{\mu}^2 \right\}. \quad (28)$$

Of course, the form of this equation depends on whether we consider the case $\psi \neq 1$ and use the normalized aggregator (23), or we consider the case $\psi = 1$ and use the normalized aggregator (24). Appendix B shows the partial differential equation that obtains in each case.

In particular, we show in Appendix B that (28) has an exact analytical solution when $\psi = 1$. This solution is

$$J(W_t, \mu_t) = I(\mu_t) \frac{W_t^{1-\gamma}}{1-\gamma}, \quad (29)$$

with

$$I(\mu_t) = \exp \left\{ A_0 + B_0 \mu_t + \frac{C_0}{2} \mu_t^2 \right\}, \quad (30)$$

where A , B , and C are functions of the primitive parameters of the model describing investment opportunities and preferences.

In the more general case $\psi \neq 1$, there is no exact analytical solution to (28). However, we can still find an approximate analytical solution following the methods described in Campbell and Viceira (2002) and Chacko and Viceira (1999). We start by guessing that the value function in this case also has the form given in (29), with

$$I(\mu_t) = H(\mu_t)^{-\left(\frac{1-\gamma}{1-\psi}\right)}. \quad (31)$$

Substitution of (31) into the Bellman equation (28) results in an ordinary differential equation for $H(\mu_t)$. This equation does not have an exact analytical solution in general. However, we show in Appendix B that taking a loglinear approximation to one of the terms in the

equation results in a new equation for $H(\mu_t)$ that admits an analytical solution. The form of this solution is an exponential-quadratic function similar to (30):

$$H(\mu_t) = \exp \left\{ A_1 + B_1 \mu_t + \frac{C_1}{2} \mu_t^2 \right\}. \quad (32)$$

The term that we need to approximate in the ordinary differential equation for $H(\mu_t)$ is $\beta^\psi H(\mu_t)^{-1}$. Substitution of (29) and (32) into the first order condition (26) shows that this term is simply the optimal consumption-wealth ratio C_t/W_t . Thus this loglinearization is equivalent to loglinearizing the optimal consumption-wealth ratio around one particular point of the state space. Campbell and Viceira (2002) and Chacko and Viceira (1999) suggest approximating this term around the unconditional mean of the log consumption wealth-ratio. This choice has the advantage that the solution is exact when the log consumption-wealth ratio is constant, that is when $\psi = 1$, and accurate if the log consumption-wealth is not too variable around its mean. It is also interesting to note that the approximation around $\psi = 1$ includes, as a special case, the approximation around the exact known solution for a log-utility investor ($\gamma = \psi \equiv 1$) suggested by Kogan and Uppal (2000). Their approximation is unlikely to be accurate for values of γ far from 1, whereas the approximation used here can be arbitrarily accurate for any value of γ provided that ψ is sufficiently close to one. Appendix B provides full details of our solution procedure.

3.3 Optimal portfolio choice

The optimal portfolio policy of the investor can be found by substituting the solution for the value function into the first-order condition (27). In the case $\psi = 1$, substitution of (29)–(30) into (27) gives

$$\alpha_t = \left(\frac{1}{\gamma}\right) \frac{\mu_t - r}{\sigma_S^2} + \left(1 - \frac{1}{\gamma}\right) \frac{\sigma_\mu}{\sigma_S} \rho(\mathcal{B}_0 + \mathcal{C}_0 \mu_t), \quad (33)$$

where $\mathcal{B}_0 = -B_0/(1 - \gamma)$ and $\mathcal{C}_0 = -C_0/(1 - \gamma)$.

In the case $\psi \neq 1$, substitution of the approximate solution (31)–(32) into (27) gives

$$\alpha_t = \left(\frac{1}{\gamma}\right) \frac{\mu_t - r}{\sigma_S^2} + \left(1 - \frac{1}{\gamma}\right) \frac{\sigma_\mu}{\sigma_S} \rho(\mathcal{B}_1 + \mathcal{C}_1 \mu_t), \quad (34)$$

where $\mathcal{B}_1 = -B_1/(1 - \psi)$ and $\mathcal{C}_1 = -C_1/(1 - \psi)$. Appendix B shows that \mathcal{B}_1 and \mathcal{C}_1 do not depend on ψ , except through a loglinearization parameter.

