Borda's Rule and Arrow's Independence Condition*

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Abstract

We argue that Arrow's (1950), (1951) independence of irrelevant alternatives condition (IIA) is unjustifiably stringent. Although, in elections, it has the desirable effect of ruling out spoilers and vote-splitting (Candidate A spoils the election for B if B beats C when all voters rank A low, but C beats B when some voters rank A high - - because A splits off support from B), it is stronger than necessary for this purpose. Worse, it makes a voting rule insensitive to voters' preference intensities. Accordingly, we propose a modified version of IIA, MIIA, that is still strong enough to rule out spoilers and, in a precise sense, is a necessary and sufficient relaxation of IIA for taking account of intensities. Rather than obtaining an impossibility result like Arrow's, we show that a continuous voting rule satisfies MIIA, Arrow's other conditions, May's (1952) axioms for majority rule, and a much weakened version of Young's (1974) consistency condition if and only if it is the Borda count (Borda 1781), i.e., rank-order voting. Because every other condition we impose is satisfied by virtually all voting rules used in practice and studied in theory, this result establishes that MIIA is the axiom uniquely distinguishing the Borda count from those other voting rules.

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1. Arrow, May, Young, and Borda

A. Arrow's IIA Condition

In (1950) and (1951), Kenneth Arrow introduced the concept of a *social welfare function* (SWF) – a mapping from profiles of individuals' preferences to social preferences.[1] The centerpiece of his analysis was the celebrated Impossibility Theorem, which establishes that, with three or more social alternatives, there exists no SWF satisfying four attractive conditions: *unrestricted domain* (U), the *Pareto Principle* (P), *non-dictatorship* (ND), and *independence of irrelevant alternatives* (IIA).

Condition U requires merely that a social welfare function be defined for all possible profiles of individual preferences (since ruling out preferences in advance could be difficult). P is the reasonable requirement that if all individuals (strictly) prefer alternative x to y, then x should be (strictly) preferred to y socially as well. ND is the weak assumption that there should not exist a single individual (a "dictator") whose strict preference always determines social preference.

These first three conditions are so undemanding that virtually any SWF studied in theory or used in practice satisfies them all. For example, consider *plurality rule* (or "first-past-the-post"), in which x is preferred to y socially if the number of individuals ranking x first is bigger than the number ranking y first.[2] Plurality rule satisfies U because it is well-defined regardless of individuals' preferences. It satisfies P because if all individuals strictly prefer x to y, then x must

[1] Formal definitions are provided in section 2.

[2] As used in elections, plurality rule (the predominant election method in the U.S. and U.K.) is, strictly speaking, a *voting rule*, not a SWF: it merely determines the *winner* (the candidate who is ranked first by a plurality of voters). By contrast, a SWF requires that *all* candidates be ranked socially (Arrow 1951 sees this as a contingency plan: if the top choice turns out not to be feasible, society can move to the second choice, etc.). See Section 5 for further discussion of voting rules.

be ranked first by more individuals than *y*.[3] Finally, it satisfies ND because if everyone else ranks *x* first, then even if the last individual strictly prefers *y* to *x*, *y* will not be ranked above *x* socially.

Alternatively, consider *instant-runoff voting* (called ranked-choice voting in the United States, preferential voting in Australia and the United Kingdom, and Hare's rule or single-transferable vote in some of the voting literature), in which *x* is preferred to *y* socially if *x* is dropped after *y* in the candidate-elimination process (the candidate dropped first is the one who is ranked first by the fewest voters; her supporters' second choices are then elevated into first place; and the process iterates). It is easy to check that it too satisfies the three conditions.

By contrast, IIA – which requires that social preferences between *x* and *y* should depend only on individuals' preferences between *x* and *y*, and not on preferences concerning some third alternative – is satisfied by very few SWFs.[4] Even so, it has a compelling justification: to prevent *spoilers* and *vote-splitting* in elections. [5]

To understand the issue, consider Scenario 1 (modified from Maskin and Sen 2016). There are three candidates – Donald Trump, Marco Rubio, and John Kasich (the example is inspired by the 2016 Republican presidential primary elections) – and three groups of voters. One group (40%) ranks Trump above Kasich above Rubio; the second (25%) places Rubio over Kasich over Trump; and the third (35%) ranks Kasich above Trump above Rubio (see Figure A).

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[3] This isn't quite accurate, because it is conceivable that *x* is *never* ranked first. But we will ignore this small qualification.

[4] One SWF that does satisfy IIA is *majority rule* (also called Condorcet voting), in which alternative *x* is socially preferred to *y* if a majority of individuals prefer *x* to *y*. However, unless individuals' preferences are restricted, social preferences with majority rule may cycle (i.e., *x* may be preferred to *y*, *y* preferred to *z*, and yet *z* preferred to *x*), as Condorcet (1785) discovered (see formula (4) below). In that case, majority rule is not actually a SWF (since its social preferences are intransitive). That is, majority rule violates U.

[5] Eliminating spoilers and vote-splitting has frequently been cited in the voting literature as a rationale for IIA. See, for example the Wikipedia article on vote-splitting https://en.wikipedia.org/wiki/Vote_splitting, especially the section on "Mathematical definitions."

40%	25%	35%
Trump	Rubio	Kasich
Kasich	Kasich	Trump
Rubio	Trump	Rubio

Figure A: Scenario 1

Many Republican primaries in 2016 used plurality rule (called "first-past-the-post" in the U.K.); so the winner was the candidate ranked first by more voters than anyone else.[6] As applied to Scenario 1, Trump is the winner with 40% of the first-place rankings. But, in fact, a large majority of voters (60%, i.e., the second and third groups) prefer Kasich to Trump. The only reason why Trump wins in Scenario 1 is that Rubio *spoils* the election for Kasich by splitting off some of his support;[7] Rubio and Kasich split the first-place votes that don't go to Trump.

An SWF that satisfies IIA avoids spoilers and vote-splitting. To see this, consider Scenario 2, which is the same as Scenario 1 except that voters in the middle group now prefer Kasich to Trump to Rubio (see Figure B).

[6] In actual plurality rule elections, citizens simply vote for a single candidate rather than rank candidates. But this leads to the same winner as long as citizens vote for their most preferred candidate.

[7] In common parlance (arising from plurality rule and runoff elections), candidate A spoils the election for B if (i) B wins when A doesn't run, and (ii) C wins when A does run (because some citizens vote for A, and these votes would otherwise have gone to B). In Arrow's (1951) framework (which we adopt here), however, there is a *fixed* set of candidates, and so we interpret a "candidate who doesn't run" as one ranked at the bottom by all voters (since a candidate ranked at the bottom has zero effect on what happens to other candidates – just like a candidate who doesn't run). Similarly, we interpret "some citizens voting for A" as their ranking A first (i.e., above B and C), since in plurality and runoff elections, one can vote for only a single candidate (presumably, one's top choice). Thus, formally, A is a spoiler for B if B beats C when all voters rank A at the bottom, but C beats B when some voters switch to ranking A at the top (with no other changes to the preference profile).

40%	25%	35%
Trump	Kasich	Kasich
Kasich	Trump	Trump
Rubio	Rubio	Rubio

Figure B: Scenario 2

Pretty much any non-pathological SWF will lead to Kasich being ranked above Trump in Scenario 2 (Kasich is not only top-ranked by 60% of voters, but is ranked second by 40%; by contrast, Trump reverses these numbers: he is ranked first by 40% and second by 60%). However, if the SWF satisfies IIA, it must also rank Kasich over Trump in Scenario 1, since each of the three groups has the same preferences between the two candidates in both scenarios. Hence, unlike plurality rule, a SWF satisfying IIA circumvents spoilers and vote-splitting: Kasich will win in Scenario 1.

But imposing IIA is too demanding: It is stronger than necessary to prevent spoilers (as we will see), and makes sensitivity to preference intensities impossible.[8] To understand this latter point, consider Scenario 3, in which there are three candidates *x*, *y*, and *z* and two groups of voters, one (45% of the electorate) who prefer *x* to *z* to *y*; and the other (55%), who prefer *y* to *x* to *z* (see Figure C).

[8] Arrow (1950), (1951) assumes that a SWF is a function only of individuals' *ordinal* preferences (for the motivation behind this assumption, see footnote 9), which means that preference intensities cannot directly be expressed in his framework. However, this does not not rule out the possibility of *inferring* intensities from ordinal data, as we argue below. And even if one takes the view that preference intensities have no place in political elections, they are central to much of welfare economics (in which the "candidates" are the policy alternatives), which is also covered by Arrow's framework.

45%	55%	Under the Borda count
x	y	x gets $3 \times 45 + 2 \times 55 = 245$ points
z	x	y gets $3 \times 55 + 1 \times 45 = 210$ points
y	z	z gets $2 \times 45 + 1 \times 55 = 145$ points

so the social ranking is $\begin{matrix} x \\ y \\ z \end{matrix}$

Figure C: Scenario 3

For this scenario, let's apply the *Borda count* (rank-order voting), in which, if there are m candidates, a candidate gets m points for every voter who ranks her first, $m-1$ points for a second-place ranking, and so on. Candidates are then ranked according to their point totals. The calculations in Figure C show that in Scenario 3, x is socially preferred to y and y is socially preferred to z. But now consider Scenario 4, where the first group's preferences are replaced by x over y over z (see Figure D).

45%	55%	Under the Borda count, the
x	y	
y	x	social ranking is now $\begin{matrix} y \\ x, \\ z \end{matrix}$ a
z	z	

violation of IIA as applied to x and y

Figure D: Scenario 4

As calculated in Figure D, the Borda social ranking becomes y over x over z. This violates IIA: in going from Scenario 3 to 4, no individual's ranking of x and y changes, yet the social ranking switches from x above y to y above x.

However, the anti-spoiler/anti-vote-splitting rationale for IIA doesn't apply to Scenarios 3 and 4. Notice that candidate z doesn't split first-place votes with y in Scenario 3; indeed, she is *never* ranked first. Moreover, her position in group 1 voters' preferences in Scenarios 3 and 4 provides potentially useful information about the intensity of those voters' preferences between x and y. In Scenario 3, z lies between x and y – suggesting that the preference gap between x and y may be substantial. In the second case, z lies below both x and y, implying that the difference between x and y is not as big. Thus, although z may not be a strong candidate herself (i.e., she is, in some sense, an "irrelevant alternative"), how individuals rank her vis à vis x and y is arguably pertinent to social preferences,[9] i.e., IIA should not apply to these scenarios.

Let us make this more precise. Imagine that from the perspective of an outside spectator (or society), an individual's utilities $u(x)$, $u(y)$, and $u(z)$ (where u captures preference intensity) are drawn from an unknown joint distribution $p(u_x, u_y, u_z)$, where $p(u_x, u_y, u_z) = \text{prob}(u(x) = u_x, u(y) = u_y, u(z) = u_z)$. Assume that p is *exchangeable* in the sense

[9] One might wonder why, instead of depending only on individuals' ordinal rankings, a SWF is not allowed to depend directly on their *cardinal* utilities, as in Benthamite utilitarianism (Bentham, 1789) or majority judgment (Balinski and Laraki, 2010). But it is not at all clear how to ascertain these utilities, even leaving aside the question of deliberate misrepresentation by individuals. Indeed, for that reason, Lionel Robbins (1932) rejected the idea of cardinal utility altogether, and Arrow (1951) followed in that tradition. Notice that in the case of ordinal preferences, there is an experiment we can perform to verify an individual's asserted ranking: if he says he prefers x to y, we can offer him the choice and see which he selects. But there is no known corresponding experiment for verifying cardinal utility - except in the case of risk preferences, where we can offer lotteries (in the von Neumann-Morgenstern 1944 procedure for constructing a utility function, utilities are cardinal in the sense that they can be interpreted as probabilities in a lottery). Yet, risk preferences are not the same thing as preference intensities. And taking account of risk preferences in social choice situations in which the alternatives entail *no* uncertainty (e.g., in an election, an alternative is simply a candidate, not a lottery) seems of dubious moral relevance (for that reason, Harsanyi's (1955) derivation of utilitarianism based on risk preferences is often criticized). Finally, even if there were an experiment for eliciting utilities, misrepresentation might interfere with it. Admittedly, there are circumstances with ordinal SWFs when individuals have the incentive to misrepresent their rankings (which is the subject of the Gibbard 1973/Satterthwaite 1975 theorem). But a cardinal SWF is subject to much greater misrepresentation because individuals have the incentive to distort even when there are only two alternatives (see Dasgupta and Maskin 2020). Thus, we are left only with the possibility of inferring preference intensities from *ordinal* preferences.

that, for any permutation σ of $\{x, y, z\}$, $p(u_x, u_y, u_z) = p(u_{\sigma(x)}, u_{\sigma(y)}, u_{\sigma(z)})$, reflecting the idea that, from a sufficiently *ex ante* perspective, the spectator can't distinguish among the three alternatives.[10]

Then,

(1) $\mathrm{E}\left[|u(x) - u(y)| \mid z \text{ between } x \text{ and } y \right] > \mathrm{E}\left[|u(x) - u(y)| \mid z \text{ not between } x \text{ and } y \right]$

That is, the expected difference between $u(x)$ and $u(y)$ is bigger if *z* lies between *x* and *y* in the individual's ranking than if it does not.

