Appendix

Appendix Contents. Appendix A presents the continuous-time specification of diagnostic expectations used in this paper. Appendix B provides model details, proofs, and additional results. Appendix C describes the equilibrium derivation, and Appendix D outlines the numerical methods used to solve the model. Appendix E gives additional details and extensions to this paper's model of diagnostic expectations, including a discrete-time formulation.

A Diagnostic Expectations Appendix

A.1 Diagnostic Expectations in Continuous Time

This section provides a microfoundation for the reduced-form expectations process outlined in Section 2.2. A goal for this paper's model of diagnostic expectations is to be a portable extension of existing models ["PEEMish"](Rabin, 2013). The expectations model is designed such that rational models can be augmented with diagnostic expectations using a single additional state variable.

Diagnostic expectations are applied to the log of the capital stock. Log capital evolves according to $dk_t = (i_t - \delta - \frac{\sigma^2}{2})dt + \sigma dZ_t$. Diagnostic expectations are applied to log capital for two reasons. Psychologically, it is consistent with Weber's Law that shocks are perceived as percentage changes rather than level changes. Mathematically, working with log capital ensures that $\mathcal{I}_t$ is stationary because the diffusion coefficient for log capital is constant.

Step 1: Defining the Background Context. Following the terminology of Bordalo et al. (2018a, henceforth BGS), the first step is to define the "background context" for capital. The background context is a counterfactual level of the log capital stock. It forms the dynamic "reference class" used to characterize representativeness.

The background context reflects the absence of recent information. Equation (3) introduces $\mathcal{I}_t \equiv \int_0^t e^{-\kappa(t-s)}\sigma dZ_s$ as a measure of recent information. This implies the following definition of the background context.

Definition 3. *Let G_t^- denote the background context of log capital at time t. G_t^- is defined as follows:*

$$G_t^- = k_t - \mathcal{I}_t.$$

Step 2: Modeling Expectations Given the Background Context. The next step is to specify how agents form expectations. Because time is continuous, expectations must be specified for future periods $t+\tau$, for all $\tau > 0$. The current period is t. Let $h(k_{t+\tau}|k_t, e_t, \mathcal{I}_t)$ denote the true distribution of log capital at time $t+\tau$ conditional on state variables k_t, e_t, and $\mathcal{I}_t$. Let $h(k_{t+\tau}|G_t^-, e_t, \mathcal{I}_t)$ denote the true distribution of log capital at time $t+\tau$ conditional on current state variables e_t and $\mathcal{I}_t$, but now using counterfactual log capital level G_t^-.

Let $k'_{t+\tau}$ denote one possible realization of log capital at time $t+\tau$. Following BGS and Gennaioli and Shleifer (2010), the "representativeness" of future state $k'_{t+\tau}$ is given by the following likelihood ratio:

$$\frac{h(k'_{t+\tau}|k_t, e_t, \mathcal{I}_t)}{h(k'_{t+\tau}|G_t^-, e_t, \mathcal{I}_t)}. \tag{25}$$

The most representative states are the ones exhibiting the largest increase in likelihood based on recent information.

One difficulty with equation (25) is that little is known about distributions $h(k_{t+\tau}|k_t, e_t, \mathcal{I}_t)$ and $h(k_{t+\tau}|G_t^-, e_t, \mathcal{I}_t)$ because k_t is an endogenous process.[33] This difficulty is overcome by using an instantaneous prediction horizon of $\tau = dt$. Because k_t is an Itô Process it is instantaneously Gaussian. Taking $\tau \to dt$, $h(k'_{t+\tau}|k_t, e_t, \mathcal{I}_t) = \mathcal{N}\left(k_t + \left[i(e_t, \mathcal{I}_t) - \delta - \frac{\sigma^2}{2}\right] dt, \sigma^2 dt\right)$ and $h(k'_{t+\tau}|G_t^-, e_t, \mathcal{I}_t) = \mathcal{N}\left(G_t^- + \left[i(e_t, \mathcal{I}_t) - \delta - \frac{\sigma^2}{2}\right] dt, \sigma^2 dt\right)$. I now define diagnostic expectations over prediction horizon $\tau = dt$. The prediction horizon will be extended iteratively in Step 3.

Diagnostic expectations overweight states that are representative of recent news.

[33] This is in contrast to BGS, where expectations are only specified for exogenous AR(N) processes.

This is formalized by assuming that agents evaluate future levels of log capital according to the distorted density:

$$h_t^\theta(k'_{t+dt}|k_t, e_t, \mathcal{I}_t) = h(k'_{t+dt}|k_t, e_t, \mathcal{I}_t) \cdot \left[\frac{h(k'_{t+dt}|k_t, e_t, \mathcal{I}_t)}{h(k'_{t+dt}|G_t^-, e_t, \mathcal{I}_t)}\right]^{\theta dt} \frac{1}{Z}. \tag{26}$$

Equation (26) modifies a similar formula in BGS. The key adjustment for continuous time is that equation (26) defines expectations at $t + dt$, while the discrete-time formulation of BGS defines expectations at $t+1$. In equation (26), the true conditional probability $h(k'_{t+dt}|k_t, e_t, \mathcal{I}_t)$ is distorted by the representativeness term in brackets.

The extent to which representativeness distorts expectations is governed by parameter θ. θ is scaled by the prediction horizon dt because representativeness should impose only an infinitesimal distortion on the perceived distribution of capital over an infinitesimally short horizon. Otherwise, the agent would expect that k_t jumps discontinuously from t to $t + dt$.

Using equation (26), the following proposition characterizes the perceived evolution of capital.

Proposition 3. *A diagnostic agent perceives that capital evolves according to*

$$\frac{\widehat{dK_t}}{K_t} = (i_t - \delta)dt + \theta\mathcal{I}_t dt + \sigma dZ_t. \tag{27}$$

Proof. See Appendix B.5. □

Judging by representativeness biases the perceived growth rate of capital by $\theta\mathcal{I}_t$.

Step 3: The Evolution of Beliefs. Step 1 defines the background context G_t^- and Step 2 specifies diagnostic expectations of $\widehat{dK_t}$. This step models the dynamics of expectations over longer horizons. Because capital is endogenous, only the instantaneous distribution of k_t is known. Future expectations are therefore defined iteratively. In particular, repeated applications of the instantaneous Gaussian properties of k_t can be used to define expectations of the economy at $t+dt$, then $t+2dt$, then

$t + 3dt$, etc. This iterative procedure imposes that the law of iterated expectations holds with respect to distorted expectations, consistent with the BGS model.

Diagnostic agents form expectations by simulating the economy forward state-by-state. As the diagnostic agent simulates the economy forward from time t, the internal representativeness parameter at simulated future time $t + \tau$ is given by:

$$\begin{aligned}\mathcal{I}^S_{t+\tau} &\equiv \int_0^t e^{-\kappa(t+\tau-s)}\sigma dZ_s, \text{ or equivalently} \\ &= e^{-\kappa\tau}\mathcal{I}_t.\end{aligned} \tag{28}$$

The superscript S, for simulated, is used to signify that $\mathcal{I}^S_{t+\tau}$ is the agent's unconscious internal representativeness state as the agent forms expectations of the economy in period $t + \tau$. Information that was representative at time t decays at rate κ as the perceived economy is simulated forward in time.

Let $k'_{t+\tau}$ and $e'_{t+\tau}$ denote one possible realization of log capital and capital capacity at time $t + \tau$. Using equation (28), the simulated background context at $t + \tau$ can now be defined in an analogous fashion to Definition 3.

Definition 4. *Let $k'_{t+\tau}$ denote some simulated level of log capital at future time $t+\tau$. Given $k'_{t+\tau}$, the simulated background context at time $t + \tau$ is defined as follows:*

$$G'^{-}_{t+\tau} = k'_{t+\tau} - \mathcal{I}^S_{t+\tau}.$$

As above, the simulated future background context reflects the absence of recent information.

Again proceeding in an analogous fashion to Step 2, at time $t + \tau$ the agent iteratively forms expectations about $t + \tau + dt$ according to:

$$h^\theta_t(k'_{t+\tau+dt}|k'_{t+\tau}, e'_{t+\tau}, \mathcal{I}^S_{t+\tau}) = h(k'_{t+\tau+dt}|k'_{t+\tau}, e'_{t+\tau}, \mathcal{I}^S_{t+\tau}) \cdot \left[\frac{h(k'_{t+\tau+dt}|k'_{t+\tau}, e'_{t+\tau}, \mathcal{I}^S_{t+\tau})}{h(k'_{t+\tau+dt}|G'^{-}_{t+\tau}, e'_{t+\tau}, \mathcal{I}^S_{t+\tau})}\right]^{\theta dt} \frac{1}{Z}. \tag{29}$$

It follows that the diagnostic agent perceives that capital evolves according to:

$$\frac{\widehat{dK'_{t+\tau}}}{K'_{t+\tau}} = (i_{t+\tau} - \delta)dt + \theta \mathcal{I}^S_{t+\tau} dt + \sigma dZ_{t+\tau}. \tag{30}$$

Future expectations in equation (30) should be contrasted with those of a rational agent who correctly believes that capital evolves according to $\frac{dK'_{t+\tau}}{K'_{t+\tau}} = (i_{t+\tau} - \delta)dt + \sigma dZ_{t+\tau}$. Since $\mathcal{I}^S_{t+\tau} = e^{-\kappa\tau}\mathcal{I}_t$, equation (30) specifies that the effect of diagnostic expectations on the perceived growth rate of capital fades at rate κ as the agent simulates the evolution of the economy further into the future ($\tau \to \infty$). Diagnostic expectations capture the overweighting of states that are representative of current economic conditions. As the agent looks further temporally ahead, information that is diagnostic of economic conditions at time t steadily dims.

It is worth noting that equation (30) only stipulates that the diagnostic agent's perception of $k_{t+\tau}$'s drift converges to rationality as $\tau \to \infty$. Since the drift has a cumulative effect on the level of $k_{t+\tau}$, diagnostic expectations of the level of $k_{t+\tau}$ can diverge increasingly from rational expectations as τ increases (e.g., Figure 1).

Summary. This completes the microfoundation of the reduced-form beliefs process specified in Section 2.2. Extensions are given in Appendix E.2. Appendix E.1 discusses how the discrete-time analogue of this paper's expectations model relates to the original BGS model.

To summarize, expectations of the endogenous capital process are formed iteratively in order to make repeated use of the instantaneous Gaussian properties of $dk_{t+\tau}$. Step 2 defines how $\mathcal{I}_t$ affects the expected evolution of the economy from t to $t + dt$. Step 3 then defines how $\mathcal{I}^S_t$ evolves as expectations are simulated forward. In detail, Step 2 takes $k_t, e_t, \mathcal{I}_t$ as given and provides a perceived mapping into $\widehat{k}_{t+dt}$ and $\widehat{e}_{t+dt}$ given shock dZ_t. The hat-notation denotes that agents may not properly understand the evolution of these state variables. Step 3 takes $\mathcal{I}_t$ as given and provides $\mathcal{I}^S_{t+dt}$. Then, Step 2 is applied again (now taking $\widehat{k}_{t+dt}$, $\widehat{e}_{t+dt}$, and $\mathcal{I}^S_{t+dt}$ as given) to calculate $\widehat{k}_{t+2dt}$ and $\widehat{e}_{t+2dt}$ given shocks dZ_t and dZ_{t+dt}. Applying Step 3 again gives $\mathcal{I}^S_{t+2dt}$. This process is repeated to generate expectations at time $t+\tau$, for all $\tau > 0$.

Diagnostic agents make two mistakes when $\theta > 0$. First, they hold incorrect beliefs about the drift of capital. Second, they have incorrect expectations about their own future expectations because they do not understand that they are diagnostic. A comparison of equations (3) and (28) shows that diagnostic agents do not perceive that future capital quality shocks will alter the bias of their future expectations.[34]

I end by discussing why this model of diagnostic expectations can serve as a portable extension of existing rational models. First, equations (27) and (30) illustrate that state variable $\mathcal{I}_t$ alone is sufficient to characterize the state of expectations relative to rationality. Second, the evolution of $\mathcal{I}_t$ is self-contained. $\mathcal{I}_t$ can be expressed in differential form as $d\mathcal{I}_t = -\kappa\mathcal{I}_t dt + \sigma dZ_t$. Thus, state variable $\mathcal{I}_t$ plus the shock σdZ_t are sufficient to calculate $d\mathcal{I}_t$. It is these two attributes that make this formulation of diagnostic expectations portable: $\mathcal{I}_t$ alone characterizes expectations relative to rationality, and $\mathcal{I}_t$ is sufficient for its own evolution.

A.2 Diagnostic Expectations Calibration and Application

θ Calibration. The baseline calibration sets θ such that one standard deviation in $\mathcal{I}$ corresponds to an output growth bias of 0.75 percentage points. The magnitude of this bias aligns with the estimates in Bordalo et al. (2018a, henceforth BGS), Bordalo et al. (2018b, BGMS), Bordalo et al. (2019b, BGLS), Bordalo et al. (2019a, BGST), and d'Arienzo (2020).

Using data from 1968Q4 through 2016Q4, BGMS assume that the realized annual growth rate of real GDP follows an AR(1) process: $x_t = \rho x_{t-1} + u_t$. BGMS estimate $\rho = 0.87$ and $\sigma_u = 1.10$. Let θ_D denote the representativeness parameter for the discrete-time specification developed in BGS. The basic model of diagnostic expectations applied to an AR(1) process predicts that in period t, the forecast of x_{t+1} is

[34] Put differently, the realized future information parameter $\mathcal{I}_{t+\tau}$ is a random variable at time t whereas $\mathcal{I}^S_{t+\tau}$ is deterministic at time t.

biased by $\theta_D \rho u_t$ (see BGS). Using the BGMS estimates of ρ and σ_u, a one standard deviation output growth bias of 0.75 percentage points corresponds to $\theta_D = 0.78$.[35] This is consistent with the estimates of θ_D provided in BGS ($\theta_D = 0.91$), BGMS (estimates vary, with a single collective estimate of $\theta_D = 0.50$), BGLS ($\theta_D = 0.90$), BGST ($\theta_D = 1.08$), and d'Arienzo (2020) (two estimates: $\theta_D = 0.47$ and $\theta_D = 0.70$).

Calculating Sentiment with Expectations Data. This section details how to construct an empirical measure of sentiment given a calibration of κ and θ. This construction is used to calibrate κ. It is also used to calculate the empirical measure of sentiment shown in Figure 9, and to calculate the SPF-implied shocks used in Figure 10.

Sentiment is constructed using the median forecast of real GDP growth from the Survey of Professional Forecasters (SPF).[36] Let FE_t^{SPF} denote the SPF forecast error in quarter t. I define the forecast error in quarter t as the realized GDP growth rate in quarter t minus the median quarter $t-2$ forecast of the GDP growth rate in quarter t. The forecast error is defined with a two-quarter lagged prediction to account for the slow incorporation of shocks into GDP statistics (e.g., Lehman declared bankruptcy in 2008Q3, but this was not reflected in U.S. GDP until 2008Q4).

Proposition 2 provides a method for calculating sentiment using subjective shocks to economic growth. SPF forecast data is collected quarterly while equation (24) is written in continuous time. Equation (24) can be discretized at a quarterly frequency as follows:

$$\mathcal{I}_t = \sum_{j=0}^{t-1} \left(\mathcal{K} + \frac{\theta}{4} \right)^j FE_{t-1-j}^{SPF}, \tag{31}$$

where $\mathcal{K} = e^{-\kappa/4}$. A derivation of equation (31) is provided below.

Equation (31) needs to be initialized with a "period 0." I assume that sentiment

[35] Setting $0.75 = \theta(0.87)(1.1)$ yields the desired result.

[36] Expert forecasts are used for consistency with the model, as discussed in footnote 27.

is equal to 0 in 1970Q1. Starting from the initial condition, equation (31) provides a method for constructing sentiment using quarterly SPF forecast errors. These forecast errors, along with the measure of sentiment in Figure 9, are also used to calculate the shocks used in Figure 10.

