Within Siblings Analysis: Fixed Effects Models

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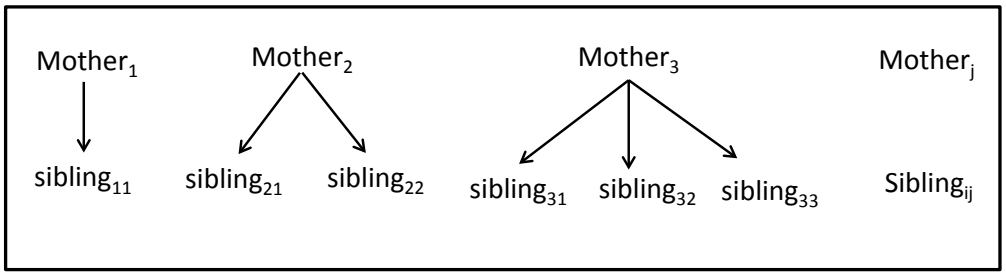
May 2, 2014

Outline

Variance-components models

Introduction

Variance-component models (VCMs) are designed to model and estimate such **within-cluster correlations** in the context of longitudinal and correlated data.



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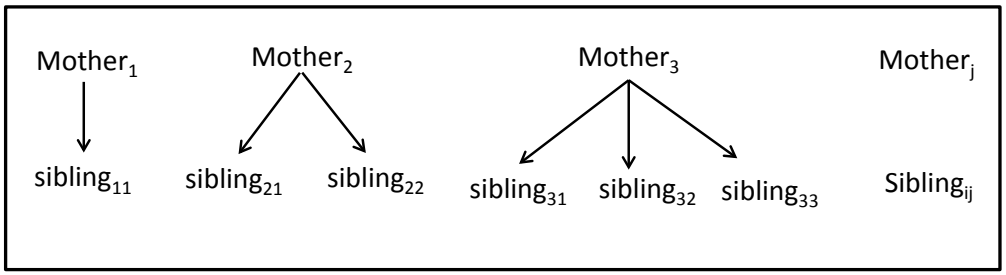

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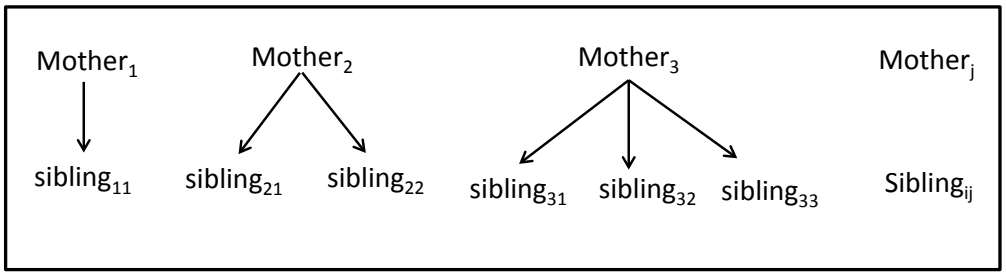

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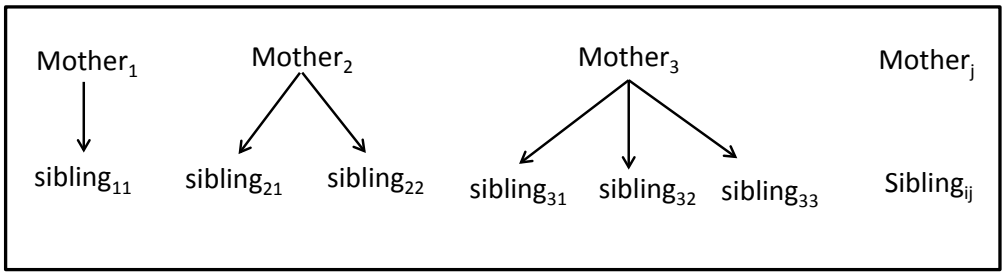