Equations (33) and (34) show that the optimal allocation to stocks is a weighted average (with weights $1/\gamma$ and $1 - 1/\gamma$) of two terms, both of them linear in the expected return on stocks μ_t . The first term is the myopic portfolio allocation to stocks, and the second term is the intertemporal hedging demand for stocks. The myopic portfolio allocation is proportional to $(1/\gamma)$, so that it approaches zero as we consider increasingly risk averse investors.

The intertemporal hedging component is proportional to $(1 - 1/\gamma)$, but it also depends

on γ through \mathcal{B} and \mathcal{C} . Appendix B shows that this term also approaches zero as $\gamma \rightarrow \infty$.³ This result follows from our assumption of a constant instantaneous real interest rate, which makes the short-term bond riskless at all investment horizons. Campbell and Viceira (2001, 2002) and Campbell, Chan and Viceira (2003) show that, with a time-varying instantaneous real interest rate, myopic portfolio demand still approaches zero as $\gamma \rightarrow \infty$, but intertemporal hedging demand does not. Instead, the intertemporal hedging portfolio is fully invested in a real perpetuity or, if that asset is not available, in the combination of available assets that most closely mimics a real perpetuity.

3.4 Numerical calibration

In this section we use the parameter values given in Table 1 to calibrate the continuous-time portfolio rule (34), and compare the resulting allocations to those implied by the discrete-time model of Campbell and Viceira (1999, 2000).⁴

Table 2 reports mean optimal portfolio allocations (Panel A), and the percentage that the

³Since intertemporal hedging demand is proportional to $(1 - 1/\gamma)$, one might be tempted to conclude that it does not approach zero in the limit as $\gamma \rightarrow \infty$. However, we need to consider that \mathcal{B}_0 (or \mathcal{B}_1) and \mathcal{C}_0 (or \mathcal{C}_1) are also functions of γ . Appendix B shows that the limit of the overall expression approaches zero as $\gamma \rightarrow \infty$.

⁴Appendix B shows that \mathcal{B}_1 and \mathcal{C}_1 depend on the loglinearization parameter $h_1 = E[c_t - w_t]$, which is endogenous. However, one can solve for h_1 using the simple numerical recursive algorithm described in Campbell and Viceira (1999).

mean intertemporal hedging portfolio allocation represents over the total mean allocation (Panel B) for investors with coefficients of relative risk aversion between 0.75 and 40, and elasticities of intertemporal substitution of consumption between 1/0.75 and 1/40. Note that the linearity of the optimal portfolio rule (33)–(34) implies that

$$E[\alpha_t] = \left(\frac{1}{\gamma}\right) \frac{\theta - r}{\sigma_S^2} + \left(1 - \frac{1}{\gamma}\right) \frac{\sigma_\mu}{\sigma_S} \rho(\mathcal{B}_i + \mathcal{C}_i\theta), \quad i = 0, 1, \quad (35)$$

where the first element of the sum is the mean myopic portfolio allocation, and the second element is the mean intertemporal hedging portfolio allocation. The numbers in Table 2 support the conclusion of Campbell and Viceira (1999, 2000) that given the historical behavior of the US stock market, intertemporal hedging motives greatly increase the average demand for stocks by investors with risk aversion greater than one. For highly conservative investors, hedging demand may represent 90% or even more of the total mean demand for stocks.

The numbers reported in Table 2 are not directly comparable to those in Campbell and Viceira (2000) because Table 2 assumes that investors can rebalance their portfolios continuously, while Campbell and Viceira assume that investors rebalance their portfolios at a quarterly frequency. Campbell and Viceira’s Table III shows that even with quarterly rebalancing the mean portfolio allocations are fairly close to the continuous-time mean allocations.⁵ But a direct comparison requires that we compute the limit of Campbell and

⁵For example, for investors with $\psi = 1$ and coefficients of relative risk aversion identical to those reported here, Campbell and Viceira (2000) report mean percentage portfolio allocations to stocks equal to {209.36%,

Viceira's solution as the frequency of rebalancing increases.