κ **Calibration.** Sentiment persistence parameter κ is calibrated to align the model-implied correlation between e_t and $\mathcal{I}_t$ with the data. I've just outlined a technique for calculating sentiment from SPF forecast data given a calibration of κ and θ. He et al. (2017) provide an empirical measure corresponding to e_t over the period 1970Q1 to 2018Q3. For any given κ and θ, this means that I can calculate the correlation between e_t and $\mathcal{I}_t$ in both the data and the model. This analysis is presented in Figure 1 below, which suggests that sentiment should have a half-life of approximately 5 years.[37]

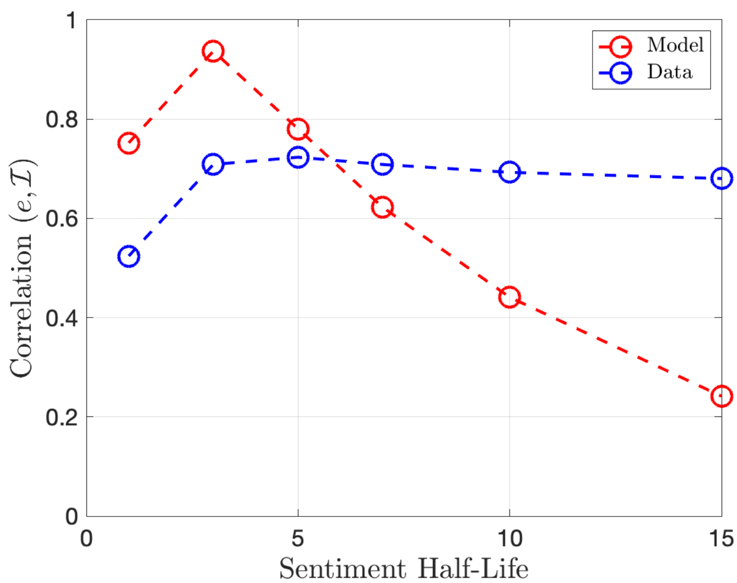

Figure 1: **Calibration of** κ. The red curve plots the correlation between e_t and $\mathcal{I}_t$ in the model as a function of κ (taking θ as given from the diagnostic expectations literature). The blue curve plots the empirical correlation between e_t (from He et al. (2017)) and $\mathcal{I}_t$ (calculated from SPF forecast errors given κ and θ).

[37] As κ is varied, θ is always set to $\frac{(\sqrt{2\kappa})0.0075}{\sigma}$. This maintains the other calibration target that one standard deviation in $\mathcal{I}$ corresponds to an output growth bias of 0.75 percentage points.

Discretization of the Model: Derivation of Equation (31). I partially rewrite the model in discrete time. The discrete-time model is written at a quarterly frequency for consistency with SPF data. Subscript t's denote model periods, such that period $t+1$ occurs one quarter after period t.

Define the capital law of motion as $K_{t+1} = K_t \exp(\upsilon_t + \sigma\epsilon_{t+1})$, where $\sigma\epsilon_t$ are quarterly capital quality shocks, υ_t captures investment and depreciation, and $\epsilon_t \sim \mathcal{N}(0, \frac{1}{4})$. The capital law of motion can be approximated as follows:

$$\frac{K_{t+1} - K_t}{K_t} = \frac{Y_{t+1} - Y_t}{Y_t} \approx \upsilon_t + \sigma\epsilon_{t+1}.$$

Next, I introduce the analogous discrete-time definition of sentiment:

$$\mathcal{I}_t = \sum_{j=0}^{t-1} \mathcal{K}^j \sigma\epsilon_{t-j}, \tag{32}$$

where $\mathcal{K} = e^{-\kappa/4}$. Under diagnostic expectations, agents expect that output (or capital) evolves approximately as follows:[38]

$$\widehat{\mathbb{E}}_t \left[\frac{Y_{t+1} - Y_t}{Y_t}\right] \approx \mathbb{E}_t \left[\frac{Y_{t+1} - Y_t}{Y_t}\right] + \frac{\theta\mathcal{I}_t}{4}.$$

Analogous to equation (5), diagnostic expectations of GDP growth consist of a rational component plus a diagnostic wedge. The dt factor in equation (5) becomes $\frac{1}{4}$ here since the model is discretized at a quarterly frequency.

[38] In logs, the law of motion for capital is $k_{t+1} = k_t + \upsilon_t + \sigma\epsilon_{t+1}$. Let $G_t^- = k_t - \mathcal{I}_t$. In discrete time, equation (26) becomes:

$$h_t^\theta(k'_{t+1}|k_t, e_t, \mathcal{I}_t) = h(k'_{t+1}|k_t, e_t, \mathcal{I}_t) \cdot \left[\frac{h(k'_{t+1}|k_t, e_t, \mathcal{I}_t)}{h(k'_{t+1}|G_t^-, e_t, \mathcal{I}_t)}\right]^{\frac{\theta}{4}} \frac{1}{Z}.$$

The main difference relative to equation (26) is that the power term of θdt becomes $\frac{\theta}{4}$ here, since the model is specified at a quarterly frequency. Since $k_{t+1}|k_t, e_t, \mathcal{I}_t \sim \mathcal{N}\left(k_t + \upsilon_t, \frac{\sigma^2}{4}\right)$ and $k_{t+1}|G_t^-, e_t, \mathcal{I}_t \sim \mathcal{N}\left(G_t^- + \upsilon_t, \frac{\sigma^2}{4}\right)$, a similar argument to that of Appendix B.5 gives $\widehat{\mathbb{E}}_t\left[k_{t+1} - k_t\right] = \upsilon_t + \frac{\theta\mathcal{I}_t}{4}$. This implies $\widehat{\mathbb{E}}_t\left[\frac{Y_{t+1}-Y_t}{Y_t}\right] \approx \mathbb{E}_t\left[\frac{Y_{t+1}-Y_t}{Y_t}\right] + \frac{\theta\mathcal{I}_t}{4}$.

At a quarterly frequency, subjective shocks are defined as

$$\sigma\widehat{\epsilon}_{t+1} \equiv \frac{Y_{t+1} - Y_t}{Y_t} - \widehat{\mathbb{E}}_t \left[\frac{Y_{t+1} - Y_t}{Y_t} \right].$$

Unlike objective shocks $\sigma\epsilon_t$, these subjective shocks are directly observable with expectations data. Using the above approximations:

$$\sigma\widehat{\epsilon}_{t+1} \approx \sigma\epsilon_{t+1} - \frac{\theta\mathcal{I}_t}{4}.$$

Equation (32) can now be redefined in terms of subjective shocks. It follows from equation (32) that $\mathcal{I}_t = \mathcal{K}\mathcal{I}_{t-1} + \sigma\epsilon_t$. Plugging in the approximation of subjective shocks gives

$$\mathcal{I}_t \approx \left(\mathcal{K} + \frac{\theta}{4} \right) \mathcal{I}_{t-1} + \sigma\widehat{\epsilon}_t.$$

Iterating backward and using the initial condition that $\mathcal{I}_0 = 0$ yields:

$$\mathcal{I}_t \approx \sum_{j=0}^{t-1} \left(\mathcal{K} + \frac{\theta}{4} \right)^j \sigma\widehat{\epsilon}_{t-j}.$$

Equation (31) is recovered by noting that $\sigma\widehat{\epsilon}_t = FE_{t-1}$. This holds since $\sigma\widehat{\epsilon}_t$ is the forecast error realized from the start of period $t-1$ to the start of period t.[39]

[39] In other words, $\sigma\widehat{\epsilon}_t$ is the forecast error that is realized over quarter $t-1$.

B Model Details

B.1 Labor Income Microfoundation

The treatment here of Frankel (1962) follows from Aghion and Howitt (2008, Chapter 2). Each individual producer faces decreasing returns to capital, but decreasing returns at the producer level are offset at the aggregate level through knowledge externalities.

At time t, there exists a measure $\mathbb{J}_t$ of intermediaries.[40] Each intermediary j operates $K_{j,t}$ units of capital and hires $L_{j,t}$ units of labor at time t. The intermediary faces the production function:

$$Y_{j,t} = \bar{A}_t K_{j,t}^{\nu} L_{j,t}^{1-\nu}, \tag{33}$$

where $\bar{A}_t$ is an endogenous aggregate productivity level. Due to knowledge spillovers, $\bar{A}_t$ depends on the total amount of capital in the economy:

$$\bar{A}_t = A \left(\int_j K_{j,t} dj \right)^{\varsigma}. \tag{34}$$

Parameter $\varsigma \in [0, 1]$ controls the level of knowledge externalities.

Let $\mathcal{W}_t$ denote the wage rate. Intermediaries hire labor as follows:

$$L_{j,t} = \underset{\ell}{\text{argmax}} \ \ \bar{A}_t K_{j,t}^{\nu} \ell^{1-\nu} - \mathcal{W}_t \ell$$

The optimal labor choice is:

$$L_{j,t} = K_{j,t} \left(\frac{\bar{A}_t (1-\nu)}{\mathcal{W}_t} \right)^{\frac{1}{\nu}}. \tag{35}$$

The next step is to impose market clearing. Specifically, $\int_j K_{j,t} dj = K_t$ and

[40] In the model, there is always a unit measure of households. The size of the financial intermediary sector varies over time due to banker entry and exit.

$\int_j L_{j,t} dj = 1$. Since all intermediaries are identical, $K_{j,t} = \frac{K_t}{\mathbb{J}_t}$ and $L_{j,t} = \frac{1}{\mathbb{J}_t}$. Imposing market clearing gives:

$$\begin{aligned}
\bar{A}_t &= AK_t^{\varsigma}, \text{ and} \\
Y_t &= \int_j Y_{j,t} dj = AK_t^{\varsigma} \int_j K_{j,t}^{\nu} L_{j,t}^{1-\nu} dj \\
&= AK_t^{\varsigma} \mathbb{J}_t \left(\frac{K_t}{\mathbb{J}_t} \right)^{\nu} \left(\frac{1}{\mathbb{J}_t} \right)^{1-\nu} \\
&= AK_t^{\varsigma+\nu}.
\end{aligned}$$

The following parameter restriction generates aggregate linearity:

$$\varsigma + \nu = 1. \tag{36}$$

Assumption (36) recovers an "AK" economy with a linear aggregate production function $Y_t = AK_t$, exactly as in the main text.

The final step is to use the market clearing conditions to solve for the wage rate. Plugging these into equation (35):

$$\begin{aligned}
\frac{1}{\mathbb{J}_t} &= \frac{K_t}{\mathbb{J}_t} \left(\frac{AK_t^{\varsigma}(1-\nu)}{\mathcal{W}_t} \right)^{\frac{1}{\nu}} \\
\mathcal{W}_t &= AK_t^{\varsigma+\nu}(1-\nu) \\
\mathcal{W}_t &= (1-\nu)AK_t, \qquad (37)
\end{aligned}$$

where the last line follows from (36). The knowledge externalities model built here provides a simple microfoundation for equation (11) in the main text. The benefits of introducing a labor income margin are discussed below.

B.2 Quantitative Benefits of the Labor Income Margin

This model's benchmark calibration sets $A = \frac{1}{3}$ and $\nu = 0.41$. The original HK model, which does not feature labor income, sets $A = 0.133$.[41] Since the average investment rate is roughly 10%, $A = 0.133$ implies that investment typically accounts for more than $\frac{2}{3}$ of the economy's output in the original HK model. Consumption accounts for less than $\frac{1}{3}$ of output. Though some parsimony of the original HK model is lost, the benefit of introducing a simple labor income margin is that it allows for a more realistic consumption-output share.

Generating a realistic consumption-output ratio yields two benefits. First, it allows for a more standard calibration of EIS parameter ζ. In the HK model, consumption accounts for only a small share of output. This implies that changes to the investment rate will cause consumption growth to swing wildly. This is particularly true in periods of financial distress, when the investment rate is sensitive to e_t. HK calibrate $\zeta = 0.13$ (EIS > 7) in order to prevent these swings in consumption from generating excessive interest rate volatility.

Second, the labor income extension generates a more realistic ratio of housing expenditures to total consumption. Recall that aggregate consumption is a Cobb-Douglas aggregator over the output good and housing services: $C_t = (c_t^y)^{1-\phi}(c_t^h)^{\phi}$. Parameter ϕ governs the ratio of housing expenditures to total consumption. To match the housing-wealth ratio of 45%, HK set $\phi = 0.6$. This implies that housing services compose 60% of expenditures.

In order to match the same housing-wealth ratio of 45% in this paper, I calibrate $\phi = 0.2$.[42] In addition to matching the housing-wealth ratio, $\phi = 0.2$ is also consistent with the ratio of housing expenditures to total consumption observed empirically.[43]

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[41] The labor income channel in this model can be shut down by setting $\nu = 1$.

[42] Matching the same housing-wealth ratio requires the equilibrium value of rental payments (D_t) to remain similar to the original HK model. Due to the labor income extension, my model features a larger share of output goods relative to housing services (i.e., $A \uparrow$). From equation (10), more output goods implies that a lower calibration of ϕ is needed in order to maintain D_t.

[43] For example, Table 2.4.5 of the 2019 NIPA reports total PCE of \$14563 (billion). Housing and utilities account for \$2670 (billion). The housing share of total consumption is 18%. This share is

B.3 Why Two Assets

At first glance it is puzzling that the model includes two assets, K_t and H, since these assets are perfectly conditionally correlated. The model attempts to jointly match key macroeconomic and financial market data. As a macroeconomic model it aims to generate empirically-plausible levels of investment volatility. As a finance model, enough asset price volatility is needed to produce quantitatively significant nonlinearities during periods of financial distress. These two goals present a well-known problem. Market values of capital are much more volatile than investment, both across firms and over time. In a standard q-theory model where investment is closely linked with asset prices, these two facts can only be reconciled with unreasonably high adjustment costs (e.g., Campbell, 2017, Ch. 7).

By introducing two assets, HK circumvent this issue. The two leftmost panels of Figure 2 show that p_t is more sensitive to e_t than q_t. This is because the supply of houses is fixed while K_t is procyclical.[44] Investment i_t is a function of q_t, so the lower variance of q_t allows for the model to match empirical investment volatility under reasonable adjustment costs. Since the intermediary holds both types of assets, the additional volatility provided by p_t generates an intermediary pricing kernel that is volatile enough to produce significant nonlinearities in financial intermediation. As Section 3.2 highlights, the two-asset model does a good job of matching empirical investment volatility as well as the overall risk-return profile of intermediary equity.

stable over time (Davis and Van Nieuwerburgh, 2015).

[44] In particular, recall that P_t is the present-discounted value of perceived future rental payments. When e_t is low, investment rate i_t is also expected to be low for a long period of time. This implies that the expected growth rate of output — and therefore rental payments — is expected to be low, too. Through this investment channel, the growth rate of rental payments is highly correlated with e_t. Hence, housing price p_t is sensitive to e_t.

B.4 Boundary Conditions

Boundary conditions are needed to solve for price functions $q(e, \mathcal{I})$ and $p(e, \mathcal{I})$. As $e \to \infty$ the equity issuance constraint ceases to affect the equilibrium price and policy functions. This implies $\lim_{e \to \infty} q_e(e, \mathcal{I}) = \lim_{e \to \infty} p_e(e, \mathcal{I}) = 0$.

A lower reflecting boundary is imposed by assuming entry into the intermediary sector deep in crisis times (term $d\psi_t$ in equation (8)). There exists an exogenous minimum reputation level $\underline{e}$ such that new intermediaries enter the economy whenever e_t hits $\underline{e}$.[45] Entry is costly because new intermediaries must acquire the skills to operate capital. Specifically, the economy must destroy $\beta > 0$ units of capital in order for entry to increase aggregate reputation $\mathcal{E}_t$ by one unit.[46] The assumption of a reflecting barrier at $\underline{e}$ implies that prices q and P must have a zero derivative with respect to e at $\underline{e}$. If this were not the case, an arbitrageur could bet on a unidirectional change in asset prices at the reflecting barrier. This implies $q_e(\underline{e}, \mathcal{I}) = 0$ and $p_e(\underline{e}, \mathcal{I}) = \frac{p(\underline{e}, \mathcal{I})\beta}{1+\underline{e}\beta}$.

To derive the lower boundary condition for p, start at the boundary with an aggregate reputation of $\underline{\mathcal{E}} = \underline{e}K$. Consider a further shock to reputation of z which sends reputation to $\underline{\mathcal{E}} - z < \underline{e}K$. After this shock, the reflecting boundary implies that capital will immediately be converted into reputation to restore $\underline{e}$. Specifically, let x denote the amount of new reputation required to restore $\underline{e}$. x is given by:[47]

$$x = \frac{z}{1 + \underline{e}\beta}.$$

Shock z requires the destruction of βx capital to restore reputation to $\underline{e}$.

This capital destruction equation can be used to derive the boundary condition for price function P. The condition that $P_e(\underline{e}, \mathcal{I}) = 0$ implies that $p = \frac{P}{K}$ will have

[45] Loosely, this captures (unmodeled) government intervention deep in crises.

[46] Equations (2) and (4) are altered at $\underline{e}$ to include this form of capital destruction.

