Appendix for: An Exploration of Optimal Stabilization Policy

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1 The Equilibrium under Flexible Prices (Section 3 of the paper)

Equations (9) through (13) in the paper fully describe the flexible-price equilibrium for a given fiscal policy. We state them here for reference, and we include in equation (10) the investment subsidy (at rate s) used later in the paper.

$$Y_t = A_t K_t \text{ for all } t,$$

$$\frac{u'(C_1)}{\beta u'(C_2)} = (1+i_1)\frac{P_1}{P_2},$$

$$A_2 = (1-s)\frac{P_1}{P_2}(1+i_1),$$

$$C_1 = A_1K_1 - K_2 - G_1,$$
$$C_2 = A_2K_2 - G_2,$$

and the government's budget constraint is:

$$G_2 = (1+i_1)\frac{P_1}{P_2}(T_1 - G_1 - sI_1) + T_2.$$

To solve for the economy's equilibrium, combine the second, third, fourth, and fifth of these to obtain:

$$A_2 = (1-s)\frac{u'(A_1K_1 - K_2 - G_1)}{\beta u'(A_2K_2 - G_2)}.$$

This expression implicitly defines K_2 as a function of parameters and K_1, which is given. Using the CRRA utility function specified in Section 3.1 of the paper, we obtain:

$$K_2 = \frac{1}{1 + \left(\frac{(1-s)}{\beta A_2}\right)^{\sigma} A_2(1-g_2)} A_1K_1(1-g_1).$$

This is expression (16) in the paper. This solution for K_2 can be used to derive values for the remaining *real* endogenous variables in the economy. Nominal variables are determined separately by equilibrium in the money market as stated in Section 2.3 of the paper and restated here:

$$P_tC_t = M_t \text{ for all } t.$$

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1.1 An Aside on Labor (Section 3.3 of the paper)

Suppose the production function includes labor L_t and exogenous labor productivity ω_t as follows:

$$Y_t = A_t \left(K_t + \omega_t L_t\right).$$

The other equilibrium conditions corresponding to equations (10) through (13) from Section 3.1, setting the investment subsidy to zero for simplicity, are:

$$A_2 = \frac{P_1}{P_2}\left(1+i_1\right),$$

$$\frac{u'\left(C_1\right)}{\beta u'\left(C_2\right)} = \left(1+i_1\right)\frac{P_1}{P_2},$$

$$C_1 = A_1\left(K_1+\omega_1 L_1\right) - K_2 - G_1,$$

$$C_2 = A_2\left(K_2+\omega_2 L_2\right) - G_2,$$

where the government's budget constraint remains:

$$G_2 = \left(1+i_1\right)\frac{P_1}{P_2}\left(T_1 - G_1\right) + T_2.$$

To solve for the economy's equilibrium, start by combining equations as in the baseline model to obtain:

$$K_2 = \frac{A_1\left(K_1+\omega_1 L_1\right)\left(1-g_1\right) - \omega_2 L_2 \left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\left(1-g_2\right)}{\left(1+\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\left(1-g_2\right)\right)}.$$

This result can be rearranged to yield:

$$K_2 + \omega_2 L_2 = \frac{A_1\left(K_1+\omega_1 L_1\right)\left(1-g_1\right) + \omega_2 L_2}{1+\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\left(1-g_2\right)}.$$

Substituting into the equations for consumption, we obtain:

$$\frac{\left(\frac{1}{A_2\beta}\right)^{\sigma}}{\left(1+\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\left(1-g_2\right)\right)} A_2\left(1-g_2\right)\left(A_1\left(1-g_1\right)\left(K_1+\omega_1 L_1\right)+\omega_2 L_2\right) = C_1,$$

$$\frac{1}{\left(1+\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\left(1-g_2\right)\right)} A_2\left(1-g_2\right)\left(A_1\left(1-g_1\right)\left(K_1+\omega_1 L_1\right)+\omega_2 L_2\right) = C_2.$$

These expressions for consumption can be substituted into the government's fiscal policy problem. The government solves:

$$\max_{g_1,g_2}\left[\begin{array}{r} u\left(\frac{\left(\frac{1}{A_2\beta}\right)^{\sigma}}{1+\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2(1-g_2)} A_2\left(1-g_2\right)\left(A_1\left(1-g_1\right)\left(K_1+\omega_1 L_1\right)+\omega_2 L_2\right)\right) + v\left(g_1 A_1\left(K_1+\omega_1 L_1\right)\right) \\ +\beta\left[u\left(\frac{1}{\left(1+\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2(1-g_2)\right)} A_2\left(1-g_2\right)\left(A_1\left(1-g_1\right)\left(K_1+\omega_1 L_1\right)+\omega_2 L_2\right)\right)\right. \\ \left. + v\left(g_2 A_2 \frac{A_1(K_1+\omega_1 L_1)(1-g_1)+\omega_2 L_2}{1+\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2(1-g_2)}\right)\right] \end{array}\right].$$

The first-order conditions of this problem are:

$$
\begin{aligned}
FOC_{g_2} \quad &: \quad \frac{\left(\frac{1}{A_2\beta}\right)^{\sigma}}{1+\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\left(1-g_2\right)} u'\left(C_1\right)\left[\begin{array}{c} \frac{\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2}{1+\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\left(1-g_2\right)} A_2\left(1-g_2\right)\left(A_1\left(1-g_1\right)\left(K_1+\omega_1 L_1\right)+\omega_2 L_2\right) \\ -A_2\left(A_1\left(1-g_1\right)\left(K_1+\omega_1 L_1\right)\right) \end{array}\right] \\
&= \quad \frac{1}{1+\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\left(1-g_2\right)} \beta\left(\begin{array}{c} u'\left(C_2\right)\left[\begin{array}{l} \frac{\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2}{1+\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\left(1-g_2\right)} A_2\left(1-g_2\right)\left(A_1\left(1-g_1\right)\left(K_1+\omega_1 L_1\right)+\omega_2 L_2\right) \\ -A_2 A_1\left(1-g_1\right)\left(K_1+\omega_1 L_1\right) \end{array}\right] \\ +v'\left(G_2\right) A_2\left(A_1\left(K_1+\omega_1 L_1\right)\left(1-g_1\right)+\omega_2 L_2\right)\left(\frac{1+\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2}{1+\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\left(1-g_2\right)}\right) \end{array}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
FOC_{g_1} \quad &: \quad u'\left(C_1\right) \frac{-\left(\frac{1}{A_2\beta}\right)^{\sigma}}{1+\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\left(1-g_2\right)} A_2\left(1-g_2\right) A_1\left(K_1+\omega_1 L_1\right)+v'\left(G_1\right) A_1\left(K_1+\omega_1 L_1\right) \\
&\quad +\beta\left(u'\left(C_2\right) \frac{-A_2\left(1-g_2\right) A_1\left(K_1+\omega_1 L_1\right)}{1+\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\left(1-g_2\right)}-v'\left(G_2\right) g_2 A_2 \frac{A_1\left(K_1+\omega_1 L_1\right)}{1+\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\left(1-g_2\right)}\right) \\
&= \quad 0.
\end{aligned}
$$

Simplifying, these conditions can be written as:

$$
\begin{aligned}
FOC_{g_2} \quad &: \quad \left(\frac{1}{A_2\beta}\right)^{\sigma} A_2 A_1\left(1-g_1\right) u'\left(C_1\right)\left[\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\left(1-g_2\right) \omega_2 L_2-\left(K_1+\omega_1 L_1\right)\right] \\
&= \quad \beta\left(\begin{array}{c} A_2 A_1\left(1-g_1\right) u'\left(C_2\right)\left[\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\left(1-g_2\right) \omega_2 L_2-\left(K_1+\omega_1 L_1\right)\right] \\ +v'\left(G_2\right) A_2\left(A_1\left(K_1+\omega_1 L_1\right)\left(1-g_1\right)+\omega_2 L_2\right)\left(1+\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\right) \end{array}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
FOC_{g_1} \quad &: \quad -u'\left(C_1\right)\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\left(1-g_2\right)+\left(1+\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\left(1-g_2\right)\right) v'\left(G_1\right) \\
&= \quad \beta\left(u'\left(C_2\right) A_2\left(1-g_2\right)+v'\left(G_2\right) g_2 A_2\right).
\end{aligned}
$$

These first-order conditions are satisfied when the same optimality conditions as in the baseline case (i.e., the household Euler condition 11, the government Euler condition 22, and the private-public consumption condition 23) are met. These conditions then yield expressions for the equilibrium values of the economy's variables:

$$
C_1=\frac{\left(\frac{1}{A_2\beta}\right)^{\sigma}}{(1+\theta)\left(1+\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\right)} A_2\left(A_1\left(K_1+\omega_1 L_1\right)+\omega_2 L_2\right),
$$

$$
C_2=\frac{1}{(1+\theta)\left(1+\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\right)} A_2\left(A_1\left(K_1+\omega_1 L_1\right)+\omega_2 L_2\right),
$$

$$
I_1=K_2=\frac{\left(A_1\left(K_1+\omega_1 L_1\right)-\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2 \omega_2 L_2\right)}{\left(1+\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\right)},
$$

$$
K_2+\omega_2 L_2=\frac{1}{\left(1+\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\right)} A_1\left(K_1+\omega_1 L_1\right)+\omega_2 L_2,
$$

$$G_1 = \frac{\theta \left(\frac{1}{A_2\beta}\right)^{\sigma}}{(1+\theta)\left(1+\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\right)} A_2 \left(A_1\left(K_1+\omega_1 L_1\right)+\omega_2 L_2\right),$$

$$G_2 = \frac{\theta}{(1+\theta)\left(1+\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\right)} A_2 \left(A_1\left(K_1+\omega_1 L_1\right)+\omega_2 L_2\right).$$

The only change required to the expressions for consumption and government purchases is that the term A_1K_1 is replaced with the expression $(A_1(K_1+\omega_1L_1)+\omega_2L_2)$ wherever it appears. The same modification converts I_1, which is equal to K_2 in the baseline model, to the sum $(K_2+\omega_2L_2)$ in this model with labor. In the expressions for output, K_t is replaced with $(K_t+\omega_tL_t)$.

2 The Equilibrium under Short-run Sticky Prices (Section 4 of the paper)

As stated in the paper, equation (9) from the flexible-price equilibrium may no longer hold with sticky prices. In particular,

$$Y_t = \begin{cases} C_t + I_t + G_t \text{ for } t=1 \\ A_tK_t \quad \text{for } t=2 \end{cases}.$$

The equilibrium equations are as follows.

The expression for the firm's profit substitutes Y_1 for A_1K_1, but the same first-order condition as expression (10) holds:

$$A_2 = (1-s)\frac{P_1}{P_2}(1+i_1)$$

The household Euler condition is the same as expression (11) from the flexible-price case:

$$\frac{u'(C_1)}{\beta u'(C_2)} = (1+i_1)\frac{P_1}{P_2}$$

The analogue to expression (12) is as shown in the paper,

$$Y_1 - K_2 - G_1 = C_1$$

while expression (13) is unchanged from the flexible-price case:

$$A_2K_2 - G_2 = C_2,$$

To solve, start with the Euler equation for the household, which can be written (assuming CRRA utility):

$$C_1 = \left(\frac{1}{\beta(1+i_1)}\frac{P_2}{P_1}\right)^{\sigma} C_2.$$

Now, we use the analogue to expression (13) to express the sum of nominal consumption as:

$$P_1C_1 + P_2C_2 = P_1C_1 + P_2\left(A_2K_2\left(1-g_2\right)\right).$$

Substitute into the left-hand side of this expression the expression for C_2 from the household's Euler condition to obtain:

$$P_1C_1 + \frac{P_2}{\left(\frac{1}{\beta(1+i_1)}\frac{P_2}{P_1}\right)^{\sigma}}C_1 = P_1C_1 + P_2\left(A_2K_2\left(1-g_2\right)\right).$$

Then, rearrange the firm's first-order condition to obtain:

$$P_2 = \frac{(1-s)(1+i_1)}{A_2}P_1,$$

which is expression (36) from the paper (including the investment subsidy). Use this expression to simplify the previous expression, yielding:

$$P_1 C_1 \left(1 + \frac{(1-s)(1+i_1)}{A_2}\left(\frac{A_2\beta}{(1-s)}\right)^{\sigma}\right) = P_1 C_1 + P_2 \left(A_2 K_2 (1-g_2)\right).$$

Solve this expression for K_2, again applying the firm's first-order condition:

$$K_2 = \frac{C_1}{\left(\frac{(1-s)}{\beta A_2}\right)^{\sigma} A_2 (1-g_2)}.$$

Next, use this result for K_2 in the analogue to expression (13) and simplify to obtain:

$$C_2 = \left(\frac{A_2\beta}{(1-s)}\right)^{\sigma} C_1,$$

Now we incorporate the nominal variables. We combine the expression

$$P_2 C_2 = M_2$$

with the firm's first-order condition to obtain:

$$C_2 = \frac{A_2}{(1-s)} \frac{M_2}{(1+i_1) P_1}.$$

This result implies:

$$C_1 = \left(\frac{(1-s)}{\beta A_2}\right)^{\sigma} \frac{A_2}{(1-s)} \frac{M_2}{(1+i_1) P_1},$$

and thus

$$I_1 = K_2 = \frac{1}{(1-g_2)(1-s)} \frac{M_2}{(1+i_1) P_1}.$$

These are the expressions (32), (31), and (33) in the paper.