To see why (1) holds, assume that $u(x) > u(y)$, and note that the left-hand side of (1) (where $u(x) > u(z) > u(y)$) is then

(2) $\sum_{r>t>s} (r-s) p(r,s,t) \Big/ \sum_{r>t>s} p(r,s,t)$.

By contrast, if, say, $u(x) > u(y) > u(z)$ (the argument is symmetric if $u(z) > u(x) > u(y)$), the right-hand side is

(3) $\sum_{r>s>t} (r-s) p(r,s,t) \Big/ \sum_{r>s>t} p(r,s,t)$.

Now, from the exchangeability of *p*, the denominators of (2) and (3) are equal. Fix r, s, t such that $r > t > s$. From exchangeability, $p(r,s,t)$ in (2) equals $p(r,t,s)$ in (3). But the coefficient of $p(r,s,t)$ in (2) is $r - s$, whereas that of $p(r,t,s)$ in (3) is $r - t$. Since the former is bigger than the latter, (1) is established.

[10] If the SWF is chosen well before it is used, then society won't at that point even know the identities of the alternatives that will arise in those uses. Thus, even if it turns out that, in some uses, the utilities corresponding to alternatives are asymmetrically distributed, each of the *permuted* distributions will be equally likely from an *ex ante* perspective, and so our exchangeability assumption will hold.

Thus, the spectator can make inferences about an individual's expected preference intensity differences from purely ordinal data – despite knowing nothing (other than exchangeability) about the distribution that the intensities are drawn from. Notice, however, that (1) is the *only* such inference that the spectator can make without more information about the distribution. For example, the spectator can't compare

$$\mathrm{E}\left[|u(x)-u(y)| \,\middle|\, z \text{ above } x \text{ and } y\right] \text{ with } \mathrm{E}\left[|u(x)-u(y)| \,\middle|\, z \text{ below } x \text{ and } y\right].$$

In view of all this, we propose the following relaxation of IIA.[11] Under *modified independence of irrelevant alternatives* (MIIA), if given two alternatives x and y and two profiles of individuals' preferences, (i) each individual ranks x and y the same way in the first profile as in the second, and (ii) each individual ranks the same number of alternatives *between* x and y in the first profile as in the second, then the social ranking of x and y must be the same for both profiles.

If we imposed only requirement (i), then MIIA would be identical to IIA. Requirement (ii) is the one that permits preference intensities to figure in social rankings. Specifically, notice that, since z lies between x and y in group 1's preferences in Scenario 3 but not in Scenario 4, MIIA does *not* require the social rankings of x and y to be the same in the two scenarios. That is, accounting for preference intensities is permissible under MIIA.

Even so, MIIA is strong enough to rule out spoilers and vote-splitting (i.e., a SWF satisfying MIIA cannot exhibit the phenomenon of footnote 7). In particular, it rules out plurality rule: in neither Scenario 1 nor Scenario 2 do group 2 voters rank Rubio between Kasich and

[11] Other authors who have considered variants of IIA include Brandl and Brandt (2020), Dhillon and Mertens (1999), Eden (2020), Fleurbaey, Hansson (1973), Mayston (1974), Suzumura, and Tadenuma (2005) and (2005a), Osborne (1976), Roberts (2009), Saari (1998), Young (1988), and Young and Levenglick (1978).

Trump. Therefore, MIIA implies that the social ranking of Kasich and Trump must be the *same* in the two scenarios, contradicting plurality rule.

Runoff voting[12] is also ruled out by MIIA. Under that voting rule, a candidate wins immediately if he is ranked first by a majority of voters.[13] But failing that, the two top vote-getters go to a runoff. Notice, that if we change Scenario 1 so that the middle group constitutes 35% of the electorate and the third group constitutes 25%, then Trump (with 40% of the votes) and Rubio (with 35%) go to the runoff (and Kasich, with only 25%, is left out). Trump then wins in the runoff, because a majority of voters prefer him to Rubio. If we change Scenario 2 correspondingly (so that the 25% and 35% groups are interchanged), then Kasich wins in the first round with an outright majority (of 60%). Thus, runoff voting violates MIIA (and so does instant runoff voting) for essentially the same reason that plurality rule does.

From our previous discussion, observe that, as a relaxation of IIA, MIIA is not only strong enough to rule out spoilers but—when the SWF is adopted before its applications are known—necessary and sufficient for making all possible inferences about utility differences (necessary in the sense that IIA must be relaxed at least this much to take account of (1), and sufficient in the sense that no greater relaxation MIIA would yield any further intrapersonal comparisons of intensities).

We now turn to the other (much weaker) axioms we will invoke.

B. May's Axioms for Majority Rule

When there are just two alternatives, majority rule is far and away the most widely used democratic method for choosing between them. Indeed, almost all other commonly used voting

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[12] Used in France, Brazil, and many other countries for presidential elections and in some U.S. states for congressional elections.

[13] Like plurality rule, runoff voting in *practice* is usually administered so that a voter just picks one candidate rather than ranking them all (see footnote 6).

rules – e.g., plurality rule, runoff voting, instant runoff voting, and the Borda count – reduce to majority rule in this case.

Kenneth May (1952) crystallized why majority rule is so compelling in the two-alternative case by showing that it is the only voting rule satisfying *anonymity* (A), *neutrality* (N), and *positive responsiveness* (PR). Axiom A is the requirement that all individuals be treated equally, i.e., that if they exchange preferences with one another (so that individual *j* gets *i*'s preferences, individual *k* get *j*'s, and so on), social preferences remain the same. N demands that all alternatives be treated equally, i.e., that if the alternatives are permuted and individuals' preferences are changed accordingly, then social preferences are changed in the same way. And PR requires that if alternative *x* rises relative to *y* in some individual's preference ordering, then (i) *x* doesn't fall relative to *y* in the social ordering, and (ii) if *x* and *y* were previously tied socially, *x* is now strictly above *y*. [14]

C. Young's Consistency

Peyton Young (1974) provided a well-known characterization of the Borda count in which the central axiom is a *consistency*[15] condition: if *x* is a top social alternative for each of several different populations, then *x* must be a top social alternative for the union of those populations.

This is a very strong condition. Indeed, Young (1975) shows that, together with U, A, and N, it implies that the SWF must be a scoring rule: there are *m* nonnegative numbers $a_1,...,a_m$ such that each time an alternative is ranked first it gets a_1 points, each time it is ranked second a_2 points, etc. Alternatives are then ranked socially according to their point totals. The set of all

[14] May (1952) expressed the A, N, and PR axioms only for the case of two alternatives. In section 2 we give formal extensions for three or more alternatives (See also Dasgupta and Maskin 2020). Our formulations are weak enough so that they apply to practically every SWF in the literature.
[15] Moulin (1988) calls this axiom "reinforcement."

scoring rules includes both the Borda count and plurality rule (for which $a_1 > 0$ and $a_2 = \cdots = a_m = 0$).

We shall invoke a far weaker axiom called *ranking consistency* (RC), which requires only that if the *entire social ranking* is strict and *identical* for each of several disjoint populations, then its (unique) top-ranked alternative must be socially top-ranked for the union. In fact, RC is so mild that (as far as we can tell) it is satisfied by every standard voting method used in practice and nearly every one studied in the literature (see Section 2).

D. Borda's Rule and Condorcet Cycles: A Special Case

The main result of this paper establishes that a continuous[16] SWF satisfies U, MIIA, A, N, PR, and RC (the other Arrow conditions – P and ND – are redundant) if and only if it is the Borda count.[17] Checking that the Borda count satisfies the six axioms[18] and continuity is straightforward.[19] We have already noted that U and RC are almost universally satisfied by SWFs in the literature. The same is true of A, N, PR, and continuity. Thus, our main result[20]

[16] An SWF is continuous if the set of profiles for which x is weakly preferred socially to y is closed. We express this formally in Section 3.
[17] Saari (2000) and (2000a) provide a vigorous defense of the Borda count based on its geometric properties.
[18] Notice that the Borda count implies a particular way of making interpersonal comparisons, e.g., if individual 1 ranks *x* two positions above *y*, that preference is exactly cancelled by two individuals who rank *y* one position above *x*. Observe, however, that none of our axioms speaks directly to interpersonal comparisons at all; such comparisons are an emergent property of the *joint* imposition of the axioms.
[19] To see that the Borda count satisfies MIIA, note that if two profiles satisfy the hypotheses of the condition, then the difference between the number of points a given voter contributes to *x* and the number she contributes to *y* must be the *same* for the two profiles (because the number of alternatives ranked between *x* and *y* is the same). Thus, the differences between the total Borda scores of *x* and *y* – and hence their social rankings – are the same. To see that the Borda count satisfies RC, imagine that the Borda ranking for *x* and *y* is the same for each of several disjoint subpopulations. Because the Borda scores for the union of the subpopulations are just the sums of those for the individual subpopulations, the Borda ranking of *x* and *y* for the union population must coincide with that for the subpopulations.
[20] The result is "tight" in the sense that dropping any single condition renders it invalid. Majority rule satisfies everything but U. Runoff voting and instant runoff voting satisfy everything but MIIA. The weighted Borda count (where different individuals have different weights) satisfies everything but A. The asymmetric Borda count (where *x*, say, gets $k+1-t$ points every time someone ranks it in position *t* but *y* gets $k+2-t$ points for this position)

implies that, for standard voting rules, MIIA is the axiom that uniquely distinguishes the Borda count.

To illustrate a central idea of the proof, let us focus on the case of three alternatives x, y, and z and suppose that F is a SWF satisfying the six axioms. We will show that when F is restricted to the domain of preferences $\left\{\begin{matrix} x & y & z \\ y, & z, & x \\ z & x & y \end{matrix}\right\}$ (i.e., when we consider only profiles with preferences drawn from this domain), it must coincide with the Borda count.

Consider, first, the profile in which 1/3 of individuals have ranking $\begin{matrix} x \\ y \\ z \end{matrix}$; 1/3 have ranking $\begin{matrix} y \\ z \\ x \end{matrix}$; and 1/3 have ranking $\begin{matrix} z \\ x \\ y \end{matrix}$.[21] We claim that the social ranking of x and y that F assigns to this profile is *social indifference*:

(4) $$\begin{matrix} \underline{1/3} & \underline{1/3} & \underline{1/3} \\ x & z & y \\ y & x & z \\ z & y & x \end{matrix} \xrightarrow{F} x \sim y$$

If (4) doesn't hold, then either

(5) $$\begin{matrix} \underline{1/3} & \underline{1/3} & \underline{1/3} \\ x & z & y \\ y & x & z \\ z & y & x \end{matrix} \xrightarrow{F} \begin{matrix} x \\ y \end{matrix}$$

or

violates only N. The SWF in which all alternatives are deemed socially indifferent regardless of individuals' rankings satisfies everything but PR. For RC, consider the following SWF devised by G. Gendler for the case $|X| = 3$: For a given profile, rank x above y socially if and only if $(6+\beta_2+\beta_{-1})(2\beta_2-\beta_{-1})+(6+\beta_1+\beta_{-2})(\beta_1-2\beta_{-2})>0$, where β_k for $k=1,2$, is the fraction of individuals in the profile who rank x k places above y and β_{-k} is the corresponding fraction who rank y k places above x. Gendler (2023) shows that this SWF satisfies all the axioms except RC. After the proof of the main theorem, we give an example of a non-Borda SWF that satisfies all axioms except continuity.