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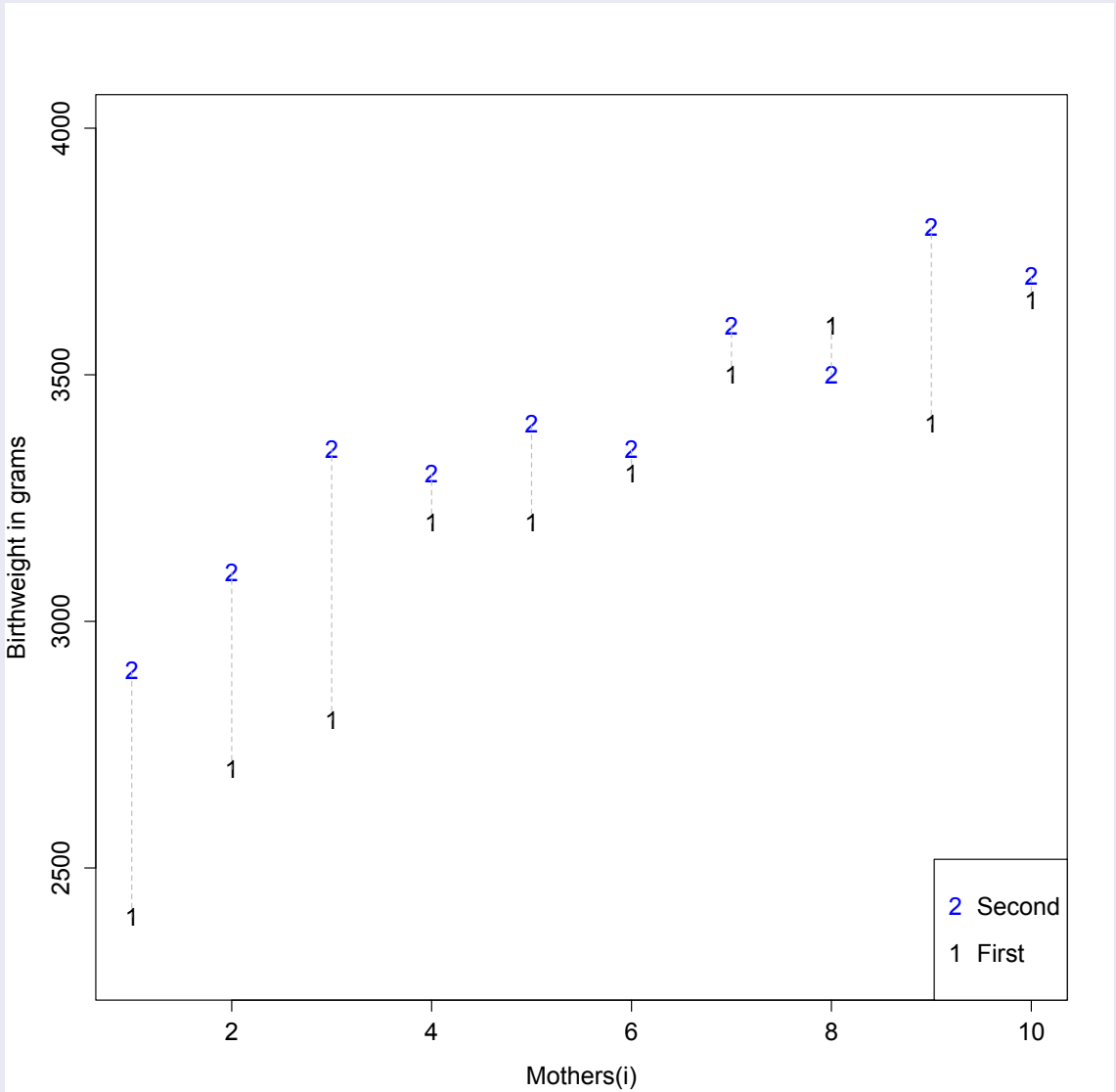
The simplest case: 2k (k = sibling) pairs of repeated measures



The simplest paired design: N Pairs of Siblings

The Model

$$Y_{ij} = \alpha + \beta X_{ij} + U_i + e_{ij}$$

Where:

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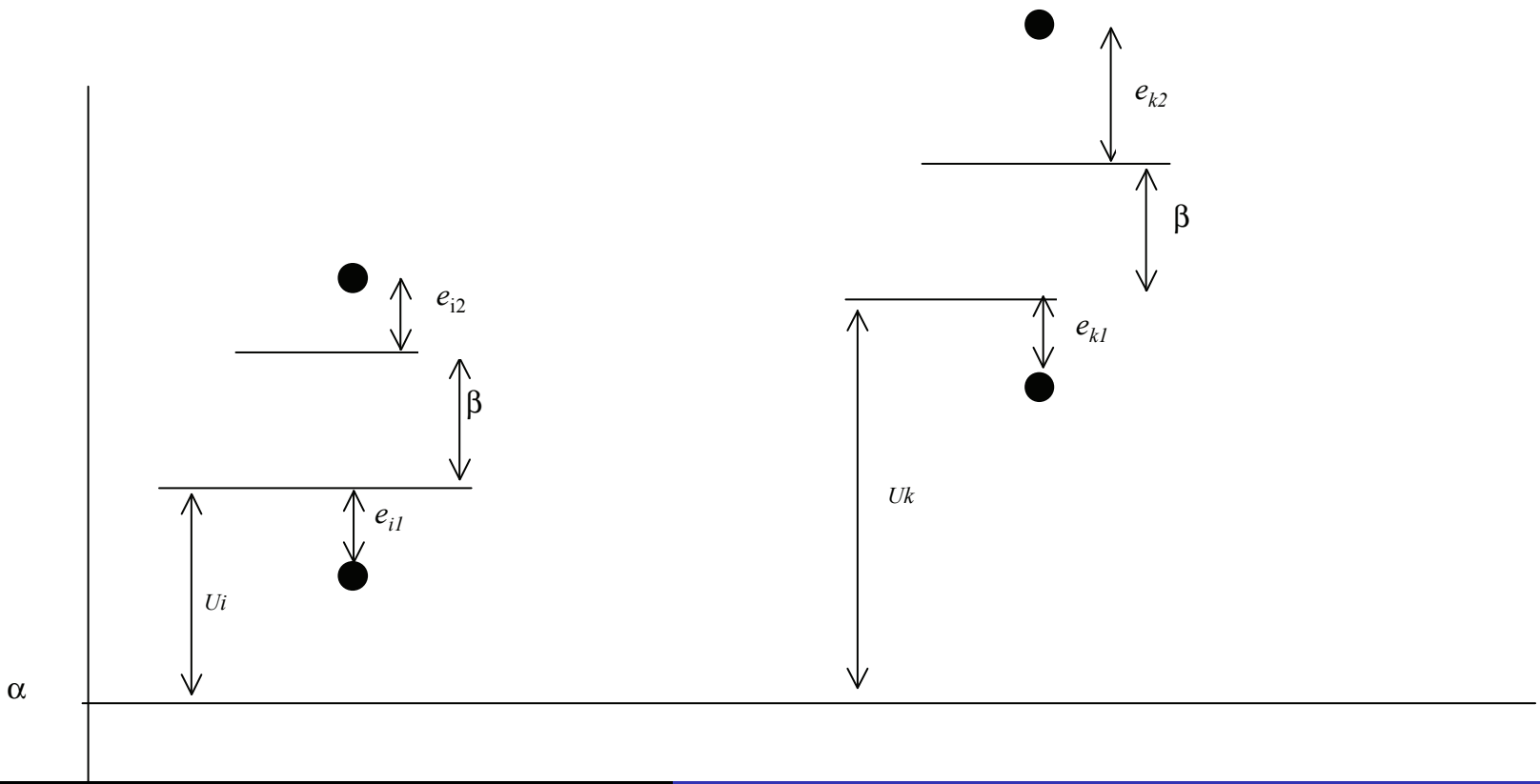

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Within-mothers differences (note that we can also use between mothers dummies variables):

- $Y_{i1} - Y_{i0} = (\alpha + \beta + U_i + e_{i1}) - (\alpha + U_i + e_{i0}) = \beta + (e_{i1} - e_{i0})$
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ICC as link between paired and independent analysis

Take message

Within-cluster (mothers) correlation increases precision for estimating within-siblings effects. So, we can obtain greater benefit pairing if ICC (ρ) is larger.

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Analytical methods for more than 2k siblings

Mean-centered transformation 'Demeaning'

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Where

Mean-centered outcome and covariates

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