[47] x is defined implicitly by $\underline{e} = \frac{\mathcal{E} - z + x}{K - \beta x}$. The numerator of this equation is the level of initial reputation minus the shock and plus x of new reputation. The denominator is the initial level of capital minus the βx of capital destroyed to produce x reputation. Capital will be destroyed until $\underline{e}$ (the minimum level of $e = \mathcal{E}/K$) is restored.

a non-zero slope at $\underline{e}$.[48] Consider the above reputation shock of z. Immediately after this shock, the price of housing is $P = p\left(\underline{e} - \frac{z}{K}, \mathcal{I}'\right) K$. This must equal the price of housing immediately after capital is spent to rebuild reputation, given by $P = p\left(\underline{e}, \mathcal{I}'\right)(K - \beta x)$. In the continuous-time limit with arbitrarily small shocks, $p\left(\underline{e} - \frac{z}{K}, \mathcal{I}'\right) K$ can be rewritten as $p(\underline{e}, \mathcal{I}')K - p_e\left(\underline{e}, \mathcal{I}'\right) z$. Combining:

$$p(\underline{e}, \mathcal{I}')K - p_e\left(\underline{e}, \mathcal{I}'\right) z = p(\underline{e}, \mathcal{I}')\left(K - \beta\frac{z}{1 + \underline{e}\beta}\right)$$
$$p_e(\underline{e}, \mathcal{I}') = \frac{p(\underline{e}, \mathcal{I}')\beta}{1 + \underline{e}\beta}.$$

This gives the boundary condition for p at $\underline{e}$. Details on how the boundary conditions are imposed numerically are provided in Appendix D.

[48]This is due to capital destruction, which makes the denominator of $\frac{P}{K}$ change at the entry boundary.

B.5 Proofs

Proof of Proposition 2. By definition, $\mathcal{I}_t \equiv \int_0^t e^{-\kappa(t-s)}\sigma dZ_s$. Subjective shocks are defined as $\sigma\widehat{dZ_t} = -\theta\mathcal{I}_t dt + \sigma dZ_t$. The law of motion for $\mathcal{I}_t$ is given by $d\mathcal{I}_t = -\kappa\mathcal{I}_t dt + \sigma dZ_t$. Plugging in the definition of subjective shocks gives

$$d\mathcal{I}_t = (-\kappa + \theta)\mathcal{I}_t dt + \sigma\widehat{dZ_t}.$$

Let $f(\mathcal{I}_t, t) = \mathcal{I}_t e^{(\kappa-\theta)t}$. Using Itô's lemma:

$$\begin{aligned} df(\mathcal{I}_t, t) &= e^{(\kappa-\theta)t}\left(d\mathcal{I}_t\right) + (\kappa - \theta)\mathcal{I}_t e^{(\kappa-\theta)t} dt \\ &= e^{(\kappa-\theta)t}\left((-\kappa + \theta)\mathcal{I}_t dt + \sigma\widehat{dZ_t}\right) + (\kappa - \theta)\mathcal{I}_t e^{(\kappa-\theta)t} dt \\ &= e^{(\kappa-\theta)t}\sigma\widehat{dZ_t}. \end{aligned}$$

Given an initial condition of $f(\mathcal{I}_0, 0) = 0$, we get:

$$\begin{aligned} f(\mathcal{I}_t, t) &= \int_0^t df(\mathcal{I}_s, s) \\ \mathcal{I}_t e^{(\kappa-\theta)t} &= \int_0^t e^{(\kappa-\theta)s}\sigma\widehat{dZ_s} \\ \mathcal{I}_t &= \int_0^t e^{(-\kappa+\theta)(t-s)}\sigma\widehat{dZ_s}. \end{aligned}$$

This completes the proof.

Proof of Proposition 3 (in Appendix A.1). Consider the evolution of log capital k_t over a horizon of $\tau > 0$, holding investment fixed at $\bar{i} = i_t$. With fixed investment:

$$\begin{aligned} k_{t+\tau} &= k_t + \int_t^{t+\tau} (\bar{i} - \delta - \frac{\sigma^2}{2}) ds + \int_t^{t+\tau} \sigma dZ_s \\ &= k_t + \tau(\bar{i} - \delta - \frac{\sigma^2}{2}) + \sigma(Z_{t+\tau} - Z_t). \end{aligned}$$

Since $\{Z_t\}$ is a standard Brownian motion, $k_{t+\tau} \sim \mathcal{N}\left(k_t + \tau(\bar{i} - \delta - \frac{\sigma^2}{2}),\ \sigma^2\tau\right)$.

For the arbitrary prediction horizon of τ, one can rewrite equation (26) as:

$$h_t^\theta(k'_{t+\tau}|k_t, e_t, \mathcal{I}_t) = h(k'_{t+\tau}|k_t, e_t, \mathcal{I}_t) \cdot \left[\frac{h(k'_{t+\tau}|k_t, e_t, \mathcal{I}_t)}{h(k'_{t+\tau}|G_t^-, e_t, \mathcal{I}_t)}\right]^{\theta\tau} \frac{1}{Z}.$$

Given a fixed investment level of $\bar{i}$, this implies:

$$h_t^\theta(k'_{t+\tau}|k_t, e_t, \mathcal{I}_t) = \mathcal{N}\left(k_t + \tau(\bar{i} - \delta - \frac{\sigma^2}{2}) + \theta(k_t - G_t^-)\tau,\ \sigma^2\tau\right).$$

The Bordalo et al. (2018a) Appendix provides algebraic details for this step. Equivalently, the agent perceives that

$$\widehat{k_{t+\tau}} - k_t = \int_t^{t+\tau} (\bar{i} - \delta - \frac{\sigma^2}{2}) ds + \int_t^{t+\tau} \theta(k_t - G_t^-) ds + \int_t^{t+\tau} \sigma dZ_s.$$

In the limit as $\tau \to dt$:

$$\widehat{dk_t} = (i_t - \delta - \frac{\sigma^2}{2}) dt + \theta(k_t - G_t^-) dt + \sigma dZ_t.$$

Definition 3 gives $\mathcal{I}_t = k_t - G_t^-$. Applying Itô's lemma to $K_t = exp(k_t)$ yields:

$$\frac{\widehat{dK_t}}{K_t} = (i_t - \delta) dt + \theta \mathcal{I}_t dt + \sigma dZ_t.$$

This completes the proof.

B.6 Robustness

This section examines robustness to behavioral parameters θ and κ, and the EIS parameter ζ. The baseline calibration sets $\theta \times SD(\mathcal{I}) = 0.75\%$, the half-life of $\mathcal{I}$ to 5 years, and $\zeta = \frac{2}{3}$ (EIS = 1.5). I analyze the following six parameter perturbations: (i) $\theta \times SD(\mathcal{I}) = 1.5\%$, (ii) $\theta \times SD(\mathcal{I}) = \frac{0.75\%}{2}$, (iii) sentiment half-life of 20 years, (iv) sentiment half-life of 1 year, (v) $\zeta = 1$, and (vi) $\zeta = \frac{1}{2}$.

For perturbations to θ and ζ, all other parameters are kept at their baseline calibration in Table 1. For perturbations to the persistence of sentiment (κ), I change θ accordingly to ensure that $\theta \times SD(\mathcal{I})$ remains equal to 0.75%. This isolates the effect of persistence.

Sentiment-Driven Financial Crises.

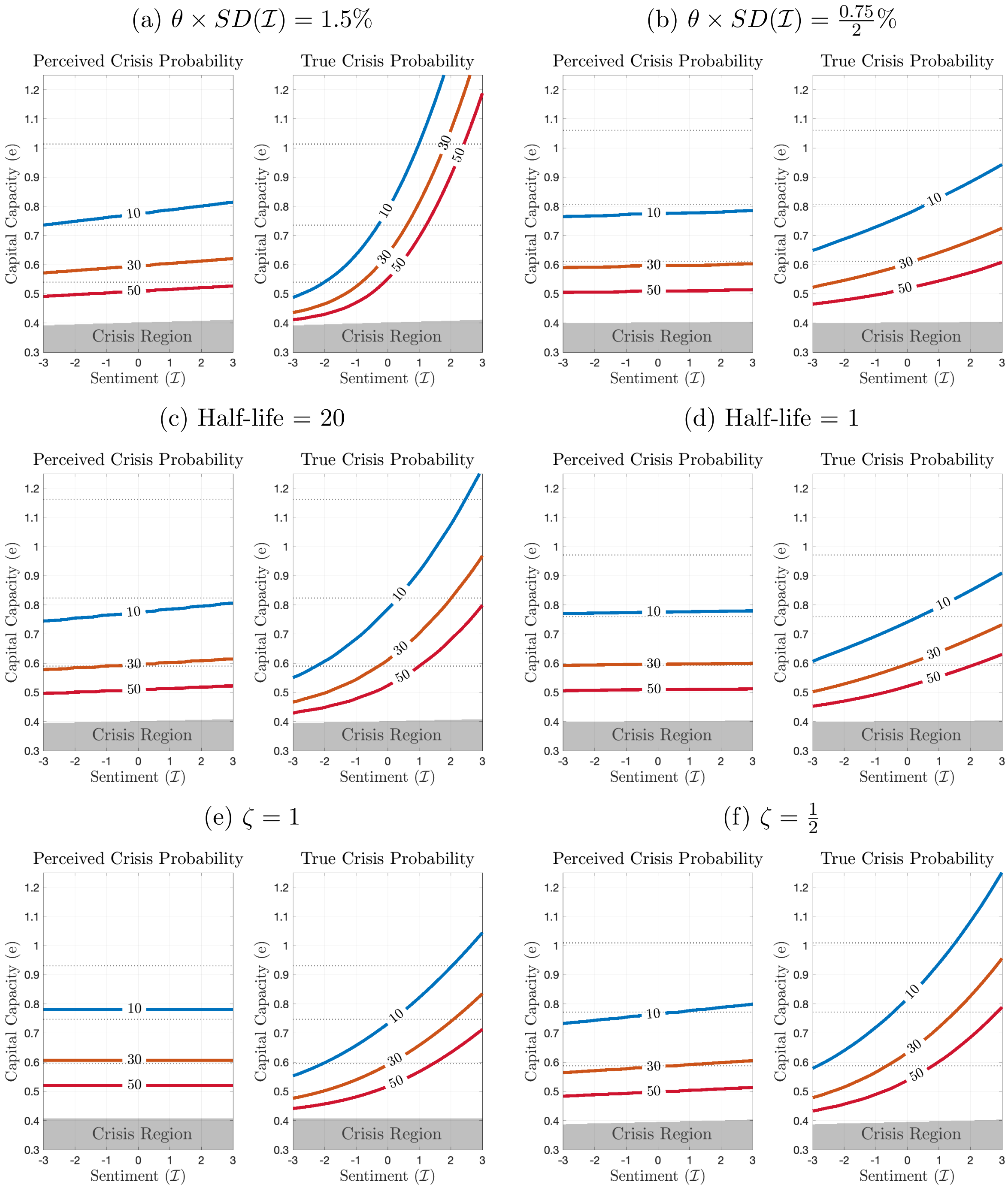


Figure 2: **Crisis hitting probabilities**. See Figure 4 for a description. In order from left-to-right, top-to-bottom: $\theta \times SD(\mathcal{I}) = 1.5\%$, $\theta \times SD(\mathcal{I}) = \frac{0.75}{2}\%$, half-life = 20, half-life = 1, $\zeta = 1$, $\zeta = \frac{1}{2}$.

Boom-Bust Investment Cycles.

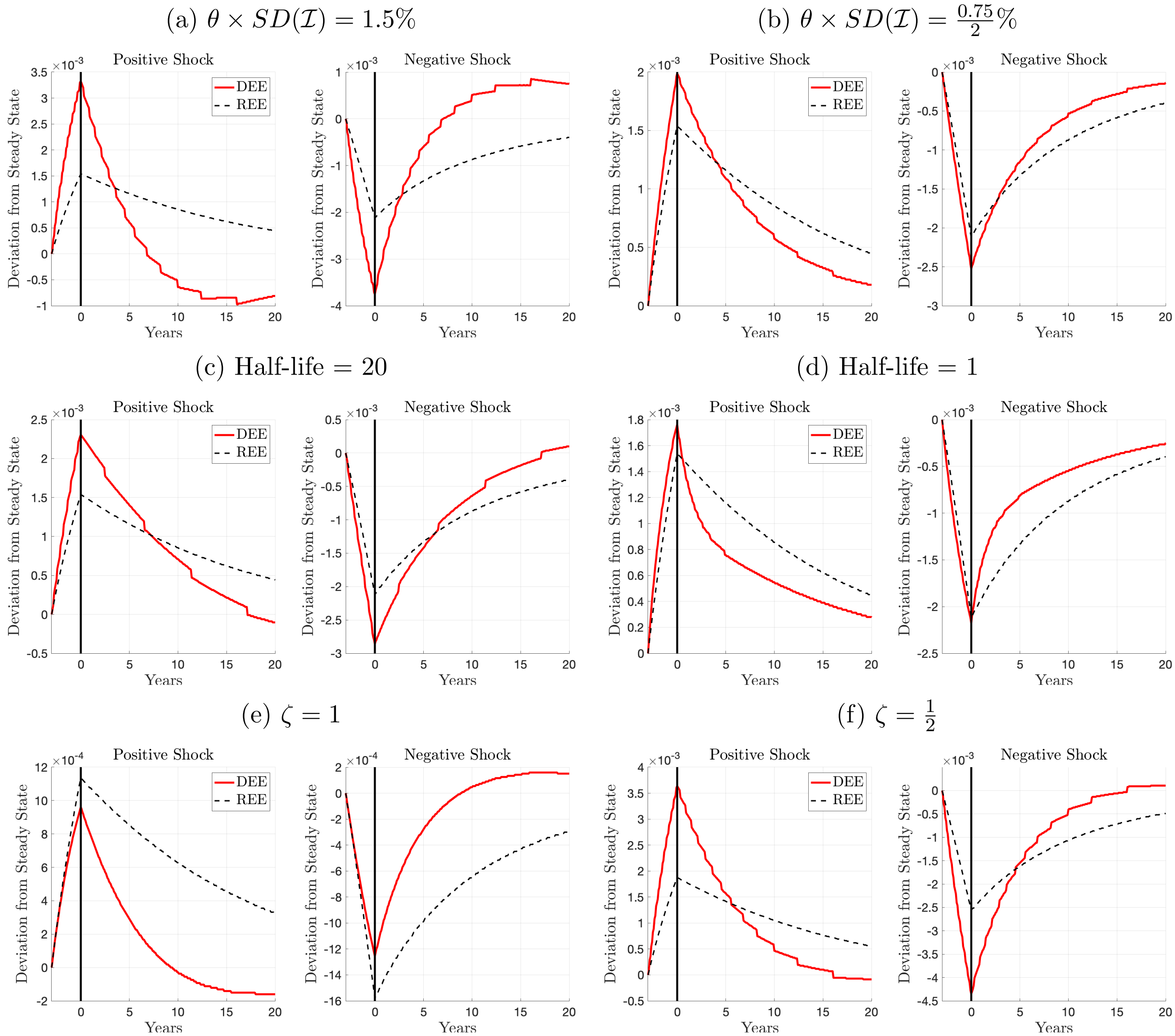


Figure 3: **Investment rate IRFs**. See Figure 6 for a description. In order from left-to-right, top-to-bottom: $\theta \times SD(\mathcal{I}) = 1.5\%$, $\theta \times SD(\mathcal{I}) = \frac{0.75}{2}\%$, half-life = 20, half-life = 1, $\zeta = 1$, $\zeta = \frac{1}{2}$.

Financial Market Stability from Beliefs. Appendix Table 3 lists crisis probabilities for the DEE and the REE across the six alternate calibrations.

	DEE Prob(Crisis)	**REE Prob(Crisis)**	$\frac{\textbf{DEE Prob(Crisis)}}{\textbf{REE Prob(Crisis)}}$
Baseline	3.22 %	3.83 %	0.84
$\theta \times SD(\mathcal{I}) = 1.5\%$	6.33 %	3.83 %	1.65
$\theta \times SD(\mathcal{I}) = \frac{0.75\%}{2}$	3.10 %	3.83 %	0.81
Half-life = 20	4.71 %	3.83 %	1.23
Half-life = 1	2.74 %	3.83 %	0.72
$\zeta = 1$	2.80 %	5.28 %	0.53
$\zeta = \frac{1}{2}$	3.43 %	2.80 %	1.23

Table 3: **Financial market stability from beliefs: robustness**.