3 Optimal Fiscal Policy when Monetary Policy is Restricted (Section 6 of the paper)

First, we consider only government purchases. Then we turn to the investment subsidy

3.1 Government purchases (Section 6.3 of the paper)

Expression (38) from the paper is the monetary policy position that generates full employment in the sticky-price model. For this section, the zero lower bound on the nominal interest rate has been reached, so $i_1 = 0$. With restricted monetary policy, M_2 equals its pre-shock full-employment level $\hat{M}_2$. Therefore, the following condition describes the ratio of the terminal money supply to the fixed first-period price level:

$$\frac{\hat{M}_2}{P_1} = (1+i_1) \frac{(1-\hat{g}_1)(1-\hat{g}_2)}{1 + \left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma} \hat{A}_2 (1-\hat{g}_2)} A_1 K_1.$$

This expression includes the expected levels of fiscal policy prior to the shock. Assuming they were set at the pre-shock optimum, these levels are implied by expressions (26) through (28) of the paper, and are:

$$\hat{g}_1 = \frac{\theta \left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma} \hat{A}_2}{(1+\theta)\left(1 + \left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma} \hat{A}_2\right)},$$

and

$$\hat{g}_2 = \frac{\theta}{1+\theta}.$$

Substituting these expressions into the previous expression, we obtain:

$$\frac{\hat{M}_2}{P_1} = \frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma}\hat{A}_2\right)} A_1 K_1.$$

Using expressions (31) through (34) from the paper, we substitute in this expression and obtain the equilibrium quantities with sticky prices and unspecified post-shock fiscal policy (g_1 and g_2):

$$C_1 = \left(\frac{1}{\beta A_2}\right)^{\sigma} A_2 \frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma}\hat{A}_2\right)} A_1 K_1,$$

$$C_2 = A_2 \frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma}\hat{A}_2\right)} A_1 K_1,$$

$$I_1 = K_2 = \frac{1}{(1-g_2)} \frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma}\hat{A}_2\right)} A_1 K_1,$$

$$Y_1 = \frac{1+\left(\frac{1}{\beta A_2}\right)^{\sigma} A_2 (1-g_2)}{(1-g_2)} \frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma}\hat{A}_2\right)} A_1 K_1 + g_1 A_1 K_1.$$

Using these results, we can state the government's problem:

$$\max_{\{g_t\}_{t=1}^{2}} \left[\begin{array}{c} u\left(\left(\frac{1}{\beta A_2}\right)^{\sigma} A_2 \frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma}\hat{A}_2\right)} A_1 K_1\right) + v\left(g_1 A_1 K_1\right) \\ +\beta\left(u\left(A_2 \frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma}\hat{A}_2\right)} A_1 K_1\right) + v\left(g_2 A_2 \frac{1}{(1-g_2)} \frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma}\hat{A}_2\right)} A_1 K_1\right)\right) \end{array} \right],$$

where the government's budget constraint, given as expression (52) in the paper, is incorporated into the agents' private decisions. Note that the pattern of taxes is immaterial, given the Ricardian nature of the model.

The government's solution is subject to the aggregate supply constraint on first-period aggregate demand:

$$Y_1 \leq A_1 K_1.$$

Using the expressions for first-period consumption, investment, and government purchases, this constraint can be written as:

$$\frac{1+\left(\frac{1}{\beta A_2}\right)^{\sigma} A_2 (1-g_2)}{(1-g_2)} \frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma}\hat{A}_2\right)} A_1 K_1 + g_1 A_1 K_1 \leq A_1 K_1.$$

The technology shock pushes aggregate demand below aggregate supply. As long as government purchases do not generate full employment, this constraint is nonbinding and the first-order condition for G_1 is:

$$v'(G_1) = 0.$$

This implies that the government uses fiscal policy to reach full employment, as noted in Section 6.3.

When government purchases cause full employment, this constraint binds and the first-order conditions for government purchases are:

$$FOC_{g_1}\colon\ A_1K_1v'\left(g_1A_1K_1\right)=\lambda A_1K_1,$$

and

$$\begin{aligned}FOC_{g_2} \quad &: \quad \beta v'\left(G_2\right)\left[\begin{array}{c} A_2\frac{1}{(1-g_2)}\frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}A_1K_1 \\ +g_2A_2\frac{1}{(1-g_2)^2}\frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}A_1K_1\end{array}\right] \\ &= \quad \lambda\left[\frac{-\left(\frac{1}{\beta A_2}\right)^{\sigma}A_2\left(1-g_2\right)+\left(1+\left(\frac{1}{\beta A_2}\right)^{\sigma}A_2\left(1-g_2\right)\right)}{\left(\left(1-\hat{g}_2\right)\right)^2}\right]\frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}A_1K_1.\end{aligned}$$

Simplifying, these yield the government-purchases Euler condition:

$$v'\left(G_1\right)=A_2\beta v'\left(G_2\right)$$

as stated in the paper.

We can then solve for G_1 and G_2. The government purchases Euler condition, using CRRA utility, is:

$$G_1\left(A_2\beta\right)^{\sigma}=G_2.$$

Combine this with full employment in the second period and the expression for C_2 to obtain:

$$\left(1-g_2\right)=\left(\frac{\frac{A_2(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}A_1K_1}{G_2+\frac{A_2(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}A_1K_1}\right).$$

This implies the following expression for first-period aggregate demand:

$$Y_1=\frac{1+\left(\frac{1}{\beta A_2}\right)^{\sigma}A_2\left(\frac{\frac{A_2(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}A_1K_1}{G_1(A_2\beta)^{\sigma}+\frac{A_2(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}A_1K_1}\right)}{\left(\frac{\frac{A_2(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}A_1K_1}{G_1(A_2\beta)^{\sigma}+\frac{A_2(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}A_1K_1}\right)}\frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}A_1K_1+G_1.$$

When full employment is reached, $Y_1=A_1K_1$, so this expression implies:

$$G_1=\frac{\left(\frac{1}{\beta A_2}\right)^{\sigma}A_2}{\left(1+\left(\frac{1}{\beta A_2}\right)^{\sigma}A_2\right)}\left(1-\frac{\left(1+\hat{\imath}_1\right)\left(1+\left(\frac{1}{\beta A_2}\right)^{\sigma}A_2\right)}{\left(1+\theta\right)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}\right)A_1K_1, \tag{1}$$

and the government-purchases Euler condition then implies:

$$G_2=\frac{A_2}{\left(1+\left(\frac{1}{\beta A_2}\right)^{\sigma}A_2\right)}\left(1-\frac{\left(1+\hat{\imath}_1\right)\left(1+\left(\frac{1}{\beta A_2}\right)^{\sigma}A_2\right)}{\left(1+\theta\right)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}\right)A_1K_1.$$

Finally, using these results and the expressions for the equilibrium consumption and investment levels under sticky prices, we obtain the expressions for equilibrium under sticky prices with optimal fiscal policy (when restricted to government purchases):

$$C_1^{sticky} = (1+\hat{\imath}_1)\frac{\left(\frac{1}{\beta A_2}\right)^{\sigma} A_2}{(1+\theta)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma}\hat{A}_2\right)} A_1 K_1,$$

$$C_2^{sticky} = (1+\hat{\imath}_1)\frac{A_2}{(1+\theta)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma}\hat{A}_2\right)} A_1 K_1,$$

$$I_1^{sticky} = K_2^{sticky} = \frac{1}{\left(1+\left(\frac{1}{\beta A_2}\right)^{\sigma} A_2\right)} A_1 K_1,$$

$$Y_1^{sticky} = A_1 K_1,$$

$$Y_2^{sticky} = \frac{A_2}{\left(1+\left(\frac{1}{\beta A_2}\right)^{\sigma} A_2\right)} A_1 K_1.$$

These imply the inequalities shown in Section 6.3 of the paper.

3.2 Investment subsidy (Section 6.4 of the paper)

The same pre-shock monetary and fiscal policy holds here as with government purchases, because the optimal pre-shock investment subsidy is zero. Therefore, as in the previous subsection:

$$\hat{g}_1 = \frac{\theta\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma}\hat{A}_2}{(1+\theta)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma}\hat{A}_2\right)},$$

$$\hat{g}_2 = \frac{\theta}{1+\theta},$$

and

$$\frac{\hat{M}_2}{P_1} = \frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma}\hat{A}_2\right)} A_1 K_1.$$

Using the same process as in the previous subsection, we can derive the following expressions for the equilibrium with general fiscal policy:

$$C_1 = \left(\frac{(1-s_1)}{\beta A_2}\right)^{\sigma}\frac{A_2}{(1-s_1)}\frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma}\hat{A}_2\right)} A_1 K_1,$$

$$C_2 = \frac{A_2}{(1-s_1)}\frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma}\hat{A}_2\right)} A_1 K_1$$

$$I_1 = K_2 = \frac{1}{(1-g_2)(1-s_1)}\frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma}\hat{A}_2\right)} A_1 K_1.$$

$$Y_1 = \frac{1+\left(\frac{(1-s_1)}{\beta A_2}\right)^{\sigma} A_2 (1-g_2)}{(1-g_2)}\frac{\frac{(1+\hat{\imath}_1)}{(1-s_1)}}{(1+\theta)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma}\hat{A}_2\right)} A_1 K_1 + g_1 A_1 K_1,$$

Using these results, we can state the government's problem:

$$\max_{\{g_t,\tau_t\}_{t=1}^{2},s}\left[\begin{array}{c} u\left(\left(\frac{(1-s_1)}{\beta A_2}\right)^{\sigma}\frac{A_2}{(1-s_1)}\frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}A_1K_1\right)+v\left(g_1A_1K_1\right) \\ +\beta\left(u\left(\frac{A_2}{(1-s_1)}\frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}A_1K_1\right)+v\left(g_2A_2\frac{1}{(1-g_2)(1-s_1)}\frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}A_1K_1\right)\right)\end{array}\right],$$

As in the previous subsection, the government's choice is subject to:

$$Y_1 \leq A_1K_1,$$

which can be written as:

$$\frac{1+\left(\frac{(1-s_1)}{\beta A_2}\right)^{\sigma}A_2\left(1-g_2\right)}{(1-g_2)}\frac{\frac{(1+\hat{\imath}_1)}{(1-s_1)}}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}A_1K_1+g_1A_1K_1\leq A_1K_1.$$

Start with the assumption that the constraint is slack. Then, the first-order condition for s_1 implies that the household's utility rises with s_1, as every argument of the household's welfare function rises with or is unaffected by s_1. This is why the paper states that a positive investment subsidy is welfare improving for unchanged or optimal government spending when output is below its full-employment level.

At full employment, we can show by construction that the flexible-price equilibrium is achievable when $\sigma \to 0$. The flexible-price values for government purchases are:

$$g_1^{flex}=\frac{\theta\left(\frac{1}{\beta A_2}\right)^{\sigma}A_2}{(1+\theta)\left(1+\left(\frac{1}{\beta A_2}\right)^{\sigma}A_2\right)},$$

$$1-g_2^{flex}=\frac{1}{1+\theta}$$

Substituting these into the constraint on the government yields:

$$\frac{\left(1+\theta+\left(\frac{(1-s_1)}{\beta A_2}\right)^{\sigma}A_2\right)\frac{(1+\hat{\imath}_1)}{(1-s_1)}}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}+\frac{\theta\left(\frac{1}{\beta A_2}\right)^{\sigma}A_2}{(1+\theta)\left(1+\left(\frac{1}{\beta A_2}\right)^{\sigma}A_2\right)}=1.$$

Now impose $\sigma \to 0$. This expression simplifies to:

$$(1-s_1)=\frac{(1+A_2)}{\left(1+\hat{A}_2\right)}\left(1+\hat{\imath}_1\right).$$

Recall that the optimal monetary policy sets:

$$\frac{M_2}{(1+i_1)P_1}=\frac{1}{(1+\theta)\left(1+\left(\frac{1}{\beta A_2}\right)^{\sigma}A_2\right)}A_1K_1.$$

When restricted to conventional monetary policy, we know that:

$$\frac{\hat{M}_2}{P_1}=\frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}A_1K_1.$$

This implies that the required nominal interest rate satisfies:

$$\frac{1}{(1+i_1)}\frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}A_1K_1=\frac{1}{(1+\theta)\left(1+\left(\frac{1}{\beta A_2}\right)^{\sigma}A_2\right)}A_1K_1$$

or, with $\sigma\to 0$,

$$(1+i_1)=\frac{(1+A_2)}{\left(1+\hat{A}_2\right)}(1+\hat{\imath}_1).$$

This yields the equivalence shown in expression (54) of the paper.