We accomplish this in two steps. First, using the system of equations (16)-(21), we recover discrete-time parameters at any desired frequency from the continuous-time parameters shown in Table 1. This ensures that we use parameter values that are mutually consistent at any frequency. Second, we recompute Campbell and Viceira's discrete-time solution. We have conducted this exercise, and found that at a daily rebalancing frequency, the mean portfolio allocations generated by the Campbell-Viceira model are virtually identical to those reported in Table 2.⁶

These results show numerically that Campbell and Viceira's (1999, 2000) discrete-time solution converges in the limit to the continuous-time solution. Since the continuous-time solution is exact when $\psi = 1$, these results also confirm their claim that their discrete-time solution is exact for the case $\psi = 1$ up to a discrete-time approximation to the log return on wealth.

An analytical proof of this convergence result for general preference parameters is straightforward conceptually, but extremely tedious algebraically. Nevertheless, we illustrate here

222.74%, 235.35%, 239.36%, 230.09%, 179.16%, 121.21%, 78.58%}.

⁶As an example, consider the case of investors with $\psi = 1$ and coefficients of relative risk aversion identical to those reported here. The model of Campbell and Viceira (1999) generates mean percentage portfolio allocations to stocks when investors can rebalance daily equal to {198.19%, 211.31%, 223.84%, 228.06%, 220.43%, 173.67%, 123.69%, 77.77%}.

how the proof proceeds in the special case of unit relative risk aversion. The model solution in that case is simple enough that we can show in one step how the discrete-time mean portfolio allocation in Proposition I in Campbell and Viceira (1999) converges to the continuous-time mean allocation (35). Taking limits as $\Delta t \rightarrow 0$ on Proposition I of Campbell and Viceira (1999) with $\gamma = 1$, we have

$$\begin{aligned}
\lim_{\Delta t \rightarrow 0} \mathbb{E} [\alpha_{t_n, \gamma=1}] &= \lim_{\Delta t \rightarrow 0} \frac{\mu + \frac{1}{2} \text{Var}_{t_n} (\varepsilon_{t_n + \Delta t})}{\text{Var}_{t_n} (\varepsilon_{t_n + \Delta t})} \\
&= \frac{\left(\theta - \frac{\sigma_S^2}{2} - r \right) + \frac{1}{2} \left[\left(\sigma_S^2 + \frac{2\rho\sigma_S\sigma_\mu}{\kappa} + \frac{\sigma_\mu^2}{\kappa^2} \right) - \frac{2\rho\sigma_S\sigma_\mu}{\kappa} - \frac{\sigma_\mu^2}{\kappa^2} \right]}{\left(\sigma_S^2 + \frac{2\rho\sigma_S\sigma_\mu}{\kappa} + \frac{\sigma_\mu^2}{\kappa^2} \right) - \frac{2\rho\sigma_S\sigma_\mu}{\kappa} - \frac{\sigma_\mu^2}{\kappa^2}} \\
&= \frac{\theta - r}{\sigma_S^2},
\end{aligned}$$

which is the mean portfolio allocation implied by (35) with $\gamma = 1$. The second equality follows from equations (17) and (21). The proof in the general case follows similar steps.

3.5 A pitfall, and its implications for portfolio choice

Anyone used to working with the discrete-time representation of a univariate continuous-time process will find natural and intuitive the relation between the intercept and slope of the continuous-time VAR and its discrete-time representation implied by equations (16)–(18). However, equations (19)–(21) show that the equivalence relation for the variance-covariance matrix of innovations is less obvious. Using an intuitive extension of the usual matching rules for a univariate process, one might be tempted to identify variance-covariance parameters

using the following expressions:

$$\text{Var}_{t_n}(\varepsilon_{n+1}) \approx \sigma_S^2 \Delta t \quad (36)$$

$$\text{Cov}_{t_n}(\varepsilon_{t_n+\Delta t}, \eta_{t_n+\Delta t}) \approx \rho \sigma_S \sigma_\mu \Delta t \quad (37)$$

$$\text{Var}_{t_n}(\eta_{t_n+\Delta t}) \approx \sigma_\mu^2 \Delta t. \quad (38)$$