[21] From A, we don't need to worry about which individuals have which preferences.

$$(6)\qquad \begin{array}{ccc} \underline{1/3} & \underline{1/3} & \underline{1/3} \\ x & z & y \\ y & x & z \\ z & y & x \end{array} \xrightarrow{F} \begin{array}{c} y \\ x \end{array}$$

If (5) holds, then apply permutation σ – with $\sigma(x)=y,\ \sigma(y)=z,$ and $\sigma(z)=x$ – to (5). From N, we obtain

$$(7)\qquad \begin{array}{ccc} \underline{1/3} & \underline{1/3} & \underline{1/3} \\ y & x & z \\ z & y & x \\ x & z & y \end{array} \xrightarrow{F} \begin{array}{c} y \\ z \end{array}$$

Applying σ to (7) and invoking N, we obtain

$$(8)\qquad \begin{array}{ccc} \underline{1/3} & \underline{1/3} & \underline{1/3} \\ z & y & x \\ x & z & y \\ y & x & z \end{array} \xrightarrow{F} \begin{array}{c} z \\ x \end{array}$$

But the profiles in (5), (7), and (8) are the same except for permutations of individuals' preferences, and so, from A, give rise to the same social ranking under *F*, which in view of (5), (7), and (8) must be

$$\begin{array}{c} x \\ y \\ z \\ x \end{array}\ ,$$

violating transitivity. The analogous contradiction arises if (6) holds. Hence, (4) must hold after all. From MIIA and (4), we have

$$(9)\qquad \begin{array}{ccc} \underline{a} & \underline{b} & \underline{1/3} \\ x & z & y \\ y & x & z \\ z & y & x \end{array} \xrightarrow{F} x \sim y, \text{ for all } a \geq 0 \text{ and } b \geq 0 \text{ such that } a+b=2/3$$

From PR and (9), we have

$$(10)\qquad \begin{array}{ccc} \underline{a} & \underline{b} & \underline{1-a-b} \\ x & z & y \\ y & x & z \\ z & y & x \end{array} \xrightarrow{F} \begin{array}{c} x \\ y \end{array}, \text{ where } a+b>2/3, \text{ and } a,b,1-a-b \geq 0,$$

and

$$(11)\quad \begin{array}{ccc} \underline{a} & \underline{b} & \underline{1-a-b} \\ x & z & y \\ y & x & z \\ z & y & x \end{array} \xrightarrow{F} \begin{array}{c} y \\ x \end{array}, \text{ where } a+b<2/3 \text{ , and } a,b \geq 0 .$$

But (9), (10), and (11) collectively imply that x is socially preferred to y if and only if x's Borda score exceeds y's Borda score,[22] i.e., F is the Borda count[23]. Q.E.D

The domain $\left\{ \begin{array}{ccc} x & z & y \\ y, & x, & z \\ z & y & x \end{array} \right\}$ is called a Condorcet cycle because, as Condorcet (1785) showed, majority rule may cycle for profiles on this domain (indeed, it cycles for the profile in (4)). This domain is the focus of much of the social choice literature, e.g., Arrow (1951) makes crucial use of Condorcet cycles in the proof of the Impossibility Theorem; and Barbie et al (2006) show that it is essentially the unique domain (for three alternatives) on which the Borda count is strategy-proof. One immediate implication of our result in this section is that the Borda count is the unique voting rule satisfying A, N, PR, and IIA on a Condorcet cycle domain - - since IIA and MIIA are equivalent on this domain.

E. Outline

In Section 2, we lay out the model and the axioms. Section 3 introduces the critical concept of an indifference curve for a SWF. In Section 4, we show that a SWF satisfying our

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[22] Alternative x's Borda score in (9) – (11) is $3a+2b+1-a-b$, and y's Borda score is $3(1-a-b)+2a+b$. Hence, x is Borda-ranked above y if and only if

$$3a+2b+1-a-b > 3(1-a-b)+2a+b,$$

which reduces to $a+b>2/3$, i.e., we obtain formula (10). Similarly, for (11).

[23] Notice that, for this special case, we did not need to invoke axiom RC. This is because for profiles on this domain, we can infer social indifference between x and y using A, N, and MIIA alone (as we do for the profile (4)). Similarly, the symmetry of (4) dispenses with the need for a continuity assumption. For the general proof, we will show, roughly speaking, that any profile can be decomposed into subprofiles for each of which such symmetry considerations are enough to imply that the social ranking is Borda. We then apply RC and continuity to conclude that the same is true of the social ranking for the combined profile.

axioms must be the Borda count. Section 5 concludes the paper by discussing a few open questions. The proofs in Sections 3 and 4 are given for the case of three alternatives. The Appendix generalizes the arguments to any number of alternatives.

2. Formal Model and Definitions

Consider a society consisting of a continuum of individuals[24] (indexed by $i \in [0,1]$) and a finite set of social alternatives X, with $|X| = m+1$.[25] For each individual *i*, let $\Re_i$ be a set of possible *strict* rankings[26] of X for individual *i,* and let $\succ_i$ be a typical element of $\Re_i$ ($x \succ_i y$ means that individual *i* strictly prefers alternative *x* to *y*). A *profile* for $[0,1]$ is a specification of every individual's ranking, i.e., profile $\succ_{\cdot}$ is an element of $\underset{i\in[0,1]}{\times}\Re_i$. With a continuum of individuals, we can't literally count the number of individuals with a particular preference; we have to work with proportions instead. For that purpose, let μ be Lebesgue measure on $[0,1]$. Given profile $\succ_{\cdot}$ on $[0,1]$, interpret $\mu(\{i | x \succ_i y\})$ as the proportion of individuals who prefer *x* to *y*.[27] To apply our ranking consistency condition, we need to consider subpopulations of [0,1]. Accordingly, we define a *social welfare function* (SWF) *F* to be a mapping such that for all subsets *C* of positive measure $C \subseteq [0,1]$, $F : \underset{i\in C}{\times}\Re_i \to \Re$, where $\Re$ is the set of all possible social rankings (here we *do*

[24] In assuming a continuum, we are following Dasgupta and Maskin (2008) and (2020). Those earlier papers invoked this assumption primarily to ensure that ties (social indifference) are nongeneric. The assumption plays that role in this paper too, but more importantly, it guarantees together with our continuity assumption that ties actually *occur*. Indeed, our proof technique relies critically on analyzing a SWF's indifference curve, i.e., the set of profiles for which there are ties.

[25] $|X|$ is the number of alternatives in *X*.

[26] Thus, we rule out the possibility that an individual can be indifferent between two alternatives. However, we conjecture that our results extend to the case where she can be indifferent (see Section 5).

[27] To be accurate, we must restrict attention to profiles $\succ_{\cdot}$ for which $\{i | x \succ_i y\}$ is a measurable set.

allow for indifference and the typical element is $\succsim$). That is, for each profile $\succ_{\cdot}$ on *C*, *F* assigns a social ranking $F(\succ_{\cdot})$.

The Arrow conditions for a SWF *F* are:

Unrestricted Domain (U): The SWF must determine social preferences for all possible preferences that individuals might have. Formally, for all $i \in [0,1]$, $\Re_i$ consists of *all* strict orderings of *X*.

Pareto Property (P): If all individuals (strictly) prefer *x* to *y*, then *x* must be strictly socially preferred to *y* . Formally, for all $C \subseteq [0,1]$ of positive measure, all profiles $\succ_{\cdot} \in \times_{i \in C} \Re_i$, and all $x, y \in X$, if $x \succ_i y$ for all *i*, then $x \succ_F y$, where $\succsim_F = F(\succ_{\cdot})$.

Nondictatorship (ND): There exists no individual who always gets his way in the sense that if he prefers *x* to *y*, then *x* must be socially preferred to *y*, regardless of others' preferences. Formally, for all $C \subseteq [0,1]$ of positive measure, there does *not* exist $i^* \in C$ such that for all $\succ_{\cdot} \in \times_{i \in C} \Re_i$ and all $x, y \in X$, if $x \succ_{i^*} y$, then $x \succ_F y$, where $\succsim_F = F(\succ_{\cdot})$.

Independence of Irrelevant Alternatives (IIA): Social preferences between *x* and *y* should depend only on individuals' preferences between *x* and *y*, and not on their preferences concerning some third alternative. Formally, for all $C \subseteq [0,1]$ of positive measure, all $\succ_{\cdot}, \succ'_{\cdot} \in \times_{i \in C} \Re_i$ and all $x, y \in X$, if, for all i, $x \succ_i y \Leftrightarrow x \succ'_i y$, then $\succsim_F$ ranks *x* and *y* the same way that $\succsim'_F$ does, where $\succsim_F = F(\succ_{\cdot})$ and $\succsim'_F = F(\succ'_{\cdot})$.

Because we have argued that IIA is too strong, we are interested in the following relaxation:

Modified IIA: If, given two profiles and two alternatives, each individual (i) ranks the two alternatives the same way in both profiles and (ii) ranks the same number of other alternatives *between* the two alternatives in both profiles, then the social preference between *x* and *y* should be the same for both profiles. Formally, for all $C \subseteq [0,1]$ of positive measure, all $\succ_{\cdot}, \succ'_{\cdot} \in \times_{i \in C} \Re_i$, and all $x, y \in X$, if, for all *i*, $x \succ_i y \Leftrightarrow x \succ'_i y$, $\left|\{z | x \succ_i z \succ_i y\}\right| = \left|\{z | x \succ'_i z \succ'_i y\}\right|$, and $\left|\{z | y \succ_i z \succ_i x\}\right| = \left|\{z | y \succ'_i z \succ'_i x\}\right|$, then $\succsim_F$ and $\succsim'_F$ rank *x* and *y* the same way, where $\succsim_F = F(\succ_{\cdot})$ and $\succsim'_F = F(\succ'_{\cdot})$.[28]

May (1952) characterizes majority rule axiomatically in the case $|X| = 2$. We will consider natural extensions of his axioms to three or more alternatives:

Anonymity (A): If we permute a preference profile so that individual *j* gets *i*'s preferences, *k* gets *j*'s preferences, etc., then the social ranking remains the same. Formally, fix $C \subseteq [0,1]$ of positive measure and a (measure-preserving)[29] permutation of society $\pi : C \to C$. For any profile $\succ_{\cdot} \in \times_{i \in C} \Re_i$, let $\succ^{\pi}_{\cdot}$ be the profile such that, for all *i*, $\succ^{\pi}_i = \succ_{\pi(i)}$. Then $F(\succ^{\pi}_{\cdot}) = F(\succ_{\cdot})$.

Neutrality (N): Suppose that we permute the alternatives so that *x* becomes *y*, *y* becomes *z*, etc., and we change individuals' preferences in the corresponding way. Then, if *x* was socially ranked above *y* originally, now *y* is socially ranked above *z*. Formally, for any permutation $\sigma : X \to X$, $C \subseteq [0,1]$ of positive measure, and any profile $\succ_{\cdot} \in \times_{ieC} \Re_i$, let $\succ^{\sigma}_{\cdot}$ be the profile such that, for all

[28] There is a similar condition developed in Maskin (2020).

[29] Because we are working with a continuum of individuals, we must explicitly assume that $\mu(\{i | x \succ_i y\}) = \mu(\{i | x \succ_{\pi(i)} y\})$, which holds automatically with a finite number of individuals.

$x, y \in X$ and all $i \in C$, $x \succ_i y \Leftrightarrow \sigma(x) \succ_i^\sigma \sigma(y)$. Then, $x \succsim_F y \Leftrightarrow \sigma(x) \succsim_F^\sigma \sigma(y)$ for all $x, y \in X$, where $\succsim_F = F(\succ_\cdot)$ and $\succsim_F^\sigma = F(\succ_\cdot^\sigma)$.

Positive Responsiveness (PR) [30]: Consider alternatives *x* and *y* and a profile such that each individual top-ranks either x or y.[31] If we change some individuals' rankings so that either (a) if *y* was preferred to *x*, now *x* is preferred to *y*, or (b) *x* moves up (weakly) relative to all other alternatives, *y* moves down (weakly) relative to all other alternatives, and no other alternative moves, then (*i*) socially *x* moves up weakly relative to *y*, and (*ii*) if the set of individuals in (a) has positive measure and *x* and *y* were originally socially indifferent, *x* is now strictly preferred to *y* socially. Formally, suppose, for $C \subseteq [0,1]$ of positive measure, $\succ_\cdot$ and $\succ_\cdot'$ are two profiles on $\times_{i \in C} \Re_i$ such that, for some $x, y \in X$ and all $j \in C$, either $x \underset{j}{\succ} z$ for all $z \neq x$ or $y \underset{j}{\succ} w$ for all $w \neq y$, and for all $j \notin \{i | y \succ_i x \text{ and } x \succ_i' y\}$,

(*) $x \succ_j z \Rightarrow x \succ_j' z$, $w \succ_j y \Rightarrow w \succ_j' y$, and $r \succ_j s \Leftrightarrow r \succ_j' s$ for all $z \neq x$, $w \neq y$ and

$r, s \in X - \{x, y\}$.