4 Unconventional Monetary Policy in a Model with Three Periods (Section 7 of the paper)

The analysis of this section follows the same process as that of the two-period model, so we omit the details and report the results instead. The post-shock equilibrium with optimal fiscal policy, when monetary policy is sufficient to restore the flexible-price equilibrium, can be summarized with the following set of equations, where $\frac{M_3}{P_1(1+i_1)(1+i_2)}$ denotes the post-shock monetary policy stance.

$$C_1=\left(\frac{1}{\beta^2A_2A_3}\right)^{\sigma}A_2A_3\frac{M_3}{P_1(1+i_1)(1+i_2)},$$

$$C_2=\left(\frac{1}{\beta A_3}\right)^{\sigma}A_2A_3\frac{M_3}{P_1(1+i_1)(1+i_2)},$$

$$C_3=A_2A_3\frac{M_3}{P_1(1+i_1)(1+i_2)},$$

$$I_1=K_2=\frac{1+\left(\frac{1}{\beta A_3}\right)^{\sigma}A_3(1-g_3)}{(1-g_2)(1-g_3)}\frac{M_3}{P_1(1+i_1)(1+i_2)},$$

$$I_2=K_3=\frac{A_2}{(1-g_3)}\frac{M_3}{P_1(1+i_1)(1+i_2)},$$

$$Y_1=\frac{1+\left(\frac{1}{\beta A_3}\right)^{\sigma}A_3(1-g_3)+\left(\frac{1}{\beta^2A_2A_3}\right)^{\sigma}A_2A_3(1-g_2)(1-g_3)}{(1-g_2)(1-g_3)(1-g_1)}\frac{M_3}{P_1(1+i_1)(1+i_2)},$$

$$P_2=\frac{(1+i_1)}{A_2}P_1,$$

$$P_3=\frac{(1+i_2)}{A_3}P.$$

The pre-shock optimal monetary policy satisfies:

$$\frac{\hat{M}_3}{P_1(1+\hat{\imath}_1)(1+\hat{\imath}_2)}=\frac{(1-\hat{g}_1)(1-\hat{g}_2)(1-\hat{g}_3)}{\left(1+\left(\frac{1}{\beta A_3}\right)^{\sigma}A_3(1-\hat{g}_3)+\left(\frac{1}{\beta^2\hat{A}_2A_3}\right)^{\sigma}\hat{A}_2A_3(1-\hat{g}_2)(1-\hat{g}_3)\right)}A_1K_1.$$

pre-shock optimal government spending levels, are determined by the optimality condition:

$$u'(C_t)=v'(G_t)\ \text{ for all } t.$$

Assuming CRRA utility from government purchases and private consumption, this condition and the results above imply:

$$\hat{g}_1 = \frac{\theta}{1+\theta} \frac{\left(\frac{1}{\beta^2 \hat{A}_2 A_3}\right)^{\sigma} \hat{A}_2 A_3}{\left(1 + \left(\frac{1}{\beta A_3}\right)^{\sigma} A_3 + \left(\frac{1}{\beta^2 \hat{A}_2 A_3}\right)^{\sigma} \hat{A}_2 A_3\right)},$$

$$\hat{g}_2 = \frac{\theta}{1+\theta} \frac{\left(\frac{1}{\beta A_3}\right)^{\sigma} A_3}{\left(1 + \left(\frac{1}{\beta A_3}\right)^{\sigma} A_3\right)},$$

$$\hat{g}_3 = \frac{\theta}{1+\theta}.$$

Naturally, full-employment monetary policy with the optimal fiscal policy in place after the shock is:

$$\frac{M_3}{P_1 (1+i_1)(1+i_2)} = \frac{(1-g_1)(1-g_2)(1-g_3)}{1 + \left(\frac{1}{\beta A_3}\right)^{\sigma} A_3 (1-g_3) + \left(\frac{1}{\beta^2 A_2 A_3}\right)^{\sigma} A_2 A_3 (1-g_2)(1-g_3)} A_1 K_1,$$

where the post-shock optimal levels of government spending are equal to the flexible-price levels after the shock.

The conventional monetary policy response to weak aggregate demand is to lower the short-term nominal interest rate i_1. For now, assume that this conventional response is the monetary authority's only response, so that the future short-term interest rate i_2 and the long-term money supply M_3 remain unchanged. Fiscal policy is at its flexible-price optimum. With these assumptions, we can solve for first-period output after the shock:

$$Y_1 = \frac{1 + \left(\frac{1}{\beta A_3}\right)^{\sigma} A_3 (1-\hat{g}_3) + \left(\frac{1}{\beta^2 \hat{A}_2 A_3}\right)^{\sigma} \hat{A}_2 A_3 (1-\hat{g}_2)(1-\hat{g}_3)}{1 + \left(\frac{1}{\beta A_3}\right)^{\sigma} A_3 (1-g_3) + \left(\frac{1}{\beta^2 A_2 A_3}\right)^{\sigma} A_2 A_3 (1-g_2)(1-g_3)} \frac{(1+\hat{\imath}_1)}{(1+i_1)} \frac{(1-g_1)(1-g_2)(1-g_3)}{(1-\hat{g}_1)(1-\hat{g}_2)(1-\hat{g}_3)} A_1 K_1.$$

Substitute in the following expressions for optimal fiscal policy:

$$1 - g_1 = \frac{(1+\theta)\left(1 + \left(\frac{1}{\beta A_3}\right)^{\sigma} A_3\right) + \left(\frac{1}{\beta^2 A_2 A_3}\right)^{\sigma} A_2 A_3}{(1+\theta)\left(1 + \left(\frac{1}{\beta A_3}\right)^{\sigma} A_3 + \left(\frac{1}{\beta^2 A_2 A_3}\right)^{\sigma} A_2 A_3\right)},$$

$$1 - g_2 = \frac{1 + \theta + \left(\frac{1}{\beta A_3}\right)^{\sigma} A_3}{(1+\theta)\left(1 + \left(\frac{1}{\beta A_3}\right)^{\sigma} A_3\right)},$$

$$1 - g_3 = \frac{1}{1+\theta},$$

so

$$Y_1 = \frac{(1+\theta)\left(1 + \left(\frac{1}{\beta A_3}\right)^{\sigma} A_3 + \left(\frac{1}{\beta^2 A_2 A_3}\right)^{\sigma} A_2 A_3\right)}{(1+\theta)\left(1 + \left(\frac{1}{\beta A_3}\right)^{\sigma} A_3 + \left(\frac{1}{\beta^2 \hat{A}_2 A_3}\right)^{\sigma} \hat{A}_2 A_3\right)} \frac{(1+\hat{\imath}_1)(1+\hat{\imath}_2)}{(1+i_1)(1+i_2)} A_1 K_1.$$

As in the baseline model analysis, manipulating this expression yields a threshold value for A_2 above which conventional policy (changing i_1 by setting $i_1 = 0$ without changing i_2) is sufficient to restore the flexible-price equilibrium. We denote this threshold $A_2|_{conventional}$, and it is:

$$A_2|_{conventional} = \left(\frac{\left(\frac{1}{\beta^2 \hat{A}_2 A_3}\right)^{\sigma} \hat{A}_2 A_3 - \hat{\imath}_1 \left(1 + \left(\frac{1}{\beta A_3}\right)^{\sigma} A_3\right)}{(1+i_1)\left(\frac{1}{\beta^2}\right)^{\sigma} (A_3)^{1-\sigma}} \right)^{\frac{1}{1-\sigma}}.$$

If A_2 falls below $A_2|_{conventional}$, the monetary authority will be unable to obtain the flexible-price equilibrium with conventional monetary policy. It will set $i_1 = 0$, its zero lower bound, but output will be below potential. In the baseline model, the monetary authority could stimulate the economy further only by affecting the long-term level of the money supply. With three periods, however, the future short-term interest rate i_2 can be used to provide monetary stimulus.

Formally, suppose $A_2 < A_2|_{conventional}$, $i_1 = 0$, and $M_3 = \hat{M}_3$, the last of which reflects that the monetary authority holds the long-term money supply unchanged in this scenario. Output is then

$$Y_1 = \frac{(1+\theta)\left(1+\left(\frac{1}{\beta A_3}\right)^{\sigma} A_3 + \left(\frac{1}{\beta^2 A_2 A_3}\right)^{\sigma} A_2 A_3\right)}{(1+\theta)\left(1+\left(\frac{1}{\beta A_3}\right)^{\sigma} A_3 + \left(\frac{1}{\beta^2 \hat{A}_2 A_3}\right)^{\sigma} \hat{A}_2 A_3\right)} (1+\hat{\imath}_1) \frac{(1+\hat{\imath}_2)}{(1+i_2)} A_1 K_1.$$

Manipulating this expression yields a new threshold value for A_2 above which the unconventional monetary policy that affects the future short-term interest rate is effective. That is, we set $i_2 = 0$ and solve for full employment. This yields:

$$A_2|_{long-term-interest} = \left(\frac{\left(\frac{1}{\beta^2 \hat{A}_2 A_3}\right)^{\sigma} \hat{A}_2 A_3 - \left((1+\hat{\imath}_1)(1+\hat{\imath}_2) - 1\right)\left(1+\left(\frac{1}{\beta A_3}\right)^{\sigma} A_3\right)}{(1+\hat{\imath}_1)(1+\hat{\imath}_2)\left(\frac{1}{\beta^2}\right)^{\sigma}(A_3)^{1-\sigma}}\right)^{\frac{1}{1-\sigma}}.$$

Comparing these threshold expressions implies

$$A_2|_{long-term-interest} < A_2|_{conventional},$$

establishing that the addition of a future short-term interest rate as a policy instrument allows the monetary authority to unilaterally restore the flexible-price equilibrium for larger negative shocks to aggregate demand.

As in the baseline model, if the shock is even larger, the central bank may always adjust the long-term money supply, here M_3, to restore the flexible-price equilibrium.

5 Government Investment (Section 8 of the paper)

In this section, government expenditures may include investment in productive capital K_t^G. The private sector is unchanged from the baseline model.

Even with no investment subsidy, fiscal policy now has three components: g_t^C is the share of full-employment output that goes to government consumption purchases in period t, τ_t is the share of full-employment output collected as lumpsum tax revenue, and $g_t^I = \frac{G_t^I}{Y_t}$ is the share of output going to government investment. The government's new budget constraint is:

$$P_1\left(T_1 - G_1^C - G_t^I\right) + \frac{1}{1+i_1} P_2\left(T_2 - G_2^C\right) = 0.$$

Aggregate demand now includes government investment:

$$Y_t = C_t + I_t + G_t^C + G_t^I \text{ for all } t,$$

and aggregate supply is now a function of both private capital K_t^F and public capital K_t^G:

$$Y_t \leq \left(A_t^F K_t^F + A_t^G \kappa\left(K_t^G\right)\right),$$

where the function $\kappa(\cdot)$ reflects that the two forms of capital are not perfect substitutes in production.

The equilibrium can be derived as follows. The result of the firm's profit maximization is unchanged:

$$\frac{u'(C_1)}{\beta u'(C_2)} = (1+i_1)\frac{P_1}{P_2},$$

as is the household's Euler equation:

$$A_2^F = (1+i_1)\frac{P_1}{P_2},$$

while the remaining equations modify the baseline model in a natural way:

$$\left(A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right)\right)\left(1-g_1^C\right) - K_2^F - K_2^G = C_1,$$

$$\left(A_2^F K_2^F + A_2^G \kappa\left(K_2^G\right)\right)\left(1-g_2^C\right) = C_2,$$

where

$$G_2^C = (1+i_1)\frac{P_1}{P_2}\left(T_1 - G_1^C - K_2^G\right) + T_2.$$

Solving as in the baseline model yields:

$$K_2^F = \frac{\left(A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right)\right)\left(1-g_1^C\right) - \left(\frac{1}{A_2^F \beta}\right)^\sigma A_2^G\left(1-g_2^C\right)\kappa\left(K_2^G\right) - K_2^G}{\left(1+\left(\frac{1}{A_2^F\beta}\right)^\sigma A_2^F\left(1-g_2^C\right)\right)}.$$

Then,

$$C_1 = \left(\frac{1}{A_2^F\beta}\right)^\sigma A_2^F\left(1-g_2^C\right)\frac{\left(A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right)\right)\left(1-g_1^C\right) + \frac{A_2^G}{A_2^F}\kappa\left(K_2^G\right) - K_2^G}{\left(1+\left(\frac{1}{A_2^F\beta}\right)^\sigma A_2^F\left(1-g_2^C\right)\right)},$$

$$C_2 = A_2^F\left(1-g_2^C\right)\left(\frac{\left(A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right)\right)\left(1-g_1^C\right) + \frac{A_2^G}{A_2^F}\kappa\left(K_2^G\right) - K_2^G}{\left(1+\left(\frac{1}{A_2^F\beta}\right)^\sigma A_2^F\left(1-g_2^C\right)\right)}\right),$$

and

$$\left(A_2^F K_2^F + A_2^G \kappa\left(K_2^G\right)\right) = \frac{A_2^F\left(A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right)\right)\left(1-g_1^C\right) + A_2^G\kappa\left(K_2^G\right) - A_2^F K_2^G}{\left(1+\left(\frac{1}{A_2^F\beta}\right)^\sigma A_2^F\left(1-g_2^C\right)\right)}.$$

The government's problem is then

$$\max_{K_2^G, g_1^C, g_2^C}\left[\begin{array}{c} u\left(\left(\frac{1}{A_2^F\beta}\right)^\sigma A_2^F\left(1-g_2^C\right)\frac{\left(A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right)\right)\left(1-g_1^C\right) + \frac{A_2^G}{A_2^F}\kappa\left(K_2^G\right) - K_2^G}{\left(1+\left(\frac{1}{A_2^F\beta}\right)^\sigma A_2^F\left(1-g_2^C\right)\right)}\right) + v\left(g_1^C\left(A_1^F K_1^F + A_1^G\kappa\left(K_1^G\right)\right)\right) \\ +\beta\left(\begin{array}{c} u\left(A_2^F\left(1-g_2^C\right)\left(\frac{\left(A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right)\right)\left(1-g_1^C\right) + \frac{A_2^G}{A_2^F}\kappa\left(K_2^G\right) - K_2^G}{\left(1+\left(\frac{1}{A_2^F\beta}\right)^\sigma A_2^F\left(1-g_2^C\right)\right)}\right)\right) \\ +v\left(g_2^C A_2^F\left(\frac{\left(A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right)\right)\left(1-g_1^C\right) + \frac{A_2^G}{A_2^F}\kappa\left(K_2^G\right) - K_2^G}{\left(1+\left(\frac{1}{A_2^F\beta}\right)^\sigma A_2^F\left(1-g_2^C\right)\right)}\right)\right)\end{array}\right)\end{array}\right].$$