Equations (36)–(38) are very different from the correct equations (19)–(21), although equation (36) is a first-order Taylor expansion of the correct expression for $\text{Var}_{t_n}(\varepsilon_{n+1})$ given in (21). The use of equations (36)–(38) is particularly dangerous when $\Delta t \neq 1$, as might be the case when one is using annualized parameters and quarterly data. In this case, portfolio allocations based on matching parameters using equations (36)–(38) can be quite different from allocations based on the correct equations (19)–(21). In our calibration exercise, incorrectly matched parameters generate intertemporal hedging demands when $\Delta t = 0.25$ that are substantially lower than the correct ones. For example, for investors with $\psi = 1$ and $\gamma = 10, 20$ and 40 , the total mean portfolio allocation is only 32%, 16% and 8%, and the percentage fraction of total portfolio demand due to intertemporal hedging is only 30%, 32% and 33%; these figures are an order of magnitude lower than those shown in Table 2.⁷ Thus a researcher using (36)–(38) to match parameters would wrongly conclude that empirically intertemporal hedging demands are much less important when investors can continuously rebalance their portfolios.

⁷This gross underestimation occurs for all parameter values. A table mimicking Table II, except that it uses incorrect parameter values, is readily available upon request from the corresponding author.

4 Conclusion

This paper solves a continuous-time consumption and portfolio choice problem with a constant interest rate and a time-varying equity premium. The model for asset returns is a continuous-time version of the model studied by Campbell and Viceira (1999). This model has an exact analytical solution when the investor has unit elasticity of intertemporal substitution in consumption and an approximate analytical solution otherwise. For calibration purposes, we also derive the discrete-time VAR representation for asset returns. This time-aggregation result is necessary to recover the parameters of the model from discrete-time VAR estimates. We show that intuitive discrete-time representations of univariate continuous-time processes do not translate immediately to multivariate processes which are cross-sectionally correlated.

Our calibration results show that our portfolio choice model is the limit, as the frequency of rebalancing increases, of its discrete-time counterpart. Thus it exhibits similar properties. In particular, given the historical experience in the US stock market, intertemporal hedging motives greatly increase the average demand for stocks by investors who are more risk averse than a logarithmic investor. For highly conservative investors, hedging may represent 90% or even more of the total mean demand for stocks.

5 Appendix A

We find $\exp(As)$ by use of an induction proof. We first prove by induction that the matrix A^n is given by

$$A^n = \begin{pmatrix} 0 & (-\kappa)^{n-1} \\ 0 & (-\kappa)^n \end{pmatrix}. \quad (39)$$

To prove this result, assume that A^n is given by (39). Then A^{n+1} is given by

$$\begin{aligned} A^{n+1} &= A^n A \\ &= \begin{pmatrix} 0 & (-\kappa)^{n-1} \\ 0 & (-\kappa)^n \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & -\kappa \end{pmatrix} \\ &= \begin{pmatrix} 0 & (-\kappa)^n \\ 0 & (-\kappa)^{n+1} \end{pmatrix}, \end{aligned}$$

which is the desired result.

The matrix $\exp(As)$ is given by $I + As + \cdots + A^n s^n/n! + \cdots$. Equation (39) allows us to write the exponential matrix as

$$\begin{aligned}
\exp(As) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{n!} \sum_{n=1}^{\infty} \begin{pmatrix} 0 & (-\kappa)^{n-1} s^n \\ 0 & (-\kappa)^n s^n \end{pmatrix} \\
&= \begin{pmatrix} 1 & \frac{1}{n!} \sum_{n=1}^{\infty} (-\kappa)^{n-1} s^n \\ 0 & \frac{1}{n!} \sum_{n=0}^{\infty} (-\kappa s)^n \end{pmatrix} \\
&= \begin{pmatrix} 1 & \frac{1}{n!} \sum_{n=1}^{\infty} (-\kappa)^{n-1} s^n \\ 0 & \exp(-\kappa s) \end{pmatrix}.
\end{aligned}$$

Now, notice that

$$\frac{d}{ds} \left(\frac{1}{n!} \sum_{n=1}^{\infty} (-\kappa)^{n-1} s^n \right) = \frac{1}{n!} \sum_{n=0}^{\infty} (-\kappa s)^n = \exp(-\kappa s),$$

so that

$$\begin{aligned}
\frac{1}{n!} \sum_{n=1}^{\infty} (-\kappa)^{n-1} s^n &= \int \exp(-\kappa s) ds \\
&= \frac{-1}{\kappa} \exp(-\kappa s) + C.
\end{aligned}$$

Since at $s = 0$ we have that $(\sum_{n=1}^{\infty} (-\kappa)^{n-1} s^n) / n! = 0$, it follows that for the equation to hold at $s = 0$ we must have that $C = \frac{1}{\kappa}$.

