11 On the Fair Allocation of Indivisible Goods

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1 INTRODUCTION

Kenneth Arrow had a major hand in establishing two of the fundamental results of general equilibrium theory: the existence of a competitive equilibrium (Arrow and Debreu 1954) and the decentralizability of a Pareto optimum (Arrow 1951). In this chapter, we are concerned with general equilibrium in a setting where some goods are indivisible. Hence, several hypotheses that are standard in equilibrium analysis are not satisfied. None the less, as we will see, the classic Arrow-Debreu techniques can be suitably modified to overcome this difficulty.

Specifically, we are interested here in the existence of fair allocations with indivisible goods. Following Foley (1967), an allocation of goods across consumers is equitable if no consumer prefers another's consumption bundle to his own. Schmeidler and Yaari (1971) and Varian (1974) define an equitable allocation to be *fair* if it is also Pareto efficient.

When preferences and goods are well behaved, one can establish the existence of a fair allocation in a pure exchange economy by simply observing that a competitive allocation is fair when agents have the same initial endowments. It is natural to try the same method of proof when goods are indivisible. To give agents equal endowments, of course, it may be necessary to assign them fractional shares of some goods; an agent may thus own $1/n$ of a house. This assignment of endowments itself causes no conceptual difficulty but, unfortunately, may not generate a competitive equilibrium. Indeed, a fair allocation

* Kenneth Arrow introduced me to the subject of general equilibrium. Indeed, the inspiration that his work in this area and in social choice provided had much to do with my going into economics in the first place. It is a pleasure, therefore, to contribute a chapter to a volume in his honour on a topic that he pioneered.

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itself may not exist unless there is also available a certain amount of a perfectly divisible good.

We show in Theorems 2 and 3 that, given enough of the divisible good, an equal endowment competitive equilibrium (possibly including a system of taxes and subsidies) exists and hence so does a fair allocation. The proof relies on the standard Arrow–Debreu technique of choosing prices that maximize the value of aggregate excess demand and finding a fixed point of the cross product of this correspondence and excess demand.

A complication of using this technique is that, because of the indivisibilities, aggregate excess demand may be neither convex-valued nor upper hemicontinuous, contrary to the requirements of the standard fixed point lemma. The first difficulty can be overcome by working with convexified excess demand and then appealing to Birkoff's theorem on doubly stochastic matrices to show that a zero of this construct is a zero of ordinary aggregate excess demand. The second problem can be avoided by introducing the taxes and subsidies mentioned above.

The concept of fairness we have been considering has an '*ex post*' flavour; *after* trade has occurred, there should be no scope for further trade, and no agent should prefer another's consumption bundle to his own. Suppose, however, that instead of buying an entire unit of an indivisible good, agents could purchase *probabilities* of receiving the good. Because probabilities are fully divisible, a market in probabilities would be free of the problems plaguing existence that we noted above. Under standard assumptions on preferences, fair allocations will exist in this probabilistic framework, as Hylland and Zeckhauser (1978) have noted. The concept of fairness here, however, is *ex ante*. After the allocation has been realized, some agents may well envy others.

In Section 2 we present the model, concepts, and notation. In Section 3 we then offer two existence theorems (Theorems 2 and 3) for fair allocations.

2 THE MODEL

Let us suppose that there are n agents and 1 unit each of n indivisible goods. There is also a perfectly divisible good whose aggregate endowment is X. We shall call this divisible good 'money'. Each agent can consume at most one of the indivisible goods. Agent i's preferences over the indivisible goods and money are representable by the utility function

$$u_i(x,k),$$

where x is a non-negative quantity of money, and $k \in \{0,1,\ldots,n\}$ refers to indivisible good k. (If $k=0$, then the agent consumes no indivisible good.) The function u_i is continuous and increasing in x. It is natural to assume that the indivisible goods are desirable:

$$u_i(x,k) > u_i(x,0) \text{ for all } k \geqslant 1 \text{ and all } i \tag{11.1}$$

An allocation is simply an assignment of goods to agents – i.e., a vector $((x_1,k_1),\ldots,(x_n,k_n))$ such that $\Sigma x_i = X$ and no two k_is are the same. Using Foley's terminology, we will call an allocation (x,k) *equitable* if, for all i and j, $u_i(x_i,k_i) \geqslant u_i(x_j,k_j)$. Thus, in an equitable allocation, no agent 'envies' the consumption bundle of another. As defined by Schmeidler–Yaari and Varian, an equitable allocation is *fair* if it is Pareto optimal.