The first-order conditions of this problem are:

$$FOC_{K_2^G}:\left[\begin{array}{c} u'\left(C_1\right)\left(\frac{1}{A_2^F\beta}\right)^\sigma A_2^F\left(1-g_2^C\right) \\ +\beta\left(u'\left(C_2\right)A_2^F\left(1-g_2^C\right) + v'\left(G_2^C\right)g_2^C A_2^F\right)\end{array}\right]\left(\frac{\frac{A_2^G}{A_2^F}\kappa'\left(K_2^G\right) - 1}{\left(1+\left(\frac{1}{A_2^F\beta}\right)^\sigma A_2^F\left(1-g_2^C\right)\right)}\right) = 0,$$

$$
\begin{aligned}
FOC_{g_1^C} \quad &: \quad \left[u'\left(C_1\right)\left(\frac{1}{A_2^F\beta}\right)^{\sigma} A_2^F\left(1-g_2^C\right)+\beta\left(u'\left(C_2\right)A_2^F\left(1-g_2^C\right)+v'\left(G_2^C\right)g_2^C A_2^F\right)\right] \\
&\quad * \frac{A_1^F K_1^F + A_1^G\kappa\left(K_1^G\right)}{\left(1+\left(\frac{1}{A_2^F\beta}\right)^{\sigma} A_2^F\left(1-g_2^C\right)\right)} \\
&= \quad v'\left(G_1^C\right)\left(A_1^F K_1^F + A_1^G\kappa\left(K_1^G\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
FOC_{g_2^C} \quad &: \quad \left[u'\left(C_1\right)\left(\frac{1}{A_2^F\beta}\right)^{\sigma} A_2^F+\beta u'\left(C_2\right)A_2^F-\beta v'\left(G_2^C\right)A_2^F\right] \\
&\quad * \left[\left(\frac{\left(A_1^F K_1^F + A_1^G\kappa\left(K_1^G\right)\right)\left(1-g_1^C\right)+\frac{A_2^G}{A_2^F}\kappa\left(K_2^G\right)-K_2^G}{\left(1+\left(\frac{1}{A_2^F\beta}\right)^{\sigma} A_2^F\left(1-g_2^C\right)\right)}\right)\right. \\
&\qquad \left. -\left(\frac{1}{A_2^F\beta}\right)^{\sigma} A_2^F\left(1-g_2^C\right)\left(\frac{\left(A_1^F K_1^F + A_1^G\kappa\left(K_1^G\right)\right)\left(1-g_1^C\right)+\frac{A_2^G}{A_2^F}\kappa\left(K_2^G\right)-K_2^G}{\left(1+\left(\frac{1}{A_2^F\beta}\right)^{\sigma} A_2^F\left(1-g_2^C\right)\right)^2}\right)\right] \\
= \quad &\beta v'\left(G_2^C\right)A_2^F\left(\frac{1}{A_2^F\beta}\right)^{\sigma} A_2^F\left(\frac{\left(A_1^F K_1^F + A_1^G\kappa\left(K_1^G\right)\right)\left(1-g_1^C\right)+\frac{A_2^G}{A_2^F}\kappa\left(K_2^G\right)-K_2^G}{\left(1+\left(\frac{1}{A_2^F\beta}\right)^{\sigma} A_2^F\left(1-g_2^C\right)\right)^2}\right).
\end{aligned}
$$

The first of these conditions implies, assuming that marginal utilities of consumption and government services are always positive,

$$\kappa'\left(K_2^G\right)=\frac{A_2^F}{A_2^G}.$$

The second implies, using the household's Euler condition $\left(\frac{u'(C_1)}{\beta u'(C_2)}=A_2^F\right)$,

$$\left(1+\left(\frac{1}{A_2^F\beta}\right)^{\sigma} A_2^F\right)\beta u'\left(C_2\right)A_2^F\left(1-g_2^C\right)=\left(1+\left(\frac{1}{A_2^F\beta}\right)^{\sigma} A_2^F\left(1-g_2^C\right)\right)v'\left(G_1^C\right)-\beta A_2^F g_2^C v'\left(G_2^C\right).$$

This holds when the following conditions are satisfied:

$$\frac{v'\left(G_1^C\right)}{\beta v'\left(G_2^C\right)}=A_2^F,$$

$$u'\left(C_2\right)=v'\left(G_2^C\right),$$

and

$$u'\left(C_1\right)=v'\left(G_1^C\right).$$

Therefore, the classical conditions continue to hold in the flexible-price equilibrium.

Using the classical conditions just derived and CRRA utility of both government and private consumption, we obtain:

$$\frac{A_2^F\left(A_1^F K_1^F + A_1^G\kappa\left(K_1^G\right)\right)\left(1-g_1^C\right)+A_2^G\kappa\left(K_2^G\right)-A_2^F K_2^G-\theta C_2}{A_2^F\left(A_1^F K_1^F + A_1^G\kappa\left(K_1^G\right)\right)\left(1-g_1^C\right)+A_2^G\kappa\left(K_2^G\right)-A_2^F K_2^G+\left(\frac{1}{A_2^F\beta}\right)^{\sigma} A_2^F\left(C_2\right)\theta}=\left(1-g_2^C\right),$$

$$\frac{A_1^F K_1^F + A_1^G\kappa\left(K_1^G\right)-\theta C_1}{\left(A_1^F K_1^F + A_1^G\kappa\left(K_1^G\right)\right)}=1-g_1^C.$$

Combining these,

$$\frac{A_2^F\left(A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right)\right)\left(1-g_1^C\right) + A_2^G \kappa\left(K_2^G\right) - A_2^F K_2^G - \theta C_2}{A_2^F\left(A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right) - \theta C_1\right) + A_2^G \kappa\left(K_2^G\right) - A_2^F K_2^G + \left(\frac{1}{A_2^F \beta}\right)^{\sigma} A_2^F \theta C_2} = \left(1-g_2^C\right),$$

and we can then derive:

$$C_1 = \left(\frac{1}{A_2^F \beta}\right)^{\sigma} A_2^F \left(1-g_2^C\right) \frac{A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right) + \frac{A_2^G}{A_2^F}\kappa\left(K_2^G\right) - K_2^G}{1 + \left(\frac{1}{A_2^F \beta}\right)^{\sigma} A_2^F \left(1-g_2^C\right)(1+\theta)}.$$

Next, given that:

$$C_2 = \frac{C_1}{\left(\frac{1}{A_2^F \beta}\right)^{\sigma}},$$

and

$$\left(A_2^F K_2^F + A_2^G \kappa\left(K_2^G\right)\right) = \frac{A_2^F C_1}{\left(\frac{1}{A_2^F \beta}\right)^{\sigma} A_2^F \left(1-g_2^C\right)},$$

we can write:

$$C_2 = A_2^F \left(1-g_2^C\right) \frac{A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right) + \frac{A_2^G}{A_2^F}\kappa\left(K_2^G\right) - K_2^G}{1 + \left(\frac{1}{A_2^F \beta}\right)^{\sigma} A_2^F \left(1-g_2^C\right)(1+\theta)},$$

and

$$\left(A_2^F K_2^F + A_2^G \kappa\left(K_2^G\right)\right) = \frac{A_2^F\left(A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right)\right) + A_2^G \kappa\left(K_2^G\right) - A_2^F K_2^G}{1 + \left(\frac{1}{A_2^F \beta}\right)^{\sigma} A_2^F \left(1-g_2^C\right)(1+\theta)}.$$

We can show that this implies:

$$\left(1-g_2^C\right) = \frac{1}{1+\theta},$$

as then the following expressions obtain:

$$C_1 = \frac{\left(\frac{1}{A_2^F \beta}\right)^{\sigma} A_2^F}{(1+\theta)\left(1 + \left(\frac{1}{A_2^F \beta}\right)^{\sigma} A_2^F\right)}\left(A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right) + \frac{A_2^G}{A_2^F}\kappa\left(K_2^G\right) - K_2^G\right),$$

$$C_2 = \frac{A_2^F}{(1+\theta)\left(1 + \left(\frac{1}{A_2^F \beta}\right)^{\sigma} A_2^F\right)}\left(A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right) + \frac{A_2^G}{A_2^F}\kappa\left(K_2^G\right) - K_2^G\right),$$

$$\left(A_2^F K_2^F + A_2^G \kappa\left(K_2^G\right)\right) = \frac{A_2^F}{1 + \left(\frac{1}{A_2^F \beta}\right)^{\sigma} A_2^F}\left(A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right) + \frac{A_2^G}{A_2^F}\kappa\left(K_2^G\right) - K_2^G\right),$$

and by the optimality conditions:

$$G_1^C = \theta \frac{\left(\frac{1}{A_2^F \beta}\right)^{\sigma} A_2^F}{(1+\theta)\left(1 + \left(\frac{1}{A_2^F \beta}\right)^{\sigma} A_2^F\right)}\left(A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right) + \frac{A_2^G}{A_2^F}\kappa\left(K_2^G\right) - K_2^G\right),$$

$$G_2^C = \theta \frac{A_2^F}{(1+\theta)\left(1 + \left(\frac{1}{A_2^F \beta}\right)^{\sigma} A_2^F\right)}\left(A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right) + \frac{A_2^G}{A_2^F}\kappa\left(K_2^G\right) - K_2^G\right),$$

so that

$$g_2 = \frac{\theta}{1+\theta}.$$

5.1 Sticky prices

The expressions describing equilibrium are:

$$Y_t \leq \left(A_t^F K_t^F + A_t^G \kappa\left(K_t^G\right)\right),$$

$$\frac{u'\left(C_1\right)}{\beta u'\left(C_2\right)} = \left(1+i_1\right)\frac{P_1}{P_2},$$

$$A_2^F = \left(1+i_1\right)\frac{P_1}{P_2},$$

$$Y_1 - G_1^C - K_2^F - K_2^G = C_1,$$

$$\left(A_2^F K_2^F + A_2^G \kappa\left(K_2^G\right)\right)\left(1-g_2^C\right) = C_2,$$

$$G_2^C = \left(1+i_1\right)\frac{P_1}{P_2}\left(T_1 - G_1^C - K_2^G\right) + T_2.$$

Suppose technology is shocked such that conventional monetary policy is insufficient to restore the flexible-price equilibrium. Then $i_1 = 0$ and the Euler and profit-maximization conditions together give us:

$$C_2 = \left(A_2^F \beta\right)^\sigma C_1,$$

$$P_2 = \frac{1}{A_2^F} P_1.$$

As before, we can use the sum of nominal consumptions across the two periods to generate the following expression for K_2^F:

$$K_2^F = \frac{1}{\left(\frac{1}{\beta A_2^F}\right)^\sigma A_2^F\left(1-g_2^C\right)} C_1 - \frac{A_2^G}{A_2^F}\kappa\left(K_2^G\right).$$

The equilibrium values are

$$C_1 = \left(\frac{1}{\beta A_2^F}\right)^\sigma A_2^F \frac{M_2}{P_1},$$

$$C_2 = A_2^F \frac{M_2}{P_1},$$

$$K_2^F = \frac{1}{\left(1-g_2^C\right)}\frac{M_2}{P_1} - \frac{A_2^G}{A_2^F}\kappa\left(K_2^G\right),$$

$$I_1 + G_1^I = K_2^F + K_2^G = \frac{1}{\left(1-g_2^C\right)}\frac{M_2}{P_1} + K_2^G - \frac{A_2^G}{A_2^F}\kappa\left(K_2^G\right),$$

$$P_2 = \frac{1}{A_2^F} P_1,$$

$$Y_1 = \frac{1 + \left(\frac{1}{\beta A_2^F}\right)^\sigma A_2^F\left(1-g_2^C\right)}{\left(1-g_2^C\right)}\frac{M_2}{P_1} + K_2^G - \frac{A_2^G}{A_2^F}\kappa\left(K_2^G\right) + G_1^C.$$

The government's problem is as follows, once full employment is reached and where $\frac{\hat{M}_2}{P_1}$ is pre-shock monetary policy.

$$\max_{K_2^G, g_1^C, g_2^C} \left[\begin{array}{c} u\left(\left(\frac{1}{\beta A_2^F}\right)^{\sigma} A_2^F \frac{\hat{M}_2}{P_1}\right) + v\left(g_1^C\left(A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right)\right)\right) \\ +\beta\left(u\left(A_2^F \frac{\hat{M}_2}{P_1}\right) + v\left(g_2^C A_2^F \frac{1}{\left(1-g_2^C\right)} \frac{\hat{M}_2}{P_1}\right)\right) \end{array} \right],$$

subject to:

$$\left(A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right)\right) = \frac{1}{\left(1-g_1^C\right)} \left(\frac{1+\left(\frac{1}{\beta A_2^F}\right)^{\sigma} A_2^F\left(1-g_2^C\right)}{\left(1-g_2^C\right)} \frac{\hat{M}_2}{P_1} + K_2^G - \frac{A_2^G}{A_2^F} \kappa\left(K_2^G\right) \right),$$

and subject to fixed monetary policy $\frac{\hat{M}_2}{P_1}$.

The government's first-order conditions are:

$$FOC_{K_2^G} : \frac{\frac{A_2^G}{A_2^F} \kappa'\left(K_2^G\right) - 1}{\left(1-g_1^C\right)} \lambda = 0,$$

implying

$$\kappa'\left(K_2^G\right) = \frac{A_2^F}{A_2^G};$$

$$FOC_{g_1^C} : v'\left(G_1^C\right) \frac{\left(A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right)\right)\left(1-g_2^C\right)}{\left(1+\left(\frac{1}{\beta A_2^F}\right)^{\sigma} A_2^F\left(1-g_2^C\right)\right) \frac{\hat{M}_2}{P_1} + \left(K_2^G - \frac{A_2^G}{A_2^F}\kappa\left(K_2^G\right)\right)\left(1-g_2^C\right)} \left(1-g_1^C\right)^2 = \lambda,$$

and

$$FOC_{g_2^C} : \beta v'\left(G_2^C\right) A_2^F = \lambda \frac{1}{\left(1-g_1^C\right)}.$$

These imply:

$$FOC_{g_1^C} : v'\left(G_1^C\right)\left(1-g_1^C\right) = \lambda,$$

$$FOC_{g_2^C} : \beta v'\left(G_2^C\right) A_2^F \left(1-g_1^C\right) = \lambda,$$

and thus

$$v'\left(G_1^C\right) = \beta A_2^F v'\left(G_2^C\right),$$

the classical condition.