First, Appendix Table 3 evaluates the effect of doubling and halving θ. The doubling of θ makes diagnostic expectations destabilizing relative to rationality. With higher θ, intermediaries make larger asset pricing mistakes. This puts downward pressure on e_t when $\mathcal{I}_t$ is high, and vice-versa. Overall, the doubling of θ reduces the correlation between e_t and $\mathcal{I}_t$ to 0.2. This makes it more likely that the economy enters the part of the state space where sentiment is elevated and the financial sector is distressed. This increases the probability of crises in the DEE. Alternatively, diagnostic expectations become more stabilizing when θ is halved. Lowering θ increases the correlation between e_t and $\mathcal{I}_t$ to 0.9. This makes it even less likely for the economy to enter states where the financial sector is distressed and yet $\mathcal{I}_t > 0$.[49]

The persistence of sentiment also matters for the effect of diagnostic expectations on financial crises. Crises become more likely to occur as the persistence of sentiment increases. Again, this effect operates through the correlation between e_t and $\mathcal{I}_t$. When sentiment has a half-life of 1 year, the correlation is 0.75. This drops to 0.1 for a half-life of 20 years. Decreasing κ reduces the correlation between e_t and $\mathcal{I}_t$ by making sentiment load less and less on the most recent shocks.

Finally, Appendix Table 3 evaluates the effect of ζ on financial crises. ζ is the

[49]Though not illustrated in Appendix Table 3, the relationship between θ and the probability of crises is non-monotonic. As $\theta \to 0$, the probability of a crisis in the DEE converges from below to the probability of a crisis in the REE.

inverse of the EIS. Unlike perturbations to the behavioral parameters, perturbations to ζ affect both the REE and the DEE. Lowering ζ decreases the probability of crises in the REE. This result mirrors the "volatility paradox" of Brunnermeier and Sannikov (2014). Lowering ζ increases asset price volatility, so intermediaries demand higher risk premia.[50] In equilibrium, this decreases the probability of crises.

ζ works in the opposite direction in the DEE. Under diagnostic expectations, lower ζ increases the probability of crises. In the DEE, asset prices become increasingly sensitive to $\mathcal{I}_t$ as ζ decreases.[51] However, diagnostic agents do not anticipate this sentiment-driven price volatility and therefore they do not demand a higher risk premium as compensation. Since lowering ζ increases asset price volatility without increasing the corresponding risk premium, this additional (uncompensated) risk increases the probability of financial crises.

[50] Lowering ζ decreases the sensitivity of the risk-free rate to variation in e_t. This means that asset prices must fluctuate instead. For example, consider a decline in e_t which causes intermediaries to demand a higher risk premium. This can be realized either through a decline in the risk-free rate r_t or a decline in asset prices. Thus, variation in r_t can insulate asset prices from variation in e_t.

[51] As ζ decreases the interest rate r_t becomes less sensitive to changes in expected consumption growth. Consider an increase in sentiment. This increases perceived future cash flows. If r_t is insensitive to variation in $\mathcal{I}_t$ then this increase in perceived cash flows will be passed through to higher asset prices q_t and P_t. Thus, lowering ζ creates larger asset price variation in $\mathcal{I}_t$.

B.7 Empirical Tests in the Recalibrated REE

In Section 6.1 the calibration of the REE is identical to the calibration of the DEE, except for $\theta = 0$. Here, I recalibrate the REE in accordance with Table 1. The analysis of Section 6.1 is repeated using the recalibrated REE.

Updated REE Calibration. Table 4 presents the updated calibration of the REE. Four parameters are changed from the main text: η, $\underline{e}$, β, and ϕ. Bank exit rate η is updated to re-establish a crisis probability of 3%. Lower entry barrier $\underline{e}$ is changed to maintain a maximum Sharpe ratio of 6.5. Entry cost β is updated to maintain land price growth volatility of roughly 11.9%. ϕ is updated to maintain a housing-wealth ratio of 45%.

	Parameter	**Choice**	**Target**
Updated Parameters			
η	Bank Exit Rate	0.139	Prob(Crisis)
$\underline{e}$	Lower Entry Barrier	0.112	Max $\mathcal{I} = 0$ Sharpe Ratio
β	Entry Cost	4.5	Land Price Volatility
ϕ	Housing Expenditure Share	0.204	Housing-Wealth Ratio
Unconditional Simulated Moments		REE (New)	REE (Old)
	Mean$\left(\frac{\text{Investment}}{\text{Capital}}\right)$	10.04%	9.93%
	Mean$\left(\frac{\text{Consumption}}{\text{Output}}\right)$	69.88%	70.43%
	Mean(Realized Sharpe Ratio)	0.54	0.52
	Mean(Realized Intermediary Risk Premium)	15.93%	14.63%
	Probability of Crisis	3.05%	3.83%
	Volatility(Land Price Growth)	11.64%	10.08
	Volatility(Interest Rate)	0.46%	0.41%
Non-Distress Simulated Moments			
	Volatility(Investment Growth)	3.49%	3.98%
	Volatility(Consumption Growth)	2.75%	2.57%
	Volatility(Output Growth)	2.97%	2.98%
	Mean$\left(\frac{\text{Housing Wealth}}{\text{Total Wealth}}\right)$	44.60%	46.79%

Table 4: **REE updated calibration**. See Table 1 for a description. Column REE (New) lists the simulated moments in the recalibrated REE. Column REE (Old) lists the simulated moments for the REE calibration used in the main text.

Persistence in the Recalibrated REE. Appendix Table 5 uses the recalibrated REE to reproduce Table 2 of the main text. The recalibrated REE features a slightly lower persistence of financial distress, but results are similar.

e_t Percentile	REE (New)	REE (Old)
5	0.46	0.42
10	1.13	1.08
25	3.74	3.72
50	10.11	10.42
75	24.84	26.41

Table 5: **Average crisis recovery time (in years)**. See Table 2 for a description. This table compares the persistence of financial fragility in the two REE calibrations.

Appendix Figure 4 uses the recalibrated REE to reproduce Figure 8 of the main text. Autocorrelations in the recalibrated REE are presented with solid black lines. The two calibrations produce very similar results.

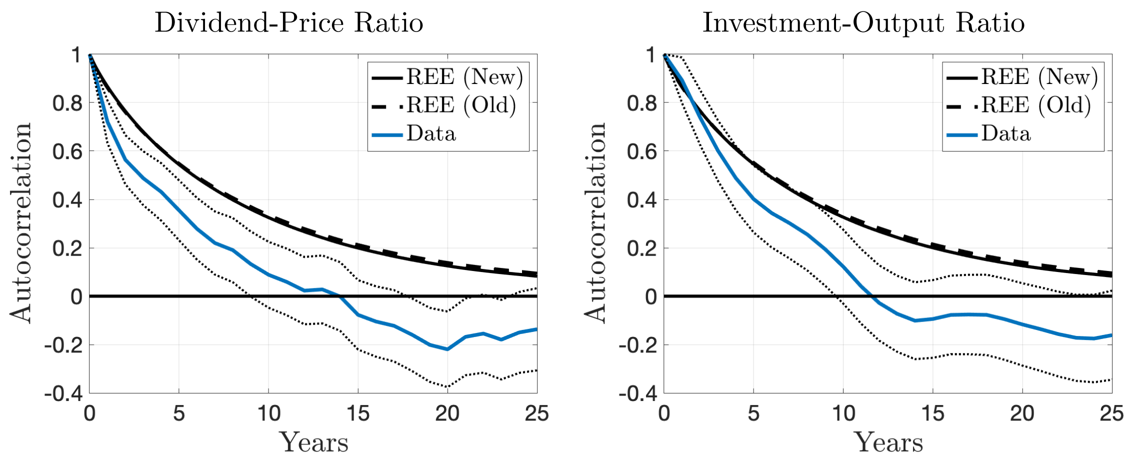
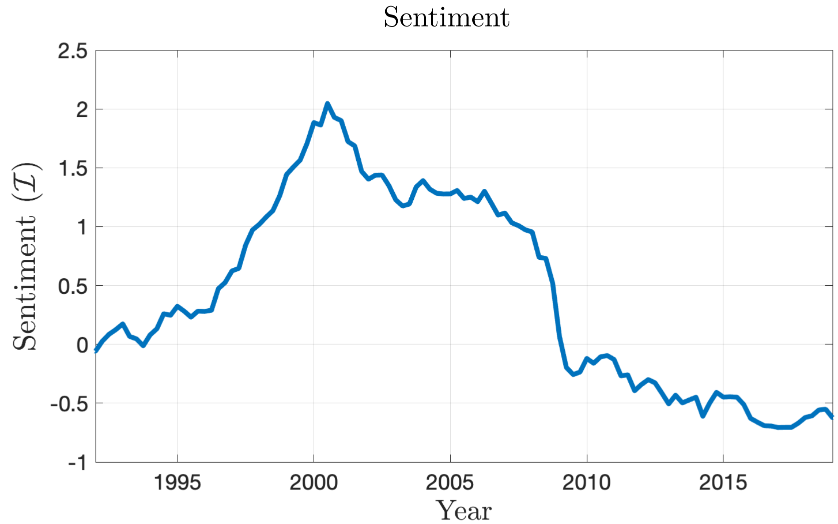


Figure 4: **Persistence: data and model**. See Figure 8 for a description. This figure compares the persistence of macro-financial aggregates in the two REE calibrations.

B.8 Simulating the 2007-2008 Financial Crisis: Additional Analysis



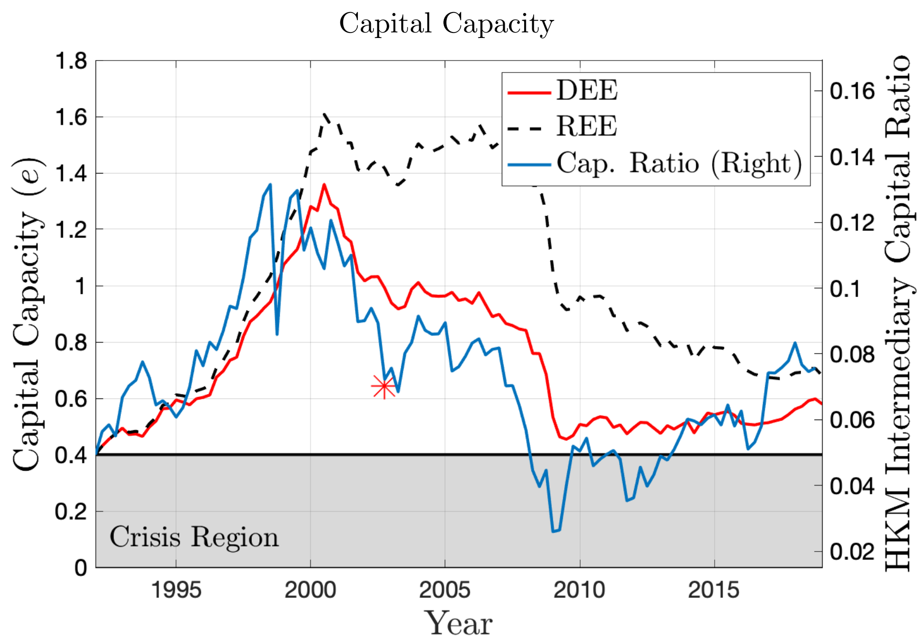


Figure 5: **A long-run analysis of the 2007-2008 Financial Crisis**. See Section 6.3 for a description. For reference, the top panel shows the path of sentiment from 1992 through 2018. The bottom panel plots the path of capital capacity e_t in the DEE and the REE (left axis), as well as the corresponding empirical measure from He et al. (2017) (right axis). The red star marks the e_t initialization of the second simulation, as shown in Figure 10 of the main text.

B.9 Additional Tables and Figures

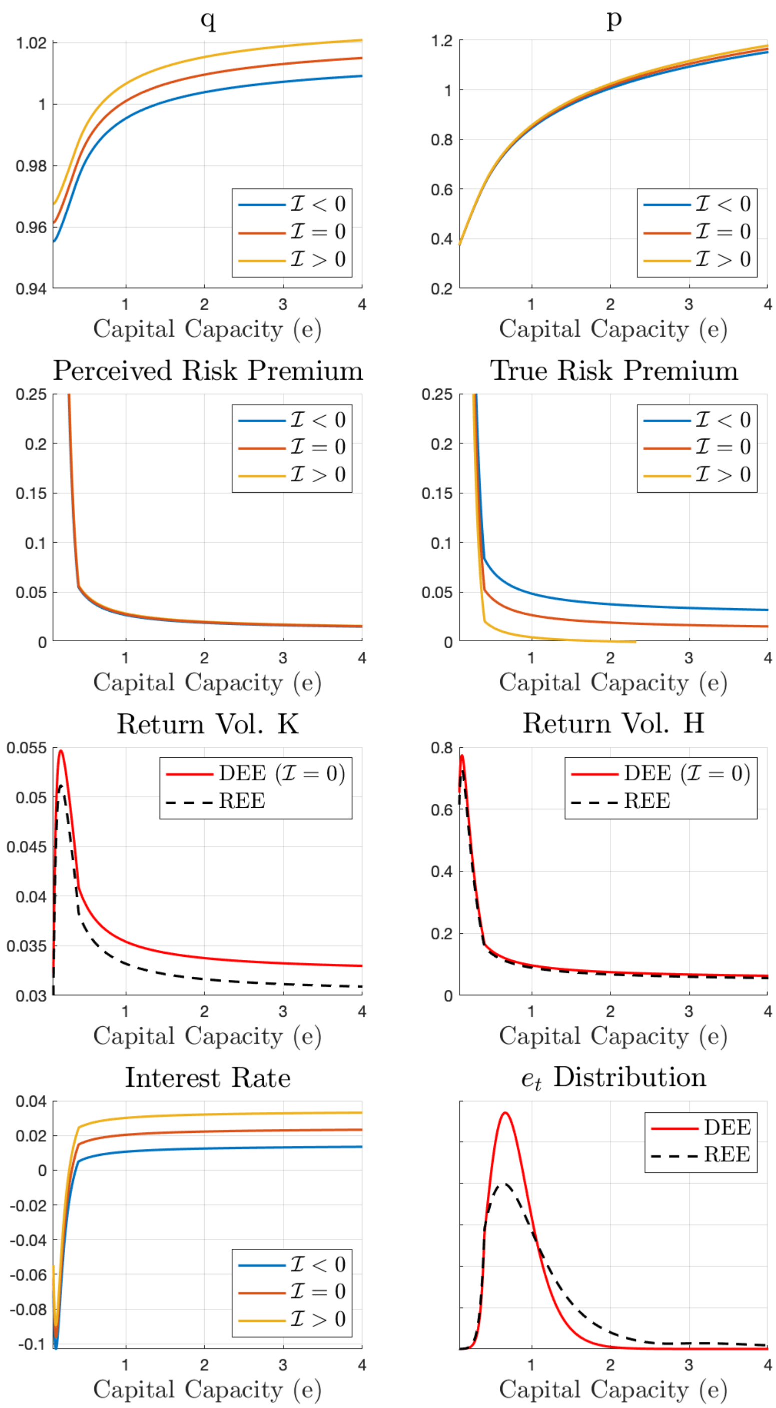


Figure 6: **Price and policy functions**. See Figure 2 for a description. This chart also contains information about delevered risk premia, asset volatilities (σ^k and σ^h), and interest rate r_t.

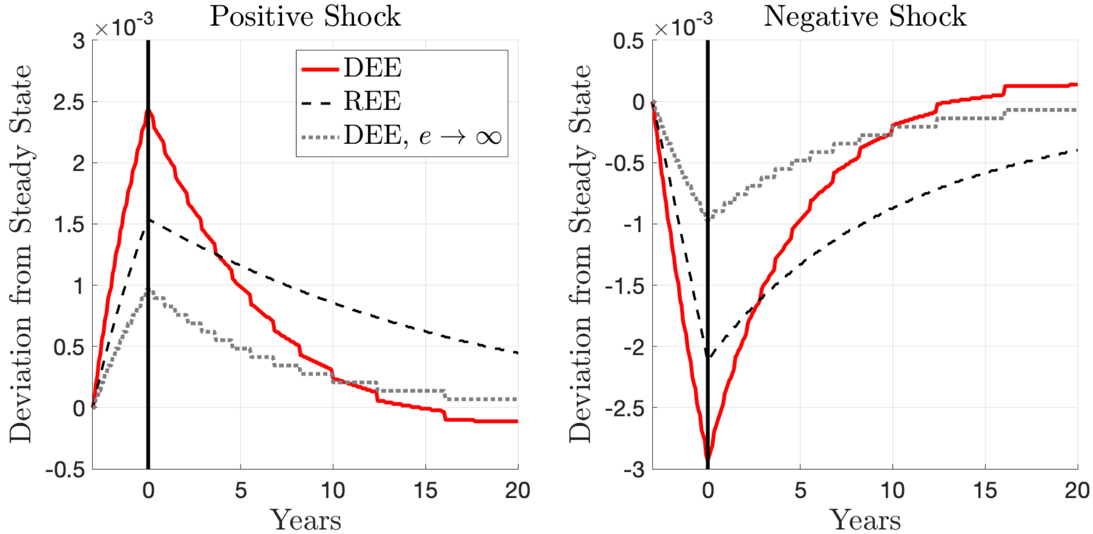


Figure 7: **Investment rate IRFs**. See Figure 6 for a description. This figure also includes the investment IRFs of an economy with diagnostic expectations but no financial frictions (dotted gray curve). Sentiment produces short-run momentum, but there is no feedback from behavioral frictions to financial frictions to generate sharp reversals thereafter.

