Supermajority Politics: Equilibrium Range, Policy Diversity, Utilitarian Welfare, and Political Compromise

Aseem Mahajan, Roland Pongou, Jean-Baptiste Tondji*

Department of Government, Harvard University, 1737 Cambridge Street, K229, Cambridge, Massachusetts, 02138, USA

Department of Economics, University of Ottawa, 120 University Private, Social Sciences Building, Ottawa, Ontario, K1N 6N5, Canada

Department of Economics, The University of Texas Rio Grande Valley, 1201 W. University Dr., ECOBE 216, Edinburg, Texas, 78539, USA

Abstract

The standard Bowen model of political competition with single-peaked preferences (Bowen, 1943) predicts party convergence to the median voter's ideal policy, with the number of equilibrium policies not exceeding two. This result assumes majority rule and unidimensional policy space. We extend this model to *static* and *dynamic* political economies where the voting rule is a *supermajority rule*, and the policy space is *totally ordered*. Voters' strategic behavior is captured by the core in static environments and by the largest consistent set in dynamic environments. In these settings, we determine the exact number of equilibria and show that it is an increasing *correspondence* of the supermajority's size. This result has implications for the depth of policy diversity across structurally identical supermajoritarian political economies. We also examine the equilibrium effects of supermajority rules on utilitarian welfare and political compromise under uncertainty.

Keywords: Decision analysis, Supermajority rules, Single-peakedness, Number of equilibria, Utilitarian welfare, Policy diversity, Political compromise, Uncertainty

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*Corresponding author

Email addresses: mahajan@g.harvard.edu (Aseem Mahajan), rpongou@uottawa.ca (Roland Pongou), jeanbaptiste.tondji@utrgv.edu (Jean-Baptiste Tondji)

1. Introduction

The standard Bowen model of political competition with single-peaked preferences under majority rule (Bowen, 1943; Black, 1948; Downs, 1957) generally predicts party convergence to the ideal policy supported by the median voter. This fundamental result assumes *majority rule* and a *unidimensional* policy space. We extend this model to a spatial model of political competition between an incumbent policy and an alternative under two different decision-making environments, one *static* and the other *dynamic*. For adoption, the alternative must obtain the support of a supermajority of voters who, by assumption, hold single-peaked preferences over a *totally ordered* policy space. In the *static* setup, we focus on the set of equilibrium policies in the **core** (Black, 1948; Gillies, 1959). *Equilibrium policies* are those which, if already the status quo, are never defeated in a pairwise supermajoritarian election against alternatives in the policy space.[1] In the *dynamic* setup, agents make amendments sequentially, and the game can go on indefinitely. In this setting, we focus on the set of equilibrium policies in the **largest consistent set** (Chwe, 1994). This solution concept assumes farsightedness. To our knowledge, our study is novel in examining the issue of the number of equilibrium in a dynamic political setting where agents are farsighted. In both cases, our main finding is to determine the **number of equilibria** and show how it depends on the supermajority's size. We discuss implications for the depth of policy diversity and divergence across structurally identical political economies and develop illustrations of immigration policies, efficiency in providing public goods, and political compromise.

We consider a voting body, $N = \{1, 2, ..., n\}$, composed of a finite number of agents and endowed with a *supermajority rule*, $\mathcal{L}_\alpha$, and a non-empty *totally ordered* policy space, Z. The policy space Z represents the set of possible policies—the number of points or ideological approaches—to a given policy problem. We assume that Z is *totally ordered* by a binary relation denoted $\geq^Z$ (i.e., $\geq^Z$ is reflexive, transitive, antisymmetric, and complete), and we denote by $>^Z$ the strict part

[1]The core, like the Nash equilibrium, is regarded as a pioneer solution concept. It is the equilibrium concept used in the pioneering works of Bowen (1943), Black (1948), and Downs (1957), although it is not called such in these studies. Note, however, that all our results are highly robust to alternative solution concepts such as the top cycle (Schwartz, 1976), the uncovered set (Miller, 1980), and the Banks set (Banks, 1985). Proofs are available upon request.

of this relation. We assume that agents' preferences are *single-peaked* with respect to the strict order $>^Z$ on Z. The *supermajority rule* $\mathcal{L}_\alpha$ is a function which maps each coalition $C \subseteq N$ into 1 or 0. Given a threshold $\alpha \in [\frac{1}{2}, 1]$, $\mathcal{L}_\alpha(C) = 1$ when either the coalition C consists of more than αn members if $\alpha < 1$, or C consists consists of n members if $\alpha = 1$; we say such C is a *winning coalition*, and it holds power to amend a policy under consideration in the decision-making process. Coalitions for which $\mathcal{L}_\alpha(C) = 0$ are *losing* coalitions, and they do not hold the right to amend policies.

Several studies have analyzed different properties of supermajority rules. Requiring sufficiently large supermajorities to modify the status-quo may restrict policy change to Pareto improvements (Buchanan and Tullock, 1962, Ch. 12) and protect citizens with imperfect information from unrepresentative legislators (Graham and Bernhardt, 2015). Whether the supermajority strengthens or weakens majority hold-up against minorities remains unsettled. Buchanan and Tullock (1962, Ch. 7, 12) argues that supermajority rules increase minority bargaining power, affording them protection from a majority seeking to expropriate their resources.[2] Writing on the representation of racial minorities in electoral systems, Guinier (1994, Ch.4) similarly argues that supermajority systems empower minorities and may encourage cross-racial coalition-building. In contrast, McGinnis and Rappaport (1998) argues that supermajority rules may prevent minorities from overturning inequitable policies. The added "inertia" from supermajority rules may also lend stability to electoral systems: Caplin and Nalebuff (1988) illustrate how a 64% majority rule can prevent electoral cycling, and Barbera and Jackson (2004) demonstrate how using qualified supermajorities to amend constitutions expands the set of self-stable constitutions for a society with fixed preferences. In addition to stabilizing policies, supermajority rules may serve as commitment devices against dynamic inconsistency problems (Messner and Polborn, 2004, 2012). Recent literature has considered supermajority rules in designing optimal voting mechanisms to foster voting incentives, participation, and utilitarian welfare (Krishna and Morgan, 2015; Gershkov et al., 2017; Faravelli and Man, 2021), and in facilitating deliberative democracy following a structured dialogical design methodology (Laouris and Romm, 2022). An emerging literature is

[2]Distributive concerns raised by Buchanan and Tullock (1962) are reflected in a large literature evaluating supermajority rules and the provision of public goods (Tullock, 1959; Buchanan and Tullock, 1962; McGinnis and Rappaport, 1998; Knight, 2000; Lee et al., 2014; Lee, 2015, 2016).

evaluating the properties of voting rules using tools from machine learning and neural networks; see, e.g., Burka et al. (2022) and the references therein.

These articles focus on a particular rationale for supermajority rules, and each adopts a specific framework highlighting the theory of interest. In line with these previous works, we analyze the connection between supermajoritarian political competition models and the cardinality of the set of their predicted outcomes.

Precisely, we examine the minimum and the maximum number of equilibrium policies under a pairwise supermajoritarian election between a status quo policy, z_0, chosen by Nature, a lottery or an agent, and an alternative policy z_1 chosen by an agent from the set $Z \setminus \{z_0\}$. Agents have equal probabilities of being selected, by Nature or a lottery, to make proposals against the status quo z_0. If z_0 wins, meaning that no winning coalition under $\mathcal{L}_\alpha$ chooses z_1 over z_0, then it remains in place, and the contest ends. If z_0 loses (z_1 wins), then z_1 replaces z_0, and the contest ends. An equilibrium policy is never defeated in the election process. We find that the *minimum number* of equilibrium policies is a constant function of the supermajority's size. However, the *maximum number* of equilibria is an increasing function of the supermajority needed to pass legislation and is a function of the way the incumbent policy is selected. This number represents the depth of policy diversity across structurally *identical* political economies under supermajority rules. More precisely, Theorem 1 shows that if Nature randomly selects a legislator to propose a policy such that only the peaks of agents in Z are considered in the political contest, the maximum number of policies is finite, and it is a non-decreasing function of the supermajority's size α.[3]. It follows that the number of equilibrium policies is a non-decreasing *correspondence* of the supermajority's size needed to pass a policy; see Figure 1. As a byproduct of Theorem 1, Corollary 1 provides the existence of a unique equilibrium under majority rule ($\alpha = \frac{1}{2}$) when there is an odd number of voters and the existence of, at most, two equilibria when there are an even number of voters. The familiar Median Voter Theorem (MVT) (Black, 1948; Downs, 1957) is a particular case of our result, extending it to a more general setting. Although we generalize the Median Voter

[3]The idea of selecting legislators and policymakers by a lottery system, also called "sortition", is old; it dates to the fourth century BC and is still practiced today; see, e.g., Manin (1997), Wantchekon and Neeman (2002), Procaccia (2019), and Landemore (2022)

Theorem as a particular case of our result, this is not our main finding. Our main goal is to count the number of equilibria as a function of the supermajority rule and discuss its empirical implications. Theorem 2 also proves that if Nature randomly selects the incumbent policy such that the whole policy space Z is considered in the political contest, the set of equilibrium policies becomes a continuum—a convex and compact subset of the policy space—and we determine its exact bounds. Theorem 3 shows that these findings are robust in that they continue to hold when legislators display farsighted behavior in the dynamic setting.

We also address the question of which rules maximize utilitarian welfare when Nature randomly selects the status quo policy in the decision-making process. Proposition 1 determines the range of supermajority size that maximizes the expected utilitarian welfare. This proposition implies that the majority rule is the unique rule that maximizes expected utilitarian welfare in large populations.

A practical implication of our analysis is that economies that are *identical* in terms of their policy spaces, voters' preferences, and voting rules may end up diverging in terms of their policy choice; see, e.g., an illustration on immigration policies in Section 4.1. Theorems 1, 2, and 3 provide the possibility to *quantify* the extent of this divergence. In particular, under the majority rule, no divergence is possible unless the number of voters is even. Under a (pure) supermajority rule, policies can diverge, and the policy gap is an increasing function of the supermajority's size. Our results can also determine the number of "competing" political parties in a given election under supermajority rules.