We can show that the private-public intratemporal optimality conditions fail (i.e., that the strict inequalities in Section 8 of the paper obtain). Recall that the government solves:

$$\max_{K_2^G, g_1^C, g_2^C} \left[\begin{array}{c} u\left(\left(\frac{1}{\beta A_2^F}\right)^{\sigma} A_2^F \frac{\hat{M}_2}{P_1}\right) + v\left(g_1^C\left(A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right)\right)\right) \\ +\beta\left(u\left(A_2^F \frac{\hat{M}_2}{P_1}\right) + v\left(g_2^C A_2^F \frac{1}{\left(1-g_2^C\right)} \frac{\hat{M}_2}{P_1}\right)\right) \end{array} \right],$$

subject to:

$$\left(A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right)\right) = \frac{1}{\left(1-g_1^C\right)} \left(\frac{1+\left(\frac{1}{\beta A_2^F}\right)^{\sigma} A_2^F\left(1-g_2^C\right)}{\left(1-g_2^C\right)} \frac{\hat{M}_2}{P_1} + K_2^G - \frac{A_2^G}{A_2^F} \kappa\left(K_2^G\right) \right),$$

If this constraint does not bind, the first-order condition for second-period government spending is:

$$FOC_{g_2^C} : \beta v'\left(G_2^C\right) A_2^F = 0.$$

As the marginal utility of G_2^C is always positive, this implies that the constraint will always bind.

When government purchases cause full employment, this constraint binds, and the first-order conditions for government purchases are:

$$FOC_{g_1^C} : v'\left(G_1^C\right) = \lambda,$$

$$FOC_{g_2^C} : \beta v'\left(G_2^C\right) A_2^F = \lambda,$$

and thus

$$v'\left(G_1^C\right) = \beta A_2^F v'\left(G_2^C\right),$$

as stated in the paper.

We can then solve for G_1 and G_2. The government purchases Euler condition, using CRRA utility, is:

$$G_1^C \left(A_2^F \beta\right)^\sigma = G_2^C.$$

Combine full employment in the second period

$$C_2 + G_2^C = \left(A_2^F K_2^F + A_2^G \kappa\left(K_2^G\right)\right),$$

and the expressions

$$C_2 = A_2^F \frac{\hat{M}_2}{P_1},$$

and

$$K_2^F = \frac{1}{\left(1 - g_2^C\right)} \frac{\hat{M}_2}{P_1} - \frac{A_2^G}{A_2^F} \kappa\left(K_2^G\right),$$

to obtain:

$$A_2^F \frac{\hat{M}_2}{P_1} + G_2^C = \left(A_2^F \left(\frac{1}{\left(1 - g_2^C\right)} \frac{\hat{M}_2}{P_1} - \frac{A_2^G}{A_2^F} \kappa\left(K_2^G\right)\right) + A_2^G \kappa\left(K_2^G\right)\right),$$

or

$$\left(1 - g_2^C\right) = \frac{A_2^F \frac{\hat{M}_2}{P_1}}{G_2^C + A_2^F \frac{\hat{M}_2}{P_1}}.$$

This, plus the government spending Euler condition, implies that the following expression for first-period aggregate demand:

$$Y_1 = \frac{1 + \left(\frac{1}{\beta A_2^F}\right)^\sigma A_2^F \left(1 - g_2^C\right)}{\left(1 - g_2^C\right)} \frac{\hat{M}_2}{P_1} + K_2^G - \frac{A_2^G}{A_2^F} \kappa\left(K_2^G\right) + G_1^C,$$

can be written as:

$$Y_1 = \frac{1 + \left(\frac{1}{\beta A_2^F}\right)^\sigma A_2^F \frac{A_2^F \frac{\hat{M}_2}{P_1}}{G_1^C \left(A_2^F \beta\right)^\sigma + A_2^F \frac{\hat{M}_2}{P_1}}}{\frac{A_2^F \frac{\hat{M}_2}{P_1}}{G_1^C \left(A_2^F \beta\right)^\sigma + A_2^F \frac{\hat{M}_2}{P_1}}} \frac{\hat{M}_2}{P_1} + G_1^C + K_2^G - \frac{A_2^G}{A_2^F} \kappa\left(K_2^G\right),$$

Note that pre-shock full-employment monetary policy is:

$$\frac{\hat{M}_2}{P_1} = \left(\left(\frac{1 + \hat{\imath}_1}{(1+\theta)\left(1 + \left(\frac{1}{\hat{A}_2^F \beta}\right)^\sigma \hat{A}_2^F\right)}\right) \left(A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right) + \frac{A_2^G}{A_2^F} \kappa\left(K_2^G\right) - K_2^G\right)\right)$$

where we substituted in for pre-shock optimal government spending:

$$\hat{G}_1^C = \theta \frac{\left(\frac{1}{\hat{A}_2^F \beta}\right)^\sigma \hat{A}_2^F}{(1+\theta)\left(1+\left(\frac{1}{\hat{A}_2^F \beta}\right)^\sigma \hat{A}_2^F\right)} \left(A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right) + \frac{A_2^G}{\hat{A}_2^F}\kappa\left(K_2^G\right) - K_2^G\right),$$

and

$$\left(1-\hat{g}_2^C\right) = \frac{1}{1+\theta}.$$

When full employment is reached, $Y_1 = \left(A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right)\right)$, so we set this equal to the expression for aggregate demand, using the expression for $\frac{\hat{M}_2}{P_1}$. After some simplification, this yields an expression for first-period government consumption:

$$G_1^C = \frac{1}{\left(1+\frac{1}{A_2^F}\left(A_2^F\beta\right)^\sigma\right)} \left(1 - \frac{(1+\hat{\imath}_1)\left(1+\left(\frac{1}{\beta A_2^F}\right)^\sigma A_2^F\right)}{(1+\theta)\left(1+\left(\frac{1}{\hat{A}_2^F \beta}\right)^\sigma \hat{A}_2^F\right)}\right) \left(A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right) + \frac{A_2^G}{A_2^F}\kappa\left(K_2^G\right) - K_2^G\right).$$

The government-purchases Euler condition then implies:

$$G_2^C = \frac{\left(A_2^F\beta\right)^\sigma}{\left(1+\frac{1}{A_2^F}\left(A_2^F\beta\right)^\sigma\right)} \left(1 - \frac{(1+\hat{\imath}_1)\left(1+\left(\frac{1}{\beta A_2^F}\right)^\sigma A_2^F\right)}{(1+\theta)\left(1+\left(\frac{1}{\hat{A}_2^F \beta}\right)^\sigma \hat{A}_2^F\right)}\right) \left(A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right) + \frac{A_2^G}{A_2^F}\kappa\left(K_2^G\right) - K_2^G\right)$$

Finally, we compare these expressions to the values of consumption, using the expression for $\frac{\hat{M}_2}{P_1}$:

$$C_1 = \left(\frac{1}{\beta A_2^F}\right)^\sigma A_2^F \left(\left(\frac{1+\hat{\imath}_1}{(1+\theta)\left(1+\left(\frac{1}{\hat{A}_2^F \beta}\right)^\sigma \hat{A}_2^F\right)}\right) \left(A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right) + \frac{A_2^G}{A_2^F}\kappa\left(K_2^G\right) - K_2^G\right)\right),$$

$$C_2 = A_2^F \left(\left(\frac{1+\hat{\imath}_1}{(1+\theta)\left(1+\left(\frac{1}{\hat{A}_2^F \beta}\right)^\sigma \hat{A}_2^F\right)}\right) \left(A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right) + \frac{A_2^G}{A_2^F}\kappa\left(K_2^G\right) - K_2^G\right)\right).$$

We are trying to show that

$$u'\left(C_1\right) > v'\left(G_1^C\right).$$

With the CRRA utility function, this holds if:

$$C_1 < \frac{1}{\theta} G_1^C.$$

In turn, this inequality holds if:

$$\begin{aligned} &\left(\frac{1}{\beta A_2^F}\right)^\sigma A_2^F \left(\left(\frac{1+\hat{\imath}_1}{(1+\theta)\left(1+\left(\frac{1}{\hat{A}_2^F \beta}\right)^\sigma \hat{A}_2^F\right)}\right) \left(A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right) + \frac{A_2^G}{A_2^F}\kappa\left(K_2^G\right) - K_2^G\right)\right) \\ < \;& \frac{1}{\theta} \frac{1}{\left(1+\frac{1}{A_2^F}\left(A_2^F\beta\right)^\sigma\right)} \left(1 - \frac{(1+\hat{\imath}_1)\left(1+\left(\frac{1}{\beta A_2^F}\right)^\sigma A_2^F\right)}{(1+\theta)\left(1+\left(\frac{1}{\hat{A}_2^F \beta}\right)^\sigma \hat{A}_2^F\right)}\right) \left(A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right) + \frac{A_2^G}{A_2^F}\kappa\left(K_2^G\right) - K_2^G\right). \end{aligned}$$

Simplifying, the intended inequality holds if:

$$(1+\hat{\imath}_1) < \frac{1+\left(\frac{1}{\hat{A}_2^F\beta}\right)^{\sigma}\hat{A}_2^F}{1+\left(\frac{1}{\beta A_2^F}\right)^{\sigma}A_2^F}.$$

We can rearrange this inequality to be:

$$A_2^F < \left(\frac{\left(\frac{1}{\hat{A}_2^F\beta}\right)^{\sigma}\hat{A}_2^F-\hat{\imath}_1}{\left(\frac{1}{\beta}\right)^{\sigma}(1+\hat{\imath}_1)}\right)^{\frac{1}{1-\sigma}}.$$

This is exactly the threshold for conventional policy to be insufficient, analogous to the expression in Section 5.1 of the paper. Thus, the inequality we were trying to prove,

$$u'(C_1) > v'\left(G_1^C\right),$$

Combined with

$$v'\left(G_1^C\right) = \beta A_2^F v'\left(G_2^C\right),$$

$$A_2^F\beta u'(C_2) = u'(C_1),$$

this inequality implies:

$$u'(C_2) > v'\left(G_2^C\right).$$

Thus, the private-public optimality conditions that hold in the flexible-price equilibrium do not hold in the case of sticky prices.

5.1.1 An investment subsidy

Including an investment subsidy on private investment, the government's problem is, once full employment is reached:

$$\max_{K_2^G,g_1^C,g_2^C}\left[\begin{array}{c} u\left(\left(\frac{1-s}{\beta A_2^F}\right)^{\sigma}\frac{A_2^F}{1-s}\frac{\hat{M}_2}{P_1}\right)+v\left(g_1^C\left(A_1^FK_1^F+A_1^G\kappa\left(K_1^G\right)\right)\right) \\ +\beta\left(u\left(\frac{A_2^F}{1-s}\frac{\hat{M}_2}{P_1}\right)+v\left(g_2^CA_2^F\frac{1}{\left(1-g_2^C\right)(1-s)}\frac{\hat{M}_2}{P_1}\right)\right)\end{array}\right],$$

subject to:

$$\left(A_1^FK_1^F+A_1^G\kappa\left(K_1^G\right)\right) = \frac{1+\left(\frac{1-s}{\beta A_2^F}\right)^{\sigma}A_2^F\left(1-g_2^C\right)}{\left(1-g_2^C\right)(1-s)}\frac{\hat{M}_2}{P_1}+K_2^G-\frac{A_2^G}{A_2^F}\kappa\left(K_2^G\right)+G_1^C.$$

Again, we look for whether, when $\sigma\to 0$, the investment subsidy can generate the flexible-price equilibrium. Impose this limit condition on the constraint to obtain:

$$(1-s) = \frac{1+A_2^F\left(1-g_2^C\right)}{\left(1-g_2^C\right)\left(\left(A_1^FK_1^F+A_1^G\kappa\left(K_1^G\right)\right)+\frac{A_2^G}{A_2^F}\kappa\left(K_2^G\right)-K_2^G-G_1^C\right)}\frac{\hat{M}_2}{P_1}.$$

Insert the optimal G_1

$$G_1^C = \frac{A_2^F}{(1+\theta)\left(1+A_2^F\right)}\left(A_1^FK_1^F+A_1^G\kappa\left(K_1^G\right)+\frac{A_2^G}{A_2^F}\kappa\left(K_2^G\right)-K_2^G\right),$$

and $1 - g_2 = \frac{1}{1+\theta}$ from the the flexible-price case:

$$(1-s) = \frac{(1+\theta)\left(1+A_2^F\right)}{\left(A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right) + \frac{A_2^G}{A_2^F}\kappa\left(K_2^G\right) - K_2^G\right)} \frac{\hat{M}_2}{P_1}.$$

Recall the equation for output in the sticky-price analysis just preceding. If we re-insert the nominal rate, allowing it to go below zero, this is:

$$Y_1 = \frac{1 + \left(\frac{1}{\beta A_2^F}\right)^{\sigma} A_2^F \left(1 - g_2^C\right)}{\left(1 - g_2^C\right)} \frac{\hat{M}_2}{(1+i_1)\, P_1} + K_2^G - \frac{A_2^G}{A_2^F}\kappa\left(K_2^G\right) + G_1^C,$$

and the flexible-price equilibrium rate given sticky long-term monetary policy is:

$$(1+i_1) = \frac{(1+\theta)\left(1+A_2^F\right)}{\left(A_1^F K_1^F + A_1^G \kappa\left(K_1^G\right) + \frac{A_2^G}{A_2^F}\kappa\left(K_2^G\right) - K_2^G\right)} \frac{\hat{M}_2}{P_1},$$

so again the investment subsidy is just the opposite of the optimal negative interest rate.