Fair allocations may not exist in this model. For example, suppose that $n=3$ and $X=0$ – i.e., there is no money. If, say, all agents strictly prefer indivisible good 1 to the others, then no equitable allocation, much less a fair one, is possible, since the agent getting good 1 will be envied by the other two.

This example tells us that to obtain a fair allocation, it must be possible to compensate agents for being assigned suboptimal indivisible goods. In particular, we thus need to assume that a sufficient quantity of the divisible good can always make up for an inferior indivisible good:

$$\text{there exists } y \text{ such that for all } i, j, \text{ and } k, \text{ and } x \leqslant X/n,$$
$$u_i(x+y,j) > u_i(x,k) \tag{11.2}$$

There must also be a minimal quantity of the divisible good available to enable such compensation to be paid.

The example is also instructive as an illustration of how the standard technique for proving existence can go wrong. As we mentioned in the introduction, one establishes the existence of fair allocations with divisible goods and well-behaved preferences by noting that a competitive allocation starting from equal division of the aggregate endowment is fair. In our example, equal division entails all agents' receiving a one-third share of each good as an endowment. A competitive equilibrium corresponding to these endowments thus cannot exist, and so it is not surprising that the excess demand correspondences do not satisfy the standard Arrow–Debreu requirements. Indeed, they violate two conditions.

First, they may not be convex-valued. Suppose, for instance, that an agent is indifferent between goods 2 and 3. For the price vector (p^1,p^2,p^3), where p^k is the price of good k and $p^1 > p^2 = p^3$, the agent's only possible excess demands are

$$(-1/3, 2/3, -1/3) \quad \text{and} \quad (-1/3, -1{,}3, 2/3) \tag{11.3}$$

(since he cannot afford to buy good 1), an obvious violation of convex-valuedness. As we will see on page 346, however, this failure has no bearing on existence, since we can work just as well with the convex hull of the excess correspondence.

The more serious failure is that of upper hemicontinuity. As long as $p^1 > p^2 = p^3$, an agent's excess demand is given by (11.3). But when p^1 converges from above to p^2, his unique excess demand vector becomes $(2/3, -1/3, -1/3)$, violating upper hemicontinuity. It is this violation that is 'responsible' for the non-existence of equilibrium.

Interestingly, the failure of upper hemicontinuity does not prevent the existence of competitive equilibrium when each agent is initially endowed with an entire indivisible good.

> *Theorem 1* (Shapley–Scarf 1974): If $X = 0$ and agent i is initially endowed with indivisible good i, a competitive equilibrium exists.

In our three-agent example, where all agents prefer good 1 to the other two, we can construct an equilibrium as follows when agent i is endowed with good i. We first assign agent 1 the final allocation consisting of good 1 and take p^1 to be bigger than either p^2 or p^3; this ensures that neither agent 2 nor 3 can afford good 1. If agents 2 and 3 strictly prefer the other's initial endowment to their own, then the competitive equilibrium should switch their endowments and set $p^2 = p^3$. In all other cases, the competitive allocation is just the initial allocation; and if agent 2 prefers good 2 to 3, then we can take $p^2 > p^3$, and otherwise set $p^2 < p^3$.

3 FAIR ALLOCATIONS

One way of ensuring upper hemicontinuity when each agent i is endowed with money as well as with shares of indivisible goods is to suppose that he prefers a bundle with X/n units of money and an indivisible good to any bundle without money. That is,

$$u_i(X/n,k) > u_i(0,j) \text{ for all } i, k, \text{ and } j \tag{11.4}$$

An analogous assumption is used by Quinzii (1984) in her proof of the existence of competitive equilibrium with indivisibilities and money. Indeed, the following preliminary result is closely related to her Theorem 3.

> *Theorem 2:* If utility functions are increasing and continuous in x, and for all i and k, u_i satisfies (11.1) and (11.4), then a fair allocation exists.

Remark

This result dispenses with Quinzii's requirement that utility functions go to infinity with x. The proof is a direct fixed point argument *à la* Arrow–Debreu, whereas Quinzii's proof consists of demonstrating that the core of this model is non-empty and then showing that any core allocation is a competitive equilibrium.