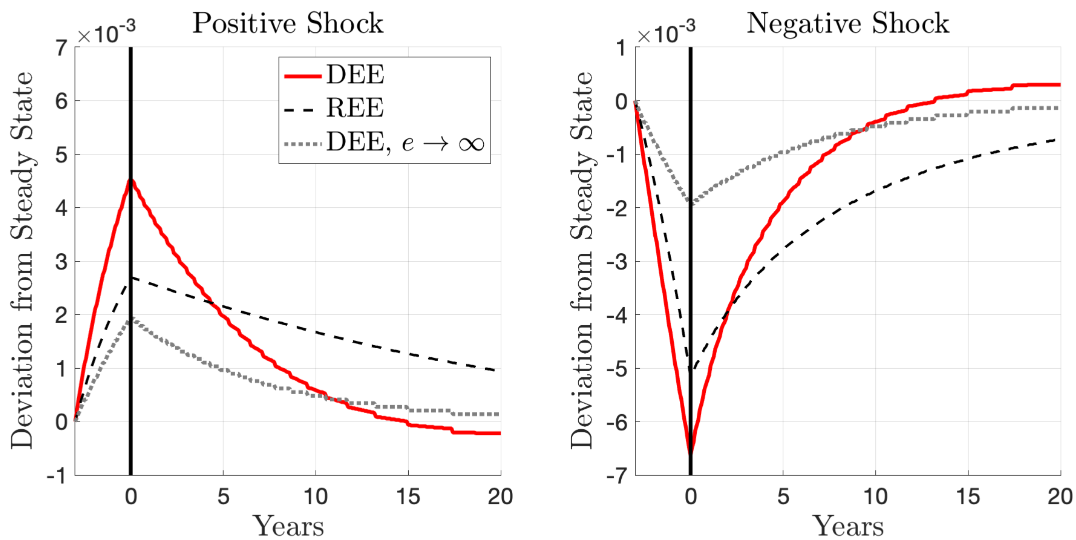


Figure 8: **Investment rate IRFs:** $2SD$ **shock**. See Figure 6 for a description. This figure replicates the investment rate IRF analysis, but doubles the size of the initial impulse. Due to the model's nonlinearity, the larger shock sequence produces a starker asymmetry between the positive and negative shock cases.

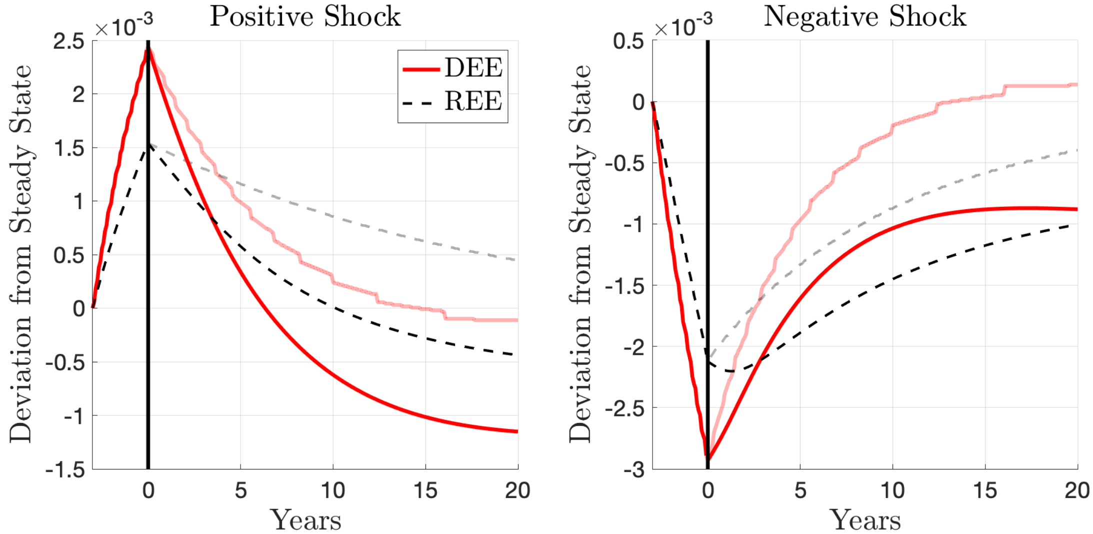


Figure 9: **Investment rate IRFs:** $\mathbb{E}_0[i_\tau]$. The baseline analysis in Figure 6 sets shocks to 0 for all $\tau \geq 0$ (shown here with transparent lines for reference). This figure instead plots the expected future investment rate $\mathbb{E}_0[i_\tau]$. Because this model is nonlinear, these two approaches are not equivalent. In particular, nonlinear financial frictions mean that the long-run expected investment rate, $\lim_{\tau\to\infty}\mathbb{E}_0[i_\tau]$, is less than the investment rate in the stochastic steady state. The expected investment rate is calculated using the Feynman-Kac formula (see Appendix D.3 for numerical details).

e_t Percentile	DEE	REE
5	0.25	0.25
10	0.67	0.83
25	1.83	2.83
50	4.00	8.08
75	8.17	20.08

Table 6: **Median crisis recovery time (in years)**. See Table 2 for a description. This table lists the median time (in years) that it takes for e_t to recover from a financial crisis to its X^{th} percentile.

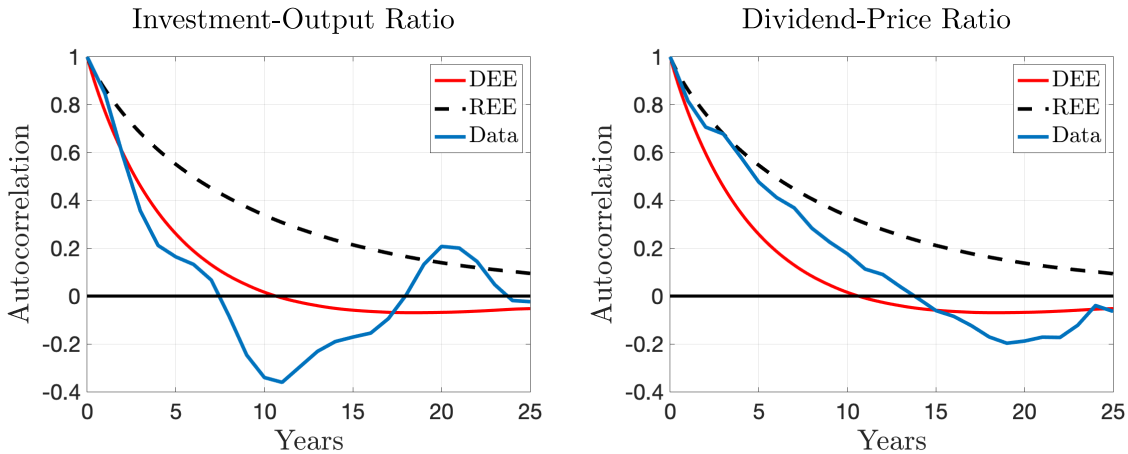


Figure 10: **Persistence: data and model**. This figure replicates Figure 8 using only U.S. data from 1950 – 2016.

C Equilibrium Solution Details

Here I detail how to solve for a Diagnostic Expectations Equilibrium (DEE) that is Markov in state variables $e_t, \mathcal{I}_t$, and K_t. Brunnermeier and Sannikov (2016) provide an excellent treatment of the techniques used here.

C.1 The Diagnostic Expectations Equilibrium

I postulate that agents perceive that q_t and p_t evolve as follows:

$$\frac{\widehat{dq_t}}{q_t} = \widehat{\mu_t^q}dt + \widehat{\sigma_t^q}dZ_t \tag{38}$$

$$\frac{\widehat{dp_t}}{p_t} = \widehat{\mu_t^p}dt + \widehat{\sigma_t^p}dZ_t. \tag{39}$$

Using these postulated price processes, the perceived laws of motion for the three state variables are:

$$\frac{\widehat{dK_t}}{K_t} = (i_t - \delta)dt + \theta\mathcal{I}_t dt + \sigma dZ_t \tag{40}$$

$$\begin{aligned}\frac{\widehat{de_t}}{e_t} &= \left(r_t + \alpha_t^h\widehat{\pi_t^h} + \alpha_t^k\widehat{\pi_t^k}\right)dt - (i_t - \delta + \theta\mathcal{I}_t)dt + (\sigma^2 - \sigma(\alpha_t^h\widehat{\sigma_t^h} + \alpha_t^k\widehat{\sigma_t^k}))dt \\ &\quad - \eta dt + d\psi_t + (\alpha_t^h\widehat{\sigma_t^h} + \alpha_t^k\widehat{\sigma_t^k} - \sigma)dZ_t \end{aligned}\tag{41}$$

$$\frac{d\mathcal{I}_t^S}{\mathcal{I}_t} = -\kappa dt \tag{42}$$

Equations (40) and (42) were derived in Appendix A. Equation (41) can be derived using Itô's lemma to expand $e_t = \frac{\mathcal{E}_t}{K_t}$ under the perceived processes $\frac{\widehat{d\mathcal{E}_t}}{\mathcal{E}_t} = \widehat{d\widetilde{R}_t} - \eta dt + d\psi_t$ and $\frac{\widehat{dK_t}}{K_t} = (i_t - \delta)dt + \theta\mathcal{I}_t dt + \sigma dZ_t$. For simplicity, I rewrite the perceived evolution of e_t as:

$$\frac{\widehat{de_t}}{e_t} = \widehat{\mu_t^e}dt + \widehat{\sigma_t^e}dZ_t.$$

Price Processes. Let $q(e,\mathcal{I})$ and $p(e,\mathcal{I})$ denote the two price functions. Applying Itô's lemma using the perceived laws of motion for the two state variables gives:

$$\widehat{\mu_t^q} = \frac{q_e(e_t,\mathcal{I}_t)}{q_t} e_t \widehat{\mu_t^e} - \frac{q_{\mathcal{I}}(e_t,\mathcal{I}_t)}{q_t}\kappa\mathcal{I}_t + \frac{1}{2}\frac{q_{ee}(e_t,\mathcal{I}_t)}{q_t}(e_t\widehat{\sigma_t^e})^2 \tag{43}$$

$$\widehat{\sigma_t^q} = \frac{q_e(e_t,\mathcal{I}_t)}{q_t} e_t \widehat{\sigma_t^e} \tag{44}$$

$$\widehat{\mu_t^p} = \frac{p_e(e_t,\mathcal{I}_t)}{p_t} e_t \widehat{\mu_t^e} - \frac{p_{\mathcal{I}}(e_t,\mathcal{I}_t)}{p_t}\kappa\mathcal{I}_t + \frac{1}{2}\frac{p_{ee}(e_t,\mathcal{I}_t)}{p_t}(e_t\widehat{\sigma_t^e})^2 \tag{45}$$

$$\widehat{\sigma_t^p} = \frac{p_e(e_t,\mathcal{I}_t)}{p_t} e_t \widehat{\sigma_t^e} \tag{46}$$

These formulas will prove useful throughout. Equations (43) and (45) are second-order PDEs for the price functions, which I will solve numerically. To do so, I need to pin down $\widehat{\mu_t^q}, \widehat{\mu_t^p}, \widehat{\mu_t^e}$, and $\widehat{\sigma_t^e}$, leaving only the price functions undetermined. This is where I now turn.

Market Clearing and Returns. From goods market clearing equation (21):

$$\begin{aligned}
Y_t &= C_t^y + \Phi(i_t, K_t) \\
A &= \frac{C_t^y}{K_t} + i_t + \frac{\xi}{2}(i_t-\delta)^2 \\
&= \frac{1-\phi}{\phi}\frac{D_t}{K_t} + i_t + \frac{\xi}{2}(i_t-\delta)^2, \text{ using (10) and } C_t^h = 1 \\
\frac{D_t}{K_t} &= \frac{\phi}{1-\phi}\left[A - i_t - \frac{\xi}{2}(i_t-\delta)^2\right].
\end{aligned}$$

Since equation (12) gives $i_t = \delta + \frac{q_t - 1}{\xi}$, this pins down $\frac{D_t}{K_t}$ as a function of q_t.

D_t is the dividend paid on housing, and this expression can now be plugged into housing returns as follows:

$$\begin{aligned}\widehat{dR_t^h} &= \frac{\widehat{dP_t} + D_t dt}{P_t} \\ &= \frac{\widehat{d(p_t K_t)}}{p_t K_t} + \frac{D_t}{p_t K_t} dt \\ &= \frac{\widehat{d(p_t K_t)}}{p_t K_t} + \frac{\frac{\phi}{1-\phi}\left[A - i_t - \frac{\xi}{2}(i_t - \delta)^2\right]}{p_t} dt.\end{aligned}$$

Applying Itô's Lemma to the first term:

$$\begin{aligned}\widehat{dR_t^h} &= \left[\widehat{\mu_t^p} + i_t - \delta + \theta \mathcal{I}_t + \sigma \widehat{\sigma_t^p} + \frac{\phi}{1-\phi} \frac{A - i_t - \frac{\xi}{2}(i_t - \delta)^2}{p_t}\right] dt + (\sigma + \widehat{\sigma_t^p}) dZ_t \\ &= (\widehat{\pi_t^h} + r_t) dt + \widehat{\sigma_t^h} dZ_t.\end{aligned}$$

Equation (17) gives a similar process for capital returns. Repeating (17) here:

$$\begin{aligned}\widehat{dR_t^k} &= \left(\frac{\nu A}{q_t} + \widehat{\mu_t^q} - \delta + \theta \mathcal{I}_t + \sigma \widehat{\sigma_t^q}\right) dt + (\sigma + \widehat{\sigma_t^q}) dZ_t \\ &= (\widehat{\pi_t^k} + r_t) dt + \widehat{\sigma_t^k} dZ_t.\end{aligned}$$

The final return process to derive is the risk-free interest rate r_t. Starting again from market clearing:

$$\begin{aligned}C_t^y &= Y_t - \Phi(i_t, K_t) \\ &= Y_t - i_t K_t - \frac{\xi}{2}(i_t - \delta)^2 K_t \\ &= \left(A - \delta - \frac{q_t - 1}{\xi} - \frac{(q_t - 1)^2}{2\xi}\right) K_t, \text{ using (12).}\end{aligned}$$

Now deriving the perceived evolution of C_t^y using Itô's Lemma:

$$\begin{aligned}
\widehat{dC_t^y} &= \left(A - \delta - \frac{q_t - 1}{\xi} - \frac{(q_t-1)^2}{2\xi}\right) K_t((i_t - \delta + \theta \mathcal{I}_t)dt + \sigma dZ_t) \\
&\quad - \frac{q_t \widehat{dq_t}}{\xi} K_t - \frac{K_t}{2\xi}(q_t \widehat{\sigma_t^q})^2 dt - \frac{q_t^2}{\xi} K_t \sigma \widehat{\sigma_t^q} dt \\
\frac{\widehat{dC_t^y}}{C_t^y} &= (i_t - \delta + \theta \mathcal{I}_t)dt - \frac{\frac{1}{\xi} q_t^2 (\widehat{\mu_t^q} + \frac{1}{2}\widehat{\sigma_t^q}^2 + \sigma \widehat{\sigma_t^q})}{A - \delta - \frac{q_t-1}{\xi} - \frac{(q_t-1)^2}{2\xi}} dt + \left(\sigma - \frac{\frac{1}{\xi} q_t^2 \widehat{\sigma_t^q}}{A - \delta - \frac{q_t-1}{\xi} - \frac{(q_t-1)^2}{2\xi}}\right) dZ_t
\end{aligned}$$

Plugging this into the interest rate formula (14):

$$r_t = \rho + \zeta \left[i_t - \delta + \theta \mathcal{I}_t - \frac{\frac{1}{\xi} q_t^2 (\widehat{\mu_t^q} + \frac{1}{2}\widehat{\sigma_t^q}^2 + \sigma \widehat{\sigma_t^q})}{A - \delta - \frac{q_t-1}{\xi} - \frac{(q_t-1)^2}{2\xi}} \right] - \frac{\zeta(\zeta+1)}{2} \left[\sigma - \frac{\frac{1}{\xi} q_t^2 \widehat{\sigma_t^q}}{A - \delta - \frac{q_t-1}{\xi} - \frac{(q_t-1)^2}{2\xi}} \right]^2 \tag{47}$$