6 Tax Policy in a Non-Ricardian Setting (Section 9 of the paper)

In this extension, the representative household acts like both a maximizing agent and a rule-of-thumb (RoT) agent. In particular, a share $(1-\lambda)$ of the household's consumption in a given period is determined by what the maximizing household above would choose, while a share λ is set equal to a fraction ρ of current disposable income. The maximizing household chooses as above. The RoT household sets

$$C_t^R = \rho\left(Y_t - T_t\right),$$

and overall consumption is

$$C_t = (1-\lambda)\, C_t^M + \lambda C_t^R.$$

6.1 Flexible prices

In a flexible-price setting, prices adjust to guarantee full employment in each period. Therefore,

$$Y_t = A_t K_t \quad \text{for all } t.$$

The household's utility maximization yields the following intertemporal Euler equations:

$$\frac{u'\left(C_1^M\right)}{\beta u'\left(C_2^M\right)} = (1+i_1)\frac{P_1}{P_2}$$

The other conditions are as in the baseline model.

To solve for the economy's equilibrium, start by combining expressions as in the baseline model's analysis, plus the condition:

$$C_t^M = \frac{C_t - \lambda C_t^R}{(1-\lambda)}.$$

This yields:

$$A_2 = (1-s)\frac{u'\left(\frac{A_1K_1(1-g_1) - K_2 - \lambda\rho A_1 K_1 (1-\tau_1)}{(1-\lambda)}\right)}{\beta u'\left(\frac{A_2K_2(1-g_2) - \lambda\rho A_2 K_2 (1-\tau_2)}{(1-\lambda)}\right)}.$$

This expression implicitly solves for K_2 as a function of parameters.

The solution for K_2 implied by this expression in turn yields values for the remaining *real* endogenous variables in the economy. Nominal variables are determined separately by equilibrium in the money market. This economy exhibits monetary neutrality.

With CRRA utility, we obtain

$$K_2 = \frac{(1-g_1) - \lambda\rho(1-\tau_1)}{1 + \left(\frac{(1-s)}{A_2\beta}\right)^{\sigma} A_2\left((1-g_2) - \lambda\rho(1-\tau_2)\right)} A_1 K_1.$$

Before proceeding, we set the value of ρ such that both types of consumers choose the same levels of consumption when fiscal policy is at its optimum levels. The value:

$$\rho = \frac{\left(\frac{1}{\beta A_2}\right)^{\sigma} A_2}{\left(1 + \theta + \left(\frac{1}{\beta A_2}\right)^{\sigma} A_2\right)},$$

causes

$$C_1^M = \frac{\left(\frac{1}{\beta A_2}\right)^{\sigma} A_2}{(1+\theta)\left(1 + \left(\frac{1}{\beta A_2}\right)^{\sigma} A_2\right)} A_1 K_1,$$

$$C_1^R = \frac{\left(\frac{1}{\beta A_2}\right)^{\sigma} A_2}{(1+\theta)\left(1 + \left(\frac{1}{\beta A_2}\right)^{\sigma} A_2\right)} A_1 K_1,$$

and

$$K_2 = \frac{1}{\left(1 + \left(\frac{1}{\beta A_2}\right)^{\sigma} A_2\right)} A_1 K_1,$$

when

$$\tau_1 = g_1 = \frac{\theta\left(\frac{1}{\beta A_2}\right)^{\sigma} A_2}{(1+\theta)\left(1 + \left(\frac{1}{\beta A_2}\right)^{\sigma} A_2\right)},$$

$$\tau_2 = g_2 = \frac{\theta}{(1+\theta)},$$

$$s = 0.$$

For general fiscal policy, the equilibrium quantities are, using this value for ρ :

$$K_2 = \frac{(1-g_1) - \lambda\rho(1-\tau_1)}{1 + \left(\frac{(1-s)}{A_2\beta}\right)^{\sigma} A_2\left((1-g_2) - \lambda\rho(1-\tau_2)\right)} A_1 K_1,$$

$$C_1 = \left((1-g_1) - \frac{(1-g_1) - \lambda\rho(1-\tau_1)}{1 + \left(\frac{(1-s)}{A_2\beta}\right)^{\sigma} A_2\left((1-g_2) - \lambda\rho(1-\tau_2)\right)}\right) A_1 K_1,$$

$$C_2 = A_2(1-g_2)\frac{(1-g_1) - \lambda\rho(1-\tau_1)}{1 + \left(\frac{(1-s)}{A_2\beta}\right)^{\sigma} A_2\left((1-g_2) - \lambda\rho(1-\tau_2)\right)} A_1 K_1,$$

If there is a shock to technology, what is the policy that generates full employment with flexible prices? Do we need active fiscal policy? Below, we solve for the taxes that yield the flexible-price equilibrium quantities in this model with sticky prices but unrestricted monetary policy. Those values are the same as are required with flexible prices, and they imply a budget surplus in the first period. Intuitively, the rule-of-thumb households consume too much when output is stabilized, so the government raises taxes to reach optimality.

6.2 Sticky prices

With sticky prices, the expression implicitly defining K_2 is:

$$A_2 = (1-s)\frac{u'\left(\frac{C_1-\lambda\rho(Y_1-T_1)}{(1-\lambda)}\right)}{\beta u'\left(\frac{A_2K_2(1-g_2)-\lambda\rho A_2K_2(1-\tau_2)}{(1-\lambda)}\right)}.$$

This yields, with CRRA utility and in combination with

$$Y_1 = (C_1 + K_2 + G_1),$$

the expression:

$$K_2 = \frac{(1-\lambda\rho)C_1 - \lambda\rho(G_1-T_1)}{\lambda\rho + \left(\frac{(1-s)}{A_2\beta}\right)^{\sigma} A_2((1-g_2)-\lambda\rho(1-\tau_2))}.$$

To derive an expression for C_1, we have to apply the nominal side of the model. We know we can set

$$P_2 = \frac{M_2}{C_2}$$

without loss of generality, as the monetary authority need not print extra money in period one once the zero rate is reached. The economy is at full employment in period 2, so:

$$C_2 = \frac{(1-\lambda\rho)C_1 - \lambda\rho(G_1-T_1)}{\lambda\rho + \left(\frac{(1-s)}{A_2\beta}\right)^{\sigma} A_2((1-g_2)-\lambda\rho(1-\tau_2))} A_2(1-g_2).$$

The Euler equation for the household gives:

$$P_2 = \left(\frac{(1-\lambda\rho)C_1 - \lambda\rho(G_1-T_1) - \lambda\rho K_2}{A_2K_2((1-g_2)-\lambda\rho(1-\tau_2))}\right)^{\frac{1}{\sigma}} \beta(1+i_1)P_1.$$

Using the expression for K_2, we obtain:

$$P_2 = \frac{(1-s)(1+i_1)}{A_2}P_1.$$

The money market condition implies

$$C_2 = \frac{A_2M_2}{(1-s)(1+i_1)P_1}.$$

Using these results in the equation for C_2 as a function of C_1,

$$C_1 = \frac{\lambda\rho}{(1-\lambda\rho)}(G_1-T_1) + \frac{1}{(1-\lambda\rho)}\frac{\lambda\rho + \left(\frac{(1-s)}{A_2\beta}\right)^{\sigma} A_2((1-g_2)-\lambda\rho(1-\tau_2))}{(1-s)(1-g_2)}\frac{M_2}{(1+i_1)P_1},$$

and

$$K_2 = \frac{1}{(1-s)(1-g_2)}\frac{M_2}{(1+i_1)P_1}.$$

We can then write:

$$Y_1 = \frac{1}{(1-\lambda\rho)}G_1 - \frac{\lambda\rho}{(1-\lambda\rho)}T_1 + \frac{1}{(1-\lambda\rho)}\frac{1 + \left(\frac{(1-s)}{A_2\beta}\right)^{\sigma} A_2((1-g_2)-\lambda\rho(1-\tau_2))}{(1-s)(1-g_2)}\frac{M_2}{(1+i_1)P_1}.$$

Note that if $(1-\lambda)=1$, this is:

$$Y_1 = G_1 + \frac{1+\left(\frac{(1-s)}{A_2\beta}\right)^{\sigma} A_2 (1-g_2)}{(1-s)(1-g_2)} \frac{M_2}{(1+i_1)P_1},$$

as in the Ricardian case. Optimal monetary policy pre-shock is:

$$\frac{\hat{M}_2}{(1+\hat{\imath}_1)P_1} = \frac{1}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)} A_1 K_1.$$

6.3 Unrestricted MP

Suppose the central bank can fully commit to monetary policy. It wants to achieve flexible-price equilibrium values? The equilibrium quantities derived above are:

$$C_1 = \frac{\lambda\rho}{(1-\lambda\rho)}(G_1 - T_1) + \frac{1}{(1-\lambda\rho)} \frac{\lambda\rho + \left(\frac{1}{A_2\beta}\right)^{\sigma} A_2 ((1-g_2) - \lambda\rho(1-\tau_2))}{(1-g_2)} \frac{M_2}{(1+i_1)P_1},$$

$$C_2 = A_2 \frac{M_2}{(1+i_1)P_1},$$

$$I_1 = K_2 = \frac{1}{(1-g_2)} \frac{M_2}{(1+i_1)P_1},$$

$$Y_1 = \frac{1}{(1-\lambda\rho)}\left(G_1 - \lambda\rho T_1 + \left(\frac{1+\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2 ((1-g_2)-\lambda\rho(1-\tau_2))}{(1-g_2)}\right)\frac{M_2}{(1+i_1)P_1}\right).$$

Note that

$$G_2 = A_2 \frac{g_2}{(1-g_2)} \frac{M_2}{(1+i_1)P_1},$$

so

$$1-g_2 = \frac{A_2 \frac{M_2}{(1+i_1)P_1}}{\left(G_2 + A_2 \frac{M_2}{(1+i_1)P_1}\right)},$$

and

$$T_2 = A_2 \frac{\tau_2}{(1-g_2)} \frac{M_2}{(1+i_1)P_1}.$$

Therefore, we can write:

$$Y_1 = \frac{1}{1-\lambda\rho} G_1 - \frac{\lambda\rho}{1-\lambda\rho} T_1 + \frac{1-\lambda\rho\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2}{A_2(1-\lambda\rho)} G_2 + \frac{\lambda\rho\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2}{A_2(1-\lambda\rho)} T_2 + \frac{1+(1-\lambda\rho)\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2}{1-\lambda\rho} \frac{M_2}{(1+i_1)P_1}.$$

For reference, the flexible-price equilibrium values are:

$$C_1^{flex} = \frac{\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2}{(1+\theta)\left(1+\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\right)} A_1 K_1,$$

$$C_2^{flex} = \frac{A_2}{(1+\theta)\left(1+\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\right)} A_1 K_1,$$

$$I_1^{flex} = K_2^{flex} = \frac{1}{\left(1 + \left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\right)} A_1 K_1,$$

$$G_1^{flex} = \frac{\theta \left(\frac{1}{A_2\beta}\right)^{\sigma} A_2}{(1+\theta)\left(1 + \left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\right)} A_1 K_1,$$

$$G_2^{flex} = \frac{\theta A_2}{(1+\theta)\left(1 + \left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\right)} A_1 K_1.$$

The first step is to use C_2 to get the optimal monetary policy directly:

$$C_2^{ROT-MP} = C_2^{flex},$$

or

$$\frac{M_2}{(1+i_1) P_1} = \frac{1}{(1+\theta)\left(1 + \left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\right)} A_1 K_1.$$

We also know we want

$$G_1^{ROT-MP} = \frac{\theta \left(\frac{1}{A_2\beta}\right)^{\sigma} A_2}{(1+\theta)\left(1 + \left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\right)} A_1 K_1,$$

and

$$G_2^{ROT-MP} = \frac{\theta A_2}{(1+\theta)\left(1 + \left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\right)} A_1 K_1,$$

and

$$K_2^{ROT-MP} = \frac{1}{\left(1 + \left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\right)} A_1 K_1,$$

so

$$1 - g_2^{ROT-MP} = \frac{1}{1+\theta}.$$

Now use C_1 and the government budget constraint. C_1 should equal its flexible-price level:

$$C_1^{ROT-MP} = C_1^{flex},$$

which is:

$$\frac{\lambda\rho}{(1-\lambda\rho)}(G_1 - T_1) + \frac{1}{(1-\lambda\rho)} \frac{\lambda\rho + \left(\frac{1}{A_2\beta}\right)^{\sigma} A_2 \left((1-g_2) - \lambda\rho(1-\tau_2)\right)}{(1-g_2)} \frac{M_2}{(1+i_1) P_1} = \frac{\left(\frac{1}{A_2\beta}\right)^{\sigma} A_2}{(1+\theta)\left(1 + \left(\frac{1}{A_2\beta}\right)^{\sigma} A_2\right)} A_1 K_1.$$

Plugging in for G_1, g_2, and monetary policy, we get

$$T_1 = \frac{1 + \tau_2 \left(\frac{1}{A_2\beta}\right)^{\sigma} A_2}{1 + \left(\frac{1}{A_2\beta}\right)^{\sigma} A_2} A_1 K_1.$$

or

$$T_1 = \frac{1}{1 + \left(\frac{1}{A_2\beta}\right)^{\sigma} A_2} A_1 K_1 + \left(\frac{1}{A_2\beta}\right)^{\sigma} T_2.$$

Next, the government's budget constraint can be written:

$$G_2 = A_2 (T_1 - G_1) + T_2,$$

or, with substitutions:

$$\frac{\left(\theta\left(\frac{1}{A_2\beta}\right)^{\sigma}A_2-1\right)A_2}{(1+\theta)\left(1+\left(\frac{1}{A_2\beta}\right)^{\sigma}A_2\right)\left(1+\left(\frac{1}{A_2\beta}\right)^{\sigma}A_2\right)}A_1K_1=T_2^{ROT-MP},$$

This implies

$$T_1=\frac{(1+\theta)+\theta\left(\frac{1}{A_2\beta}\right)^{\sigma}A_2\left(1+\left(\frac{1}{A_2\beta}\right)^{\sigma}A_2\right)}{(1+\theta)\left(1+\left(\frac{1}{A_2\beta}\right)^{\sigma}A_2\right)\left(1+\left(\frac{1}{A_2\beta}\right)^{\sigma}A_2\right)}A_1K_1,$$

so

$$\tau_2=\frac{\theta\left(\frac{1}{A_2\beta}\right)^{\sigma}A_2-1}{(1+\theta)\left(1+\left(\frac{1}{A_2\beta}\right)^{\sigma}A_2\right)}.$$

Note that this policy requires raising taxes in the first-period relative to a balanced budget. Intuitively, the rule-of-thumb households consume too much when output is stabilized through monetary policy, so the government should raise taxes to reach optimality.