Duverger (1963) was among the first scholars to examine the relationship between electoral systems and party structures in national elections. Duverger focused primarily on the plurality rule and proportional representation electoral systems under strategic voting, whereas our focus is on *supermajority rules*.[4] Duverger proposed what are known today as Duverger's law and Duverger's hypothesis. Duverger's law predicts that two major parties will form under the plurality rule (Duverger, 1963, p. 217), and Duverger's hypothesis states that "the simple-majority system

[4]Generally, strategic voting in single-member districts in national elections refers to a voter deserting a more preferred candidate with a poor chance of winning a political contest for a less preferred candidate with a better chance at winning. Although we do not consider strategic voting in our static framework, we assume that voters are farsighted in the dynamic setting and do not necessarily vote sincerely.

with the second ballot and proportional representation favors multi-partyism" (Duverger, 1963, p. 239). Our analysis transposes Duverger's ideas to supermajority rules in static and dynamic models of political competition in legislatures. We find, in particular, that the majority rule favors a two-party system, which offers an analogy to the prediction of Duverger's law. However, we find a different prediction under a larger majority requirement: the maximum equilibrium number of political parties is a non-decreasing function of the supermajority's size.[5], Fujiwara et al. (2011), and Forand and Maheshri (2015), among others.

In addition to policy diversity, we discuss in Section 4.2 the relationship between the range of equilibrium policies and the socially optimal provision of public goods. We find that the social optimum—the policy that maximizes aggregate utilitarian welfare—is generally unstable under majority rule. Then, we determine the minimal supermajority rule that guarantees its stability. Our analysis implies that supermajority rules lead to policy diversity and protect the social optimum from defeat in a pairwise supermajoritarian competition. In Section 4.3, we also illustrate our findings with the issue of political compromise. If a political party prefers an incumbent policy to alternatives, how much should it compromise to ensure it does not suffer defeat in a pairwise supermajoritarian competition? Under majority rule, the MVT suggests convergence toward the median voter's ideal point. Parties more ideologically distant from the median voter must make more significant compromises to avoid defeat. We extend this insight to any supermajority rule and determine the *minimal level* of compromise that a political party should accept to become successful. We find that this minimal level decreases with respect to the size of the supermajority needed to replace the incumbent policy. In other words, the greater the supermajority needed to adopt a new policy, the less the original policy's proposer or supporters must compromise to ensure that it is reenacted. An implication that follows directly from this analysis is that political compromise is maximal under majority rule. Therefore, our analysis highlights two new properties of majority rule: it maximizes expected utilitarian welfare when Nature chooses the incumbent policy and maximizes political compromise.

[5] We note that several models of voting, including empirical studies, have formalized and tested Duverger's arguments and have found mixed results; see, Riker (1982), Palfrey (1989), Feddersen et al. (1990), Cox (1994, 1997), Fey (1997), Myerson (1999), Gallagher and Mitchell (2005), Benoit (2006), Clough (2007), Callander and Wilson (2007)

By focusing on the **number of equilibria** in a model of spatial political competition, we depart from the extant literature that has primarily studied the question of equilibrium existence (see, e.g., Feldman and Serrano (2006) for a thorough overview of these findings), but has completely overlooked the issue of the number of equilibria. More generally, supermajority rules have been studied in terms of their equilibrium properties (see, for instance, Fey (2003), Tchantcho et al. (2010), Peleg (1978), and Freixas and Kurz (2019)) and as basis for generating more complex voting rules (see, e.g., Taylor and Zwicker (1993), Freixas (2004); Freixas and Puente (2008), Guemmegne and Pongou (2014), and Kurz et al. (2020)). Our study also contributes to the literature that uses static and dynamic cooperative game models and their applications in operational research. For a brief overview of the wealth of knowledge in this growing field of research, we refer to the studies of Wang and Parlar (1989), Nagarajan and Sošić (2008), Sošić (2011), Fiestras-Janeiro et al. (2011), Guajardo and Rönnqvist (2015), Adler et al. (2020), Li and Chen (2020), and Laouris and Romm (2022). Our analysis departs from this literature by focusing on the number of equilibria instead and deriving implications for the depth of policy diversity and political compromise across structurally similar supermajoritarian political economies. In so doing, we also extend classical results to a more general environment.

Although we mainly analyze spatial political games, the issue of the number of equilibria has also been examined in exchange economies and strategic form games. Starting from the issue of the uniqueness of equilibria in exchange economies, Debreu (1970) highlights the possibility of a finite number of equilibria in regular exchange economies. This was followed by a treatment of uniqueness by Dierker and Dierker (1972). Then, Dierker (1972) refines Debreu's findings by showing that the number of equilibria is odd. Also, Varian (1975), in a note on Dierker's study, provided an alternative proof of Debreu (1970). Along the same lines, Nishimura (1978) shows that the previous results are independent of the assumption of preference monotonicity used by Debreu (1970) and Dierker (1972) and offers an insight which makes the findings more applicable to economics, international trade theory, and stability theory.

In strategic form games, one primary concern dates back to the works of Wilson (1971a) and Harsanyi (1973) on the computation of Nash equilibria in N-person games. Wilson (1971a, Theorem 1, p. 85) demonstrates that, apart from certain degenerate cases, in any game with pure finite strategies, the number of mixed strategy Nash equilibria is positive and odd. Harsanyi provides an alternative proof for Wilson's result and proves that in "almost all" games with pure finite strategies, the number of mixed-strategy Nash equilibria is finite and odd. However, we note that, except for a unique equilibrium solution, the previous studies do not provide a formal expression of the number of equilibria in a game. Studies that provide formulas for the number of equilibria (or at least for the lower and upper bounds), as we do in this study, include, among others, McLennan (1997) who determines the maximal generic number of pure (and mixed) strategies Nash equilibria, Von Stengel (1999) who determines a lower bound of $\frac{2.414^d}{\sqrt{d}}$ for the maximal number of Nash equilibria in $d \times d$ bimatrix games, McLennan (2005) who characterizes the mean (or expected) number of pure (and mixed) strategy Nash equilibria in random strategic form games[6], and Deutsch et al. (2011) who provide explicit and computable expressions for all possible Nash equilibria in a (bi-linear) inspection game. Closely related to the previous literature are efforts to develop algorithms to facilitate the search for Nash equilibria in a strategic form game.[7]

Our contributions also complement the seminal study by Caplin and Nalebuff (1988) and the studies inspired by their work (see, e.g., Caplin and Nalebuff (1991), Levin and Nalebuff (1995), and Barbera and Jackson (2004)), and Kline (2014), but they do not address the questions we examine in our study. Caplin and Nalebuff formalize the conjectures made by Condorcet (1785) and Arrow (1951) in "static" electoral systems, showing that voting cycles are impossible under the 64%-majority rule.[8], and the references therein. Kline studies the effects of the status quo on the existence of the core, and the Banks set (Banks, 1985). Kline provides conditions

[6]In an n-agent model of random game, the agents' payoffs are statistically independent, with each agent's payoff uniformly distributed on the unit sphere in $\mathbb{R}^S$, where $S = S_1 \times ... \times S_n$, S_i a finite pure strategy set, $i = 1, ..., n$.

[7]We can cite, among many others, Echenique (2007) who provides a simple and fast algorithm that finds all the pure strategy Nash equilibria in games with strategic complementarities (these models have several applications in operational research; see, e.g., Lippman and McCardle (1997), Cachon (2001), and Bernstein and Federgruen (2004)), Díaz-Báñez et al. (2011) who propose an algorithm to find all possible pure strategy Nash equilibria in a planar location-price game, and recently Deutsch (2021) who develops a linear-time algorithm to compute all Nash equilibria solutions for a general two-person nonzero-sum simultaneous inspection game.

[8]For a review of other studies on voting in social choice theory and additional details on electoral systems, we refer to the works of Nurmi (1986), Buchanan and Tullock (1962), Brams and Fishburn (2002), Arrow et al. (2010), Freixas et al. (2014), Polyakovskiy et al. (2016), Menezes et al. (2016), Tideman (2017), Burka et al. (2022)

under which the location of the status quo alternative determines the intersection between the Banks set and the core in supermajoritarian spatial voting games. Our dynamic approach is inspired by Chwe's study and the largest consistent set (Chwe, 1994). Moreover, Kline (2014) studies sophisticated agenda settings. Our framework includes a static and dynamic approach, and we focus on counting the number of equilibria and their implications for policy-making in supermajoritarian political games. In that respect, we think that our contribution is original.

The paper proceeds as follows. Section 2 introduces preliminary concepts. Section 3 examines the number and range of equilibrium policies under static and dynamic political settings. It also studies the welfare implications of supermajority rules. Section 4 applies our results to explain policy diversity across identical political economies, provision of public goods, and political compromise. Section 5 concludes.

2. Preliminary Concepts

We model a *political economy* as a list $\mathcal{P} \equiv \mathcal{P}(\alpha) = (N, Z, (\succeq_i), \mathcal{L}_\alpha)$, where: **(a)** $N = \{1, 2, ..., n\}$ is a voting body, composed of a finite number of agents (we assume that n is at least 2); **(b)** Z is a non-empty policy space, which is totally ordered by a binary relation $\geq^Z$ that is reflexive, transitive, antisymmetric, and complete (we denote by $>^Z$ the strict part of the binary relation $\geq^Z$); **(c)** $\succeq_i$ denotes agent i's preference relation over Z and $(\succeq_i)$ denotes a preference profile over Z; and **(d)** $\mathcal{L}_\alpha$ is a supermajority rule (or qualified majority) of size α ($\alpha \in [\frac{1}{2}, 1]$). A supermajority rule is a distribution of political decision-making power among the various coalitions of agents eligible to vote (for simplicity, we assume that each agent can vote). The aggregate function $\mathcal{L}_\alpha$ is a family of voting rules that includes from simple majority rule ($\alpha = \frac{1}{2}$) up to unanimity ($\alpha = 1$).

For any policies $x, y \in Z$, the intervals $[x, y]$ (and $]x, y[$) are subsets of Z defined as: $[x, y] = \{z \in Z : y \geq^Z z \geq^Z x\}$ (and $]x, y[= \{z \in Z : y >^Z z >^Z x\}$), respectively. For a given finite and non-empty set X, we denote by $|X|$, the cardinality of X (i.e., the number of elements contained in X), and n the cardinality of N. For $x, y \in Z$, $y \succeq_i x$ indicates that agent i weakly prefers y to x; $y \succ_i x$ indicates that agent i prefers y to x; and $y \sim_i x$ indicates that agent i is indifferent between y and x. Moreover, for $S \subseteq N$, $y \succ_S x$ indicates that $y \succ_i x$ for each $i \in S$

(we say S prefers y over x); and $y \succeq_S x$ indicates that $y \succ_i x$ for some $i \in S$ and $y \sim_j x$ for other $j \in S$ (we say S weakly prefers y over x).