Proof

Endow each agent with $1/n$ of the aggregate endowment. Let price vectors $p=(p^0,p^1,\ldots,p^n)$ be points in the $n+1$-dimensional simplex, where p^0 is the price of money. Truncate agent i's consumption set so that he cannot consume more than M units of money where $M>X$, and let D_i be the truncated consumption set. Let $d_i(\cdot)$ be agent i's ($n+1$-dimensional) truncated demand correspondence. Of course, for any $k \geq 1$, and any $y \in d_i(p)$, y^k (the component of y corresponding to good k) is either 0 or 1. From (11.4), and because u_i is continuous, $d_i(\cdot)$ is upper hemicontinuous. Take $d(\cdot)=\sum_{i=1}^{n} d_i(\cdot)$, $D=D_1+\ldots+D_n$ and let $\hat{d}(\cdot)$ be the convex hull of $d(\cdot)$. For each $d \in D$ let $f(d)=\{\hat{p} \mid \hat{p}$

maximises $p\cdot(d-(X,1,\ldots,1))\}$. The correspondence $(p,d)\to f(d)\times \tilde{d}(p)$ satisfies all the hypotheses of the Kakutani lemma. It therefore has a fixed point $(\bar{p},\bar{d})$. Now, because each agent must satisfy his budget constraint, we have

$$\bar{p}\cdot(\bar{d}-(X,1,\ldots,1))\leqslant 0 \tag{11.5}$$

If $\bar{p}^0=0$, then because u_i is increasing in x, each agent will demand M, and so $\bar{d}^0-X>0$. By definition of f, this implies that $\bar{p}\cdot(\bar{d}-(X,1,\ldots,1))>0$, which contradicts (11.5). Hence $\bar{p}^0>0$, and so (11.5) holds with equality. Now if $\bar{d}^j>1$ for some $j\geqslant 1$, then again by definition of f, $\bar{p}\cdot)\bar{d}-(X,1,\ldots,1)>0$, contradicting (11.5). Hence $\bar{d}^j\leqslant 1$ for all $j\geqslant 1$. If $d^j<1$ for some j, then because (11.5) holds with equality, $\bar{p}^j=0$. Since $\sum_{k=1}^{n}\bar{d}^k<n$, there exists $d\epsilon d(\bar{p})$ for which some agent i does not demand any indivisible good – i.e., he consumes just his endowment. But because $u_i(X/n,j)>u_i(X/n,0)$, agent i is better off demanding good j. Hence, $\bar{d}^j=1$ for all $j\geqslant 1$. Because (11.5) holds with equality and $\bar{p}^0>0$, we conclude that $\bar{d}=(X,1,\ldots,1)$ and so

$$\bar{d}^*=(1,\ldots,1)$$

where $\bar{d}^*$ omits the first component of $\bar{d}$. By definition of $\bar{d}^*$, there exists an $n\times n$ doubly stochastic matrix (Z_i^j) – i.e., a non-negative matrix where row and column sums are 1 and whose ith row and jth column entry is Z_i^j – such that for each i the row Z_i^* is a convex combination of agent i's demand* vectors (demand vectors that omit the demand for money) when prices are $\bar{p}$. From Birkhoff's Theorem (see Liu 1968), moreover, we can express (Z_i^j) as a convex combination of the set of $n\times n$ permutation matrices (matrices with exactly one '1' in each row and column and otherwise consisting of zeros). Each permutation matrix appearing in the convex combination thus corresponds to a set of demand* vectors (one for each agent) for prices $\bar{p}$, and, furthermore, these vectors sum to $(1,\ldots,1)$. They thus constitute truncated equilibrium demands (where M truncates demand).

Finally, consider a sequence of Ms tending to infinity. A subsequence of the corresponding truncated equilibria converges. Suppose that p is the limit price vector and that (x_i,k_i) is the limit consumption bundle for agent i. A conventional continuity argument and the monotonicity of u_i establish that (x_i, k_i) is agent i's utility-maximizing consumption bundle when faced with prices p.

QED

Condition (11.4) is a much stronger condition on preferences than necessary to prove the existence of a fair allocation; we can make do with (11.2), implied by (11.4) (when $\lim_{x \to \infty} u_i(x,k) = \infty$).