Intermediary Optimality. From equation (41), it is the case that $\alpha_t^k \widehat{\sigma_t^k} + \alpha_t^h \widehat{\sigma_t^h} = \widehat{\sigma_t^e} + \sigma$. Using this in equation (20) gives:

$$\frac{\widehat{\pi_t^h}}{\widehat{\sigma_t^h}} = \frac{\widehat{\pi_t^k}}{\widehat{\sigma_t^k}} = \gamma(\widehat{\sigma_t^e} + \sigma).$$

Recall $\widehat{\sigma_t^k} = \sigma + \widehat{\sigma_t^q}$ and $\widehat{\sigma_t^h} = \sigma + \widehat{\sigma_t^p}$. Combining the banker's optimality condition with the perceived return on capital:

$$\begin{aligned}
\gamma(\widehat{\sigma_t^e} + \sigma) &= \frac{\widehat{\pi_t^k}}{\widehat{\sigma_t^k}} \\
&= \frac{\left(\frac{\nu A}{q_t} + \widehat{\mu_t^q} - \delta + \theta \mathcal{I}_t + \sigma \widehat{\sigma_t^q}\right) - r_t}{\sigma + \widehat{\sigma_t^q}} \\
&= \frac{\left(\frac{\nu A}{q_t} + \widehat{\mu_t^q} - \delta + \theta \mathcal{I}_t + \sigma \frac{q_e(e_t, \mathcal{I}_t)}{q_t} e_t \widehat{\sigma_t^e}\right) - r_t}{\sigma + \frac{q_e(e_t, \mathcal{I}_t)}{q_t} e_t \widehat{\sigma_t^e}}, \text{ using equation (44)}
\end{aligned}$$

$$\gamma(\widehat{\sigma_t^e} + \sigma)\left(\sigma + \frac{q_e(e_t, \mathcal{I}_t)}{q_t} e_t \widehat{\sigma_t^e}\right) = \left(\frac{\nu A}{q_t} + \widehat{\mu_t^q} - \delta + \theta \mathcal{I}_t + \sigma \frac{q_e(e_t, \mathcal{I}_t)}{q_t} e_t \widehat{\sigma_t^e}\right) - r_t \tag{48}$$

Proceeding similarly for housing returns:

$$\begin{aligned}
\gamma(\widehat{\sigma_t^e}+\sigma) &= \frac{\widehat{\pi_t^h}}{\widehat{\sigma_t^h}} \\
&= \frac{\widehat{\mu_t^p}+i_t-\delta+\theta\mathcal{I}_t+\sigma\widehat{\sigma_t^p}+\frac{D_t}{p_tK_t}-r_t}{\sigma+\widehat{\sigma_t^p}} \\
&= \frac{\widehat{\mu_t^p}+i_t-\delta+\theta\mathcal{I}_t+\sigma\frac{p_e(e_t,\mathcal{I}_t)}{p_t}e_t\widehat{\sigma_t^e}+\frac{D_t}{p_tK_t}-r_t}{\sigma+\frac{p_e(e_t,\mathcal{I}_t)}{p_t}e_t\widehat{\sigma_t^e}}, \text{ using (46)} \\
\gamma(\widehat{\sigma_t^e}+\sigma)\left(\sigma+\frac{p_e(e_t,\mathcal{I}_t)}{p_t}e_t\widehat{\sigma_t^e}\right) &= \widehat{\mu_t^p}+i_t-\delta+\theta\mathcal{I}_t+\sigma\frac{p_e(e_t,\mathcal{I}_t)}{p_t}e_t\widehat{\sigma_t^e}+\frac{D_t}{p_tK_t}-r_t
\end{aligned} \tag{49}$$

Pinning down $\widehat{\sigma_t^e}$. Equations (48) and (49) express $\widehat{\mu_t^q}$ and $\widehat{\mu_t^p}$ in terms of prices, state variables, and $\widehat{\sigma_t^e}$. The final step is to pin down $\widehat{\sigma_t^e}$:

$$\begin{aligned}
\widehat{\sigma_t^e} &= \alpha_t^h\widehat{\sigma_t^h}+\alpha_t^k\widehat{\sigma_t^k}-\sigma \\
&= \alpha_t^h(\sigma+\widehat{\sigma_t^p})+\alpha_t^k(\sigma+\widehat{\sigma_t^q})-\sigma \\
&= \frac{K_t}{E_t}\left[p_t(\sigma+\widehat{\sigma_t^p})+q_t(\sigma+\widehat{\sigma_t^q})\right]-\sigma, \text{ using (22) and (23)} \\
&= \frac{K_t}{E_t}\left[p_t(\sigma+\frac{p_e(e_t,\mathcal{I}_t)}{p_t}e_t\widehat{\sigma_t^e})+q_t(\sigma+\frac{q_e(e_t,\mathcal{I}_t)}{q_t}e_t\widehat{\sigma_t^e})\right]-\sigma \\
&= \frac{K_t}{E_t}\left[(p_t+q_t-\frac{E_t}{K_t})\sigma+(p_e(e_t,\mathcal{I}_t)+q_e(e_t,\mathcal{I}_t))e_t\widehat{\sigma_t^e}\right] \\
e_t\widehat{\sigma_t^e}\left(\frac{1}{e_t}-\frac{K_t}{E_t}(p_e(e_t,\mathcal{I}_t)+q_e(e_t,\mathcal{I}_t))\right) &= \frac{K_t}{E_t}(p_t+q_t-\frac{E_t}{K_t})\sigma \\
e_t\widehat{\sigma_t^e} &= \frac{K_t}{E_t}\frac{(p_t+q_t-\frac{E_t}{K_t})\sigma}{\frac{1}{e_t}-\frac{K_t}{E_t}p_e(e_t,\mathcal{I}_t)-\frac{K_t}{E_t}q_e(e_t,\mathcal{I}_t)} \\
e_t\widehat{\sigma_t^e} &= \frac{(p_t+q_t-\frac{E_t}{K_t})\sigma}{\frac{E_t}{K_t}\frac{1}{e_t}-p_e(e_t,\mathcal{I}_t)-q_e(e_t,\mathcal{I}_t)}.
\end{aligned} \tag{50}$$

Recall that $E_t = \min\{\mathcal{E}_t,(1-\lambda)(q_tK_t+p_tK_t)\}$, or equivalently $\frac{E_t}{K_t} = \min\{e_t,(1-\lambda)(q_t+p_t)\}$. Thus, equation (50) expresses $\widehat{\sigma_t^e}$ in terms of the two price functions and the two state variables e_t and $\mathcal{I}_t$.

Solving for Prices. I can now solve for price functions $q(e,\mathcal{I})$ and $p(e,\mathcal{I})$. Specifically, they are given by the following system of second-order PDEs:

$$q_t\widehat{\mu_t^q} = q_e(e_t,\mathcal{I}_t)e_t\widehat{\mu_t^e} - q_{\mathcal{I}}(e_t,\mathcal{I}_t)\kappa\mathcal{I}_t + \frac{1}{2}q_{ee}(e_t,\mathcal{I}_t)(e_t\widehat{\sigma_t^e})^2$$
$$p_t\widehat{\mu_t^p} = p_e(e_t,\mathcal{I}_t)e_t\widehat{\mu_t^e} - p_{\mathcal{I}}(e_t,\mathcal{I}_t)\kappa\mathcal{I}_t + \frac{1}{2}p_{ee}(e_t,\mathcal{I}_t)(e_t\widehat{\sigma_t^e})^2$$

All terms in this system of second-order PDEs have now been expressed in terms of state variables e_t and $\mathcal{I}_t$, exogenous parameters, and price functions. Specifically, r, $\widehat{\mu_t^q}$, and $\widehat{\mu_t^p}$ are given by equations (47), (48) and (49). $\widehat{\mu_t^e}$ is given by (41), noting that $\widehat{dR_t}$ is itself a function of prices r, $\widehat{\mu_t^q}$, $\widehat{\mu_t^p}$, $\widehat{\sigma_t^q}$ and $\widehat{\sigma_t^p}$. Equations (44) and (46) give $\widehat{\sigma_t^q}$ and $\widehat{\sigma_t^p}$ in terms of the price functions and $\widehat{\sigma_t^e}$. Equation (50) closes the loop by solving for $\widehat{\sigma_t^e}$ in terms of the two price functions.

This system of PDEs is solved numerically. Details are in Appendix D.

C.2 True Laws of Motion

As with the perceived laws of motion, I begin by postulating that q_t and p_t truly evolve according to:

$$\frac{dq_t}{q_t} = \mu_t^q dt + \sigma_t^q dZ_t \tag{51}$$
$$\frac{dp_t}{p_t} = \mu_t^p dt + \sigma_t^p dZ_t. \tag{52}$$

The true evolution of the three state variables is:

$$\frac{dK_t}{K_t} = (i_t - \delta)dt + \sigma dZ_t, \tag{53}$$
$$\frac{de_t}{e_t} = (r_t + \alpha_t^h\pi_t^h + \alpha_t^k\pi_t^k)dt - (i_t-\delta)dt + (\sigma^2 - \sigma(\alpha_t^h\sigma_t^h + \alpha_t^k\sigma_t^k))dt$$
$$- \eta dt + d\psi_t + (\alpha_t^h\sigma_t^h + \alpha_t^k\sigma_t^k - \sigma)dZ_t \tag{54}$$
$$d\mathcal{I}_t = -\kappa\mathcal{I}_t dt + \sigma dZ_t \tag{55}$$

As above, equation (54) can be derived using Itô's lemma to expand $e_t = \frac{\mathcal{E}_t}{K_t}$ under the true processes $\frac{d\mathcal{E}_t}{\mathcal{E}_t} = d\widetilde{R}_t - \eta dt + d\psi_t$ and $\frac{dK_t}{K_t} = (i_t - \delta)dt + \sigma dZ_t$. For simplicity, I rewrite the true evolution of e_t as:

$$\frac{de_t}{e_t} = \mu_t^e dt + \sigma_t^e dZ_t.$$

Price Processes. The methods developed above show how to solve for price functions $q(e, \mathcal{I})$ and $p(e, \mathcal{I})$. Applying Itô's lemma to these price functions using the true laws of motion for the two state variables gives:

$$\mu_t^q = \frac{q_e(e_t, \mathcal{I}_t)}{q_t} e_t \mu_t^e - \frac{q_{\mathcal{I}}(e_t, \mathcal{I}_t)}{q_t} \kappa \mathcal{I}_t + \frac{q_{e\mathcal{I}}(e_t, \mathcal{I}_t)}{q_t} \sigma (e_t \sigma_t^e) + \frac{1}{2} \frac{q_{ee}(e_t, \mathcal{I}_t)}{q_t} (e_t \sigma_t^e)^2 + \frac{1}{2} \frac{q_{\mathcal{I}\mathcal{I}}(e_t, \mathcal{I}_t)}{q_t} \sigma^2 \tag{56}$$

$$\sigma_t^q = \frac{q_e(e_t, \mathcal{I}_t)}{q_t} e_t \sigma_t^e + \frac{q_{\mathcal{I}}(e_t, \mathcal{I}_t)}{q_t} \sigma \tag{57}$$

$$\mu_t^p = \frac{p_e(e_t, \mathcal{I}_t)}{p_t} e_t \mu_t^e - \frac{p_{\mathcal{I}}(e_t, \mathcal{I}_t)}{p_t} \kappa \mathcal{I}_t + \frac{p_{e\mathcal{I}}(e_t, \mathcal{I}_t)}{p_t} \sigma (e_t \sigma_t^e) + \frac{1}{2} \frac{p_{ee}(e_t, \mathcal{I}_t)}{p_t} (e_t \sigma_t^e)^2 + \frac{1}{2} \frac{p_{\mathcal{I}\mathcal{I}}(e_t, \mathcal{I}_t)}{p_t} \sigma^2 \tag{58}$$

$$\sigma_t^p = \frac{p_e(e_t, \mathcal{I}_t)}{p_t} e_t \sigma_t^e + \frac{p_{\mathcal{I}}(e_t, \mathcal{I}_t)}{p_t} \sigma \tag{59}$$

Market Clearing and Returns. Following similar steps as above, the true housing return process is given by:

$$\begin{aligned} dR_t^h &= \frac{dP_t + D_t dt}{P_t} \\ &= \frac{d(p_t K_t)}{p_t K_t} + \frac{D_t}{p_t K_t} dt \\ &= \left[\mu_t^p + i_t - \delta + \sigma \sigma_t^p + \frac{\phi}{1-\phi} \frac{A - i_t - \frac{\xi}{2}(i_t - \delta)^2}{p_t} \right] dt + (\sigma + \sigma_t^p) dZ_t \\ &= (\pi_t^h + r_t) dt + \sigma_t^h dZ_t. \end{aligned}$$

The true process for capital returns is given by:

$$
\begin{aligned}
dR_t^k &= \left(\frac{\nu A}{q_t} + \mu_t^q - \delta + \sigma\sigma_t^q\right)dt + (\sigma + \sigma_t^q)dZ_t \\
&= (\pi_t^k + r_t)dt + \sigma_t^k dZ_t
\end{aligned}
$$

Pinning down σ_t^e. Solving for the true volatility of e_t:

$$
\begin{aligned}
\sigma_t^e &= \alpha_t^h\sigma_t^h + \alpha_t^k\sigma_t^k - \sigma \\
&= \alpha_t^h(\sigma + \sigma_t^p) + \alpha_t^k(\sigma + \sigma_t^q) - \sigma \\
&= \frac{K_t}{E_t}\left[p_t(\sigma + \sigma_t^p) + q_t(\sigma + \sigma_t^q)\right] - \sigma, \text{ using (22) and (23)} \\
&= \frac{K_t}{E_t}\left[p_t(\sigma + \frac{p_e(e_t,\mathcal{I}_t)}{p_t}e_t\sigma_t^e + \frac{p_{\mathcal{I}}(e_t,\mathcal{I}_t)}{p_t}\sigma) + q_t(\sigma + \frac{q_e(e_t,\mathcal{I}_t)}{q_t}e_t\sigma_t^e + \frac{q_{\mathcal{I}}(e_t,\mathcal{I}_t)}{q_t}\sigma)\right] - \sigma \\
&= \frac{K_t}{E_t}\left[(p_t + q_t + p_{\mathcal{I}}(e_t,\mathcal{I}_t) + q_{\mathcal{I}}(e_t,\mathcal{I}_t) - \frac{E_t}{K_t})\sigma + (p_e(e_t,\mathcal{I}_t) + q_e(e_t,\mathcal{I}_t))e_t\sigma_t^e\right] \\
e_t\sigma_t^e\left[\frac{1}{e_t} - \frac{K_t}{E_t}p_e(e_t,\mathcal{I}_t) - \frac{K_t}{E_t}q_e(e_t,\mathcal{I}_t)\right] &= \frac{K_t}{E_t}(p_t + q_t + p_{\mathcal{I}}(e_t,\mathcal{I}_t) + q_{\mathcal{I}}(e_t,\mathcal{I}_t) - \frac{E_t}{K_t})\sigma \\
e_t\sigma_t^e &= \frac{K_t}{E_t}\frac{(p_t + q_t + p_{\mathcal{I}}(e_t,\mathcal{I}_t) + q_{\mathcal{I}}(e_t,\mathcal{I}_t) - \frac{E_t}{K_t})\sigma}{\frac{1}{e_t} - \frac{K_t}{E_t}p_e(e_t,\mathcal{I}_t) - \frac{K_t}{E_t}q_e(e_t,\mathcal{I}_t)} \\
e_t\sigma_t^e &= \frac{(p_t + q_t + p_{\mathcal{I}}(e_t,\mathcal{I}_t) + q_{\mathcal{I}}(e_t,\mathcal{I}_t) - \frac{E_t}{K_t})\sigma}{\frac{E_t}{K_t}\frac{1}{e_t} - p_e(e_t,\mathcal{I}_t) - q_e(e_t,\mathcal{I}_t)}. \qquad (60)
\end{aligned}
$$

C.3 Verifying "Equity Member" Portfolio Choice

The main text assumes that the "equity member" will invest the maximal amount into the equity of the financial sector. This assumption must be verified ex-post given the resulting equilibrium.