Following the classical literature on spatial competition (see, e.g., Bowen (1943) and Black (1948)), we assume that the profile $(\succeq_i)$ is **single-peaked** with respect to the strict order $>^Z$ on Z. Each agent has an ideal policy in the policy space Z, and policies further from this ideal policy are preferred less. Formally, for each agent $i \in N$, there exists a policy $z_i^p \in Z$ such that: (1) for any other policy $z \neq z_i^p$, $z_i^p \succ_i z$; and (2) for any policy z, z', if $z >^Z z' >^Z z_i^p$, then $z' \succ_i z$, and, if $z_i^p >^Z z >^Z z'$, then $z \succ_i z'$.

3. Number and Range of Equilibrium Policies

In this section, we examine the existence and the maximum number of equilibrium policies under one-shot political games (Section 3.1) and dynamic political games (Section 3.2).[9] To perform the analysis in one-shot games, we distinguish two cases: (i) an agent is randomly chosen to propose a policy in the political contest (Section 3.1.1); or (ii) the status quo policy is chosen by Nature (Section 3.1.2). We assume that agents have equal probabilities of being selected by Nature or a lottery as an agenda setter. Controlling for temporal factors that affect agents' preferences and status-quo policies, what is the relationship between a legislative body's voting rule and policy stability? Section 3 answers this question.

3.1. One-shot Political Games

Political contests occur as follows:

1. At time $t = 0$, a policy z_0 is randomly chosen by Nature, a lottery, or an agent from the policy space Z.

2. At time $t = 1$, a contest is organized between z_0 (the status quo) and an alternative z_1, chosen exogenously by an agent from the set $Z \setminus \{z_0\}$.

[9]The domain of political games that we study is a subclass of games defined by effectivity functions (see, e.g., Wilson (1971b), Moulin and Peleg (1982), Peleg (1984), Chwe (1994), and Fotso et al. (2017) who provide a brief survey of such games).

a) If z_0 wins, meaning that no winning coalition under $\mathcal{L}_\alpha$ chooses z_1 over z_0, then it remains in place, and the contest ends.

b) If z_0 loses (z_1 wins), then z_1 replaces z_0, and the contest ends.

The rational behavior in the one-shot political game above is straightforward. Each agent chooses between the status quo policy and a political alternative. The incentive driving agents to vote for an opposition policy is that it is preferable to the status quo. Formalizing this behavior, let $\gg_i$ denote the incentive by which agent i decides to support an opposition policy z_1 over the status quo z_0. If agent i prefers z_1 over z_0 (i.e., $z_1 \succ_i z_0$), agent i will vote for z_1 over z_0, denoted as $z_1 \gg_i z_0$. The policy z_1 wins the pairwise supermajoritarian election if there exists a winning coalition C that supports z_1 over z_0 ($z_1 \gg_C z_0$). We can now introduce the equilibrium set, defined as follows.[10]

Definition 1 Let $\mathcal{P}(\alpha) = (N, Z, (\succeq_i), \mathcal{L}_\alpha)$ be a political economy and C be a winning coalition.

1. z' defeats z (or $z' \gg z$) thanks to C (i.e., $z' \gg_C z$) if C prefers z' over z (i.e, $z' \succ_C z$).
2. z is defeated if there exists a policy z' and a winning coalition C' such that z' defeats z thanks to C'.
3. The *core* or *equilibrium set* $\mathcal{E}(\mathcal{P}(\alpha))$ consists of all undefeated policies. $\square$

An equilibrium policy is one that, if chosen as the status quo, could not be defeated or replaced by another policy. In a pairwise contest between two policies, say z and z', the former receives votes from agents whose ideal points are closer to z than z', and vice versa. Each agent's payoff depends on the distance between her ideal peak and the winning policy.

3.1.1. Nature Randomly Selects a Proposer

At the time $t = 0$, an agent is randomly selected, by Nature or a lottery, to propose a policy. Agents are identical with equal probabilities of being selected. The proposer chooses the status quo. Given single-peakedness and rational behavior, each agent's best choice is to propose the

[10] For a brief review on formalizing and testing rationality concepts in static and dynamic voting games, including effectivity functions, we refer to the studies of Fotso et al. (2017) and the references therein.

closest equilibrium policy ideal to their ideal point as the status quo.[11] Theorem 1 demonstrates the existence of a policy that cannot be defeated in a pairwise supermajoritarian election and provides the maximum number of policies that can be implemented. Before enunciating the result, we introduce the following notation: for any real number x, the value $floor[x]$, and labeled as $\lfloor x \rfloor$, is the largest integer less than or equal to x.

Theorem 1 *Let $\mathcal{P}(\alpha) = (N, Z, (\succeq_i), \mathcal{L}_\alpha)$ be a political economy. Assume that only the peaks of agents in Z are considered in the political contest. If agents have single-peaked preferences over Z, then at least one equilibrium exists, and the number of equilibria is finite. Formally:*

$$1 \leq |\mathcal{E}(\mathcal{P}(\alpha))| \leq \min\{2\lfloor \alpha n \rfloor + 2 - n, n\}.$$

The maximum number of equilibria in $\mathcal{P}(\alpha)$ is n when $\alpha = 1$, and $2\lfloor \alpha n \rfloor + 2 - n$ when $\alpha \in [\frac{1}{2}, 1)$. $\square$

To prove Theorem 1, the following lemmas proved helpful.

Lemma 1 *Let $\mathcal{P}(\alpha) = (N, Z, (\succeq_i), \mathcal{L}_\alpha)$ be a political economy, with $\alpha \in [0, 1)$. There exist two peaks $z_1^*(\alpha)$ and $z_2^*(\alpha)$ such that: $z_1^*(\alpha)$ minimizes f over Z_f, and $z_2^*(\alpha)$ minimizes g over Z_g.* $\square$

Proof (Lemma 1) Let $z \in Z$ be a policy and define $S(z)$ as the number of agents for whom z is the peak. A coalition of agents S has a veto right to amend a given status quo if S is a winning coalition, i.e., $|S| > \alpha n$. Consider the functions f and g defined on the policy space Z as follows: for any policy $z' \in Z$,

$$f(z') = \sum_{z \geq^Z z'} S(z) - \alpha n, \text{ and } g(z') = \sum_{z' \geq^Z z} S(z) - \alpha n.$$

We define the following sets:

$$Z_f = \{z' \in Z : \ f(z') > 0\}, \text{ and } Z_g = \{z' \in Z : \ g(z') > 0\}.$$

[11]One can trace a similar argument from the work of Downs (1957) and the seminal essay of Riker (1982) and the references therein. Even if we assume that the proposer is not rational and he or she proposes his or her ideal point as the status quo, our findings do not change. Throughout the paper, we assume that agents are rational in their decisions.

Notice that neither Z_f, nor Z_g is empty. In fact, given that z_{min} and z_{max} are respectively the smallest and the greatest peaks of Z, then $f(z_{min}) = n - \alpha n = (1-\alpha)n > 0$ and $g(z_{max}) = n - \alpha n = (1-\alpha)n > 0$, since $\alpha < 1$, which implies that $z_{min} \in Z_f$ and $z_{max} \in Z_g$, in turn implying that $Z_f \neq \emptyset$ and $Z_g \neq \emptyset$. Given that Z_f is finite and f is a strictly decreasing function, there exists a unique peak $z_1^*(\alpha)$ which minimizes f over Z_f. In addition, for any peak $z' >^Z z_1^*(\alpha)$, $f(z') \leq 0$, which implies that $\sum\limits_{z \geq^Z z'} S(z) \leq \alpha n$. Similarly, given that Z_g is finite and g is a strictly increasing function, there exists a unique peak $z_2^*(\alpha)$ which minimizes g over Z_g. In addition, for any peak $z_2^*(\alpha) >^Z z'$, $g(z') \leq 0$, which implies that $\sum\limits_{z' \geq^Z z} S(z) \leq \alpha n$. ∎

Lemma 2 *Assume that* $z_1^*(\alpha) = z_2^*(\alpha) = z^*(\alpha)$. *Then,* $\mathcal{E}(\mathcal{P}(\alpha)) = \{z^*(\alpha)\}$. □

PROOF (LEMMA 2) We claim that the Condorcet winner is $z^*(\alpha)$. Indeed, let $z \in Z$ be a peak. If $z^*(\alpha) >^Z z$, by definition of $z^*(\alpha)$, $\sum\limits_{z \geq^Z z'} S(z') \leq \alpha n$ and $\sum\limits_{z' \geq^Z z^*(\alpha)} S(z') > \alpha n$, which implies that $z^*(\alpha)$ defeats z in a pairwise supermajoritarian election. Similarly, if $z >^Z z^*(\alpha)$, we show in the same way that $z^*(\alpha)$ defeats z. It follows that $z^*(\alpha)$ defeats any other peak z. Since there is no other option which defeats $z^*(\alpha)$, then $\mathcal{E}(\mathcal{P}(\alpha)) = \{z^*(\alpha)\}$. ∎

Lemma 3 *If* $z_1^*(\alpha) \neq z_2^*(\alpha)$, *then* $z_2^*(\alpha) >^Z z_1^*(\alpha)$. □

PROOF (LEMMA 3) Assume by contradiction that $z_1^*(\alpha) >^Z z_2^*(\alpha)$. By definition of $z_1^*(\alpha)$ and $z_2^*(\alpha)$, we have $\sum\limits_{z \geq^Z z_1^*(\alpha)} S(z) > \alpha n$ and $\sum\limits_{z_2^*(\alpha) \geq^Z z} S(z) > \alpha n$, then $\sum\limits_{z \in Z} S(z) > 2\alpha n$. Given that $\sum\limits_{z \in Z} S(z) = n$, it follows that $n > 2\alpha n$, meaning that $\alpha < \frac{1}{2}$, a contradiction, since by assumption $\frac{1}{2} \leq \alpha < 1$. Hence, the only remaining possibility is $z_2^*(\alpha) >^Z z_1^*(\alpha)$. ∎

Lemma 4 *There exists* $z^* \in]z_1^*(\alpha), z_2^*(\alpha)[$, *with* $S(z^*) \neq 0$. □