Once we drop (11.4), however, we reintroduce the problem of satisfying upper hemicontinuity. One solution is to keep the price of money sufficiently high so that an agent can afford to buy any indivisible good and desires to buy at least one. To prevent all agents from buying the same good, we devise a system of taxes and subsidies that, in equilibrium, induces each agent to select a different good. This system, however, requires that there be a sufficient aggregate endowment of money, namely,

$$X > (n-1)y \tag{11.6}$$

Theorem 3: Suppose that the aggregate endowment of money satisfies (11.6). If utility functions are increasing and continuous in x, and, for all i, u_i satisfies (11.1) and (11.2), then a fair allocation exists.

Proof

Let the price of money be 1. Normalize the other prices so that their sum is r, where r is small enough so that

$$X - (n-1)y - nr > 0 \tag{11.7}$$

$$u_i(X/n - r, k) > u_i(X/n + r, 0) \text{ for all } k \tag{11.8}$$

and

$$u_i(x + y - r, k) > u_i(x, j) \text{ for all } i, j, \text{ and } k \tag{11.9}$$

We shall imagine that each agent is endowed with ($1/n$ of the) money but nothing else. He therefore has to buy the indivisible goods from an 'auctioneer'. Let us suppose that the proceeds (which amount to r) are returned in lump-sum fashion.

Assume that an agent who buys good k receives the monetary transfer

$$T^k = \left[\frac{r}{np^*} - \frac{p^k}{p^*}\right] y \tag{11.10}$$

where $p^* = \max_i p^i$. Notice that $\Sigma T^k = 0$ and that, from (11.7), an agent can afford any good. Hence, if $e^i(\cdot)$ is agent i's demand correspondence for the indivisible goods, it is upper himicontinuous. Take $e(\cdot) = \Sigma e^i(\cdot)$. Let $\hat{e}(\cdot)$ be the convex hull of $e(\cdot)$. From (11.8) and our transfer rules, an agent is always better off buying the cheapest good than no good at all. Hence, for any price vector, $\mathbf{p}$, each agent buys exactly one good, and

$$0 \leqslant p \cdot e \leqslant nr \text{ for any } e \in \hat{e}(p) \tag{11.11}$$

For any $e \in [0,n] \times \ldots \times [0,n]$ let

$$g(e) = \{\hat{p} \mid \hat{p} \text{ maximizes } p \cdot e\}$$

The correspondence $(p,e) \rightarrow g(e) \times \hat{d}(p)$ has a fixed point $(\bar{p},\bar{e})$.

Suppose that $\bar{p}^j = 0$ for some good j. Let k be a good whose price is maximal. Then

$$\bar{p}^k \geqslant r/(n-1) \tag{11.12}$$

From (11.10), the purchaser of good j pays at least y less than the purchaser of good k. Thus, from (11.9), all agents are better off buying j than k. By the definition of g, this is a contradiction of $\bar{p}^j = 0$. We conclude that $\bar{p}^k > 0$ for all k, and so $\bar{d}^1 = \ldots = \bar{d}^n$ by definition of g. Hence,

$$(\bar{d}^1,\ldots,\bar{d}^n)=(1,\ldots,1) \tag{11.13}$$

From the same application of Birkhoff's Theorem that we made in the proof of Theorem 2, we may conclude that $(1,\ldots,1)$ is in $d(\bar{p})$. Hence, $(1,\ldots,1)$ corresponds to an equilibrium allocation and therefore to a fair allocation.

QED

It may be helpful to illustrate the role of condition (11.6) with an example. Suppose that each of two agents gets utility i from indivisible good i $(i=1,2)$ and marginal utility of 1 from money. In this case $y=2-1=1$, and so condition (11.6) requires that there be an aggregate endowment of at least 1 unit of money. Notice that condition (11.4), which would require that agents prefer the bundle $(1/2,1)$ to $(0,2)$, is violated. However, as long as (11.6) is satisfied, the allocation where one agent is assigned $((X+1)/2,1)$ and the other, $((X-1)/2,2)$, is a fair allocation.

Note: After this chapter was written I learned of the article by Svensson (see below), which establishes the existence of fair allocation with indivisibilities under rather different conditions.

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