Then, (*i*) $x \succsim_F y \Rightarrow x \succsim_F' y$ and $x \succ_F y \Rightarrow x \succ_F' y$ (where $\succsim_F = F(\succ_\cdot)$ and $\succsim_F' = F(\succ_\cdot')$), and (*ii*) if, in addition $\mu\{i | y \succ_i x \text{ and } x \succ_i y\} > 0$, then $x \sim_F y \Rightarrow x \succ_F' y$.

As defined, PR applies only to profiles in which every individual top-ranks either *x* or *y* (the two alternatives being compared); as footnote 31 explains, we make this restriction to ensure that the axiom holds for any reasonable SWF. However, a SWF satisfying U, PR and MIIA will also satisfy *ordinary PR* (for which the restriction to top-rankedness is *not* made). This is

[30] For a different generalization of PR to more than two alternatives, see Horan, Osborne, and Sanver (2019).
[31] We place this restriction on profiles to make PR weak enough to be satisfied by all SWFs mentioned in this paper. In particular, it is satisfied by plurality rule, which takes account only of individuals' top-ranked alternatives.

intuitive because, for an individual who prefers *x* to *y*, MIIA and U allow us to move *x*, *y*, and all alternatives between them up in the individual's ranking so that *x* is now top-ranked but the social ranking of *x* and *y* doesn't change. Here is a formal proof:

Lemma 0: A SWF that satisfies U, PR and MIIA also satisfies ordinary PR.

Proof: Suppose that $\succ_{\cdot}$ and $\succ'_{\cdot}$ are two profiles such that, for some $x, y \in X$ and all $j \notin \{i | y \succ_i x \text{ and } x_i \succ'_i y\}$, condition $(*)$ in the definition of PR holds but for which individuals do not necessarily rank *x* or *y* at the top. Assume that

(12) $x \succsim_F y$.

We must show that $x \succsim'_F y$. Consider profiles $\hat{\succ}_{\cdot}$ and $\hat{\succ}'_{\cdot}$, where $\hat{\succ}'_{\cdot}$ is the same as $\succ_{\cdot}$ except that, for each individual, alternatives *x*, *y*, and all alternatives between them are moved to the top of the individual's ranking (without changing the ranking of those intermediate alternatives or the alternatives not between *x* and *y*) and $\hat{\succ}'_{\cdot}$ has the corresponding relation to $\succ'_{\cdot}$. We will show that $\hat{\succ}_{\cdot}$ and $\hat{\succ}'_{\cdot}$ satisfy $(*)$ of the definition of PR. Note first that, from (12) and MIIA,

(13) $x \hat{\succsim}_F y$.

Suppose, for $j \notin \{i | y \hat{\succ}_i x \text{ and } x \hat{\succ}_i y\}$ and some, that $z \neq x$, that

(14) $x \hat{\succ}_j z$

Assume first that *z* lies between *x* and *y* (including *y*) in ranking $\succ_j$. Then from (14), $x \succ_j z \succsim_j y$ and so from $(*)$

(15) $x \succ'_j z$

But (15) implies

(16) $x \hat{\succ}'_j z$

Assume next that z does not lie between x and y in ranking $\succ_j$. If z also does not lie between x and y in ranking $\succ_j'$, then (15) implies that (16) holds. If z *does* lie between x and y in ranking $\succ_j'$, then (16) holds once again provided that $x \succ_j' y$. Thus, suppose instead that

(17) $y \succ_j' z \succ_j' x$

But then, from $(*)$ applied to x and y, either $z \succ_j y \succ_j x$ or $y \succ_j x \succ_j z$. Yet, in the former case, y rises relative to z in going from $\succ_j$ to $\succ_j'$ and in the latter case, x falls relative to z in going from $\succ_j$ to $\succ_j'$ − both violations of $(*)$. Thus, we conclude that (14) always implies (16).

Similarly, we can show that $w \hat{\succ}_j y \Rightarrow w \hat{\succ}_j' y$ for $w \neq y$ and $r \hat{\succ}_j s \Rightarrow r \hat{\succ}_j' s$ for $r, s \notin \{x, y\}$.

That is, $\hat{\succ}_.$ and $\hat{\succ}_.'$, satisfy $(*)$. PR then implies, from (13) that

$$x \hat{\succsim}_F' y$$

MIIA then implies $x \succsim_F' y$, as we needed to show. The argument is completely analogous for showing that $x \succ_F y \Rightarrow x \succ_F' y$ and, given $\mu\{i | y \succ_i x \text{ and } x \succ_i\} > 0$, that $x \sim_F y \Rightarrow x \succ_F' y$.

Hence, ordinary PR holds.

Q.E.D.

Henceforth, we shall always interpret PR as "ordinary PR."

Ranking Consistency (RC): If, given a profile of individual preferences, each of a set of disjoint subpopulations has the identical strict social ranking, then the (unique) top-ranked alternative for that ranking is also the (unique) top-ranked alternative for the union of those subpopulations.

Formally, consider a partition $C^1, \ldots, C^k$ of $[0,1]$ and a profile $\succ_.$ on $\times_{i \in [0,1]} \Re_i$ For each $h \in \{1, \ldots k\}$ let

$\succ_{\cdot}^{h}$ be the restriction of $\succ_{\cdot}$ to the individuals in C^h and suppose that $F\left(\succ_{\cdot}^{1}\right)=\cdots=F\left(\succ_{\cdot}^{k}\right)= \succ$, where $\succ$ is a strict ranking. Then, if $x \succ y$ for all $y \neq x$, we have $x \succ_{F(\succ_{\cdot})} y$ for all $y \neq x$.

We are not aware of a SWF actually used in practice that fails to satisfy RC. Indeed, RC holds for almost any SWF studied in the literature[32]. For example, besides scoring rules (which include plurality rule and the Borda count), it is satisfied by instant-runoff voting,[33] Coomb's rule (which is the same as instant-runoff voting except that instead of eliminating the candidate ranked first least often, it drops the candidate ranked last most often), ordinary runoff voting[34] and majority rule.[35] RC also holds for a wide array of Condorcet-conforming voting methods (methods that elect a Condorcet winner if one exists and otherwise rank candidates some other way). For example, the Kemeny-Young method – in which the social ranking minimizes the sum of the Kendall tau distances (the Kendall tau distance between two rankings is the total number of discordant pairs) between it and the individuals' rankings—satisfies RC,[36] as do Copeland's

[32] This is in contrast with ordinary consistency (Young 1974), which, in combination with U, A, and N, is satisfied only by scoring rules.

[33] Suppose that, given profile $\succ_{\cdot}$ on $[0,1]$, each of the subpopulations $C_1,...,C_k$ eliminates alternative x_m first in an instant run-off election, then x_{m-1}, and so on until only x_1 remains (so that the social ranking for each subpopulation is $x_1 \succ x_2 \succ \cdots \succ x_m$). Because x_m is eliminated first in each subpopulation, it must be ranked first least often in the overall population $[0,1]$, and so will be eliminated first in the instant runoff. But then the same argument applies to x_{m-1}, etc. In other words, the social ranking for $[0,1]$ is, again, $x_1 \succ \cdots \succ x_m$.

[34] For $|X| = 3$, ordinary runoff voting is the same as instant runoff voting (assuming that individuals vote according to their rankings). For $|X| > 3$, all alternatives that don't get into the runoff are considered socially indifferent to one another. Thus, in the latter case, RC is vacuously satisfied since the social ranking can't be strict.

[35] Suppose that, given profile $\succ_{\cdot}$ on $[0,1]$, the majority social ranking in each subpopulation $C_1,...,C_k$ is $x_1 \succ x_2 \succ \cdots \succ x_m$. That is, for every $r < s$, a majority of individuals in each subpopulation C_k prefer x_r to x_s. But then a majority of individuals in the overall population $[0,1]$ must also prefer x_r to x_s. And so, the same majority ranking $x_1 \succ x_2 \succ ... \succ x_m$ holds for $[0,1]$. Majority rule actually satisfies the stronger condition, ordinary consistency. But because it fails to satisfy U, it doesn't violate Young's (1978) scoring-rule theorem.

[36] See the Wikipedia article on the Kemeny-Young method https://en.wikipedia.org/wiki/Kemeny-Young_method.

method,[37] Smith's method,[38] Baldwin's method,[39] Tideman's ranked-pairs method,[40] and the minimax Condorcet method[41] (these last two satisfy RC for the case $|X| = 3$ only). Finally, we note that two other popular methods – approval voting[42] and range voting[43] – also satisfy RC (strictly speaking, neither is a SWF in our formal sense, since we require that the social ranking depend only on ordinal information about individuals' preference – and both rely on cardinal data).

We now define the Borda count precisely:

Borda Count: Alternative *x* is socially (weakly) preferred to *y* if and only if *x*'s Borda score is (weakly) bigger than *y*'s Borda score (where *x* gets *m* points every time an individual ranks it

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[37] In Copeland's method, alternatives are ranked socially according to how many other alternatives they defeat by a majority in a pairwise comparison. This method satisfies RC because if, for a given subpopulation, there is a strict social ranking of the *m* alternatives, then the top-ranked alternative must defeat each of the other $m-1$ alternatives (i.e., it is a Condorcet winner), the second-ranked alternative must defeat all but the Condorcet winner, etc. And if this same social ranking holds for all other subpopulations, then it must also hold for the overall population.

[38] The Smith set is the smallest set of alternatives each of which defeats any alternative not in the set by a majority in a pairwise comparison. Smith's method chooses the Smith set as the top indifference curve in the social ranking, the Smith set for the remaining alternatives once the top indifference curve is removed, etc. It satisfies RC because if, as RC demands, the social ranking is a strict ordering, then the ranking is the same as for majority rule.

[39] Baldwin's method is a variant of instant-runoff voting in which if no alternative is ranked first by a majority of votes, the alternative with the lowest Borda score is dropped, and the process iterates with this reduced set of alternatives. It is a Condorcet-conforming method because the alternative with the lowest Borda score can't be a Condorcet winner. It satisfies RC by argument similar to that for standard IRV.

[40] In the ranked-pairs method (Tideman 1987), for each pair of alternatives *x* and *y*, *x* is provisionally ranked above *y* socially (for a given profile) if and only if a majority of individuals prefer *x* to *y*. These pairwise rankings are then sequentially locked in: first, the ranking for the pair for which the majority is largest, then the one for the second-largest majority, etc. If, however, we reach a pair for which locking in would create a Condorcet cycle, that pair is skipped. To see that the method satisfies RC for three alternatives, assume $X = \{x, y, z\}$, and suppose that the social ranking is $x \succ y \succ z$ for each subpopulation. Within a subpopulation, the only way that the majority ranking between some pair could differ from the social ranking of that pair is if a majority prefer *z* to *x* but the majorities for *x* over *y* and *y* over *z* are bigger. But then the same must be true for the overall population: the only way that an overall majority could prefer *z* to *x* is if the majorities for *x* over *y* and *y* over *z* are bigger, so that the overall social ranking is still $x \succ y \succ z$.

[41] In the minimax Condorcet method with three alternatives, the alternatives are socially ranked according to majority rule unless there is a Condorcet cycle, in which case the social ranking corresponding to the narrowest margin is reversed. Thus, in this case, the method is the same as Tideman's.

[42] In approval voting, each individual approves or disapproves each alternative, and alternatives are ranked according to their approval totals. RC is satisfied because approvals are additive across subpopulations.

[43] In range voting, an individual "grades" each alternative on a numerical scale, and alternatives are ranked according to their total grades. RC is satisfied because total grades are additive across subpopulations.

first, $m-1$ points every time an individual ranks it second, etc.). Formally, for all $C \subseteq [0,1]$ of nonzero measure, all $x, y \in X$, and all profiles $\succ_{\cdot} \in \times_{i\in C} \Re_i$,

(**) $\quad x \succsim_{Bor} y \iff \int r_{\succ_i}(x) d\mu(i) \geq \int r_{\succ_i}(y) d\mu(i),$

where $r_{\succ_i}(x) = \left|\{ y \in X \,|\, x \succ_i y \}\right| + 1$ and $\succsim_{Bor}$ is the Borda ranking corresponding to $\succ_{\cdot}$.