PROOF (LEMMA 4) Assume the contrary. By the definition of policies $z_1^*(\alpha)$ and $z_2^*(\alpha)$, we have $\sum\limits_{z \geq^Z z_1^*(\alpha)} S(z) > \alpha n$ and $\sum\limits_{z_2^*(\alpha) \geq^Z z} S(z) > \alpha n$. These imply that $\sum\limits_{z \geq^Z z_1^*(\alpha)} S(z) + \sum\limits_{z_2^*(\alpha) \geq^Z z} S(z) > 2\alpha n$ or $\sum\limits_{z \in Z} S(z) > 2\alpha n$ leading to $\alpha < \frac{1}{2}$, which is a contradiction. It follows that there exists a policy $z^* \in]z_1^*(\alpha), z_2^*(\alpha)[$, such that $S(z^*) \neq 0$. Note that, in this case, $z_1^*(\alpha)$ and $z_2^*(\alpha)$ are such that $\sum\limits_{z_1^*(\alpha) \geq^Z z} S(z) < \alpha n$, and $\sum\limits_{z \geq^Z z_2^*(\alpha)} S(z) < \alpha n$. ∎

Lemma 5 *If* $z \in Z \setminus [z_1^*(\alpha), z_2^*(\alpha)]$, *then* $z \notin \mathcal{E}(\mathcal{P}(\alpha))$. □

Proof (Lemma 5) Consider $z \in Z$ distinct to $z_1^*(\alpha)$ and $z_2^*(\alpha)$. Assume that z is the closest peak to the left of $z_1^*(\alpha)$. In a pairwise supermajoritarian opposition between z and $z_1^*(\alpha)$, the former receives at most $\sum_{z \geq^Z z'} S(z')$ number of votes, while the latter receives at most $\sum_{z' \geq^Z z_1^*(\alpha)} S(z')$. Since, $\sum_{z \geq^Z z'} S(z') \leq \sum_{z_1^*(\alpha) \geq^Z z'} S(z') < \alpha n$, and $\sum_{z' \geq^Z z_1^*(\alpha)} S(z') > \alpha n$, then, $z_1^*(\alpha)$ wins. We can also show that $z_2^*(\alpha)$ defeats any peak z, with $z >^Z z_2^*(\alpha)$. ■

Lemma 6 *If* $z \in [z_1^*(\alpha), z_2^*(\alpha)]$, *then* $z \in \mathcal{E}(\mathcal{P}(\alpha))$. □

Proof (Lemma 6) Consider a peak $z \in \ [z_1^*(\alpha), z_2^*(\alpha)]$. Assume that there exists $z' \in]z_1^*(\alpha), z_2^*(\alpha)[$ such that z' defeats z. Without loss of generality, assume that z' is the closest peak to z with $z >^Z z'$. Policy z' defeats z implies that $\sum_{z' \geq^Z x} S(x) > \alpha n$, which is a contradiction, because by definition of $z_2^*(\alpha)$, $z_2^*(\alpha) >^Z z'$ implies that $\sum_{z' \geq^Z x} S(x) \leq \alpha n$. Thus, $\mathcal{E}(\mathcal{P}(\alpha)) = [z_1^*(\alpha), z_2^*(\alpha)]$. ■

Now, we prove Theorem 1.

Proof (Theorem 1) First, if $\alpha = 1$, the only winning coalition is the set N. Since individuals make proposals against the status quo z_0, each peak is a predicted game outcome. The maximum number of votes that an alternative policy z_1 (distinct from z_0) in a pairwise supermajoritarian opposition can receive is $n - 1$. If the supermajority rule requires n votes to win, then no alternative can be defeated, and the maximum number of predicted outcomes is the cardinality of N, i.e., n. Second, if n is odd, and $\alpha = \frac{1}{2}$, then the median peak is the unique prediction of the pairwise supermajoritarian game because it is the Condorcet winner, i.e., it defeats any other policy in a pairwise supermajoritarian opposition. Third, from Lemmas 2, 5, and 6, we show that any alternative which is not part of the interval bounded by the peaks $z_1^*(\alpha)$ and $z_2^*(\alpha)$ can be directly defeated by either $z_1^*(\alpha)$ or $z_2^*(\alpha)$, and any peak in this interval cannot be defeated. Therefore, the maximal number of equilibria is equal to the number of agents who have a peak between $z_1^*(\alpha)$ and $z_2^*(\alpha)$. Given that the proportion of agents required to form a winning coalition is at least $\frac{\lfloor \alpha n \rfloor + 1}{n}$, then the upper bound of $\mathcal{E}(\mathcal{P}(\alpha))$ is $n - 2(1 - \frac{\lfloor \alpha n \rfloor + 1}{n})n = 2\lfloor \alpha n \rfloor + 2 - n < n$. ■

As shown in Figure 1, the minimum number of equilibrium policies is one regardless of the supermajority rule. The maximum number of equilibrium policies is a non-decreasing function of the supermajority needed to replace them. It follows that, for a fixed size n of voters, the number of equilibrium policies is a non-decreasing *correspondence* of the supermajority's size α. A corollary of Theorem 1 is the following result, which derives the size of the equilibrium set under the majority rule and thus clarifies the way Theorem 1 extends the MVT when the number of agents is even.

Corollary 1 *Let $\mathcal{P}(\alpha) = (N, Z, (\succeq_i), \mathcal{L}_\alpha)$ be a political economy. Assume that only the peaks of agents in Z are considered in the political contest. If preferences are single-peaked, and policies are chosen using the majority rule ($\alpha = \frac{1}{2}$), then,*

1. *There is only one equilibrium if the size of voters is odd.*
2. *There exists at least one and at most two equilibria if the size of voters is even.* □

PROOF (COROLLARY 1) From Theorem 1, if $\alpha = \frac{1}{2}$, then the size of equilibrium set $\mathcal{E}(\mathcal{P}(\frac{1}{2}))$ depends on the size of n. If n is odd, the number $\lfloor \alpha n \rfloor = \frac{n-1}{2}$, therefore $2\lfloor \alpha n \rfloor + 2 - n = n - 1 + 2 - n = 1$, meaning that a unique equilibrium exists. It is, in fact, the ideal policy for the median voter. If n is even, there exist at most two equilibria since the number $\lfloor \alpha n \rfloor = \frac{n}{2}$ and $2\lfloor \alpha n \rfloor + 2 - n = n + 2 - n = 2$. ■

3.1.2. Nature Randomly Chooses a Status Quo

Next, suppose that Nature, rather than choosing the proposer in $t = 0$, instead chooses the status quo $z_0 \in Z$ from the set of all policies. Theorem 2 proves the existence of at least one and possibly an infinite number of equilibrium policies.

Theorem 2 *Let $\mathcal{P}(\alpha) = (N, Z, (\succeq_i), \mathcal{L}_\alpha)$ be a political economy and assume that the whole policy space Z is considered in the political contest. Let $z_1^*(\alpha)$ and $z_2^*(\alpha)$ denote, respectively, the minimal and the maximal equilibria when Nature randomly selects a proposer. Then, $\mathcal{E}(\mathcal{P}(\alpha)) = [z_1^*(\alpha), z_2^*(\alpha)]$.* □

PROOF (THEOREM 2) The proof is deduced from the proof of Theorem 1. The status quo, chosen randomly by Nature, can take any position in spatial space Z. From Theorem 1, any position between and including $z_1^*(\alpha)$ and $z_2^*(\alpha)$, is invulnerable to pairwise supermajoritarian

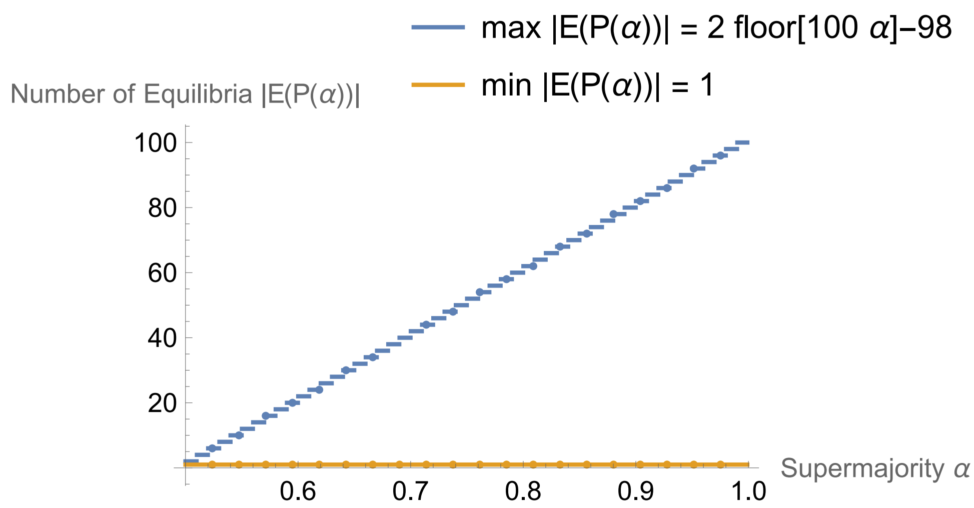

Figure 1: Number of equilibria ($|\mathcal{E}(\mathcal{P}(\alpha))|$) and the size of supermajority rule (α) in a voting body of 100 agents. Note: For each value of $\alpha \in [0.5, 1)$, the cornflower curve represents the maximum number of equilibria: $\max|\mathcal{E}(\mathcal{P}(\alpha))| = 2\lfloor 100\alpha \rfloor - 98$, and the orange curve represents the minimum number of equilibria: $\min|\mathcal{E}(\mathcal{P}(\alpha))| = 1$.

opposition. In this case, the interval bounded by the peaks $z_1^*(\alpha)$ and $z_2^*(\alpha)$ is the equilibrium set, $\mathcal{E}(\mathcal{P}(\alpha))$. ■

Under the majority rule, the equilibrium set described in Theorem 2 exhibits an interesting property. When the number of voters is odd and $\alpha = \frac{1}{2}$, $z_1^*(\alpha) = z_2^*(\alpha) = z_m^p$, and the set $[z_1^*(\alpha), z_2^*(\alpha)]$ is the singleton $\{z_m^p\}$, where z_m^p is the ideal point of the median voter. If the number of voters is even, however, the set of equilibria may be infinite. In this sense, Theorem 2 offers a complete statement of the MVT compared to Black (1948). Our findings in Theorems 1 and 2 reveal the existence and the number of equilibria in a static spatial political competition. An equivalent definition of the existence of an equilibrium in our framework is the absence of voting cycles. As mentioned in the Introduction, several studies have focused on the existence of equilibria and the possibility of voting cycles. An earlier study by Schofield et al. (1988) and recent extensions by Owen and Carreras (2022) and Martin et al. (2022) surveyed important lines of research investigating core existence in spatial voting games. Another appropriate tool in the domain of *simple games* without vetoers (voters who belong to all winning coalitions) is the so-called *Nakamura number* (Nakamura, 1979).[12] A recent study by Freixas and Kurz (2019) and the references therein offer an excellent survey of the applications of the Nakamura number in voting contexts and related problems, including cutting stock problems in operational research.[13] As we do in this paper, these previous studies offer optimistic results on the absence of cycles and the stability of group choice in political competition models.