6.4 Restricted monetary policy, using only government purchases and taxes

Plugging in for the shock and these pre-shock values, assuming $i_1=0$, and converting levels of G and T to shares, we get the values:

$$C_1=\left(\frac{\lambda\rho}{(1-\lambda\rho)}(g_1-\tau_1)+\frac{1}{(1-\lambda\rho)}\frac{\lambda\rho+\left(\frac{1}{A_2\beta}\right)^{\sigma}A_2\left((1-g_2)-\lambda\rho\left(1-\tau_2\right)\right)}{(1-g_2)}\frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}\right)A_1K_1,$$

$$C_2=\frac{A_2\left(1+\hat{\imath}_1\right)}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}A_1K_1,$$

$$I_1=K_2=\frac{1}{(1-g_2)}\frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}A_1K_1,$$

$$Y_1=\left(\frac{1}{(1-\lambda\rho)}g_1-\frac{\lambda\rho}{(1-\lambda\rho)}\tau_1+\frac{1}{(1-\lambda\rho)}\left(\frac{1+\left(\frac{1}{A_2\beta}\right)^{\sigma}A_2\left((1-g_2)-\lambda\rho\left(1-\tau_2\right)\right)}{(1-g_2)}\right)\frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}\right)A_1K_1.$$

Monetary policy pre-shock is:

$$\frac{\hat{M}_2}{(1+\hat{\imath}_1)P_1}=\frac{1}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}A_1K_1.$$

Note that there is no way to achieve the flexible-price equilibrium here, since C_2 is unaffected by G and T and is not equal to the flexible-price level of C_2 :

$$C_2=\frac{A_2}{(1+\theta)}\frac{1}{\left(1+\left(\frac{1}{A_2\beta}\right)^{\sigma}A_2\right)}A_1K_1.$$

The government's budget constraint can be written as:

$$G_2=(1+i_1)\frac{P_1}{P_2}(T_1-G_1-sI_1)+T_2,$$

or plugging in for P_2 and $s=0$, and rearranging:

$$T_2 = G_2 - A_2 \left(T_1 - G_1\right),$$

or, converting to shares vs. levels using K_2 from above,

$$\tau_2 = g_2 + \left(g_1 - \tau_1\right)\left(1 - g_2\right) \frac{\left(1+\theta\right)\left(1 + \left(\frac{1}{\beta \hat{A}_2}\right)^\sigma \hat{A}_2\right)}{\left(1+\hat{\imath}_1\right)}.$$

Then, the equilibrium values after substituting in the government budget constraint become

$$C_1 = \left(\frac{\lambda\rho\left(1 + \left(\frac{1}{A_2\beta}\right)^\sigma A_2\right)}{\left(1-\lambda\rho\right)} \left(g_1 - \tau_1\right) + \frac{\lambda\rho + \left(\frac{1}{A_2\beta}\right)^\sigma A_2 \left(1-\lambda\rho\right)\left(1-g_2\right)}{\left(1-\lambda\rho\right)\left(1-g_2\right)} \frac{\left(1+\hat{\imath}_1\right)}{\left(1+\theta\right)\left(1 + \left(\frac{1}{\beta \hat{A}_2}\right)^\sigma \hat{A}_2\right)} \right) A_1 K_1,$$

$$C_2 = \frac{A_2}{\left(1+\theta\right)} \frac{\left(1+\hat{\imath}_1\right)}{\left(1 + \left(\frac{1}{\beta \hat{A}_2}\right)^\sigma \hat{A}_2\right)} A_1 K_1,$$

$$I_1 = K_2 = \frac{1}{\left(1-g_2\right)} \frac{\left(1+\hat{\imath}_1\right)}{\left(1+\theta\right)\left(1 + \left(\frac{1}{\beta \hat{A}_2}\right)^\sigma \hat{A}_2\right)} A_1 K_1,$$

and the full-employment constraint is, for period 1:

$$1 = \left\{ \begin{array}{c} \frac{\left(1-\lambda\rho\right) + \lambda\rho\left(1 + \left(\frac{1}{A_2\beta}\right)^\sigma A_2\right)}{\left(1-\lambda\rho\right)} g_1 - \frac{\lambda\rho\left(1 + \left(\frac{1}{A_2\beta}\right)^\sigma A_2\right)}{\left(1-\lambda\rho\right)} \tau_1 \\ + \frac{1 + \left(\frac{1}{A_2\beta}\right)^\sigma A_2 \left(1-\lambda\rho\right)\left(1-g_2\right)}{\left(1-\lambda\rho\right)\left(1-g_2\right)} \frac{\left(1+\hat{\imath}_1\right)}{\left(1+\theta\right)\left(1 + \left(\frac{1}{\beta \hat{A}_2}\right)^\sigma \hat{A}_2\right)} \end{array} \right\}.$$

We can use this constraint to solve for τ_1 in terms of the other policy variables:

$$\tau_1 = \frac{1}{\lambda\rho\left(1 + \left(\frac{1}{A_2\beta}\right)^\sigma A_2\right)} \left[\left(1 + \lambda\rho\left(\frac{1}{A_2\beta}\right)^\sigma A_2\right) g_1 - \left(1-\lambda\rho\right) + \frac{1 + \left(\frac{1}{A_2\beta}\right)^\sigma A_2 \left(1-\lambda\rho\right)\left(1-g_2\right)}{\left(1-g_2\right)\left(1+\theta\right)\left(1 + \left(\frac{1}{\beta \hat{A}_2}\right)^\sigma \hat{A}_2\right)} \left(1+\hat{\imath}_1\right) \right],$$

so that

$$\left(g_1 - \tau_1\right) = \left(\begin{array}{c} \frac{1}{\lambda\rho\left(1 + \left(\frac{1}{A_2\beta}\right)^\sigma A_2\right)} \left[\left(1-\lambda\rho\right) - \frac{1 + \left(\frac{1}{A_2\beta}\right)^\sigma A_2 \left(1-\lambda\rho\right)\left(1-g_2\right)}{\left(1-g_2\right)\left(1+\theta\right)\left(1 + \left(\frac{1}{\beta \hat{A}_2}\right)^\sigma \hat{A}_2\right)} \left(1+\hat{\imath}_1\right) \right] \\ -g_1 \left(\frac{\left(1-\lambda\rho\right)}{\lambda\rho\left(1 + \left(\frac{1}{A_2\beta}\right)^\sigma A_2\right)} \right) \end{array} \right).$$

Then, the equilibrium in g_1 and g_2 is:

$$C_1 = \left(\left(1-g_1\right) - \frac{1}{\left(1-g_2\right)\left(1+\theta\right)\left(1 + \left(\frac{1}{\beta \hat{A}_2}\right)^\sigma \hat{A}_2\right)} \left(1+\hat{\imath}_1\right) \right) A_1 K_1,$$

$$C_2 = \frac{A_2}{\left(1+\theta\right)} \frac{\left(1+\hat{\imath}_1\right)}{\left(1 + \left(\frac{1}{\beta \hat{A}_2}\right)^\sigma \hat{A}_2\right)} A_1 K_1,$$

$$I_1 = K_2 = \frac{1}{\left(1-g_2\right)} \frac{\left(1+\hat{\imath}_1\right)}{\left(1+\theta\right)\left(1 + \left(\frac{1}{\beta \hat{A}_2}\right)^\sigma \hat{A}_2\right)} A_1 K_1.$$

The government's maximization problem is:

$$\max_{\{g_t,\tau_t\}_{t=1}^{2}}\left[\begin{array}{c} u\left(\left((1-g_1)-\frac{1}{(1-g_2)(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}(1+\hat{\imath}_1)\right)A_1K_1\right)+v\left(g_1A_1K_1\right) \\ +\beta\left(u\left(\frac{A_2}{(1+\theta)}\frac{(1+\hat{\imath}_1)}{\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}A_1K_1\right)+v\left(g_2A_2\frac{1}{(1-g_2)}\frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}A_1K_1\right)\right)\end{array}\right],$$

yielding the first-order condition:

$$FOC_{g_1}: u'\left(C_1\right)=v'\left(G_1\right),$$

which is the classical condition, and:

$$\begin{aligned} FOC_{g_2} \quad &: \quad u'\left(C_1\right)\frac{1}{\left(1-g_2\right)^2(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}\left(1+\hat{\imath}_1\right)A_1K_1 \\ &= \quad \beta v'\left(G_2\right)\left(\begin{array}{c} g_2A_2\frac{1}{((1-g_2))^2}\frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}A_1K_1 \\ +A_2\frac{1}{(1-g_2)}\frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}A_1K_1\end{array}\right).\end{aligned}$$

Simplifying, we obtain:

$$u'\left(C_1\right)=\beta A_2v'\left(G_2\right),$$

implying

$$v'\left(G_1\right)=\beta A_2v'\left(G_2\right).$$

Now, assuming CRRA utility, the first first-order condition implies

$$C_1=\theta g_1A_1K_1,$$

and the second implies:

$$C_1=\left(\beta A_2\right)^{(-\sigma)}A_2\frac{g_2}{(1-g_2)}\frac{\theta\left(1+\hat{\imath}_1\right)}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}A_1K_1.$$

Combined, these first-order conditions imply:

$$g_1A_1K_1=\left(\beta A_2\right)^{(-\sigma)}A_2\frac{g_2}{(1-g_2)}\frac{\left(1+\hat{\imath}_1\right)}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}A_1K_1.$$

Substituting into the expression for C_1 to solve for g_2, we obtain:

$$g_2=\frac{1}{(1+\theta)}\frac{\theta(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)-\theta\left(1+\hat{\imath}_1\right)}{\theta\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)+\left(\frac{1}{\beta A_2}\right)^{\sigma}A_2\left(1+\hat{\imath}_1\right)},$$

so

$$1-g_2=\frac{1}{(1+\theta)}\frac{\left(\theta+(1+\theta)\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)\left(1+\hat{\imath}_1\right)}{\theta\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)+\left(\frac{1}{\beta A_2}\right)^{\sigma}A_2\left(1+\hat{\imath}_1\right)},$$

so

$$\frac{g_2}{(1-g_2)}=\frac{\theta(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)-\theta\left(1+\hat{\imath}_1\right)}{\left(\theta+(1+\theta)\left(\frac{1}{\beta A_2}\right)^{\sigma}A_2\right)\left(1+\hat{\imath}_1\right)},$$

and

$$g_1 = \left(\frac{1}{\beta A_2}\right)^{\sigma} A_2 \frac{\theta\left(1+\theta\right)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma}\hat{A}_2\right)-\theta\left(1+\hat{\imath}_1\right)}{\left(\theta+\left(1+\theta\right)\left(\frac{1}{\beta A_2}\right)^{\sigma}A_2\right)\left(1+\theta\right)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma}\hat{A}_2\right)},$$

and

$$C_1^{ROT} = \frac{1}{\theta}\left(\frac{1}{\beta A_2}\right)^{\sigma} A_2 \frac{\theta\left(1+\theta\right)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma}\hat{A}_2\right)-\theta\left(1+\hat{\imath}_1\right)}{\left(\theta+\left(1+\theta\right)\left(\frac{1}{\beta A_2}\right)^{\sigma}A_2\right)\left(1+\theta\right)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma}\hat{A}_2\right)} A_1 K_1,$$

compared to the sticky-price Ricardian case:

$$C_1^{sticky} = \left(1+\hat{\imath}_1\right)\frac{\left(\frac{1}{\beta A_2}\right)^{\sigma}A_2}{\left(1+\theta\right)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma}\hat{A}_2\right)} A_1 K_1.$$

The key question is whether

$$C_1^{ROT} > C_1^{sticky}.$$

This inequality holds if:

$$1 > \frac{\left(1+\hat{\imath}_1\right)\left(1+\left(\frac{1}{\beta A_2}\right)^{\sigma}A_2\right)}{\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma}\hat{A}_2\right)},$$

which is the definition of the threshold we are considering.