The household maximizes the value function in equation (9), where $C_t = (c_t^y)^{1-\phi}(c_t^h)^\phi = (c_t^y)^{1-\phi}$, since $c^h = 1$ in equilibrium. From (9), the household accrues utility flow $\frac{(c_t^y)^{(1-\phi)(1-\gamma_h)}}{1-\gamma_h}$. Multiplying the utility function by $\frac{1}{1-\phi}$ (a positive affine transformation) shows that in equilibrium the household can be represented with power utility preferences, using relative risk aversion coefficient $\zeta = 1 - (1-\phi)(1-\gamma_h)$. Market clearing gives $c_t^y = AK_t - \Phi(i_t, K_t) = \left(A - \delta - \frac{q_t - 1}{\xi} - \frac{(q_t-1)^2}{2\xi}\right) K_t$.

Let a be an arbitrary asset with perceived mean return $\widehat{\mu^a}$ and volatility $\widehat{\sigma^a}$. In equilibrium, CRRA utility implies:

$$\frac{\widehat{\mu^a} - r_t}{\widehat{\sigma^a}} = \zeta \widehat{\sigma_t^{cy}},$$

where $\widehat{\sigma_t^{cy}}$ is the perceived volatility of $\frac{dc_t^y}{c_t^y}$ (a formula is provided in Appendix C). Thus, the household demands a perceived Sharpe ratio of $\zeta\widehat{\sigma_t^{cy}}$. Note that this equation need not hold with equality when there are portfolio restrictions placed on the household.

An investment in intermediary equity earns a perceived risk premium of $\alpha^k \widehat{\pi^k} + \alpha^h \widehat{\pi^h}$, with a perceived risk of $\alpha^k \widehat{\sigma^k} + \alpha^h \widehat{\sigma^h}$. From equation (20), intermediaries demand a risk premium of:

$$\begin{aligned} \widehat{\pi_t^k} &= \gamma(\alpha_t^k \widehat{\sigma_t^k} + \alpha_t^h \widehat{\sigma_t^h}) \widehat{\sigma_t^k} \\ \widehat{\pi_t^h} &= \gamma(\alpha_t^k \widehat{\sigma_t^k} + \alpha_t^h \widehat{\sigma_t^h}) \widehat{\sigma_t^h} \end{aligned}$$

For the portfolio as a whole, this implies:

$$\alpha^k \widehat{\pi^k} + \alpha^h \widehat{\pi^h} = \gamma(\alpha_t^k \widehat{\sigma_t^k} + \alpha_t^h \widehat{\sigma_t^h})^2.$$

The equity member will invest all wealth in intermediary equity whenever the perceived Sharpe ratio on this investment is weakly greater than $\zeta\widehat{\sigma_t^{cy}}$:

$$\begin{aligned} \frac{\gamma(\alpha_t^k \widehat{\sigma_t^k} + \alpha_t^h \widehat{\sigma_t^h})^2}{\alpha_t^k \widehat{\sigma_t^k} + \alpha_t^h \widehat{\sigma_t^h}} &> \zeta \widehat{\sigma_t^{cy}}, \text{ or equivalently} \\ \gamma(\alpha_t^k \widehat{\sigma_t^k} + \alpha_t^h \widehat{\sigma_t^h}) &> \zeta \widehat{\sigma_t^{cy}}. \end{aligned}$$

This condition is verified numerically over the entire state space.

D Numerical Methods

Before outlining the numerical methods, additional notation is required. The state variables $\{e, \mathcal{I}\}$ can be represented as a two-dimensional Itô process, denoted $\mathcal{S}$. Agents perceive that $\mathcal{S}$ evolves according to:

$$\widehat{d\mathcal{S}_t} = \begin{bmatrix} \widehat{de_t} \\ d\mathcal{I}_t^S \end{bmatrix} = \begin{bmatrix} e_t \widehat{\mu}_t^e \\ -\kappa \mathcal{I}_t \end{bmatrix} dt + \begin{bmatrix} e_t \widehat{\sigma}_t^e \\ 0 \end{bmatrix} dZ_t, \tag{61}$$

where Z_t is a one-dimensional Brownian motion. The true evolution of $\mathcal{S}$ is:

$$d\mathcal{S}_t = \begin{bmatrix} de_t \\ d\mathcal{I}_t \end{bmatrix} = \begin{bmatrix} e_t \mu_t^e \\ -\kappa \mathcal{I}_t \end{bmatrix} dt + \begin{bmatrix} e_t \sigma_t^e \\ \sigma \end{bmatrix} dZ_t. \tag{62}$$

The evolution of $\widehat{d\mathcal{S}_t}$ and $d\mathcal{S}_t$ is subject to a reflecting barrier in the e dimension at $\underline{e}$.

To simplify notation, let $\mathcal{A}$ denote the infinitesimal generator of $\mathcal{S}_t$. Let $\widehat{\mathcal{A}}$ denote the infinitesimal generator of $\widehat{\mathcal{S}}_t$.

D.1 Solving for Price Functions

As shown in Appendix C, price functions $q(e, \mathcal{I})$ and $p(e, \mathcal{I})$ compose a system of second-order PDEs:

$$\begin{aligned} q_t \widehat{\mu}_t^q &= q_e(e_t, \mathcal{I}_t) e_t \widehat{\mu}_t^e - q_{\mathcal{I}}(e_t, \mathcal{I}_t) \kappa \mathcal{I}_t + \frac{1}{2} q_{ee}(e_t, \mathcal{I}_t)(e_t \widehat{\sigma}_t^e)^2 \qquad (63) \\ &= \widehat{\mathcal{A}} q_t \\ p_t \widehat{\mu}_t^p &= p_e(e_t, \mathcal{I}_t) e_t \widehat{\mu}_t^e - p_{\mathcal{I}}(e_t, \mathcal{I}_t) \kappa \mathcal{I}_t + \frac{1}{2} p_{ee}(e_t, \mathcal{I}_t)(e_t \widehat{\sigma}_t^e)^2 \qquad (64) \\ &= \widehat{\mathcal{A}} p_t \end{aligned}$$

In order to solve for these price functions I employ finite-difference methods. The numerical methods appendix of Achdou et al. (2017) provides an excellent reference. I assume knowledge of these methods here.

Algorithm. I solve for price functions using two nested while-loops. In the outer loop, I iterate over lower boundary $\underline{e}$. In the inner loop, I take $\underline{e}$ as given and iterate over price functions p and q until convergence. Details for the inner loop are provided below. The outer loop continues to iterate over $\underline{e}$ until the resulting Sharpe ratio at $e = \underline{e}$, $\mathcal{I} = 0$ is close to the calibration target in Table 1.

I create a discretized grid over state variables e and $\mathcal{I}$.[52] Let subscript i denote the gridpoints in the e-dimension, and let subscript j denote the gridpoints in the $\mathcal{I}$-dimension. The algorithm is as follows. Let $n = 1, 2, ...$ track the current loop iteration.

1. Guess price functions $q^0_{i,j}$ and $p^0_{i,j}$ at each grid point $\{i, j\}$.

2. Solve for $\widehat{\mu^q_{i,j}}$, $\widehat{\mu^p_{i,j}}$, $\widehat{\mu^e_{i,j}}$, and $\widehat{\sigma^e_{i,j}}$ using the previous iteration's price functions of $q^{n-1}_{i,j}$ and $p^{n-1}_{i,j}$ (or the initial guess). To do so, use equation (50) to solve for $\widehat{\sigma^e_{i,j}}$, (48) to solve for $\widehat{\mu^q_{i,j}}$, (49) to solve for $\widehat{\mu^p_{i,j}}$, and (41) to solve for $\widehat{\mu^e_{i,j}}$. Next, construct the discretized infinitesimal generator, denoted $\widehat{\mathbf{A}}^{n-1}$, using $\widehat{\mu^e_{i,j}}$ and $\widehat{\sigma^e_{i,j}}$.[53] Note that $\widehat{\mathbf{A}}$ features a reflecting barrier at $\underline{e}$.[54]

3. Use an implicit scheme to solve for price functions $q^n_{i,j}$ and $p^n_{i,j}$:

$$\frac{q^n_{i,j} - q^{n-1}_{i,j}}{\Delta} + \widehat{\mu^q_{i,j}}\, q^n_{i,j} = \widehat{\mathbf{A}}^{n-1} q^n_{i,j}, \text{ which implies}$$
$$q^n_{i,j} = \left(\frac{1}{\Delta}\mathbf{I} + \text{diag}(\widehat{\mu^q_{i,j}}) - \widehat{\mathbf{A}}^{n-1}\right)^{-1} (\frac{1}{\Delta} q^{n-1}_{i,j})$$

[52]The grid over e is non-uniform.
[53]"$n-1$" notation is used because $\widehat{\mathbf{A}}$ is constructed using the price functions from iteration $n-1$.
[54]Implementation details are in Achdou et al. (2017).

Similarly,[55]

$$p_{i,j}^n = \left(\frac{1}{\Delta}\mathbf{I} + \text{diag}(\widehat{\mu_{i,j}^p}) - \widehat{\mathbf{A}}^{n-1}\right)^{-1} (\frac{1}{\Delta} p_{i,j}^{n-1})$$

These equations define $q_{i,j}^n$ and $p_{i,j}^n$ as functions of information from iteration $n-1$. Parameter Δ is the step size, and governs how quickly the price functions are updated. Convergence is not guaranteed, so Δ should not be set too large.

4. If price functions have converged within a pre-specified tolerance, stop. If not, go to step 2 and repeat.

Once the algorithm has converged, I use the final values of $q_{i,j}$ and $p_{i,j}$ to solve for the realized evolution of state variables e_t and $\mathcal{I}_t$. The realized evolution of price functions p_t and q_t can also be derived (similar to step 2).

D.2 Kolmogorov Equations

Kolmogorov Forward Equation. Readers should refer to the numerical appendix of Achdou et al. (2017) for details. Let $g_t(e, \mathcal{I})$ denote a probability density function over e and $\mathcal{I}$. The Kolmogorov forward equation (KF) gives $\frac{\partial}{\partial t} g_t$. A stationary distribution is a distribution $\bar{g}$ that solves $\frac{\partial}{\partial t}\bar{g} = 0$.

A benefit of finite-difference methods is that the KF equation comes "for free." Let $g_{i,j}^t$ denote a discretized distribution over e and $\mathcal{I}$ at time t. The perceived evolution of $g_{i,j}^t$ is given by:

$$\frac{g_{i,j}^{t+\Delta t} - g_{i,j}^t}{\Delta t} = (\widehat{\mathbf{A}})^{\mathrm{T}} g_{i,j}^{t+\Delta t} \quad \Longrightarrow \quad g^{t+\Delta t} = \left(\mathbf{I} - \Delta t (\widehat{\mathbf{A}})^{\mathrm{T}}\right)^{-1} g_{i,j}^t.$$

[55] Footnote 13 of the numerical appendix of Achdou et al. (2017) shows that a reflecting barrier imposes that the derivative at the reflecting barrier is 0. This is correct for q, as $q_e(\underline{e}, \mathcal{I}) = 0$. However, this is not correct for p, since $p_e(\underline{e}, \mathcal{I}) = \frac{p(\underline{e},\mathcal{I})\beta}{1+\underline{e}\beta}$ (see Appendix B.4 for details). In the numerical solution for p, discretized matrix $\widehat{\mathbf{A}}$ is amended to ensure that $p_{1,j}^n = p_{2,j}^n - \frac{p_{1,j}^{n-1}\beta}{1+\underline{e}\beta} \times \Delta e$, where Δe denotes the grid increment in the e-dimension.

The true evolution of $g_{i,j}^t$ is given by:

$$\frac{g_{i,j}^{t+\Delta t} - g_{i,j}^t}{\Delta t} = (\mathbf{A})^{\mathrm{T}} g_{i,j}^{t+\Delta t} \quad \Longrightarrow \quad g^{t+\Delta t} = \left(\mathbf{I} - \Delta t(\mathbf{A})^{\mathrm{T}}\right)^{-1} g_{i,j}^t.$$

Kolmogorov Backward Equation. The Kolmogorov backward equation (KB) is used to derive the hitting probabilities in Figure 4.

Let $\bar{e}_{crisis}(\mathcal{I})$ denote the upper boundary of the crisis region for sentiment level $\mathcal{I}$. For $s \in [0, T]$, let $f(e_s, \mathcal{I}_s, s)$ denote the true probability that the economy currently at $\{e_s, \mathcal{I}_s\}$ enters the crisis region between time s and T. Similarly, let $\widehat{f}(e_s, \mathcal{I}_s, s)$ denote the perceived probability. The goal is to solve for $f(e, \mathcal{I}, 0)$ and $\widehat{f}(e, \mathcal{I}, 0)$. Conditional probability function $f(e, \mathcal{I}, s)$ is the solution to:

$$\begin{aligned} 0 = \frac{\partial f(e, \mathcal{I}, s)}{\partial s} + \mathcal{A} f(e, \mathcal{I}, s),\ &\text{subject to boundary conditions} \\ (i)\ f(e, \mathcal{I}, s) = 1 \text{ if } e \leq \bar{e}_{crisis}(\mathcal{I}),\ &\text{and} \\ (ii)\ f(e, \mathcal{I}, T) = 0 \text{ if } e > \bar{e}_{crisis}(\mathcal{I})& \end{aligned}$$

This result is stated without proof. Informally, the differential equation $0 = \frac{\partial f(e,\mathcal{I},s)}{\partial s} + \mathcal{A} f(e, \mathcal{I}, s)$ arises from applying Itô's lemma to $f(e, \mathcal{I}, s)$ and setting the drift of the resulting expression equal to 0. The drift is set to 0 in the non-crisis region due to the law of iterated expectations: $f(e_s, \mathcal{I}_s, s) = \mathbb{E}_s[f(e_{s+dt}, \mathcal{I}_{s+dt}, s + dt)]$ and therefore $\mathbb{E}_s[df(e_s, \mathcal{I}_s, s)] = 0$. The first boundary condition sets f equal to 1 whenever the crisis region is hit. The second boundary condition is a terminal condition which assigns $f(e, \mathcal{I}, T) = 0$ if the crisis region is not hit at terminal period T. f is solved backwards from this terminal condition.

Using the perceived generator $\widehat{\mathcal{A}}$, function $\widehat{f}(e,\mathcal{I},s)$ is the solution to:

$$0 = \frac{\partial \widehat{f}(e,\mathcal{I},s)}{\partial s} + \widehat{\mathcal{A}}\widehat{f}(e,\mathcal{I},s), \text{ subject to boundary conditions}$$
$$(i)\ \widehat{f}(e,\mathcal{I},s) = 1 \text{ if } e \leq \bar{e}_{crisis}(\mathcal{I}), \text{ and}$$
$$(ii)\ \widehat{f}(e,\mathcal{I},T) = 0 \text{ if } e > \bar{e}_{crisis}(\mathcal{I})$$

Numerically, the discretized versions of both $\mathcal{A}$ and $\widehat{\mathcal{A}}$ were already generated when solving for price functions. Finite-difference methods can then be used to solve backward for f and $\widehat{f}$ starting from the terminal condition.

D.3 Feynman-Kac Equation

The Feynman-Kac equation is used to calculate the expected investment rate profile in Appendix Figure 9. A straightforward application of the formula implies that the conditional expectation:

$$u(e,\mathcal{I},0) = \mathbb{E}\left[i(e_\tau,\mathcal{I}_\tau) \mid e_0 = e, \mathcal{I}_0 = \mathcal{I}\right]$$

satisfies a partial differential equation:

$$0 = \frac{\partial u(e,\mathcal{I},s)}{\partial s} + \mathcal{A}u(e,\mathcal{I},s),$$

subject to the terminal condition that $u(e,\mathcal{I},\tau) = i(e,\mathcal{I})$.

Numerically, the discretized version of $\mathcal{A}$ was already generated when solving for price functions. Finite-difference methods can then be used to solve backward for $u(e,\mathcal{I},0)$ starting from the terminal condition.

E Diagnostic Expectations: Additional Details

E.1 Expectations of an AR(1): Comparison to Bordalo et al. (2018a)

Here I show how the discrete-time analogue of this paper's formulation of diagnostic expectations relates to the model of Bordalo et al. (2018a) when applied to exogenous AR(1) processes. Following the notation of BGS, let ω_t be an AR(1) process $\omega_t = b\omega_{t-1} + \epsilon_t$, where $\epsilon_t \sim \mathcal{N}(0, \sigma^2)$.