3.2. **Dynamic Political Games**

Contrary to one-shot games, agents (or coalitions) may vote indefinitely in dynamic political games. Assume that a status quo z_0 is randomly chosen from the set of policies. If no winning coalition replaces z_0, it remains in place indefinitely, and the political opposition ends. If a winning coalition S replaces z_0, say with z_1, then z_1 becomes the new status quo, and the process restarts, continuing until a policy has been reached to which no winning coalition is willing to object. Once that policy has been reached, each agent earns and consumes his or her payoff, and the political contest ends. We illustrate the predictions of such a game with the *largest consistent set* (Chwe, 1994), one of the prominent equilibrium concepts in infinite-horizon political games.[14]

Chwe (1994) defines the *largest consistent set*, an equilibrium concept for *social environments* where agents, acting in public, can freely form coalitions without binding agreements and are farsighted. Chwe assumes that agent i holds a *strict preference* relation $\succ_i$ over Z, and coalitions of agents may be endowed with the power to replace one policy with some other policies. If a coalition $S \subseteq N$ has the right to replace $z \in Z$ by some $z' \in Z$, we write $z \longrightarrow_S z'$. Following Chwe's notations, a *social environment* is represented by a list $(N, Z, \{\succ_i\}_{i \in N}, \{\longrightarrow_S\}_{S \subset N, S \neq \emptyset})$. To capture the idea of farsightedness, Chwe formalizes the notion of *indirect dominance* that was formally discussed by Harsanyi (1974) in his criticism of the von Neumann and Morgenstern (1944)'s solution concept which is based on *direct dominance*. For $z, z' \in Z$, z' is said to *indirectly dominate* z, or $z' \gg z$, if there exists a sequence of policies $z_0,\ z_1,\ \ldots,\ z_m \in Z\ (where\ z_0 =$

[12] A *simple game* is a mapping from the set of coalitions into $\{0, 1\}$, where "1" means the coalition is a winning coalition and "0" means the coalition is not a winning coalition. The *Nakamura number* of a simple game is the smallest number k such that there exist k winning coalitions with empty intersections. A recent study by Molinero et al. (2022) offers additional properties and applications of simple games in operational research.

[13] See, e.g., Gilmore and Gomory (1961) and Scheithauer and Terno (1995).

[14] For a brief review on other solution concepts in dynamic farsighted coalitional games, we refer to the studies of Nagarajan and Sošić (2008), Sošić (2011), Fiestras-Janeiro et al. (2011), Fotso et al. (2017), and Li and Chen (2020).

$z\ and\ z_m = z'$) and a sequence of winning coalitions $S_0,\ S_1,\ ..., S_{m-1}$ such that $z_i \longrightarrow_{S_i} z_{i+1}$ and $z' \succ_{S_i} z_i$ for $i = 0, 1, ..., m-1$. The case $m = 1$ yields the definition of the *direct dominance*. Chwe (1994, Proposition 2, P. 305) shows that the *largest consistent set* is non-empty if Z is finite or countably infinite, and there are no $\gg$-chains, i.e., an infinite sequences of policies z_1, z_2, z_3,... such that $i < j \Longrightarrow z_j \gg z_i$. Xue (1997, Theorem, p. 455) extends Chwe (1994, Proposition 2, p. 305)'s non-emptiness result of the *largest consistent set* by removing the countability and by weakening the condition that there is no $\gg$-chains. As discussed by Xue (1997, p. 453), such an extension allows one to apply the largest consistent set to models with a continuum of alternatives. Note however, that both Chwe (1994, Proposition 2, p. 305) and Xue (1997, p. 453) assume that agents have *strict preferences* over the policy space Z, a different assumption that we make in this paper.

In this section, we examine Chwe (1994, Proposition 2, p. 305)'s non-emptiness result of the *largest consistent set* when a supermajority rule gives the distribution of veto rights among coalitions, the policy space Z is totally ordered, and agents have single-peaked preferences over Z. For $z, z' \in Z$, $z \longrightarrow_S z'$ if and only if S is a winning coalition (i.e., $|S| > \alpha n$). Therefore, a social environment $(N, Z, \{\succ_i\}_{i\in N}, \{\longrightarrow_S\}_{S\subset N, S\neq\emptyset})$ is equivalent to a political economy $\mathcal{P}(\alpha) = (N, Z, (\succeq_i), \mathcal{L}_\alpha)$, where $\mathcal{L}_\alpha$ replaces $\{\longrightarrow_S\}_{S\subset N, S\neq\emptyset}$. We recall the definition of the *largest consistent set* below.

Definition 2 Let $\mathcal{P}(\alpha) = (N, Z, (\succeq_i), \mathcal{L}_\alpha)$ be a political economy, and X be a subset of Z.

1. X is said to be *consistent* if $x \in X$ if and only if $\forall y \in Z$ and $S \subset N$ such that $x \longrightarrow_S y$, there exists $z \in X$, where $y = z$ or $z \gg y$, and $not(x \succ_S z)$.

2. The *largest consistent set* of the political economy $\mathcal{P}(\alpha)$, denoted $LCS(\mathcal{P}(\alpha))$, is the union of all the consistent sets. □

The largest consistent set formalizes that a coalition that moves from a status quo to an alternative policy anticipates the possibility that another coalition might react. A third coalition might, in turn, react, and so on, without limit. It is therefore essential to act in a way that does not lead a coalition to regret its action ultimately, i.e., coalitions are "*fully farsighted*" (Chwe, 1994, p. 300). In Theorem 3, we show that the *largest consistent set* is non-empty, and we derive the maximum number of equilibria in the largest consistent set when Nature randomly chooses agents with equal probabilities to propose a status quo.

Theorem 3 *Let $\mathcal{P}(\alpha) = (N, Z, (\succeq_i), \mathcal{L}_\alpha)$ be a political economy.*

A. *Assume that the whole policy space Z is considered in the political contest. Then, $LCS(\mathcal{P}(\alpha)) = [z_1^*(\alpha), z_2^*(\alpha)]$.*

B. *Assume that only the peaks of agents in Z are considered in the political contest. Then, $1 \leq |LCS(\mathcal{P}(\alpha))| \leq \min\{2\lfloor \alpha n \rfloor + 2 - n, n\}$. Thus, the maximum number of equilibria in $\mathcal{P}$ is n when $\alpha = 1$, and $2\lfloor \alpha n \rfloor + 2 - n$ when $\alpha \in [\frac{1}{2}, 1)$.* □

PROOF (THEOREM 3) Let $\mathcal{P}(\alpha) = (N, Z, (\succeq_i), \mathcal{L}_\alpha)$ be a political economy.

A. Let $z \in Z$. If $z_1^*(\alpha) >^Z z$, then $z_1^*(\alpha)$ indirectly dominates z; If $z >^Z z_2^*(\alpha)$, then $z_2^*(\alpha)$ indirectly dominates z. The only alternatives that are not indirectly dominated belong to the interval $[z_1^*(\alpha), z_2^*(\alpha)]$. A subset $X \subseteq Z$ is consistent if

$$f(X) = \begin{Bmatrix} x \in Z : \forall y \in Z, \forall S,\ x \longrightarrow_S y,\ \exists z \in X,\ \text{where} \\ y = z \text{ or } z \gg y \text{ and } not(x \succ_S z) \end{Bmatrix} = X.$$

For each agent $i \in N$, we denote their ideal policy by z_i^p. By definition of $z_1^*(\alpha)$ and $z_2^*(\alpha)$, the sets $S = \{i \in N : z_i^p \geq^Z z_1^*(\alpha)\}$ and $T = \{i \in N : z_2^*(\alpha) \geq^Z z_i^p\}$ are winning coalitions. Let $z \in Z$ be a proposal: (a) if $z_1^*(\alpha) >^Z z$, then any deviation from z by any winning coalition to $z_1^*(\alpha)$ is not deterred. Similarly; (b) if $z >^Z z_2^*(\alpha)$, then any deviation from z by any winning coalition to $z_2^*(\alpha)$ is not deterred. Hence, these two cases hold that $z \notin f(Z)$. However, if $z \in [z_1^*(\alpha), z_2^*(\alpha)]$, any deviation from z is deterred. Indeed, without loss of generality, assume $x = z_1^*(\alpha)$, and consider $y \in Z$ and a winning coalition S', such that $x \longrightarrow_{S'} y$. (c) If $z_1^*(\alpha) >^Z y$, then there exists $z = z_1^*(\alpha)$, with $z_1^*(\alpha) \gg y$ via T, and $not(z_1^*(\alpha) \succ_S z_1^*(\alpha))$; (d) If $y \in]z_1^*(\alpha), z_2^*(\alpha)[$, then, there exists $z = y$, such that $not(z \succ_{S'} z_1^*(\alpha))$, with $|S'| > \alpha n$, because $z_2^*(\alpha) >^Z y$; (e) If $y >^Z z_2^*(\alpha)$, then, there exists $z = z_2^*(\alpha)$, with $z_2^*(\alpha) \gg y$ via S, and $not(z_2^*(\alpha) \succ_{S'} z_1^*(\alpha))$, with $|S'| > \alpha n$. It follows that $f(Z) = [z_1^*(\alpha), z_2^*(\alpha)]$. It is straightforward to check that $f(f(Z)) = f(Z)$; therefore $f(Z)$ is the largest consistent set, and item A. of Theorem 3 is proved.