Therefore

$$\frac{1}{n!} \sum_{n=1}^{\infty} (-\kappa)^{n-1} s^n = \frac{1}{\kappa} (1 - \exp(-\kappa s)), \tag{40}$$

from which it follows that we can write the matrix $\exp(As)$ as

$$\exp(As) = \begin{pmatrix} 1 & \frac{1}{\kappa} (1 - e^{-\kappa s}) \\ 0 & e^{-\kappa s} \end{pmatrix}. \tag{41}$$

6 Appendix B

6.1 Exact analytical solution when $\psi = 1$

Substitution of (29) and (30) into the Bellman equation (28) leads, after some simplification,

to the following equation:

$$\begin{aligned}
0 = & -\frac{1}{1-\gamma}\beta \left\{ A_0 + B_0\mu_t + \frac{C_0}{2}\mu_t^2 \right\} + \beta \log \beta + r - \beta \\
& + \frac{\kappa(\theta - \mu_t)}{1-\gamma} (B_0 + C_0\mu_t) + \frac{\sigma_\mu^2}{2(1-\gamma)} \{ C_0 + (B_0 + C_0\mu_t)^2 \} \\
& + \frac{1}{2\gamma} \left\{ \frac{(\mu_t - r)^2}{\sigma_S^2} + 2\frac{\rho\sigma_\mu(\mu_t - r)}{\sigma_S} (B_0 + C_0\mu_t) + \rho^2\sigma_\mu^2 (B_0 + C_0\mu_t)^2 \right\}.
\end{aligned}$$

We can now obtain A_0 , B_0 and C_0 from the system of recursive equations that results from collecting terms in μ_t^2 , μ_t , and constant terms:

$$0 = \frac{\sigma_\mu^2}{2} \left(1 + \frac{1-\gamma}{\gamma}\rho^2 \right) C_0^2 + \left(-\frac{\beta}{2} - \kappa + \frac{1-\gamma}{\gamma}\frac{\rho\sigma_\mu}{\sigma_S} \right) C_0 + \frac{1}{2\sigma_S^2} \left(\frac{1-\gamma}{\gamma} \right), \quad (42)$$

$$\begin{aligned}
0 = & \kappa\theta C_0 - \frac{1-\gamma}{\gamma}\frac{r}{\sigma_S^2} - \frac{1-\gamma}{\gamma}\frac{\rho r\sigma_\mu}{\sigma_S} C_0 \\
& + \left(-\kappa - \beta + \sigma_\mu^2 \left(1 + \frac{1-\gamma}{\gamma}\rho^2 \right) C_0 + \frac{1-\gamma}{\gamma}\frac{\rho\sigma_\mu}{\sigma_S} \right) B_0,
\end{aligned} \quad (43)$$

$$\begin{aligned}
0 = & -\beta A_0 + (1-\gamma)\beta \log \beta + (1-\gamma)(r - \beta) + \frac{r^2}{2\gamma\sigma_S^2} + \frac{\sigma_\mu^2}{2} C_0 \\
& + \frac{\sigma_\mu^2}{2} \left(1 + \frac{1-\gamma}{\gamma}\rho^2 \right) B_0^2 + \left(-\frac{1-\gamma}{\gamma}\frac{\sigma_\mu}{\sigma_S}\rho r + \kappa\theta \right) B.
\end{aligned} \quad (44)$$

We can solve this system by solving equation (42) and then using the result to solve (43) and finally solve (44). Equation (42) is a quadratic equation whose only unknown is C_0 .