3. The Indifference Curve of a Social Welfare Function

The proof of our characterization result makes much use of a SWF *F*'s indifference curve. To define this concept, let us start with the case of three alternatives[44] $X = \{x, y, z\}$ and fix a profile $\succ_{\cdot}$. Let $a_{xyz}(\succ_{\cdot})$ be the fraction of individuals who have ranking $\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}$. Then, if *F* satisfies A, the 6-tuple

(17) $\quad \alpha = (\alpha_{xzy}, \alpha_{yzx}, \alpha_{xyz}, \alpha_{zxy}, \alpha_{yxz}, \alpha_{zyx})$

$$= (a_{xzy}(\succ_{\cdot}), a_{yzx}(\succ_{\cdot}), a_{xyz}(\succ_{\cdot}), a_{zxy}(\succ_{\cdot}), a_{yxz}(\succ_{\cdot}), a_{zyx}(\succ_{\cdot}))$$

is a sufficient statistic for $\succ_{\cdot}$ in determining social preferences $\succsim_F$ and we can use the 6-tuple interchangeably with $\succ_{\cdot}$. In particular, we can now define what it means for SWF *F* (satisfying A) to be continuous.

Continuity: If, for any $r, s \in \{x, y, z\}$, the set of profiles $\{\alpha \,|\, r \succsim_{F(\alpha)} s\}$ is closed,[45] then *F* is continuous.

We define *F*'s *indifference curve* for *x* and *y*, I_F^{xy}, to be the set of 6-tuples for which society is indifferent between *x* and *y* according to

[44] The case $|X| > 3$ is handled in the Appendix.

[45] Just as the axioms U, A, N, PR, and RC are satisfied by virtually all SWFs in the literature, so is continuity.

$F: I_F^{xy} = \left\{\alpha \in \Delta^5 \middle| x \sim_{F(\alpha)} y \text{ for } \alpha \text{ satisfying (17)}\right\}$. For example, the Borda indifference curve is given by

(18) $$I_{Bor}^{xy} = \left\{\alpha \middle| \alpha_{xyz} + \alpha_{zxy} + 2\alpha_{xzy} = \alpha_{yxz} + \alpha_{zyx} + 2\alpha_{yzx}\right\}.$$

The indifference curve is useful in proving that a SWF *F* satisfying the axioms is the Borda count. In particular, we rely on the following simple result:

Lemma 1: Suppose that *F* satisfies U, A, N, MIIA, and PR. If, for some $x, y \in X$,

(19) $$I_{Bor}^{xy} \subseteq I_F^{xy}$$

then

(20) *F* = Borda count.

In other words, to show that *F* and the Borda count coincide, we need show only that *F*'s indifference curve contains the Borda indifference curve. And, as the proof demonstrates, this follows largely because of PR and MIIA.

Proof: Suppose that (19) holds for some $x, y \in X$ but there exist $\alpha = \left(\alpha_{xzy}, \alpha_{yzx}, \alpha_{xyz}, \alpha_{zxy}, \alpha_{yxz}, \alpha_{zyx}\right)$ for which

(21) $$F(\alpha)\big|_{\{x,y\}} \neq Bor(\alpha)\big|_{\{x,y\}},$$

where $F(\alpha)\big|_{\{x,y\}}$ is the restriction of $F(\alpha)$ to $\{x, y\}$. Without loss of generality, we can assume that

(22) $$x \succ_{Bor(\alpha)} y \text{ and } y \succsim_{F(\alpha)} x$$

(if (21) holds yet we have social indifference for the Borda count, then we contradict (19)).

Suppose that we continuously decrease α_{xzy}, a_{xyz}, and α_{zxy} while increasing α_{yxz} by the same amounts. If all of α_{xzy}, α_{xyz}, and α_{zxy} are reduced to zero, then $y \succ_{Bor} x$ for the corresponding profile. Thus, because the Borda count is continuous, we must, before then, reach a 6-tuple α^* for which

(23) $\alpha^* \in I^{xy}_{Bor}$

But from PR and (22), $y \succ_{F(\alpha^*)} x$, which, in view of (23) contradicts (19).

Q.E.D.

In section 1D we showed that a SWF satisfying U, A, and N must generate social indifference among *x*, *y*, and *z*[46] for the profile: $\alpha^o = \left(\alpha^o_{xzy},\ \alpha^o_{yzx},\ \alpha^o_{xyz},\ \alpha^o_{zxy},\ \alpha^o_{yxz},\ \alpha^o_{zyx}\right)$

$= (0,\ 1/3,\ 1/3,\ 1/3,\ 0,\ 0)$, i.e.,

(24)
$$= \begin{array}{ccc} 1/3 & 1/3 & 1/3 \\ \hline y & x & z \\ z & y & x \\ x & z & y \end{array} \xrightarrow{F} x - y - z$$

By symmetry, the same is true for $\alpha^{oo} = \left(\alpha^{oo}_{xzy},\ \alpha^{oo}_{yzx},\ \alpha^{oo}_{xyz},\ \alpha^{oo}_{zxy},\ \alpha^{oo}_{yxz},\ \alpha^{oo}_{yxz},\ \alpha^{oo}_{zyx}\right)$

$= (1/3,\ 0,\ 0,\ 0,\ 1/3,\ 1/3)$, i.e.,

(25)
$$= \begin{array}{ccc} 1/3 & 1/3 & 1/3 \\ \hline x & y & z \\ z & x & y \\ y & z & x \end{array} \xrightarrow{F} x - y - z$$

Consider next

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[46] More specifically, we demonstrated social indifference between *x* and *y*, but N then implies that we have social indifference among all three alternatives.

$$\alpha^* = \left(\alpha^*_{xzy},\ \alpha^*_{yzx},\ \alpha^*_{xyz},\ \alpha^*_{zxy},\ \alpha^*_{yxz},\ \alpha^*_{zyx}\right)$$

(26) $$= \left(1/2,\ 1/2,\ 0,\ 0,\ 0,\ 0\right)$$

$$= \begin{array}{cc} 1/2 & 1/2 \\ \hline x & y \\ z & z \\ y & x \end{array}$$

If $x \succ_{F(\alpha^*)} y$, then for permutation $\sigma(x) = y,\ \sigma(y) = x,\ \sigma(z) = z$

(27) $y \succ_{F(\alpha')} x$

for F satisfying U, A, and N, where

$$\alpha' = \begin{array}{cc} 1/2 & 1/2 \\ \hline y & x \\ z & z \\ x & y \end{array}$$

But, from A, $F(\alpha^*) = F(\alpha')$, a contradiction of (27). Hence,

(28) $x \sim_{F(\alpha^*)} y$

Finally, consider

$$\alpha^{**} = \left(\alpha^{**}_{xzy},\ \alpha^{**}_{yzx},\ \alpha^{**}_{xyz},\ \alpha^{**}_{zxy},\ \alpha^{**}_{yxz},\ \alpha^{**}_{zyx}\right)$$

(29) $$= \left(0,\ 0,\ 1/2,\ 0,\ 0,\ 1/2\right)$$

$$= \begin{array}{cc} 1/2 & 1/2 \\ \hline x & z \\ y & y \\ z & x \end{array}$$

Using the same permutation as in the previous paragraph, we can again infer that

(30) $x \sim_{F(\alpha^{**})} y$

provided that MIIA also holds.

From (28) and (30), we can deduce

(31) $x \sim_{F(\alpha^*)} y \sim_{F(\alpha^*)} z$

and

(32) $x \sim_{F(\alpha^{**})} y \sim_{F(\alpha^{**})} z$

for *F* satisfying U, A, N, and MIIA.

Now, in the following section, we will show that for any $\alpha \in I_{Bor}^{xy}$, α can "essentially" be expressed as a convex combination of α^o, α^{oo}, α^*, and α^{**}.[47] As part of our effort to show that a continuous SWF satisfying U, A, N, MIIA, PR, RC coincides with the Borda count, we next show:

Lemma 2: If *F* is continuous and satisfies U, A, N, MIIA, PR, and RC, and α is a convex combination of α^o, α^{oo}, α^*, and α^{**} as defined by (24) – (27) (i.e., $\alpha = c^o\alpha^o + c^{oo}\alpha^{oo} + c^*\alpha^* + c^{**}\alpha^{**}$, where $c^o + c^{oo} + c^* + c^{**} = 1$ and c^o, c^{oo}, c^*, $c^{**} \geq 0$), then $\alpha \in I_F^{xy}$.

Proof: As we have already noted, α^o, α^{oo}, α^*, $\alpha^{**} \in I_F^{xy}$. So, if we could directly apply RC to the convex combination α of these four profiles, we would be done. But RC is considerably weaker than this and applies only when the social rankings for the constituent profiles are strict (and identical).

Accordingly, we first perturb each of α^o, α^{oo}, α^*, α^{**} to make the corresponding social ranking $x \succ y \succ z$; we then make corresponding perturbations to obtain social ranking $y \succ x \succ z$. Using RC, we next show that a convex combination of the first perturbations yields social

[47] More precisely, we may have to move *z* from the bottom to the top of some individuals' rankings or vice versa, but from MIIA, this doesn't affect the social preference between *x* and *y*.

ranking $x \succ y$ and that a convex combination of the second yields $y \succ x$. We finally invoke continuity to conclude that, when the perturbations are sent to zero, $x \sim_{F(\alpha)} y$.

Let's now do this in detail. From (24), (25), (29), (32), MIIA, and PR, we have

$$(33) \qquad \hat{\alpha}^{o} = \begin{array}{ccc} 1/3 & 1/3 & 1/3 \\ \hline y & x & x \\ z & y & y \\ x & z & z \end{array} \xrightarrow{F} \begin{array}{c} x-y \\ z \end{array}$$

$$(34) \qquad \hat{\alpha}^{oo} = \begin{array}{ccc} 1/3 & 1/3 & 1/3 \\ \hline x & y & y \\ z & x & x \\ y & z & z \end{array} \xrightarrow{F} \begin{array}{c} x-y \\ z \end{array}$$

$$(35) \qquad \hat{\alpha}^{**} = \begin{array}{cc} 1/2 & 1/2 \\ \hline y & x \\ x & y \\ z & z \end{array} \xrightarrow{F} \begin{array}{c} x-y \\ z \end{array}$$

Now let's perturb $\hat{\alpha}^{o}$, $\hat{\alpha}^{oo}$, α^{*}, $\hat{\alpha}^{**}$ by small $\varepsilon > 0$. From PR, continuity, and MIIA we obtain

$$(36) \qquad \hat{\alpha}^{o}_{\varepsilon} = \begin{array}{ccc} 1/3-\varepsilon & 1/3+\varepsilon & 1/3 \\ \hline y & x & x \\ z & y & y \\ x & z & z \end{array} \xrightarrow{F} \begin{array}{c} x \\ y \\ z \end{array}$$

$$(37) \qquad \hat{\alpha}^{oo}_{\varepsilon} = \begin{array}{cccc} \varepsilon & 1/3 & 1/3 & 1/3-\varepsilon \\ \hline x & x & y & y \\ y & z & x & x \\ z & y & z & z \end{array} \xrightarrow{F} \begin{array}{c} x \\ y \\ z \end{array}$$

$$(38) \qquad \hat{\alpha}^{**}_{\varepsilon} = \begin{array}{cc} 1/2-\varepsilon & 1/2+\varepsilon \\ \hline y & x \\ x & y \\ z & z \end{array} \xrightarrow{F} \begin{array}{c} x \\ y \\ z \end{array}$$

From (26), (31), PR and MIIA,

(39) $$\alpha^*_{\varepsilon} = \begin{array}{ccc} \varepsilon & 1/2 & 1/2\ -\varepsilon \\ \hline x & x & y \\ y & z & z \\ z & y & x \end{array} \xrightarrow{F} \begin{array}{c} x \\ y - z \end{array}$$

From PR, continuity, and (39), we can choose

(40) $$p(\varepsilon) \in (0, \varepsilon)$$

small enough so that

(41) $$\alpha^*_{\varepsilon, p(\varepsilon)} = \begin{array}{ccc} \varepsilon + p(\varepsilon) & 1/2 - p(\varepsilon) & 1/2\ -\varepsilon \\ \hline x & x & y \\ y & z & z \\ z & y & x \end{array} \xrightarrow{F} \begin{array}{c} x \\ y \\ z \end{array}$$

Hence, from (36) – (38) and (41), RC implies

(42) $$x \succ_{F(\alpha_{\varepsilon})} y,$$

where

(43) $$\alpha_{\varepsilon} = c^{o}\hat{\alpha}^{o}_{\varepsilon} + c^{oo}\hat{\alpha}^{oo}_{\varepsilon} + c^{*}\alpha^{*}_{\varepsilon, p(\varepsilon)} + c^{**}\hat{\alpha}^{*}_{\varepsilon}$$