Now,

$$G_1^{ROT} = \left(\frac{1}{\beta A_2}\right)^{\sigma} A_2 \frac{\theta\left(1+\theta\right)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma}\hat{A}_2\right)-\theta\left(1+\hat{\imath}_1\right)}{\left(\theta+\left(1+\theta\right)\left(\frac{1}{\beta A_2}\right)^{\sigma}A_2\right)\left(1+\theta\right)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma}\hat{A}_2\right)} A_1 K_1,$$

vs. its level in the sticky-price Ricardian case:

$$G_1^{sticky} = \left(\frac{1}{\beta A_2}\right)^{\sigma} A_2 \left(\frac{\left(1+\theta\right)\left(1+\left(\frac{1}{\hat{A}_2\beta}\right)^{\sigma}\hat{A}_2\right)-\left(1+\hat{\imath}_1\right)\left(1+\left(\frac{1}{\beta A_2}\right)^{\sigma}A_2\right)}{\left(1+\left(\frac{1}{\beta A_2}\right)^{\sigma}A_2\right)\left(1+\theta\right)\left(1+\left(\frac{1}{\hat{A}_2\beta}\right)^{\sigma}\hat{A}_2\right)}\right) A_1 K_1.$$

Again, we can show that:

$$G_1^{ROT} < G_1^{sticky}.$$

Second-period g_2 is

$$G_2^{ROT} = \frac{A_2}{\left(1+\theta\right)} \frac{\theta\left(1+\theta\right)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma}\hat{A}_2\right)-\theta\left(1+\hat{\imath}_1\right)}{\left(\theta+\left(1+\theta\right)\left(\frac{1}{\beta A_2}\right)^{\sigma}A_2\right)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma}\hat{A}_2\right)} A_1 K_1,$$

vs. its level in the sticky-price Ricardian case:

$$G_2^{sticky} = \frac{A_2}{1+\left(\frac{1}{\beta A_2}\right)^{\sigma}A_2} \left(\frac{\left(1+\theta\right)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma}\hat{A}_2\right)-\left(1+\hat{\imath}_1\right)\left(1+\left(\frac{1}{\beta A_2}\right)^{\sigma}A_2\right)}{\left(1+\theta\right)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}\right) A_1 K_1.$$

We can show:

$$G_2^{ROT} < G_2^{sticky}.$$

Second-period consumption is

$$C_2^{ROT} = \frac{A_2}{(1+\theta)} \frac{(1+\hat{\imath}_1)}{\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma} \hat{A}_2\right)} A_1 K_1,$$

vs.

$$C_2^{sticky} = (1+\hat{\imath}_1) \frac{A_2}{(1+\theta)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma} \hat{A}_2\right)} A_1 K_1,$$

so

$$C_2^{ROT} = C_2^{sticky}.$$

6.5 Restricted monetary policy with an investment subsidy

Plugging in for the shock and these pre-shock values, assuming $i_1 = 0$, we get the values:

$$C_1 = \left(\frac{\lambda\rho}{(1-\lambda\rho)} (g_1 - \tau_1) + \frac{1}{(1-\lambda\rho)} \frac{\lambda\rho + \left(\frac{(1-s)}{A_2\beta}\right)^{\sigma} A_2 \left((1-g_2) - \lambda\rho(1-\tau_2)\right)}{(1-s)(1-g_2)(1+\theta)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma} \hat{A}_2\right)} (1+\hat{\imath}_1) \right) A_1 K_1,$$

$$C_2 = \frac{A_2}{(1-s)} \frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma} \hat{A}_2\right)} A_1 K_1,$$

$$I_1 = K_2 = \frac{1}{(1-s)(1-g_2)} \frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma} \hat{A}_2\right)} A_1 K_1,$$

$$Y_1 = \left(\frac{1}{(1-\lambda\rho)} g_1 - \frac{\lambda\rho}{(1-\lambda\rho)} \tau_1 + \frac{1}{(1-\lambda\rho)} \left(\frac{1 + \left(\frac{(1-s)}{A_2\beta}\right)^{\sigma} A_2 \left((1-g_2) - \lambda\rho(1-\tau_2)\right)}{(1-s)(1-g_2)(1+\theta)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma} \hat{A}_2\right)} \right) (1+\hat{\imath}_1) \right) A_1 K_1.$$

The parameter ρ remains at its optimum pre-shock.

$$\rho = \frac{\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma} \hat{A}_2}{\left(1+\theta+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma} \hat{A}_2\right)}.$$

The government's budget constraint can be written as:

$$G_2 = (1+i_1) \frac{P_1}{P_2} (T_1 - G_1 - sI_1) + T_2, \tag{2}$$

or plugging in for P_2 and rearranging:

$$A_2 = (1-s) \frac{P_1}{P_2} (1+i_1),$$

$$T_2 = G_2 - \frac{A_2}{(1-s)} (T_1 - G_1 - sI_1), \tag{3}$$

or, converting to shares vs. levels using K_2 from above,

$$\tau_2 = g_2 + (g_1 - \tau_1)(1-g_2) \frac{(1+\theta)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma} \hat{A}_2\right)}{(1+\hat{\imath}_1)} + \frac{s}{1-s}. \tag{4}$$

Then, the equilibrium values with the government budget constraint inside become

$$\begin{aligned}
C_1 &= \left[\left(1+\left(\frac{(1-s)}{A_2\beta}\right)^{\sigma}\frac{A_2}{(1-s)}\right)\frac{\lambda\rho}{(1-\lambda\rho)}(g_1-\tau_1)\right. \\
&\left.+\frac{1}{(1-\lambda\rho)}\frac{\lambda\rho+\left(\frac{(1-s)}{A_2\beta}\right)^{\sigma}A_2\left((1-\lambda\rho)(1-g_2)+\lambda\rho\frac{s}{1-s}\right)}{(1-s)(1-g_2)}\frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}\right]A_1K_1,
\end{aligned}$$

$$C_2=\frac{A_2}{(1-s)}\frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}A_1K_1,$$

$$I_1=K_2=\frac{1}{(1-s)(1-g_2)}\frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}A_1K_1,$$

$$Y_1=\left(\begin{array}{c}\frac{1+\lambda\rho\left(\frac{(1-s)}{A_2\beta}\right)^{\sigma}\frac{A_2}{(1-s)}}{(1-\lambda\rho)}g_1-\frac{\lambda\rho\left(1+\left(\frac{(1-s)}{A_2\beta}\right)^{\sigma}\frac{A_2}{(1-s)}\right)}{(1-\lambda\rho)}\tau_1\\ +\frac{1+\left(\frac{(1-s)}{A_2\beta}\right)^{\sigma}A_2\left((1-\lambda\rho)(1-g_2)+\lambda\rho\frac{s}{1-s}\right)}{(1-\lambda\rho)(1-s)(1-g_2)}\frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}\end{array}\right)A_1K_1,$$

and the full-employment constraint is, for period 1:

$$\begin{aligned}
1 &= \frac{1+\lambda\rho\left(\frac{(1-s)}{A_2\beta}\right)^{\sigma}\frac{A_2}{(1-s)}}{(1-\lambda\rho)}g_1-\frac{\lambda\rho\left(1+\left(\frac{(1-s)}{A_2\beta}\right)^{\sigma}\frac{A_2}{(1-s)}\right)}{(1-\lambda\rho)}\tau_1 \\
&+\frac{1+\left(\frac{(1-s)}{A_2\beta}\right)^{\sigma}A_2\left((1-\lambda\rho)(1-g_2)+\lambda\rho\frac{s}{1-s}\right)}{(1-\lambda\rho)(1-s)(1-g_2)}\frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}.
\end{aligned}$$

We can use this constraint to solve for τ_1 in terms of the other policy variables:

$$\begin{aligned}
\tau_1 &= \left[\left(1+\lambda\rho\left(\frac{(1-s)}{A_2\beta}\right)^{\sigma}\frac{A_2}{(1-s)}\right)g_1-(1-\lambda\rho)\right. \\
&\left.+\frac{1+\left(\frac{(1-s)}{A_2\beta}\right)^{\sigma}A_2\left((1-\lambda\rho)(1-g_2)+\lambda\rho\frac{s}{1-s}\right)}{(1-s)(1-g_2)(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}(1+\hat{\imath}_1)\right]\frac{1}{\lambda\rho\left(1+\left(\frac{(1-s)}{A_2\beta}\right)^{\sigma}\frac{A_2}{(1-s)}\right)},
\end{aligned}$$

so that

$$\begin{aligned}
(g_1-\tau_1) &= \left[(1-\lambda\rho)(1-g_1)-\frac{1+\left(\frac{(1-s)}{A_2\beta}\right)^{\sigma}A_2\left((1-\lambda\rho)(1-g_2)+\lambda\rho\frac{s}{1-s}\right)}{(1-s)(1-g_2)(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}(1+\hat{\imath}_1)\right] \\
&*\frac{1}{\lambda\rho\left(1+\left(\frac{(1-s)}{A_2\beta}\right)^{\sigma}\frac{A_2}{(1-s)}\right)},
\end{aligned}$$

so, then the equilibrium in g_1 and g_2 is:

$$C_1=\left((1-g_1)-\frac{1}{(1-s)(1-g_2)}\frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}\right)A_1K_1,$$

$$C_2=\frac{A_2}{(1-s)}\frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta\hat{A}_2}\right)^{\sigma}\hat{A}_2\right)}A_1K_1,$$

$$I_1 = K_2 = \frac{1}{(1-s)(1-g_2)} \frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma} \hat{A}_2\right)} A_1 K_1.$$

The government's problem is:

$$\max \left[\begin{array}{c} u\left(\left((1-g_1) - \frac{1}{(1-s)(1-g_2)} \frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma} \hat{A}_2\right)}\right) A_1 K_1\right) + v\left(g_1 A_1 K_1\right) \\ +\beta\left(u\left(\frac{A_2}{(1-s)} \frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma} \hat{A}_2\right)} A_1 K_1\right) + v\left(g_2 A_2 \frac{1}{(1-s)(1-g_2)} \frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma} \hat{A}_2\right)} A_1 K_1\right)\right) \end{array} \right],$$

In the paper, we claim that with these policy instruments the flexible-price equilibrium can be obtained. Here is the proof.

The flexible-price consumption levels are:

$$C_1 = \frac{\left(\frac{1}{\beta A_2}\right)^{\sigma} A_2}{(1+\theta)\left[1+\left(\frac{1}{\beta A_2}\right)^{\sigma} A_2\right]} A_1 K_1,$$

$$C_2 = \frac{A_2}{(1+\theta)\left[1+\left(\frac{1}{\beta A_2}\right)^{\sigma} A_2\right]} A_1 K_1,$$

$$I_1 = \frac{1}{\left[1+\left(\frac{1}{\beta A_2}\right)^{\sigma} A_2\right]} A_1 K_1.$$

We can show that the policy instruments can be chosen to achieve these levels. Start with C_2. The only policy instrument in C_2 is s, so that we can set:

$$\frac{A_2}{(1+\theta)\left[1+\left(\frac{1}{\beta A_2}\right)^{\sigma} A_2\right]} A_1 K_1 = \frac{A_2}{(1-s)} \frac{(1+\hat{\imath}_1)}{(1+\theta)\left(1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma} \hat{A}_2\right)} A_1 K_1,$$

and solve for:

$$(1-s) = \frac{1+\left(\frac{1}{\beta A_2}\right)^{\sigma} A_2}{1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma} \hat{A}_2} (1+\hat{\imath}_1).$$

Note that full-employment conventional monetary policy sets the short-term interest rate so that:

$$\frac{\hat{M}_2}{(1+i_1) P_1} = \frac{(1+\hat{\imath}_1)}{(1+i_1)(1+\theta)\left(1+\left(\frac{1}{\hat{A}_2 \beta}\right)^{\sigma} \hat{A}_2\right)} A_1 K_1 = \frac{1}{(1+\theta)\left(1+\left(\frac{1}{\beta A_2}\right)^{\sigma} A_2\right)} A_1 K_1,$$

where

$$\frac{M_2}{(1+i_1) P_1} = \frac{1}{(1+\theta)\left(1+\left(\frac{1}{A_2 \beta}\right)^{\sigma} A_2\right)} A_1 K_1,$$

is full-employment monetary policy after the shock if all instruments are flexible. Solving for i_1, we would need to set:

$$(1+i_1) = \frac{1+\left(\frac{1}{\beta A_2}\right)^{\sigma} A_2}{1+\left(\frac{1}{\beta \hat{A}_2}\right)^{\sigma} \hat{A}_2} (1+\hat{\imath}_1),$$

which implies:

$$s = -i_1,$$

as in the Ricardian model. Next, we know the optimal level of g_1 :

$$g_1 = \frac{\theta}{(1+\theta)} \frac{\left(\frac{1}{\beta A_2}\right)^\sigma A_2}{\left[1 + \left(\frac{1}{\beta A_2}\right)^\sigma A_2\right]},$$

and g_2 from

$$G_2 = \frac{\theta A_2}{(1+\theta)\left[1 + \left(\frac{1}{\beta A_2}\right)^\sigma A_2\right]} A_1 K_1,$$

$$I_1 = \frac{1}{\left[1 + \left(\frac{1}{\beta A_2}\right)^\sigma A_2\right]} A_1 K_1,$$

so

$$g_2 = \frac{\theta}{(1+\theta)}.$$

We can place these into the expression for C_1 to obtain:

$$C_1 = \left(\left(1 - \frac{\theta}{(1+\theta)} \frac{\left(\frac{1}{\beta A_2}\right)^\sigma A_2}{\left[1 + \left(\frac{1}{\beta A_2}\right)^\sigma A_2\right]} \right) - \frac{1}{\frac{1+\left(\frac{1}{\beta A_2}\right)^\sigma A_2}{1+\left(\frac{1}{\beta \hat{A}_2}\right)^\sigma \hat{A}_2} (1+i_1) \frac{1}{1+\theta}} \frac{(1+i_1)}{(1+\theta)\left(1 + \left(\frac{1}{\beta \hat{A}_2}\right)^\sigma \hat{A}_2\right)} \right) A_1 K_1,$$

or, simplifying:

$$C_1 = \frac{1}{(1+\theta)} \frac{\left(\frac{1}{\beta A_2}\right)^\sigma A_2}{1 + \left(\frac{1}{\beta A_2}\right)^\sigma A_2} A_1 K_1,$$

which is the optimal level. Therefore, we achieve the flexible-price equilibrium.