Thus it has two roots. Campbell and Viceira (1999) show that only one of them maximizes

expected utility. This root is the one associated with the positive root of the discriminant of the equation. This is also the only root that ensures that $C_0 = 0$ when $\gamma = 1$, that is, in the log utility case. This is a necessary condition for intertemporal hedging demand to be zero, as we know it must in the log utility case.

We can use these results to obtain the optimal portfolio policy of the investor from the first order condition (27), and the optimal consumption policy from the first order condition (26). The optimal portfolio policy is given in equation (33) in text. It is easy to see that the optimal consumption policy is $C_t/W_t = \beta$, a constant consumption-wealth ratio equal to the rate of time preference.

We now show that the intertemporal hedging component of portfolio demand approaches zero as $\gamma \rightarrow \infty$. First, note that the solution C_0 to the quadratic equation (42) has the following finite limit as $\gamma \rightarrow \infty$:

$$\lim_{\gamma \rightarrow \infty} C_0 = \frac{1}{\sigma_\mu \sigma_S \sqrt{(1 - \rho^2)}}.$$

Similarly, from equation (43) we have that the limit of B_0 as $\gamma \rightarrow \infty$ is also finite:

$$\begin{aligned} \lim_{\gamma \rightarrow \infty} B_0 &= -\frac{\frac{r}{\sigma_S^2} + \left(\kappa\theta + \frac{\rho r \sigma_\mu}{\sigma_S}\right) \lim_{\gamma \rightarrow \infty} C_0}{\left(-\kappa - \beta - \frac{\rho \sigma_\mu}{\sigma_S} + \sigma_\mu^2 (1 - \rho^2) \lim_{\gamma \rightarrow \infty} C_0\right)} \\ &= -\frac{\frac{r}{\sigma_S^2} + \frac{\kappa\theta + \frac{\rho r \sigma_\mu}{\sigma_S}}{\sigma_\mu \sigma_S \sqrt{(1 - \rho^2)}}}{\left(-\kappa - \beta - \frac{\rho \sigma_\mu}{\sigma_S} + \frac{\sigma_\mu \sqrt{(1 - \rho^2)}}{\sigma_S}\right)} \end{aligned}$$

Thus

$$\lim_{\gamma \rightarrow \infty} \mathcal{C}_0 = \lim_{\gamma \rightarrow \infty} \frac{C_0}{\gamma - 1} = 0,$$

and

$$\lim_{\gamma \rightarrow \infty} \mathcal{B}_0 = \lim_{\gamma \rightarrow \infty} \frac{B_0}{\gamma - 1} = 0,$$

which in turn implies that

$$\lim_{\gamma \rightarrow \infty} \left(1 - \frac{1}{\gamma}\right) \frac{\sigma_\mu}{\sigma_S} \rho (\mathcal{B}_0 + \mathcal{C}_0 \mu_t) = 0.$$

6.2 Approximate analytical solution when $\psi \neq 1$

Substitution of (29) and (31) into the Bellman equation (28) gives, after some simplification, the following ordinary differential equation:

$$\begin{aligned} 0 = & -\beta^\psi H^{-1} + \beta\psi + r(1 - \psi) - \frac{H_\mu}{H} \kappa (\theta - \mu_t) \\ & + \frac{\sigma_\mu^2}{2} \left(-\frac{H_{\mu\mu}}{H} + \left(1 + \frac{1 - \gamma}{1 - \psi}\right) \left(\frac{H_\mu}{H}\right)^2 \right) \\ & + \frac{1 - \psi}{2\gamma} \left(\frac{\mu_t - r}{\sigma_S}\right)^2 - \frac{1 - \gamma}{\gamma} \frac{H_\mu}{H} \rho \sigma_\mu \left(\frac{\mu_t - r}{\sigma_S}\right) + \frac{1}{2} \frac{(1 - \gamma)^2}{\gamma(1 - \psi)} \left(\frac{H_{\mu\mu}}{H}\right)^2 \rho^2 \sigma_\mu^2. \end{aligned} \quad (45)$$

This ordinary differential equation does not have an exact analytical solution, unless $\psi = 1$.