Analogously, we have

(44) $$\hat{\alpha}^{o}_{\varepsilon^-} = \begin{array}{ccc} \varepsilon^- & 1/3 & 2/3\ -\varepsilon^- \\ \hline y & y & x \\ x & z & y \\ z & x & z \end{array} \xrightarrow{F} \begin{array}{c} y \\ x \\ z \end{array}$$

(45) $$\hat{\alpha}^{oo}_{\varepsilon^-} = \begin{array}{ccc} 1/3 - \varepsilon^- & 1/3 + \varepsilon^- & 1/3 \\ \hline x & y & y \\ z & x & x \\ y & z & z \end{array} \xrightarrow{F} \begin{array}{c} y \\ x \\ z \end{array}$$

(46) $$\hat{\alpha}^{**}_{\varepsilon^-} = \begin{array}{c} 1/2+\varepsilon \\ \hline y \\ x \\ z \end{array} \quad \begin{array}{c} 1/2-\varepsilon \\ \hline x \\ y \\ z \end{array} \quad \xrightarrow{F} \quad \begin{array}{c} y \\ x \\ z \end{array}$$

and

(47) $$\alpha^{*}_{\varepsilon^-, p(\varepsilon^-)} = \begin{array}{c} \varepsilon^- + p\left(\varepsilon^-\right) \\ \hline y \\ x \\ z \end{array} \quad \begin{array}{c} 1/2-\varepsilon^- \\ \hline x \\ z \\ y \end{array} \quad \begin{array}{c} 1/2-p\left(\varepsilon^-\right) \\ \hline y \\ z \\ x \end{array} \quad \xrightarrow{F} \quad \begin{array}{c} y \\ x \\ z \end{array}$$

for all $\varepsilon^- > 0$ and $p\left(\varepsilon^-\right)$ small enough. Thus, from (44) – (47) and RC,

(48) $$y \succ_{F\left(\alpha_{\varepsilon^-}\right)} x,$$

where $\alpha_{\varepsilon_-} = c^o \hat{\alpha}^o_{\varepsilon^-} + c^{oo} \hat{\alpha}^{oo}_{\varepsilon^-} + c^* \alpha^*_{\varepsilon^-, p(\varepsilon^-)} + c^{**} \hat{\alpha}^{**}_{\varepsilon^-}$.

Now, let $\varepsilon, \varepsilon^- \to 0$. From (42), (48), and continuity, we have $\alpha \in I_F^{xy}$.

Q.E.D.

4. The Characterization Theorem

We are now ready to establish our characterization theorem.

Theorem: Continuous SWF *F* satisfies U, MIIA, A, N, PR, and RC if and only if *F* is the Borda count.

Proof: The "if" part is clear. We shall concentrate on "only if."

For $|X| = 2$, the result follows from May (1952).

Suppose that $X = \{x, y, z\}$.[48]

[48] Again, the case $|X| > 3$ is treated in the Appendix.

Because F satisfies MIIA (in addition to A), we can reduce our 6-tuples to 4-tuples as sufficient statistics for profiles $\succsim$ in determining the social preference between x and y. Specifically, let

$$\beta = (\beta_2, \beta_1, \beta_{-1}, \beta_{-2}) = (b_2(\succsim), b_1(\succsim), b_{-1}(\succsim), b_{-2}(\succsim)),$$

where, for $k = 1, 2$, $\beta_k = \beta_k(\succsim)$ is the proportion of individuals in profile $\succsim$ who rank x k places above y and $\beta_{-k} = \beta_{-k}(\succsim)$ is the proportion who rank y k places above x.

We will show that if

(49) $\beta \in I_{Bor}^{xy}$

then

(50) $\beta \in I_F^{xy}$.

Now,

(51) $I_{Bor}^{xy} = \left\{ \beta \in \Delta^3 \middle| 2\beta_2 + \beta_1 - \beta_{-1} - 2\beta_{-2} = 0 \right\}$

Notice that the extreme points of I_{Bor}^{xy} are $(\beta_2, \beta_{-2}, \beta_1, \beta_{-1}) = (0, 1/3, 2/3, 0)$, $(1/3, 0, 0, 2/3)$, $(1/2, 1/2, 0, 0)$, and $(0, 0, 1/2, 1/2)$, since those are the points in the set on the right-hand side of (51) for which the non-negativity constraints are binding in two directions. Thus, we can rewrite I_{Bor}^{xy} as

(52) $I_{Bor}^{xy} =$ Convex hull $\{(0, 1/3, 2/3, 0),\ (1/3, 0, 0, 2/3),\ (1/2, 1/2, 0, 0),\ (0, 0, 1/2, 1/2)\}$

Consider $\beta \in I_{Bor}^{xy}$. We have

(53) $\beta = c^o (0, 1/3, 2/3, 0) + c^{oo} (1/3, 0, 0, 2/3) + c^* (1/2, 1/2, 0, 0) + c^{**} (0, 0, 1/2, 1/2)$

From MIIA, we can revert to expressing this profile as a 6-tuple. Specifically, the right-hand side of (53) is consistent with

$$(54)\quad \begin{aligned} &c^{o}\left(0,1/3,1/3,1/3,0,0\right)+c^{oo}\left(1/3,0,0,0,1/3,1/3\right) \\ &\quad +c^{*}\left(1/2,1/2,0,0,0,0\right)+c^{**}\left(0,0,1/2,0,0,1/2\right), \end{aligned}$$

where each 6-tuple takes the form $\left(\alpha_{xzy},\alpha_{yzx},\alpha_{xyz},\alpha_{zxy},\alpha_{yxz},\alpha_{zyx}\right)$.

But Lemma 2 in Section 3 establishes that the profile in (54) is in I_F^{xy}. Thus, $\beta \in I_F^{xy}$, as claimed. An application of Lemma 1 now completes the proof.

Q.E.D.

To briefly summarize the proof: if a profile belongs to I_{Bor}^{xy}, then it is a convex combination of $\alpha^{o},\alpha^{oo},\alpha^{*},$ and α^{**} (modulo moving *z* from the top to the bottom of some individuals' ordering or vice versa). But, by arguments mirroring that in Section 1D, each of $\alpha^{o},\alpha^{oo},\alpha^{*}$, and α^{**} is in I_F^{xy}. Using RC, Lemma 2 then establishes that the convex combination belongs to I_F^{xy}. Finally, Lemma 1 implies that *F* and the Borda count coincide everywhere.

The continuity assumption in Lemmas 1 and 2 and the Theorem guarantees that if α^{*} is a profile on the boundary between the regions where $x \succ_{F(\alpha)} y$ and $y \succ_{F(\alpha)} x$, then $\alpha^{*} \in I_F^{xy}$. If we drop continuity, the Theorem no longer holds. Now it is true that symmetry is sometimes enough in conjunction with A, N, and MIIA (and without the need for RC) to guarantee that if $\alpha \in I_{Bor}^{xy}$, then $\alpha^{*} \in I_F^{xy}$, as when $a^{*}=\left(\alpha^{*}_{xzy},\alpha^{*}_{yzx},\alpha^{*}_{xyz},\alpha^{*}_{zxy},\alpha^{*}_{yxz},\alpha^{*}_{zyx}\right)=\left(0,1/3,1/3,1/3,0,0\right)$ or $\alpha^{*}=\left(\frac{1}{2},\frac{1}{2},0,0,0,0\right)$. However, symmetry does not hold for all profiles in I_{Bor}^{xy}, e.g., it doesn't hold for profile $\left(1/4,5/12,1/6,1/6,0,0\right)$. Thus, a SWF which coincides with Borda except at this last profile and its permutations will satisfy all axioms beside continuity.

5. Open Questions

There are at least five questions that seem worth pursuing in follow-up work:

First, we have assumed throughout that, although society can be indifferent between a pair of alternatives, *x* and *y*, individuals are never indifferent. We conjecture that if individual indifference were allowed, then the same axioms (with MIIA modified appropriately) would imply the natural extension of the Borda count, e.g., if an individual is indifferent between *x* and *y*, then instead of *x* getting *p* points and *y* getting $p-1$ (as would be the case if the individual ranked *x* immediately above *y*), the alternatives will split the point count $p+p-1=2p-1$ equally. Indeed, in his Harvard senior thesis (Kim 2023), Jeremiah Kim showed that this conjecture holds in the case $|X|=3$.

Second, we have made important use of the continuum of voters in our proof. Specifically, the continuum, together with our continuity assumption, guarantees that there will be profiles for which society is indifferent between *x* and *y*, and indifference curves play a major role in our argument. In view of this, it would be interesting to explore to what extent the characterization result extends to the case of finitely many voters.

Third, as we noted, the Theorem does not hold without continuity. Nevertheless, we suspect that without continuity we could establish the counterpart of the Theorem for strict social preferences.

Fourth, this paper studies SWFs, which rank *all* alternatives. By contrast, a voting rule simply selects the *winner* (see footnote 2). In a previous draft of this paper, we proposed a way to modify the axioms to obtain a characterization of the voting-rule version of the Borda count (i.e., the winner is the alternative with the highest Borda score). However, that draft considered only the case in which social indifference curves are linear or polynomial. Whether that characterization holds in the more general setting of the current draft has not yet been explored.

Finally, although the anonymity and neutrality axioms are quite natural in political elections, they don't apply universally (think, for example, of corporate elections where voters are weighted by their ownership stake or of amendment processes where certain alternatives – e.g., the status quo – may be privileged). It is clear that certain variants of the Borda count – e.g., where different individuals can have different weights or some particular alternatives get extra Borda points – satisfy the remaining axioms when A and N are dropped, but we do not have a full characterization of all SWFs satisfying those axioms.

Appendix: Proofs for $|X| > 3$

Indifference Curves

Suppose $|X| = m+1$, where $m \geq 2$. Choose $x, y \in X$ and fix profile $\succ_\cdot$. For any ranking $\succ$ of *X*, let $a_\succ(\succ_\cdot)$ be the fraction of individuals in $\succ_\cdot$ who have ranking $\succ$. If SWF *F* satisfies A, then the $(m+1)!$ -tuple α —where, for each $\succ$, $\alpha_\succ = a_\succ(\succ_\cdot)$—is a sufficient statistic for $\succ_\cdot$ in determining the social ranking of *x* and *y*. Next, for any $k \in \{1,...,m\}$, let $b_k(\succ_\cdot)$ be the fraction of individuals in $\succ_\cdot$ who rank x k places above *y* and $b_{-k}(\succ_\cdot)$ the fraction who rank *y k* places above *x*. If in addition to A, *F* satisfies MIIA (which we will assume henceforth), then

(A1) $$\begin{aligned}\beta &= (\beta_m,...,\beta_1,\beta_{-1},...,\beta_{-k}) \\ &= (b_m(\succ_\cdot),...,b_1(\succ_\cdot),b_{-1}(\succ_\cdot),...,b_{-k}(\succ_\cdot))\end{aligned}$$

is also a sufficient statistic for $\succ_\cdot$ in determining the social ranking of *x* and *y*. We will go back and forth between α and β as representations of $\succ_\cdot$. *F* is *continuous* if, for any $x, y \in X$, the set $\{\beta \mid x \succsim_{F(\beta)} y\}$ is closed.

F's *indifference curve* for *x* and *y* is defined as

$$I_F^{xy} = \{\beta \in \Delta^{2m-1} \mid x \sim_{F(\beta)} y\} \quad .$$

In particular, the Borda indifference curve is:

(A2) $$I_{Bor}^{xy} = \left\{\beta \;\middle|\; \sum_{k=1}^{m} k(\beta_k - \beta_{-k}) = 0\right\} \quad .$$

The counterpart of Lemma 1 in Section 3 is:

Lemma 1^*: Suppose continuous *F* satisfies U, MIIA, A, N, and PR. If, for some $x, y \in X$,

(A3) $I_F^{xy} \subseteq I_{Bor}^{xy}$,

then

(A4) F = Borda count.

Proof: Suppose (A3) holds for some x, y, but there exists $\beta = (\beta_1, \ldots \beta_{m-1}, \beta_{-1}, \ldots, \beta_{-m})$ for which

$$F(\beta) \neq Bor(\beta).$$

Then, without loss of generality, we may also assume

(A5) $x \succ_{Bor(\beta)} y$ and $y \succsim_{F(\beta)} x$

Let us continuously decrease $\beta_1, \ldots, \beta_m$ and correspondingly increase $\beta_{-1}, \ldots, \beta_{-m}$. If all of $\beta_1, \ldots, \beta_m$ are reduced to zero, then $y \succ_{Bor} x$ for the corresponding profile. Thus, before then, we must, from continuity of the Borda count, reach a profile β^* for which

(A6) $\beta^* \in I_{Bor}^{xy}$.