Discrete-Time Setup. In discrete time, information measure $\mathcal{I}_t$ is given by:

$$\mathcal{I}_t = \sum_{j=0}^{\infty} \mathcal{K}^j \epsilon_{t-j},$$

where $\mathcal{K}$ is the discount factor governing the speed of information decay.[56] The background context is:

$$G_t^- = \omega_t - \mathcal{I}_t.$$

As the agent simulates the economy forward from period t:

$$\mathcal{I}_{t+\tau}^S = \sum_{j=0}^{\infty} \mathcal{K}^{j+\tau} \epsilon_{t-j}.$$

Analogous to the continuous-time formulation, for any future value $\omega'_{t+\tau}$ the simulated future background context is:

$$G'^-_{t+\tau} = \omega'_{t+\tau} - \mathcal{I}_{t+\tau}^S.$$

[56] If the discrete-time model is written with a period frequency of Δ years, then $\mathcal{K} = e^{-\kappa\Delta}$.

Expectations of $\omega_{t+\tau}$. In this discrete-time framework, I start by expressing ω_t in changes:

$$\omega_t - \omega_{t-1} = (b-1)\omega_{t-1} + \epsilon_t.$$

One can think of the "drift" of ω_t as $(b-1)\omega_t$. For any $\mathcal{I}_t$, the diagnostic agent perceives that

$$\begin{aligned} h^\theta(\omega_{t+1}|\omega_t) &= \mathcal{N}(\omega_t + (b-1)\omega_t + \theta(\omega_t - G_t^-), \sigma^2) \\ &= \mathcal{N}(b\omega_t + \theta\mathcal{I}_t, \sigma^2) \\ &= \mathcal{N}(b\omega_t + \theta\left(\sum_{j=0}^{\infty} \mathcal{K}^j \epsilon_{t-j}\right), \sigma^2) \end{aligned}$$

Simulating forward, for any $\omega'_{t+\tau}$ the perceived evolution from $t+\tau$ to $t+\tau+1$ is:

$$\begin{aligned} h^\theta(\omega_{t+\tau+1}|\omega'_{t+\tau}) &= \mathcal{N}(\omega'_{t+\tau} + (b-1)\omega'_{t+\tau} + \theta(\omega'_{t+\tau} - G'^-_{t+\tau}), \sigma^2) \\ &= \mathcal{N}(b\omega'_{t+\tau} + \theta\mathcal{I}^S_{t+\tau}, \sigma^2) \\ &= \mathcal{N}(b\omega'_{t+\tau} + \theta\left(\sum_{j=0}^{\infty} \mathcal{K}^{j+\tau} \epsilon_{t-j}\right), \sigma^2). \end{aligned}$$

Generally, in this AR(1) context the diagnostic agent will have the following perceptions about the distribution of $\omega_{t+\tau}$:

$$
\begin{aligned}
h^\theta(\omega_{t+1}|\omega_t) &\sim \mathcal{N}\left(b\omega_t + \theta\mathcal{I}_t, \sigma^2\right) \\
h^\theta(\omega_{t+2}|\omega_t) &\sim \mathcal{N}\left(b^2\omega_t + b\theta\mathcal{I}_t + \mathcal{K}\theta\mathcal{I}_t, \sigma^2 + b^2\sigma^2\right) \\
&\vdots \\
h^\theta(\omega_{t+\tau}|\omega_t) &\sim \mathcal{N}\left(b^\tau\omega_t + \theta\mathcal{I}_t\left(\sum_{i=0}^{\tau-1} b^i\mathcal{K}^{\tau-1-i}\right),\ \sigma^2\sum_{i=0}^{\tau-1} b^{2i}\right),\ \text{or equivalently} \\
h^\theta(\omega_{t+\tau}|\omega_t) &\sim \mathcal{N}\left(b^\tau\omega_t + \theta\mathcal{I}_t\left(\frac{b^\tau - \mathcal{K}^\tau}{b-\mathcal{K}}\right),\ \sigma^2\sum_{i=0}^{\tau-1} b^{2i}\right)
\end{aligned}
$$

BGS Equivalence. To reproduce the AR(1) framework of BGS, take $\mathcal{K} \to 0$. In the limit, $G_t^- = \omega_t - \epsilon_t = b\omega_{t-1}$. In other words, background context G_t^- does not incorporate the most recent shock, but fully incorporates all shocks of further lags. Looking forward, $G'^-_{t+\tau} = \omega'_{t+\tau}$ for all $\tau \geq 1$.

Consider this paper's iterative framework for defining expectations. From t to $t+1$, expectations are biased by $\theta(\omega_t - G_t^-) = \theta\epsilon_t$. While there is no further bias from $t+1$ onward in the perceived drift of $\omega_{t+\tau}$, the perceived level of $\omega_{t+\tau}$ will still be biased.

In more detail, at time t the agent believes that the distribution at time $t+1$ is $\mathcal{N}(b\omega_t + \theta\epsilon_t, \sigma^2)$. Because there is no additional bias in the perceived drift, the distribution at time $t+2$ is $\mathcal{N}(b\left(b\omega_t + \theta\epsilon_t\right), \sigma^2 + b^2\sigma^2)$, the distribution at time $t+3$ is $\mathcal{N}(b^2\left(b\omega_t + \theta\epsilon_t\right), \sigma^2 + b^2\sigma^2 + b^4\sigma^2)$, etc. Using the formula above as $\mathcal{K} \to 0$, the distribution at time $t+\tau$ is $\mathcal{N}(b^\tau\omega_t + b^{\tau-1}\theta\epsilon_t,\ \sigma^2\sum_{i=0}^{\tau-1} b^{2i})$. This is almost identical to BGS, with the one difference being that parameter θ here is their parameter θ/b.

Moving from Discrete to Continuous Time. To pass from discrete to continuous time, one needs to set $\mathcal{K} > 0$. When $\mathcal{K} \to 0$ only the most recent shock matters. But, the concept of "only the most recent shock" varies with the length of the period.

This is especially important when passing to continuous time, where the most recent shock is an instantaneous Brownian increment. Thus, in this paper I instead adopt a sentiment function $\mathcal{I}_t$ that is based on multiple lags of past shocks (i.e., $\mathcal{K} > 0$).

E.2 Extensions of the Baseline Model

Objective versus Subjective Shocks. Information measure $\mathcal{I}_t \equiv \int_0^t e^{-\kappa(t-s)}\sigma dZ_s$ is based on objective shocks, σdZ_t. This choice is consistent with BGS, who argue that overreaction to objective news is more consistent with the psychology of diagnostic expectations.

However, it also tractable to define $\mathcal{I}_t$ based on subjective capital quality shocks. An agent with a bias of $\theta\mathcal{I}_t$ will perceive a subjective shock at time t of $\sigma \widehat{dZ}_t = -\theta\mathcal{I}_t dt + \sigma dZ_t$. In this case, one can define the new information measure as

$$\mathcal{I}_t = \int_0^t e^{-\kappa(t-s)}\sigma \widehat{dZ}_s.$$

This measure of subjective new information evolves according to:

$$\begin{aligned} d\mathcal{I}_t &= -\kappa\mathcal{I}_t dt + \sigma\widehat{dZ}_t \\ &= -(\kappa+\theta)\mathcal{I}_t dt + \sigma dZ_t. \end{aligned} \tag{65}$$

Note that the baseline measure of information in equation (3) evolves according to $d\mathcal{I}_t = -\kappa\mathcal{I}_t dt + \sigma dZ_t$. Comparing this to equation (65), $\mathcal{I}_t$ decays more quickly when defined in terms of subjective shocks. This is because an overoptimistic agent will interpret incoming shocks with a negative bias (and vice-versa for an overpessimistic agent), leading to a faster unwinding of sentiment.

Multiple Frequencies of Extrapolation. The main text assumes that sentiment evolves at a single frequency, with shocks fading from $\mathcal{I}_t$ at rate κ. However, it is likely more psychologically realistic to have sentiment operate over multiple frequencies. The empirical literature on extrapolative expectations has documented extrapolation over a variety of horizons. At a low frequency, Malmendier and Nagel (2011) find that long-term risk attitudes are shaped by lifetime macroeconomic experiences. Alternatively, Greenwood and Shleifer (2014) find that stock market expectations depend strongly on returns experienced in the past quarter.

The expectations model can be modified to capture multiple frequencies of extrapolation. For illustration, consider an agent whose sentiment is a function of a slow-moving component and a fast-moving component. The agent has a low-frequency measure of new information:

$$\mathcal{I}_t^L = \int_0^t e^{-\kappa^L (t-s)} \sigma dZ_s,$$

and a high-frequency measure of new information:

$$\mathcal{I}_t^H = \int_0^t e^{-\kappa^H (t-s)} \sigma dZ_s,$$

with $\kappa^L < \kappa^H$. The overall measure of new information can be defined as:

$$\mathcal{I}_t = \theta^L \mathcal{I}_t^L + \theta^H \mathcal{I}_t^H.$$

Parameters θ^L and θ^H determine the relative strength of low-frequency and high-frequency sentiment.

The downside of including multiple sentiment frequencies is that each frequency requires its own state variable. The main text uses a single frequency for parsimony.

E.3 Properties of Sentiment $\mathcal{I}_t$

Sentiment parameter $\mathcal{I}_t$ is an Ornstein-Uhlenbeck (OU) process. This section sketches some useful mathematical properties of OU processes. A textbook treatment is provided by Karatzas and Shreve (1998).

Conditional and Unconditional Distributions. Repeating equation (3), "recent information" parameter $\mathcal{I}_t$ is defined as:

$$\mathcal{I}_t = \int_0^t e^{-\kappa(t-s)} \sigma dZ_s.$$

The conditional distribution of $\mathcal{I}_{t+\tau}$ given $\mathcal{I}_t$ is characterized as follows:

$$\begin{aligned}\mathcal{I}_{t+\tau} &= \int_0^{t+\tau} e^{-\kappa(t+\tau-s)} \sigma dZ_s \\ &= e^{-\kappa\tau}\mathcal{I}_t + \int_t^{t+\tau} e^{-\kappa(t+\tau-s)} \sigma dZ_s.\end{aligned}$$

Since $\{Z_t\}$ is a standard Brownian motion with independent Gaussian increments, $\mathcal{I}_{t+\tau}|\mathcal{I}_t$ is itself Gaussian. The conditional mean of $\mathcal{I}_{t+\tau}$ is $e^{-\kappa\tau}\mathcal{I}_t$. By Itô isometry, the conditional variance of $\mathcal{I}_{t+\tau}$ is $\frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa\tau}\right)$.

The unconditional distribution of the OU process is obtained by taking $\tau \to \infty$. The unconditional distribution of $\mathcal{I}$ is Gaussian with mean 0 and variance $\frac{\sigma^2}{2\kappa}$.

Distribution of First Hitting Times. The first hitting time of a stochastic process is defined as the time at which a stochastic process first crosses some threshold. The distribution of first hitting times for an OU process can be characterized analytically in the special case where the threshold is the OU process' mean. So, from any initial level of sentiment $\mathcal{I}_t$, the time $t + \tau$ at which the OU process first returns to 0 has a known distribution, with a density given by:

$$h(\tau|\mathcal{I}_t) = \frac{|\mathcal{I}_t|}{\sigma\sqrt{2\pi}} \left(\frac{\kappa}{\sinh(\kappa\tau)}\right)^{\frac{3}{2}} \exp\left(-\frac{\kappa\mathcal{I}_t^2 e^{-\kappa\tau}}{2\sigma^2\sinh(\kappa\tau)} + \frac{\kappa\tau}{2}\right). \tag{66}$$

This is stated without proof, and details are provided in Alili et al. (2005).

Importance of κ for Sentiment Persistence. Let $\mathcal{I}_t^{SD} = \frac{\mathcal{I}_t}{\left(\frac{\sigma}{\sqrt{2\kappa}}\right)}$ denote sentiment normalized by its unconditional standard deviation. Upon normalizing $\mathcal{I}_t$ by its standard deviation, volatility parameter σ drops out of the equation for both the conditional distribution of $\mathcal{I}_{t+\tau}^{SD}$ and the distribution of first hitting times. These distributions become a function of only the persistence parameter κ. This shows that κ is the critical parameter governing the persistence of sentiment.

In detail, the conditional distribution of $\mathcal{I}_{t+\tau}^{SD}$ given $\mathcal{I}_t^{SD}$ is Gaussian with mean $e^{-\kappa\tau}\mathcal{I}_t^{SD}$ and variance $(1 - e^{-2\kappa\tau})$. Similarly, equation (66) can be rewritten as:

$$h(\tau|\mathcal{I}_t^{SD}) = \frac{\kappa|\mathcal{I}_t^{SD}|}{2\sqrt{\pi}} \left(\frac{1}{\sinh(\kappa\tau)}\right)^{\frac{3}{2}} \exp\left(-\frac{(\mathcal{I}_t^{SD})^2 e^{-\kappa\tau}}{4\sinh(\kappa\tau)} + \frac{\kappa\tau}{2}\right). \tag{67}$$

Equation (67) shows that the distribution of first hitting times for $\mathcal{I}_t^{SD}$ depends only on the initial level of $\mathcal{I}_t^{SD}$ and κ, but is independent of σ.

Ornstein-Uhlenbeck Figures: Baseline Calibration.

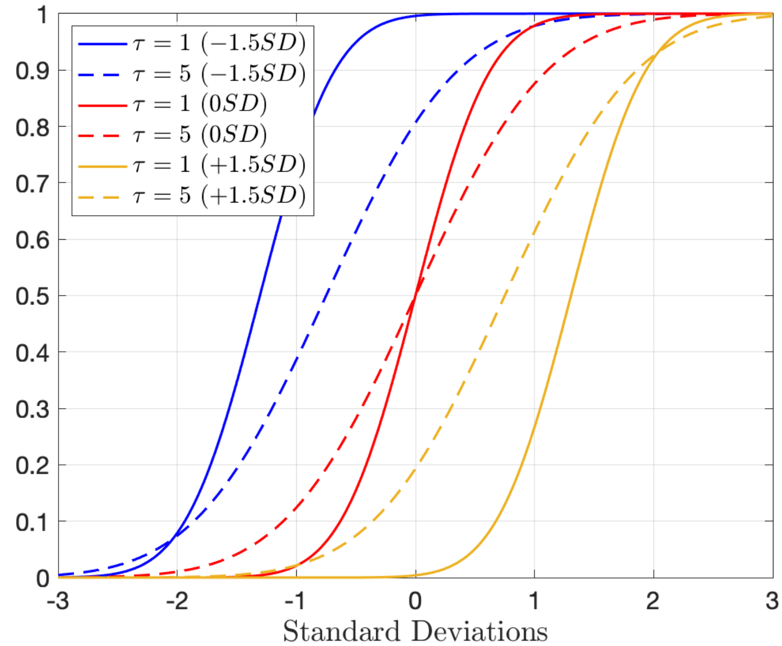


Figure 11: **Conditional CDF**. CDF of $\mathcal{I}_{t+\tau}^{SD}|\mathcal{I}_t^{SD} \in \{0SD, \pm 1.5SD\}$ for $\tau \in \{1, 5\}$.

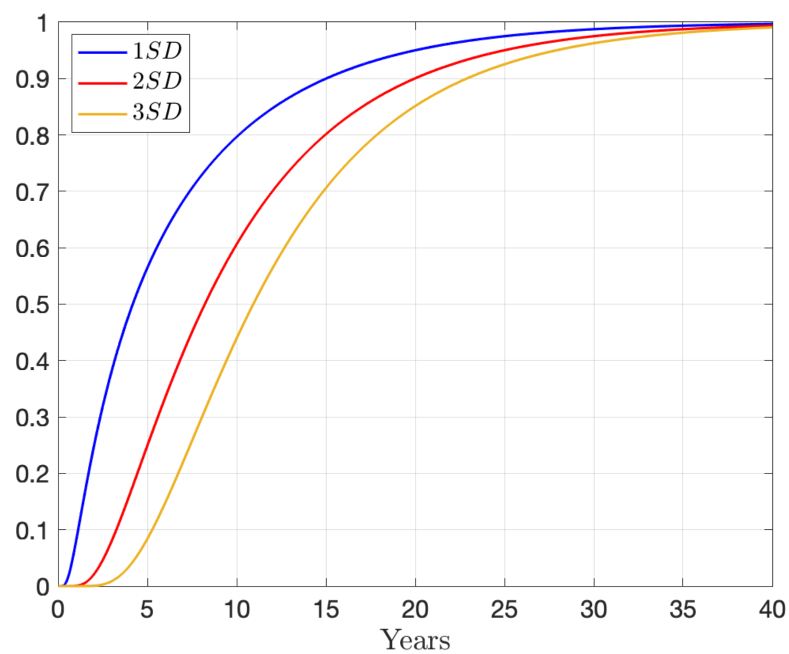


Figure 12: **1st hitting times**. Hitting time CDF for $\mathcal{I}_t \in \{\pm 1SD, \pm 2SD, \pm 3SD\}$.

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