Though there does not exist an exact analytical solution to (45), we can still find an approximate analytical solution following the methods described in Campbell and Viceira (2002).and Chacko and Viceira (1999). First, we note that substitution of the solution guess

(29)-(31) into the first order condition (26) gives

$$\frac{C_t}{W_t} = \beta^\psi H(\mu_t)^{-1}.$$

We can now use the following approximation for $\beta^\psi H^{-1}$ around the unconditional mean of the log consumption-wealth ratio:

$$\begin{aligned} \beta^\psi H(\mu_t)^{-1} &= \exp\{c_t - w_t\} \\ &\approx h_0 + h_1(c_t - w_t) \\ &= h_0 + h_1(\psi \log \beta - h_t), \end{aligned} \tag{46}$$

where $c_t = \log C_t$, $w_t = \log W_t$, $h_t = \log H(\mu_t)$, and

$$h_1 = \exp\{E[c_t - w_t]\}, \tag{47}$$

$$h_0 = h_1(1 - \log h_1). \tag{48}$$

Substitution of the approximation (46) for the first term of (45) transforms this ordinary differential equation into another one that has an exact solution, with the following exponential-quadratic form:

$$H(\mu_t) = \exp\left\{A_1 + B_1\mu_t + \frac{C_1}{2}\mu_t^2\right\}.$$

The coefficients A_1 , B_1 , and C_1 , can be obtained by solving the approximated Bellman

equation

$$\begin{aligned}
0 = & -h_0 - h_1 \left\{ \psi \log \beta - \left(A_1 + B_1 \mu_t + \frac{C_1}{2} \mu_t^2 \right) \right\} + \beta \psi + r(1 - \psi) - \kappa(\theta - \mu_t)(B_1 + C_1 \mu_t) \\
& + \frac{\sigma_\mu^2}{2} \left[-((B_1 + C_1 \mu_t)^2 + C_1) + \left(\frac{1 - \gamma}{1 - \psi} + 1 \right) (B_1 + C_1 \mu_t)^2 \right] + \frac{1 - \psi}{2\gamma} \left(\frac{\mu_t - r}{\sigma_S} \right)^2 \\
& - \frac{1 - \gamma}{\gamma} (B_1 + C_1 \mu_t) \rho \sigma_\mu \left(\frac{\mu_t - r}{\sigma_S} \right) + \frac{1 - \gamma}{2\gamma} \left(\frac{1 - \gamma}{1 - \psi} \right) (B_1 + C_1 \mu_t)^2 \rho^2 \sigma_\mu^2,
\end{aligned}$$

which implies the following system of recursive equations:

$$0 = \frac{\sigma_\mu^2}{2} \frac{1 - \gamma}{1 - \psi} \left(1 + \frac{1 - \gamma}{\gamma} \rho^2 \right) C_1^2 + \left(\frac{h_1}{2} + \kappa - \frac{1 - \gamma}{\gamma} \frac{\rho \sigma_\mu}{\sigma_S} \right) C_1 + \frac{1 - \psi}{2\gamma \sigma_S^2}, \quad (49)$$

$$\begin{aligned}
0 = & \kappa \theta C_1 + \frac{1 - \psi}{\gamma} \frac{r}{\sigma_S^2} - \frac{1 - \gamma}{\gamma} \frac{\rho r \sigma_\mu}{\sigma_S} C_1 \\
& + \left(\kappa + h_1 + \sigma_\mu^2 \frac{1 - \gamma}{1 - \psi} \left(1 + \frac{1 - \gamma}{\gamma} \rho^2 \right) C_1 - \frac{1 - \gamma}{\gamma} \frac{\rho \sigma_\mu}{\sigma_S} \right) B_1,
\end{aligned} \quad (50)$$

$$\begin{aligned}
0 = & h_1 A_1 - h_0 - h_1 \psi \log \beta + \beta \psi + r(1 - \psi) + \frac{1 - \psi}{2\gamma} \frac{r^2}{\sigma_S^2} - \frac{\sigma_\mu^2}{2} C_1 \\
& + \frac{\sigma_\mu^2}{2} \frac{1 - \gamma}{1 - \psi} \left(1 + \frac{1 - \gamma}{\gamma} \rho^2 \right) B_1^2 + \left(\frac{1 - \gamma}{\gamma} \frac{\sigma_\mu}{\sigma_S} \rho r - \kappa \theta \right) B_1.
\end{aligned} \quad (51)$$

We can solve this system by solving equation (49) and then using the result to solve (50) and finally solve (51). Equation (49) is a quadratic equation whose only unknown is C . Thus it has two roots. Campbell and Viceira (1999) show that only one of them maximizes expected utility. This root is the one associated with the positive root of the discriminant of the equation. Note also that this equation implies that $C/(1 - \psi)$ does not depend on ψ —except through the loglinearization parameter h_1 —which in turn implies, through equation (50), that $B/(1 - \psi)$ does not depend on ψ either.