But from PR and (A5), $y \succ_{F(\beta^*)} x$, which in view of (A6) violates (A3).

Q.E.D.

Fix any $x, y \in X\ (x \neq y)$ for the remainder of the Appendix. For any $r, s, \in \{1, \ldots, m\}$, let β^{rs} be the 2*m*-tuple such that $\beta_r^{rs} = s/(r+s)$, $\beta_{-s}^{rs} = r/(r+s)$, and for all $t \notin \{r, -s\}$ $\beta_t^{rs} = 0$. It is easy to verify that $\beta^{rs} \in I_{Bor}^{xy}$. In fact, the set $\{\beta^{rs}\}_{r,s\in\{1,\ldots,m\}}$ comprises the extreme points of the convex set I_{Bor}^{xy},[49] and so I_{Bor}^{xy} consists of the convex hull of $\{\beta^{rs}\}$. As part of our effort to show that $I_{Bor}^{xy} \subseteq I_F^{xy}$, we next establish

[49] Extreme points are the points in I_{Bor}^{xy} for which only two components are positive, since if three or more components are positive, we can reduce one component and correspondingly increase another while remaining in I_F^{xy}.

Lemma 2^*: If continuous *F* satisfies U, A, N, MIIA, PR, and RC, then, for all *r* and *s*, $\beta^{rs} \in I_F^{xy}$. Furthermore, any convex combination of $\left\{\beta^{rs}\right\}$ lies in I_F^{xy}.

Proof: We shall establish the lemma by induction on $|X|$. For $|X| = 3$, the result was proved in Section 3. Suppose that it is true for $|X| = m$, for some *m*. We must show that this implies the result holds for $|X| = m+1$.

Let $X = \left\{x, y, z^1, ..., z^{m-1}\right\}$ and $\hat{X} = \left\{x, y, z^1, ..., z^{m-2}\right\}$. Let $\hat{F}$ be the SWF defined on profiles $\hat{\alpha}$ of rankings over $\hat{X}$ such that, for all $v, w \in \hat{X}$,

$$v \succsim_{\hat{F}(\hat{\alpha})} w \Leftrightarrow v \succsim_{F(\alpha)} w,$$

where α is the profile of rankings on X obtained from $\hat{\alpha}$ by adding z^{m-1} to the bottom of every individual's ranking. Because *F* satisfies all the conditions of the Lemma, so does $\hat{F}$. Thus, for every $r, s < m$ and every $\hat{\beta}^{rs}$ corresponding to $\hat{X}$, the induction hypothesis implies that $\hat{\beta}^{rs} \in I_{\hat{F}}^{xy}$. Finally, by MIIA, N, and the definition of $\hat{F}$, it follows that $\beta^{rs} \in I_F^{rs}$ (MIIA and N imply that only the number of alternatives in between *x* and *y* – and not the identity of those alternatives – matters).

Next, consider β^{m1}. From MIIA, this generates the same social preference between *x* and *y* as the profile

$$\text{(A7)} \quad \begin{array}{ccccc} \frac{1}{m+1} & \frac{1}{m+1} & \frac{1}{m+1} & \cdots & \frac{1}{m+1} \\ \hline x & y & z^{m-1} & & z^1 \\ z^1 & x & y & & \vdots \\ \vdots & z^1 & x & & z^{m-1} \\ z^{m-1} & \vdots & \vdots & & y \\ y & z^{m-1} & z^{m-2} & & x \end{array}$$

By the same symmetry argument as in Section 1D, profile (A7) belongs to I_F^{xy}, and thus so does β^{m1}. By symmetry the same is true of β^{1m}.

We now want to show that $\beta^{ms} \in I_F^{xy}$ for all $s \in \{2,...,m\}$. This directly follows from A, N, and MIIA if $s = m$ (see the argument for why $\alpha^* \in I_F^{xy}$ in Section 3). Thus, assume $s < m$. Consider the profile

(A8) $\hat{\beta}^{ms} = \left(s(m+1)/(2ms+m+s)\right)\beta^{m1} + \left(m(s+1)/(2ms+m+s)\right)\beta^{1s}$

Let

$$\alpha_\varepsilon^{m1} = \begin{array}{ccc} \varepsilon & \frac{1}{m+1} & \frac{m}{m+1}-\varepsilon \\ \hline x & x & y \\ y & z^{m-1} & x \\ z^1 & \vdots & z^1 \\ \vdots & z^1 & \vdots \\ z^{m-1} & y & z^{m-1} \end{array}$$

for ε small. From $\beta^{m1} \in I_F^{xy}$ and PR,

(A9) $x \succ_{F(\alpha_\varepsilon^{m1})} y$

From $\beta^{21} \in I_F^{yz^1}$ and PR,

(A10) $y \succ_{F(\alpha_\varepsilon^{m1})} z^1$

From PR and $\beta^{11} \in I_F^{z^t z^{t+1}}$, $t = 1,...,m-2$

(A11) $z^1 \succ_{F(\alpha_\varepsilon^{m1})} z^2 \succ \cdots \succ_{F(\alpha_\varepsilon^{m1})} z^{m-1}$

By inductive hypothesis, we can find a corresponding perturbation $\alpha_{\varepsilon}^{1s}$ such that (A9) – (A11) hold when $\alpha_{\varepsilon}^{m1}$ is replaced by $\alpha_{\varepsilon}^{1s}$ (which is defined by analogy to $\alpha_{\varepsilon}^{m1}$). Hence, from RC, we obtain

(A12) $x \succ_{F\left(\hat{\alpha}_{\varepsilon}^{ms}\right)} y$,

where $\hat{\alpha}_{\varepsilon}^{ms} = \left(s\left(m+1\right)/\left(2ms+m+s\right)\right)\alpha_{\varepsilon}^{m1} + \left(m\left(s+1\right)/\left(2ms+m+s\right)\right)\alpha_{\varepsilon}^{1s}$.

Similarly, we can find perturbations $\alpha_{\varepsilon^{-}}^{m1}$ and $\alpha_{\varepsilon^{-}}^{1s}$ such that

(A13) $y \succ_{F\left(\hat{\alpha}_{\varepsilon^{-}}^{ms}\right)} x$,

where $\hat{\alpha}_{\varepsilon^{-}}^{ms} = \left(s\left(m+1\right)/\left(2ms+m+s\right)\right)\alpha_{\varepsilon^{-}}^{m1} + \left(m\left(s+1\right)/\left(2ms+m+s\right)\right)\alpha_{\varepsilon^{-}}^{1s}$. Sending ε and ε^{-} to zero and applying continuity, we deduce that

(A14) $x \sim_{F\left(\hat{\alpha}^{ms}\right)} y$,

where $\hat{\alpha}^{ms}$ is $\hat{\alpha}_{\varepsilon}^{ms}$ with $\alpha_{\varepsilon}^{ms}$ and $\alpha_{\varepsilon}^{1s}$ replaced by α_{0}^{ms} and α_{0}^{1s}. And so

(A15) $x \sim_{F\left(\hat{\beta}^{ms}\right)} y$.

Now, notice that, from (A8), $\hat{\beta}^{ms}$ can be rewritten as

(A16) $$\hat{\beta}^{ms} = \frac{m+s}{2ms+m+s}\beta^{ms} + \frac{2ms}{2ms+m+s}\beta^{11}$$

If $\beta^{ms} \notin I_{F}^{xy}$, then $y \succ_{F\left(\beta^{ms}\right)} x$ or

(A17) $x \succ_{F\left(\beta^{ms}\right)} y$

Let's assume that (A17) holds. Then, from PR, there exists $\delta > 0$ such that $\alpha_{\delta}^{ms} \in I_F^{xy}$, where

$$\alpha_{\delta}^{ms} = \begin{array}{cc} \frac{s}{s+m}-\delta & \frac{m}{s+m}+\delta \\ \hline x & y \\ z^{m-1} & z^{1} \\ \vdots & \vdots \\ z^{1} & z^{s-1} \\ y & x \\ & z^{s} \\ & \vdots \\ & z^{m-1} \end{array}$$

Now, for $\varepsilon > 0$,

$$\text{(A18)} \quad \alpha_{\delta\varepsilon}^{ms} = \begin{array}{ccc} \varepsilon & \frac{s}{s+m}-\delta & \frac{m}{s+m}+\delta-\varepsilon \\ \hline x & x & y \\ y & z^{m-1} & z^{1} \\ z^{1} & \vdots & \vdots \\ \vdots & z^{1} & z^{s-1} \\ z^{m-1} & y & x \\ & & z^{s} \\ & & \vdots \\ & & z^{m-1} \end{array} \overset{F}{\rightarrow} \begin{array}{c} x \\ y \\ z^{1} \\ \vdots \\ z^{m-1} \end{array}$$

$$\text{(A19)} \quad \alpha_{\varepsilon}^{11} = \begin{array}{cc} \frac{1}{2}+\varepsilon & \frac{1}{2}-\varepsilon \\ \hline x & y \\ y & x \\ z^{1} & z^{1} \\ \vdots & \vdots \\ z^{m-1} & z^{m-1} \end{array} \overset{F}{\rightarrow} \begin{array}{c} x \\ y \\ z^{1} \\ \vdots \\ z^{m-1} \end{array}$$

Hence, from (A18), (A19) and RC,

(A20) $x \succ_{F\left(\hat{\beta}^{ms}_{\delta\varepsilon}\right)} y$,

where

(A21) $$\hat{\beta}^{ms}_{\delta\varepsilon} = \frac{m+s}{2ms+m+s}\ \alpha^{ms}_{\delta\varepsilon} + \frac{2ms}{2ms+m+s}\ \alpha^{11}_{\varepsilon}$$

Analogously, we can find permutations $\alpha^{ms}_{\delta\varepsilon^-}$ and $\alpha^{11}_{\varepsilon^-}$ of α^{ms}_{δ} and α^{11} respectively such that

(A22) $y \succ_{F\left(\hat{\beta}^{ms}_{\delta\varepsilon^-}\right)} x$,

where $\hat{\beta}^{ms}_{\delta\varepsilon^-}$ satisfies (A21) with $\alpha^{ms}_{\delta\varepsilon^-}$ and $\alpha^{11}_{\varepsilon^-}$ replacing $\alpha^{ms}_{\delta\varepsilon}$ and $\alpha^{11}_{\varepsilon}$.

Letting $\varepsilon \to 0$ and $\varepsilon^- \to 0$ in (A20) and (A22) and applying continuity, we obtain

(A23) $x \sim_{F\left(\hat{\beta}^{ms}_{\delta}\right)} y$

But (A15) and (A23) contradict PR. Hence, we conclude that

$\beta^{ms} \in I^{xy}_F$ as claimed. Symmetrically, we obtain $\beta^{sm} \in I^{xy}_F$.

We have thus completed the inductive step for showing that, for all $r, s \in \{1,...,m\}$, $\beta^{rs} \in I^{xy}_F$.

It remains to show that all convex combinations of $\left\{\beta^{rs}\right\}$ belong to I_F^{xy}, i.e., $\beta \in I_F^{xy}$, when $\beta = \sum_{r,s} c^{rs}\beta^{rs}$, where $c^{rs} \geq 0$ for all *r, s* and $\sum_{r,s} c^{rs} = 1$. We will establish this by showing that, for each β^{rs}, a corresponding profile α^{rs} (which, like β^{rs}, yields social indifference between *x* and *y*) can be perturbed to α_ε^{rs} and $\alpha_{\varepsilon^-}^{rs}$ so that

(A24) $x \succ_{F(\alpha_\varepsilon^{rs})} y \succ_{F(\alpha_\varepsilon^{rs})} z^1 \succ_{F(\alpha_\varepsilon^{rs})} \cdots \succ_{F(\alpha_\varepsilon^{rs})} z^{m-1}$

and

(A25) $y \succ_{F(\alpha_{\varepsilon^-}^{rs})} x \succ_{F(\alpha_{\varepsilon^-}^{rs})} z^{m-1} \succ_{F(\alpha_{\varepsilon^-}^{rs})} \cdots \succ_{F(\alpha_{\varepsilon^-}^{rs})} z^1$,

respectively. Then from RC, (A24), and (A25) *x* is top-ranked socially when α_ε^{rs} is replaced by $\beta_\varepsilon = \sum c^{rs}\beta_\varepsilon^{rs}$ and *y* is top-ranked socially when $\alpha_{\varepsilon^-}^{rs}$ is replaced by $\beta_{\varepsilon^-} = \sum c^{rs}\beta_{\varepsilon^-}^{rs}$. Sending ε and ε^- to 0, we obtain from continuity that $x \sim_{F(\beta)} y$, as claimed.