B. First, if $\alpha = 1$, the only winning coalition is the set N. Given that agents propose the status quo z_0, the cardinality of the largest coalition that can propose an alternative policy z_1 against z_0 is $n-1$. If the supermajority rule requires n agents to replace the status quo, then no alternative can be indirectly dominated, and the maximum number of predicted outcomes is the cardinality of N, i.e., n. Second, if n is odd, and $\alpha = \frac{1}{2}$, then the median peak is the unique prediction of the largest consistent set because it is the Condorcet winner, i.e., it indirectly dominated any other policy in the game. Third, from part A., we show that $LCS(\mathcal{P}(\alpha)) = [z_1^*(\alpha), z_2^*(\alpha)]$. Therefore, the maximal number of equilibria is the number of agents who have a peak between $z_1^*(\alpha)$ and $z_2^*(\alpha)$. Given that the proportion of agents required to form a winning coalition is at least $\frac{\lfloor \alpha n \rfloor + 1}{n}$, then the upper bound of $LCS(\mathcal{P}(\alpha))$ is $n - 2(1 - \frac{\lfloor \alpha n \rfloor + 1}{n})n = 2\lfloor \alpha n \rfloor + 2 - n < n$. The latter concludes the proof of item B. of Theorem 3. ■

Remark 1 We note that there is an alternative proof of Theorem 3. One can prove that the direct ($\gg$) and the indirect dominance ($\gtrdot$) relations are equivalent in the domain of single-peaked preferences. Under this equivalence, Theorem 3 is a direct consequence of Theorems 1 and 2.[15] It is, however, important to observe that the direct and indirect dominance relations model different rational behaviors in different decision-making environments. One implication of the fact that the relations ($\gg$) and ($\gtrdot$) coincide in single-peaked preference domains is that any final outcome z of a dynamic political game (as described in Section 3.2) is an element of the core, even if the status quo policy z_0 ($z_0 \neq z$) in the dynamic game is a defeated policy in the corresponding one-shot political game. □

3.3. Utilitarian Social Planner

In previous sections, we derive the bounds of the equilibrium set as a function of the decision rule used to aggregate agents' preferences and voting decisions in static and dynamic political competitions. In this section, we address the question of which decision rule maximizes social welfare.

[15]We thank a referee for pointing out this remark.

Let us consider a political $\mathcal{P}(\alpha) = (N, Z, (\succeq_i), \mathcal{L}_\alpha)$, and assume that Nature randomly chooses the status quo policy in the voting procedure. Then, from Theorems 2 and 3, the equilibrium set is $\mathcal{E}(\mathcal{P}(\alpha)) = [z_1^*(\alpha), z_2^*(\alpha)]$. One might ask how policy diversity, explained by the multiplicity of equilibria as the supermajority size α increases, affects social welfare from a utilitarian perspective. To examine this question, we assume that $Z \subset \mathbb{R}$, and each agent i's preference $\succeq_i$ over Z can be represented by a strictly quasi-concave utility function, V_i. We take an ex-ante perspective and examine the behavior of a utilitarian planner who chooses a supermajority rule $\mathcal{L}_\alpha$ to maximize expected utilitarian welfare given by:

$$\overline{W}(\alpha) = \int\limits_{z \in \mathcal{E}(\mathcal{P}(\alpha))} p(z) W(z) dz,$$

where W is the usual social welfare function given by the sum of voters' utilities and defined as: $W(z) := \sum\limits_{i \in N} V_i(z)$, for $z \in Z$. For simplicity, we assume that n is odd. Under the supermajority threshold $\alpha \in [\frac{1}{2}, \frac{n+1}{2n}]$, it holds that $z_1^*(\alpha) = z_2^*(\alpha) = z_m^p$ and $\mathcal{E}(\mathcal{P}(\alpha)) = \{z_m^p\}$, where z_m^p is the peak of the median voter, and $\overline{W}(\alpha) = W(z_m^p)$. For any supermajority rule α, with $\alpha > \frac{n+1}{2n}$, it is generally the case that $z_1^*(\alpha) \neq z_2^*(\alpha) \neq z_m^p$, and $p(z) = \frac{1}{z_2^*(\alpha) - z_1^*(\alpha)}$ for each $z \in \mathcal{E}(\mathcal{P}(\alpha))$, because policies in $\mathcal{E}(\mathcal{P}(\alpha))$ have the same chance to be chosen by Nature. Therefore, $\overline{W}(\alpha) = \frac{1}{z_2^*(\alpha) - z_1^*(\alpha)} \int_{z_1^*(\alpha)}^{z_2^*(\alpha)} W(z) dz$. In summary,

$$\overline{W}(\alpha) = \begin{cases} W(z_m^p) & \text{if } \alpha \in [\frac{1}{2}, \frac{n+1}{2n}] \\ \frac{1}{z_2^*(\alpha) - z_1^*(\alpha)} \int_{z_1^*(\alpha)}^{z_2^*(\alpha)} W(z) dz & \text{if } \alpha \in (\frac{n+1}{2n}, 1] \end{cases}. \tag{1}$$

Assuming that all other elements in $\mathcal{P}(\alpha)$ remain the same except the supermajority rule $\mathcal{L}_\alpha$, which threshold α maximize $\overline{W}(\alpha)$? Proposition 1 below answers this question.[16]

Proposition 1 *The expected utilitarian welfare $\overline{W}(\alpha)$ is maximal for any $\alpha \in [\frac{1}{2}, \frac{n+1}{2n}]$. It follows that for a sufficiently large population, the majority rule is the unique rule that maximizes expected utilitarian welfare.* □

PROOF (PROPOSITION 1) Let ϵ be a small and positive number. The equilibrium set of the

[16]We thank a referee for directing us to address this question.

political economy $\mathcal{P}(\alpha+\epsilon)$ is $\mathcal{E}(\mathcal{P}(\alpha+\epsilon)) = [z_1^*(\alpha+\epsilon), z_2^*(\alpha+\epsilon)]$, which contains the set $\mathcal{E}(\mathcal{P}(\alpha)) = [z_1^*(\alpha), z_2^*(\alpha)]$ because $z_2^*(\alpha+\epsilon) \geq z_2^*(\alpha) \geq z_1^*(\alpha) \geq z_1^*(\alpha+\epsilon)$. Then, we can write $\overline{W}(\alpha+\epsilon)$ as

$$\begin{aligned}\overline{W}(\alpha+\epsilon) &= \frac{1}{z_2^*(\alpha+\epsilon) - z_1^*(\alpha+\epsilon)} \int_{z_1^*(\alpha+\epsilon)}^{z_2^*(\alpha+\epsilon)} W(z)dz \\ &= \frac{1}{z_2^*(\alpha+\epsilon) - z_1^*(\alpha+\epsilon)} \left(\int_{z_1^*(\alpha+\epsilon)}^{z_1^*(\alpha)} W(z)dz + \int_{z_1^*(\alpha)}^{z_2^*(\alpha)} W(z)dz + \int_{z_2^*(\alpha)}^{z_2^*(\alpha+\epsilon)} W(z)dz \right).\end{aligned}$$

Given that $\frac{1}{z_2^*(\alpha)-z_1^*(\alpha)} > \frac{1}{z_2^*(\alpha+\epsilon)-z_1^*(\alpha)+\epsilon}$, it follows that

$$\overline{W}(\alpha+\epsilon) < \frac{1}{z_2^*(\alpha) - z_1^*(\alpha)} \left(\int_{z_1^*(\alpha+\epsilon)}^{z_1^*(\alpha)} W(z)dz + \int_{z_1^*(\alpha)}^{z_2^*(\alpha)} W(z)dz + \int_{z_2^*(\alpha)}^{z_2^*(\alpha+\epsilon)} W(z)dz \right)$$

or

$$\overline{W}(\alpha+\epsilon) - \overline{W}(\alpha) < \frac{1}{z_2^*(\alpha) - z_1^*(\alpha)} \int_{z_1^*(\alpha+\epsilon)}^{z_1^*(\alpha)} W(z)dz + \frac{1}{z_2^*(\alpha) - z_1^*(\alpha)} \int_{z_2^*(\alpha)}^{z_2^*(\alpha+\epsilon)} W(z)dz,$$

and

$$\frac{\overline{W}(\alpha+\epsilon) - \overline{W}(\alpha)}{\epsilon} < \frac{1}{\epsilon} \frac{1}{z_2^*(\alpha) - z_1^*(\alpha)} \left(\int_{z_1^*(\alpha+\epsilon)}^{z_1^*(\alpha)} W(z)dz + \int_{z_2^*(\alpha)}^{z_2^*(\alpha+\epsilon)} W(z)dz \right). \quad (2)$$

Consequently, when $\overline{W}(\alpha) \neq \overline{W}(\alpha+\epsilon) \neq W(z_m^p)$, we have $\overline{W}'(\alpha) = \lim_{\epsilon \longrightarrow 0} \frac{\overline{W}(\alpha+\epsilon)-\overline{W}(\alpha)}{\epsilon} < 0$, because the right-hand side of equation (2) tends to zero as $\epsilon \longrightarrow 0$. Then, if $\alpha \in [\frac{1}{2}, \frac{n+1}{2n}]$, $\overline{W}'(\alpha) = 0$, and if $\alpha \in (\frac{n+1}{2n}, 1]$, $\overline{W}'(\alpha) < 0$, and the function $\overline{W}$ is a continuous and non-increasing function. Given that for any $\alpha \in [\frac{1}{2}, \frac{n+1}{2n}]$, $\overline{W}(\alpha) = W(z_m^p)$, and $W(z_m^p) = \max_{\alpha \in [\frac{1}{2}, 1]} \overline{W}(\alpha)$, we can conclude that $\overline{W}(\alpha)$ is maximal for any $\alpha \in [\frac{1}{2}, \frac{n+1}{2n}]$. Therefore, for a sufficiently large n, the majority rule ($\alpha = \frac{1}{2}$) is the unique rule that maximizes $\overline{W}$. ∎

The analysis shows that when Nature chooses the status quo randomly, the majority rule maximizes expected utilitarian welfare. This finding complements studies highlighting other interesting properties of the majority rule; see, e.g., May (1952), and Dasgupta and Maskin (2008). We find another interesting property of the majority rule among the illustrations developed in Section 4.

For instance, in Section 4.3, we show that the majority rule is the unique rule that maximizes political compromise.

4. Illustrations

Having presented Theorems 1, 2, and 3, we now use the results to illustrate policy diversity across identical political economies. In Section 4.1, we propose an illustration demonstrating how two countries with identical political, economic, and cultural preferences over immigration resettlement could implement different policies.