We now show that the intertemporal hedging component of portfolio demand approaches zero as $\gamma \rightarrow \infty$. First, note that the solution C_1 to the quadratic equation (42) converges to zero as $\gamma \rightarrow \infty$. To see this, note that the limit of the numerator in the solution is finite, while the denominator diverges to ∞ . Thus $\lim_{\gamma \rightarrow \infty} C_1 = 0$. From equation (43), it is immediate to see that this implies that B_1 also approaches zero as $\gamma \rightarrow \infty$. Therefore,

$$\lim_{\gamma \rightarrow \infty} C_1 = \lim_{\gamma \rightarrow \infty} \frac{C_1}{\psi - 1} = 0,$$

and

$$\lim_{\gamma \rightarrow \infty} B_1 = \lim_{\gamma \rightarrow \infty} \frac{B_1}{\psi - 1} = 0,$$

so

$$\lim_{\gamma \rightarrow \infty} \left(1 - \frac{1}{\gamma}\right) \frac{\sigma_\mu}{\sigma_S} \rho(B_1 + C_1 \mu_t) = 0.$$

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TABLE 1
Continuous-Time VAR: Parameter Values

Model:

$$dB_t/B_t = rdt,$$

$$dS_t/S_t = \mu_t dt + \sigma_S d\tilde{Z}_S,$$

$$d\mu_t = \kappa(\theta - \mu_t)dt + \sigma_\mu d\tilde{Z}_\mu,$$

$$d\tilde{Z}_S d\tilde{Z}_\mu = \rho dt$$

Parameter Values:

r	0.0818e-2
κ	4.3875e-2
θ	1.3980e-2
σ_S	7.8959e-2
σ_μ	0.5738e-2
ρ	-0.9626

Note: These parameter values are based on quarterly estimates of the discrete-time equivalent process reported in Campbell and Viceira (2000). They are obtained using equations (16)-(21) in text, with $\Delta t = 1$.

TABLE 2
Mean Optimal Percentage Allocation to Stocks and
Percentage Mean Hedging Demand Over Mean Total Demand

R.R.A.	E.I.S.							
(A) Mean optimal percentage allocation to stocks:								
	1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	180.31	198.02	209.80	214.47	220.42	223.53	224.49	224.97
1.00	211.12	211.12	211.12	211.12	211.12	211.12	211.12	211.12
1.50	239.07	223.64	211.64	206.76	200.57	197.39	196.40	195.92
2.00	248.85	227.85	210.53	203.36	194.22	189.52	188.07	187.36
4.00	241.99	220.24	200.14	191.27	179.59	173.46	171.56	170.63
10.0	183.24	173.55	163.80	159.06	152.29	148.46	147.23	146.62
20.0	125.04	123.63	122.17	121.42	120.29	119.61	119.38	119.27
40.0	75.57	77.74	80.00	81.17	82.98	84.12	84.51	84.71
(B) Fraction due to hedging demand (percentage):								
	1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	-56.12	-42.16	-34.17	-31.26	-27.71	-25.94	-25.39	-25.13
1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
1.50	41.13	37.06	33.49	31.93	29.82	28.69	28.34	28.16
2.00	57.58	53.67	49.86	48.09	45.65	44.30	43.87	43.66
4.00	78.19	76.03	73.63	72.40	70.61	69.57	69.23	69.07
10.0	88.48	87.84	87.11	86.73	86.14	85.78	85.66	85.60
20.0	91.56	91.46	91.36	91.31	91.22	91.17	91.16	91.15
40.0	93.02	93.21	93.40	93.50	93.64	93.73	93.75	93.77