Now, to obtain α_ε^{rs}, first assume that $r > s$. In fact, by inductive hypothesis, we may assume that $r = m$. Then take $\alpha_\varepsilon^{ms} = \alpha_{\delta\varepsilon}^{ms}$ as defined in (A18) when $\delta = 0$. Then, from (A18), (A24) holds. For $r < s$, we can assume $s = m$ by inductive hypothesis. Take

$$\alpha_{\varepsilon}^{rm} = \begin{array}{cc} \frac{r}{m+r}-\varepsilon & \frac{m}{m+r}+\varepsilon \\ \hline y & x \\ z^{m-1} & z^{1} \\ \vdots & \vdots \\ z^{1} & z^{r-1} \\ x & y \\ & z^{r} \\ & \vdots \\ & z^{m-1} \end{array}$$

Then from PR and $\beta^{rm} \in I_F^{xy}$,

(A26) $x \succ_{F(\alpha_{\varepsilon}^{rm})} y$

Now, in $\alpha_{\varepsilon}^{rm}$ the proportion of individuals who rank y $m-1$ places above z^1 is (slightly less than) $\frac{r}{r+m}$ while that who rank z^1 $r-1$ places above y is (slightly more than) $\frac{m}{r+m}$. Since $(r-1)/(m+r-2) < r/(r+m)$ and $\beta^{(m-1)(r-1)} \in I_F^{yz^1}$, PR implies that

(A27) $y \succ_{F(\alpha_{\varepsilon}^{rm})} z^1$

In $\alpha_{\varepsilon}^{rm}$, the proportion of individuals who rank z^{t+1} 1 place above z^t (for $t = 1,...,r-2,r,...,m-2$) is (slightly less than) $\frac{r}{r+m}$, while that who rank z^t 1 place above z^{t+1} (for $t = 1,...,r-2,r,...,m-2$) is (slightly more than) $\frac{m}{r+m}$. Since $m > r$ and $\beta^{11} \in I_F^{z^t z^{t+1}}$

(A28) $z^t \succ_{F(\alpha_{\varepsilon}^{rm})} z^{t+1}$, $t = 1,...,r-2,r,...,m-2$

Finally, in α_ε^{rm}, the proportion of individuals who rank z^r 1 place above z^{r-1} is (slightly less than) $\frac{r}{r+m}$, while that who rank z^{r-1} two places above z^r is (slightly more than) $\frac{m}{r+m}$. From $m > r$, $\beta^{12} \in I_F^{z^r z^{r-1}}$, and PR

(A29) $z^{r-1} \succ_{F\left(\alpha_\varepsilon^{rm}\right)} z^r$

Combining (A26) – (A29), we obtain (A24). The argument for (A25) is symmetric.

It remains to consider the case of α_ε^{rr}. Again, by inductive hypothesis, we can take $r = m.$ Let

$$\alpha_\varepsilon^{mm} = \begin{array}{ccc} \varepsilon & \frac{1}{2} & \frac{1}{2}-\varepsilon \\ \hline x & x & y \\ y & z^{m-1} & z^1 \\ z^1 & \vdots & \vdots \\ \vdots & z^1 & z^{m-1} \\ z^{m-1} & y & x \end{array}$$

Now for $\varepsilon = 0$,

(A30) $x \sim_{F(\alpha_0^{m})} y \sim_{F(\alpha_0^{mm})} z^1 \sim \cdots \sim_{F(\alpha_0^{mm})} z^{m-1}$

since $\beta^{mm} \in I_F^{xy}$. From PR, and (A30),

(A31) $x \succ_{F(\alpha_\varepsilon^{mm})} y$

and, from MIIA and (A30)

(A32) $y \sim_{F\left(\alpha_\varepsilon^{mm}\right)} z^1 \sim_{F\left(\alpha_\varepsilon^{mm}\right)} \cdots \sim_{F\left(\alpha_\varepsilon^{mm}\right)} z^{m-1}$.

Now for $p \in (0,\varepsilon)$, (A31), (A32), and PR imply

(A33)

$$\alpha^{mm}_{\varepsilon,p} = \begin{array}{ccc} \varepsilon + p & \frac{1}{2} - p & \frac{1}{2} - \varepsilon \\ \hline x & x & y \\ y & z^{m-1} & z^1 \\ z^1 & \vdots & \vdots \\ \vdots & z^1 & z^{m-1} \\ z^{m-1} & y & x \end{array} \xrightarrow{F} \begin{array}{c} y \\ z^1 \\ \vdots \\ z^{m-1} \end{array}$$

As we argue in deriving (42), continuity and PR imply that there exists $p = p(\varepsilon)$ small enough so that

(A34) $x \succ_{F(\alpha^{mm}_{\varepsilon,p(\varepsilon)})} y$,

and so (A24) holds when $(r,s) = (m,m)$ and $\alpha^{mm}_{\varepsilon,p(\varepsilon)}$ replaces $\alpha^{mm}_{\varepsilon}$.

Analogously, we can take

(A35) $$\alpha^{mm}_{\varepsilon^-,p(\varepsilon^-)} = \begin{array}{ccc} \varepsilon^- + p(\varepsilon^-) & \frac{1}{2} - \varepsilon^- & \frac{1}{2} - p(\varepsilon^-) \\ \hline y & x & y \\ x & z^{m-1} & z^1 \\ z^{m-1} & \vdots & \vdots \\ \vdots & z^1 & z^{m-1} \\ z^1 & y & x \end{array} \xrightarrow{F} \begin{array}{c} y \\ x \\ z^{m-1} \\ \vdots \\ z^1 \end{array}$$

giving us (A25) and completing the proof.

Q.E.D.

Proof of Characterization Theorem

From Lemma 1^*, we must show that if, for $x, y \in X$,

(A36) $\beta \in I^{xy}_{Bor}$

then

(A37) $\beta \in I^{xy}_{F}$.

Now,

$$I_{Bor}^{xy} = \left\{ \beta \in \Delta^{2m-1} \middle| \sum_{j=1}^{m} j(\beta_j - \beta_{-j}) = 0 \right\},$$

where $|X| = m+1$. As noted in footnote 49 the extreme points of the boundary of I_{Bor}^{xy} are $\{\beta^{rs}\}, r, s \in \{1, ..., m\}$, where β^{rs} is defined above (just before Lemma 2*). Hence, I_{Bor}^{xy} can be rewritten as

(A38) $I_{Bor}^{xy} =$ Convex hull $\{\beta^{rs}\}_{r,s \in \{1,...,m\}}$

In view of (A38), (A37) follows directly from (A36) and Lemma 2*. The result now follows from Lemma 1^*.

Q.E.D.

References

Arrow, Kenneth J. 1950. “A Difficulty in the Concept of Social Welfare.” *Journal of Political Economy*, 58 (4): 328-346.

Arrow, Kenneth J. 1951. *Social Choice and Individual Values*. New York: John Wiley & Sons.

Balinski, Michel and Rida Laraki. 2010. *Majority Judgment: Measuring, Ranking, and Electing*. Cambridge, MA: MIT Press.

Barbie, Martin, Clemens Puppe, and Attila Tasnádi. 2006. “Non-manipulable Domains for the Borda Count.” *Economic Theory*, 27 (2): 411-430.

Bentham, Jeremy. 1789. *An Introduction to the Principles of Morals and Legislation.* London: T. Payne, and Son.

Borda, Jean-Charles. 1781. “Mémoire sur les élections au scrutin.” *Hiswire de I'Academie Royale des Sciences*, 657-665.

Brandl, Florian, and Felix Brandt. 2020. “Arrovian Aggregation of Convex Preferences.” *Econometrica*, 88 (2): 799-844.

Condorcet, Marie Jean A.N.C.. 1785. *Essai sur l’application de l’analyse à la pluralité des voix*. Imprimerie Royale.

Dasgupta, Partha, and Eric Maskin. 2008. “On the Robustness of Majority Rule,” *Journal of the European Economic Association*, 6(5): 949-973.

Dasgupta, Partha, and Eric Maskin. 2020. “Strategy-Proofness, IIA, and Majority Rule.” *American Economic Review: Insights*, 2(4): 459–73.

Dhillon, Amrita, and Jean-Franois Mertens. 1999. “Relative Utilitarianism.” *Econometrica*, 67(3): 471-498.

Eden, Maya. 2020. “Aggregating Welfare Gains.” *CEPR* Discussion Paper DP14783.

Fleurbaey, Marc, Kotaro Suzumura, and Koichi Tadenuma. 2005. “Arrovian aggregation in economic environments: How much should we know about indifference surfaces?”, *Journal of Economic Theory*, 124: 22-44.

Fleurbaey, Marc, Kotaro Suzumura, and Koichi Tadenuma. 2005a. “The informational basis of the theory of fair allocation.” *Social Choice and Welfare*, 24(2): 311-341.

Gendler, Gabriel. 2023. “Non-Borda Social Welfare Functions under Relaxed IIA Conditions,” working paper.

Gibbard, Allan. 1973. “Manipulation of Voting Schemes: A General Result.” *Econometrica*, 41(4): 587-601.

Hansson Bengt. 1973. “The independence condition in the theory of social choice.” *Theory and Decision*, 4:25–49

Harsanyi, John C. 1955. “Cardinal Welfare, Individualistic Ethics, and Interpersonal Comparisons of Utility.” *Journal of Political Economy*, 63(4):309-21.

Horan, Sean, Osborne, Martin J., and M. Remzi Sanver. 2019. “Positively Responsive Collective Choice Rules And Majority Rule: A Generalization of May's Theorem To Many Alternatives.” *International Economic Review*, 60(4): 1489-1504.

Kim, Jeremiah. 2023. “Impossibility and Voting: Extending the Borda Characterization Theorem to Non-Strict Voter Preferences.” Harvard Senior Thesis.

Maskin, Eric. 2020. “A Modified Version of Arrow’s IIA Condition.” *Social Choice and Welfare*, 54: 203-209.

Maskin, Eric, and Amartya Sen. 2016. “How Majority Rule Might Have Stopped Donald Trump.” *New York Times*. https://www.nytimes.com/2016/05/01/opinion/sunday/how-majority-rule-might-have-stopped-donald-trump.html.

May, Kenneth O. 1952. “A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decisions.” *Econometrica*, 20: 680-684.

Mayston David J. 1974. *The Idea of Social Choice*. London: Macmillan.

Moulin, Hervé. 1988. “Condorcet's Principle Implies the No Show Paradox.” *Journal of Economic Theory*, 45 (1): 53–64

Osborne, Dale K. 1976. “Irrelevant alternatives and social welfare.” *Econometrica*, 1001-1015.

Robbins, Lionel. 1932. *An Essay on the Nature and Significance of Economic Science.* London: Macmillan.

Roberts, Kevin. 2009. “Irrelevant Alternatives” in Basu, Kaushik; Kanbur, Ravi (eds.). *Arguments for a Better World: Essays in Honor of Amartya Sen*. Oxford: Oxford University Press. 231-249.

Saari, Donald G. 1998. “Connecting and Resolving Sen’s and Arrow’s Theorems.” *Social Choice Welfare*, 15: 239-261.

Saari, Donald G. 2000. “Mathematical Structure of Voting Paradoxes: I. Pairwise Votes.” *Economic Theory*, 15: 1-53.

Saari, Donald G. 2000a. “Mathematical Structure of Voting Paradoxes: II. Positional Voting.” *Economic Theory*, 15 (1): 55-102.

Satterthwaite, Mark. 1975. “Strategy-proofness and Arrow’s Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions.” *Journal of Economic Theory* 10(2): 187-217.

Tideman, T.N. 1987. “Independence of Clones as a Criterion for Voting Rules.” *Social Choice and Welfare*, 4 (3): 185–206.

von Neumann, John, and Oskar Morgenstern. 1944. *Theory of Games and Economic Behavior,* Princeton University Press.

Young, H. Peyton. 1974. “An Axiomatization of Borda’s Rule.” *Journal of Economic Theory*, 9(1): 43-52.

Young, H. Peyton. 1975. "Social Choice Scoring Functions." *SIAM Journal on Applied Mathematics* 28(4): 824-838.

Young, H. Peyton. 1988. “Condorcet’s Theory of Voting.” *The American Political Science Review,* 82 (4): 1231-124.

Young, H. Peyton, and Arthur Levenglick. 1978. “A Consistent Extension of Condorcet’s Election Principle.” *SIAM Journal on Applied Mathematics*, 35(2): 285-300.