4.1. Equilibrium Number and Policy Diversity

In this illustration, the political economy $\mathcal{P}(\alpha) = (N, Z, (\succeq_i), \mathcal{L}_\alpha)$ represents the government of a country that is developing a refugee resettlement program to help asylum seekers. We assume that this decision belongs to the legislators (in N) who represent the country's citizens, and the country derives utility from the number of refugees ($z \in Z$) it admits. The utility can be in terms of the national and international "warm glow" it receives or the skills or cultural diversity brought by the refugees. We assume that $Z =]0, +\infty[$ and represent a legislator i's preference, $\succeq_i$, over Z by a utility function, V_i. In the legislature, the decision is made under the supermajority rule, $\mathcal{L}_\alpha$. For simplicity, we assume that the net utility received by each legislator i from z refugees being admitted is $V_i(z) = v_i \ln(z) - \frac{z}{n}$, where $\frac{1}{n}$ is the fraction of the total cost of refugee admission incurred by each constituency (assuming n constituencies) in the country, and v_i is legislator i's valuation of the number of refugees. Observe that V_i is single-peaked, and so voter i's peak is obtained by solving $V'(z_i) = 0$, leading to the solution $z_i^p = nv_i$. Suppose nine legislators ($n = 9$) collectively choose the number of refugees to be admitted following either the static or the dynamic voting procedure described in Section 3. We assume that Nature randomly selects a legislator to propose a policy to the legislature. Using Theorems 1 and 3, the maximum number of equilibria is

$$\max|\mathcal{E}(\mathcal{P}(\alpha))| = \begin{cases} 1 & \text{if } \alpha \in [\frac{1}{2}, \frac{5}{9}] \\ 3 & \text{if } \alpha \in (\frac{5}{9}, \frac{2}{3}] \\ 5 & \text{if } \alpha \in (\frac{2}{3}, \frac{7}{9}] \\ 7 & \text{if } \alpha \in (\frac{7}{9}, \frac{8}{9}] \\ 9 & \text{if } \alpha \in (\frac{8}{9}, 1] \end{cases}.$$

We assume that $v_i = i$, where $i = 1, 2, ..., 9$. Then, the legislators' peaks are: $z_1^p = 9$, $z_2^p = 18$, $z_3^p = 27$, $z_4^p = 36$, $z_5^p = 45$, $z_6^p = 54$, $z_7^p = 63$, $z_8^p = 72$, and $z_9^p = 81$. The median voter is the legislator with valuation $v_i = 5$. The peak $z_5^p = z_1^*(\frac{1}{2}) = z_2^*(\frac{1}{2}) = 45$ defeats all other peaks in a pairwise majoritarian election ($\alpha = \frac{1}{2}$), and becomes the only peak which is not defeated. Therefore, under majority rule, the country grants permanent residency to 45 refugees.

Now, suppose that the legislators choose the number of refugees using a two-thirds supermajority rule ($\alpha = \frac{2}{3}$). Any proposal in the set $\{27, 36, 45, 54, 63\}$ cannot be defeated in a pairwise supermajoritarian contest because all alternatives will fail to win support from the necessary supermajoritarian coalition. These proposals are shielded from the possibility of an amendment on the legislative floor. Moreover, as illustrated in Figure 2, any outcome in the set $\{9, 18, 72, 81\}$ can be defeated by either $z_1^*(\frac{2}{3}) = 27$ or $z_2^*(\frac{2}{3}) = 63$. It follows that two countries that are identical regarding the number of legislators, legislators' preferences, and voting rules are likely to diverge in policy choice under the two-thirds supermajority rule. For example, depending on the random voter chosen to propose a policy, one country may grant permanent residency to only 27 refugees while the other may grant this privilege to 54 refugees. Under majority rule, both countries will converge in their policy and grant permanent residency to 45 refugees.[17]

[17] Beyond explaining policy developments in international negotiations, our findings are also validated by McGrath et al. (2018), who conducted a comparative study across U.S. states. Leveraging cross-country variation in state legislative override requirements, they find that legislatures with higher override requirements demonstrate less ability to override an executive veto. Mapping the legislative process to our model, state governors first propose budgets, and then legislatures pass their own. The budget is then sent to the governor for approval and, if vetoed, can only be enacted if a legislative supermajority overrides the veto. The supermajority thresholds used in the study — which, in this case, are the proportions of the legislature needed to override an executive veto — vary between $\frac{1}{2}$ and $\frac{2}{3}$. (Note, however, that three U.S. states with a $\frac{3}{5}$ or majority veto override were excluded from some models because they also had supermajority budgetary requirements (McGrath et al., 2018, p. 165).) Following our results, budgets passed in U.S. states with higher override requirements were substantially closer to

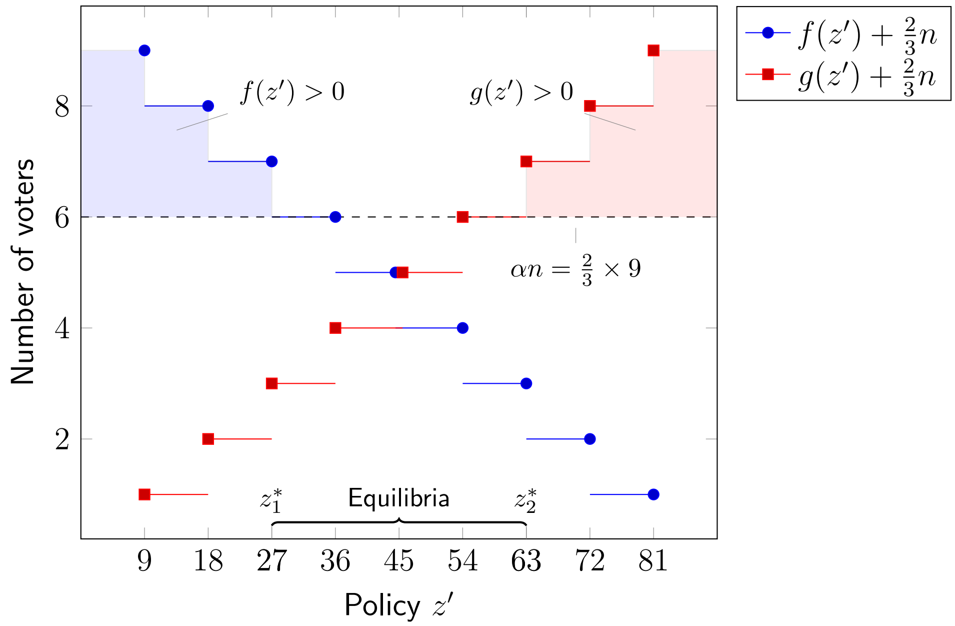

Figure 2: Equilibrium Range for refugee resettlement program when $\alpha = 2/3$. Note: Equilibrium points are those between $z_1^*(\frac{2}{3}) = 27$ and $z_2^*(\frac{2}{3}) = 63$ inclusive. For all z', $f(z') + \frac{2}{3}n = \sum_{z \geq^Z z'} S(z)$ and $g(z') + \frac{2}{3}n = \sum_{z' \geq^Z z} S(z)$.

4.2. Provision of Public Goods: Social Optimum and Equilibrium

In this section, the political economy $\mathcal{P}(\alpha) = (N, Z, (\succeq_i), \mathcal{L}_\alpha)$ represents a community, N, in which agents must decide the level of provision of a public, $z \in Z$, in a pairwise political contest following the supermajority rule, $\mathcal{L}_\alpha$, and a voting procedure described in Section 3. For simplicity, we assume that Z is one-dimensional. As in Section 4.1, we represent agent i's preference $\succeq_i$ over Z by a utility function, V_i. The current level of the public good is z, and agents vote to increase or decrease it. A voting equilibrium z^* is any amount of public good that belongs to the equilibrium set $\mathcal{E}(\mathcal{P}(\alpha))$. An outcome z_e is a **social optimum** (or an efficient provision of the public good) if $z_e \in \arg\max_{z \in Z} W(z)$, where $W(z) = \sum_{i \in N} V_i(z)$ is the social welfare at z.

Does there exist a supermajority rule $\mathcal{L}_\alpha$ that guarantees the social optimum $z_e \in \mathcal{E}(\mathcal{P}(\alpha))$? What is the minimal α among such rules? To address these questions, we first show that the social optimum outcome belongs to the range $[z_{min}, z_{max}]$ (see Figure 3).

By contradiction, assume that $z_{min} > z_e$. Then, in a pairwise opposition between z_e and z_{min},

those proposed by the governor, with the most substantial effects in states where executives' preferences diverged sharply from those of legislative veto players.

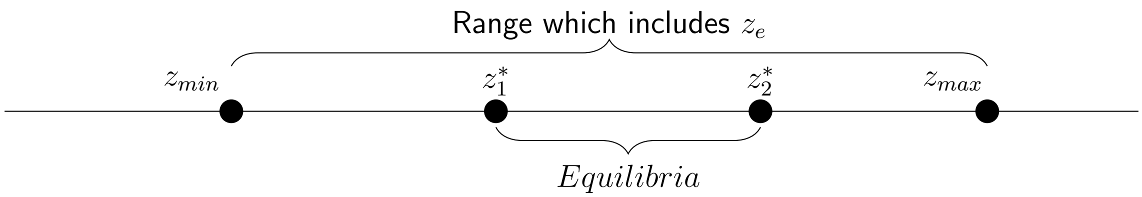

Figure 3: The social optimum and voting equilibrium.

the former receives zero share of the votes, meaning that each agent prefers z_{min} to z_e, or $V_i(z_{min}) > V_i(z_e),\ \forall i \in N$. The latter expression leads to $W(z_{min}) > W(z_e)$, which is absurd by the definition of z_e. We obtain the same conclusion if we assume that $z_e > z_{max}$. If the decision is made by using the unanimity rule ($\alpha = 1$), then the social optimum $z_e \in \mathcal{E}(\mathcal{P}(1))$, because there will never be enough agents who can form a winning coalition to defeat z_e in a pairwise political competition. Assuming that Nature chooses the status quo, and the minimum size of the majority required to pass a decision is less than the size of all voters (unanimity rule), then the social optimum could be an equilibrium. Moreover, our findings suggest that the size of the equilibrium set increases as the size of the majority required to pass social decisions increases. Then, depending on agents' preferences, there always exists a minimum threshold $\alpha_{min} \in [\frac{1}{2}, 1]$, such that $z_e \in \mathcal{E}(\mathcal{P}(\alpha_{min}))$. However, if Nature randomly chooses a proposer, and the social optimum does not coincide with any ideal policy closer to the proposer's peak, then it will not have a chance to be submitted for a vote, whatever the threshold required by the supermajority rule. For that reason, there will not exist a majority threshold α under which the social optimum, z_e, is undefeated in our political competition models.

For the sake of illustration and using the numerical example in Section 4.1, suppose that $|N| = 5$, and an agent i's net utility takes the form: $V_i(z) = v_i \ln z - \frac{z}{5}$, where the factor v_i represents agent i's valuation or taste for the public good. If an outcome z emerges as the voting equilibrium, each agent will pay a fraction $\frac{1}{5}$ of the additional cost. It is straightforward to show that V_i is single-peaked. Agents' peaks are the sequence (z_i^p) such that: $z_i^p = 5v_i$, where $i = 1, 2, 3, 4, 5$. The social welfare at each alternative z is given by $W(z) = \sum_{i=1}^{5} v_i \ln z - z$, so that the efficient level of public good is $z_e = \sum_{i=1}^{5} v_i$. Arbitrarily taking $v_1 = 6$, $v_2 = 2$, $v_3 = 3$, $v_4 = 4$ and $v_5 = 8$, agents' peaks are $z_1^p = 30$, $z_2^p = 10$, $z_3^p = 15$, $z_4^p = 20$ and $z_5^p = 40$, and the social optimum z_e is 23, with $W(z_e) = 49.11637$. The voting equilibrium level of the public good depends on the size of the majority required to pass the decision. Using Theorems 2 and 3, and assuming that Nature randomly chooses the status quo, the equilibrium set is $\mathcal{E}(\mathcal{P}(\alpha)) = [z_1^*(\alpha), z_2^*(\alpha)]$, where the bounds $z_1^*(\alpha)$ and $z_1^*(\alpha)$ depend on the supermajority rule α. Under any supermajority rule $\alpha \in [\frac{1}{2}, \frac{3}{5}]$, for instance, we observe that $z_4^p = z_m^p$, the median peak, defeats any other agent's peak in a pairwise political contest and so the only voting equilibrium is z_4^p. However, the social optimum z_e is not a voting equilibrium, because $z_e \neq z_4^p$. Nevertheless, for any supermajority $\alpha \in (\frac{3}{5}, \frac{4}{5}]$, the equilibrium range is given by the interval $[z_3^p = 15, z_1^p = 30]$, including the social optimum $z_e = 23$. Similarly, when $\alpha \in (\frac{4}{5}, 1]$, the equilibrium set is $[z_2^p = 10, z_5^p = 40]$, which also includes z_e. It follows that the minimal supermajority rule that guarantees z_e in $\mathcal{E}(\mathcal{P}(\alpha_{min}))$ is $\alpha_{min} = \frac{3}{5}$. Evaluating expected utilitarian welfare, it holds that

$$\overline{W}(\alpha) = \begin{cases} W(z_4^p) \approxeq 48.90184 & \text{if } \alpha \in [\frac{1}{2}, \frac{3}{5}] \\ \frac{1}{15}\int_{15}^{30}(23 \ln z - z)dz \approxeq 48.6699 & \text{if } \alpha \in (\frac{3}{5}, \frac{4}{5}] \\ \frac{1}{30}\int_{10}^{40}(23 \ln z - z)dz \approxeq 47.47248 & \text{if } \alpha \in (\frac{4}{5}, 1] \end{cases} . \tag{3}$$

Using the system in (3), we note that for any supermajority $\alpha \in [\frac{1}{2}, \frac{3}{5}]$, $\overline{W}(\alpha)$ is maximal, and $\max_{\alpha \in [\frac{1}{2}, 1]} \overline{W}(\alpha) = W(z_4^p)$.

4.3. Political Compromise

Successful reforms in polarizing policy domains—gun control, abortion, healthcare, and immigration—require legislators to make mutual sacrifices and willfully compromise their core values, principles, or interests. Today, growing cleavages between parties in many developed and democratic countries have hampered political compromise, as those at the opposite ends of the ideological spectrum find policy near the median voter's ideal point increasingly unappealing.[18] In this illustration, we do not seek to provide the sources of political compromise in democratic settings but rather to rationalize the political compromise observed in democratic legislatures. Let

[18] Gutmann and Thompson (2010) attribute the success of the 2017 Tax Cuts and Jobs Act in the United States Congress to successful bipartisan compromise and the passage of the 1986 Tax Reform Act and the 2010 Patient Protection and Affordable Care Act to mutual sacrifice and mutual opposition by Democratic and Republican leadership. Additionally, a 2019 survey by the Pew Research Center finds that most U.S. adults support more political compromise and respect toward opposing political views from *opposing* political parties (Pew Research Center, 2019).

$\mathcal{P}(\alpha) = (N, Z, (\succeq_i), \mathcal{L}_\alpha)$ be a political economy, and assume that Z is endowed with the order topology induced by the total order $\geq^Z$, and we denote by d, a *distance* defined on Z. We assume that Nature randomly chooses an agent i to propose a policy in the political contest and let z_i^p be the ideal policy of i. The **level of political compromise** of proposer i is the distance pc_i defined as: $pc_i(\alpha) = \min_{z \in \mathcal{E}(\mathcal{P}(\alpha))} \{d(z, z_i^p)\}$.

An agent i will propose their ideal point z_i^p only if it is an equilibrium point; if their ideal point is not an equilibrium, they will propose the closest equilibrium point to their ideal point; this is because proposing their ideal point will result in a defeat. Therefore, the level of political compromise for an agent i is the distance between their ideal point and the closest equilibrium point.

For principled politicians whose ideological platforms are located either to the left of equilibrium $z_1^*(\alpha)$ or to the right of equilibrium $z_2^*(\alpha)$, Figure 4 illustrates the necessary compromise they must make, as the incumbents, to avoid defeat in all pairwise supermajoritarian political contests.

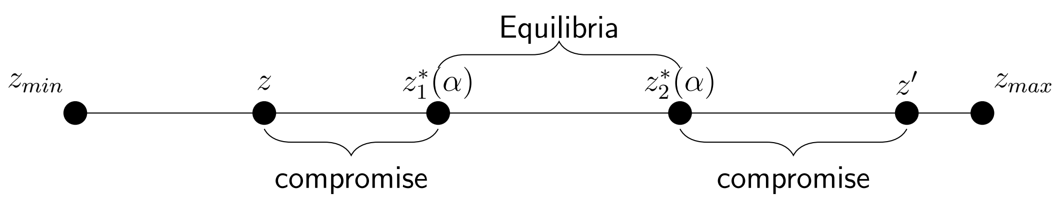

Figure 4: Political compromise.

A political party with an ideal policy (or fundamental ideological identity) "z" must compromise by moving toward the closest equilibrium to its political platform to avoid defeat in a pairwise election against $z_2^*(\alpha)$. It is rational, then, for this party to run on platform $z_1^*(\alpha)$ if it enters the race. The same strategy applies to the politician with ideal point z', who must compromise by running on $z_2^*(\alpha)$. Compromising is rational, as it increases the likelihood of challenging the status quo in a pairwise supermajoritarian election. Indeed, for a given proposer i, the optimal amount of compromise is the minimal distance between i's ideal policy to the equilibrium point that maximizes i's preferences. Given that preferences are single-peaked, the minimal distance is obtained from the closest equilibrium policy to i's ideal point. Our findings in Theorems 1, 2, and 3 show that the size of the equilibrium set varies increasingly as the size of the majority α required by the decision rule increases. This reduces the distances of political ideologies to possible equilibria and therefore diminishes the level of political compromise, $pc_i(\alpha)$, for each proposer i as the size α increases. Therefore, if $\overline{pc}(\alpha) = \frac{1}{n}\sum_{i=1}^{n} pc_i(\alpha)$ represents the average level of political compromise in the political economy $\mathcal{P}(\alpha)$, then, we can show that $\overline{pc}$ is maximal under majority rule. In other words, only the majority rule maximizes political compromise in a supermajoritarian political economy.

5. Concluding remarks

In this study, we derive the minimum and the maximum number of equilibrium policies in static and dynamic political games under supermajority rules when agents have single-peaked preferences over a totally ordered policy space. Voters' strategic behavior is captured by the *core* (Black, 1948; Downs, 1957) in static environments, and by the *largest consistent set* (Chwe, 1994) in dynamic environments. We fully characterize the relationship between these numbers and a voting body's supermajority rule, showing that the minimum number is one regardless of the rule, and the maximum number increases in a nontrivial manner in the size of the supermajority coalition needed to change policy. The well-known Median Voter Theorem, which predicts party convergence to the median voter's ideal policy, is a particular case of our results. Our findings explain why highly divergent policies may persist, even across democracies with identical political preferences and voting rules. Policy divergence increases as we move further from majority rule. Moreover, in deriving the minimum and the maximum number of equilibrium policies in a supermajoritarian setting, our results translate Duverger's propositions on institutions and political parties. In only imposing the assumption that voters hold single-peaked preferences over a totally ordered policy space, our model is quite general and applies to various policies beyond those chosen from a unidimensional set.

Our theory generalizes voting dynamics in other theoretical work (e.g., Dixit et al. (2000)), and its implications align with voting behavior in institutions ranging from state legislatures (e.g., McGrath et al. (2018)) to international institutions (e.g., (Stone, 2009)). Additionally, we contribute to existing social choice literature. Focusing on supermajority voting rules — a topic that has, to date, received limited attention — the article raises and answers novel questions. What is the relationship between supermajority thresholds and the number of equilibrium policies? And how does this relationship manifest in the diversity of policies across institutions with one threshold instead of another? Which rules maximize utilitarian welfare? Which rules maximize political compromise?

Our model also offers avenues for future empirical and theoretical research. Further extensions can consider proposal or amendment costs that vary based on legislators' ideal points or the location of the proposed policy or amendment. The model is also amenable to accommodating "decision-costs" from policy gridlock (Buchanan and Tullock, 1962) and introducing uncertainty in legislators' policy preferences. The latter extension would draw connections between policy diversity and the extensive literature examining the Condorcet Jury Theorem.[19]

Empirically, the model offers several testable predictions. Do reductions in amendment thresholds — such as revisions to the United States Senate requirements to invoke cloture, decrease policy diversity, and increase the extent to which proposers (or political parties) compromise? And, comparing legislative bodies whose members have similar preferences, do those requiring high supermajoritarian thresholds to amend proposals generate more diverse policies than those with low thresholds? And how does the distribution of agenda power mediate the relationship between policy diversity and voting rules? Despite the challenges in finding variation in voting rules across otherwise comparable legislative bodies (Cameron, 2009), recent research has employed innovative data to discern such relationships, both globally and domestically; see, e.g., Blake and Payton (2015), McGrath et al. (2018), and Brutger and Li (2019).

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[19] The question of choosing the optimal system considering uncertainty was first formulated by Condorcet (1793). Also, see Nitzan and Paroush (2017) for further details and extensions.

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