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Decompositions of minimum rank matrices

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Dedicated to Avi Berman, Moshe Goldberg and Raphi Loewy for their lifetime contributions to the theory of matrices

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ABSTRACT

Let F be a field, let G be an undirected graph on n vertices, and let $\mathcal{S}^F(G)$ be the class of all F-valued symmetric $n \times n$ matrices whose nonzero off-diagonal entries occur in exactly the positions corresponding to the edges of G. For each graph G, there is an associated minimum rank class $\mathcal{MR}^F(G)$ consisting of all matrices $A \in \mathcal{S}^F(G)$ with rank $A = \text{mr}^F(G)$. For most graphs G with connectivity 1 or 2, we give explicit decompositions of matrices in $\mathcal{MR}^F(G)$ into sums of minimum rank matrices of simpler graphs (usually proper subgraphs) related to G. Our results can be thought of as generalizations of well-known formulae for the minimum rank of a graph with a cut vertex and of a graph with a 2-separation. We conclude by also showing that for these graphs, matrices in $\mathcal{MR}^F(G)$ can be constructed from matrices of simpler graphs; moreover, we give analogues for positive semidefinite matrices.

1. Introduction

The minimum rank problem in combinatorial matrix theory is concerned with determining the minimum possible rank over all symmetric matrices with a specified zero/nonzero off-diagonal

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pattern. Our aim in this paper is to also determine the structure of all matrices attaining the minimum rank for a large number of patterns.

In order to state this problem precisely, we introduce the relevant graph-theoretic notation. Let F be a field and let S_n be the set of all symmetric $n \times n$ matrices over F. Given $A \in S_n$, define $G(A) = (V, E)$ to be the (simple, undirected) graph with vertex set $V = \{1, 2, \ldots, n\}$ and edge set $E = \{\{i, j\} | a_{ij} \neq 0, i \neq j\}$. Given any graph G on n vertices, let

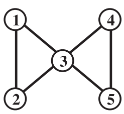
$$\mathcal{S}^F(G) = \{A \in S_n \mid G(A) = G\}$$

$$\mathrm{mr}^F(G) = \min\{\mathrm{rank}\, A \mid A \in \mathcal{S}^F(G)\}$$

$$\mathcal{MR}^F(G) = \{A \in \mathcal{S}^F(G) \mid \mathrm{rank}\, A = \mathrm{mr}^F(G)\}.$$

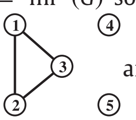
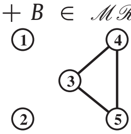
All of our results and most of our arguments do not depend on the field F, so we often suppress it in later use of these definitions. We adopt the convention of including the F in statements of theorems to emphasize field independence while excluding the F from proofs except where the particular field becomes of importance. The minimum ranks of many graphs are well-known (see, e.g., www.aimmath.org/pastworkshops/matrixspectrum.html) and in examples we will usually state the minimum rank of a graph without explanation.

Much less is known about $\mathcal{MR}^F(G)$. For the field F_2 of two elements, $\mathcal{MR}^{F_2}(G)$ is given explicitly for a few small graphs in Lemma 16 and Proposition 17 of [3]. For many graphs G, it is well understood how to construct matrices in $\mathcal{MR}^F(G)$ by considering appropriate subgraphs. The next two examples illustrate this for graphs with connectivity one and two.

Example 1.1. Let $\bowtie$ be the bowtie graph  obtained by identifying two K_3's at a vertex.

Clearly a matrix in $\mathcal{MR}^F(K_3)$ is the all-ones matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and it is natural to embed this matrix in two 5×5 matrices and add them to obtain a matrix in $\mathcal{MR}^F(\bowtie)$. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Then $\mathrm{rank}\,(A + B) = 2 = \mathrm{mr}^F(G)$ so that $A + B \in \mathcal{MR}^F(\bowtie)$. Note that $A \in \mathcal{MR}^F(G_1)$ and $B \in \mathcal{MR}^F(G_2)$ where G_1 is  and G_2 is  and $\mathrm{mr}^F(G_1) = \mathrm{mr}^F(G_2) = 1$.

This motivates the following definition.

Definition 1.2. Given a proper subgraph H of a graph G, let $\widetilde{H}$ be the graph with vertex set $V(G)$ and edge set $E(H)$.

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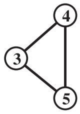
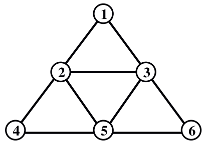
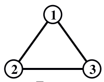
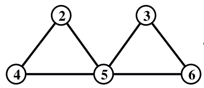
So if H is the subgraph  of the bowtie graph in Example 1.1, then $\widetilde{H}$ is the graph G_2 above.

Note that for any graph G with subgraph H, since H and $\widetilde{H}$ differ only by some number of isolated vertices, $\mathrm{mr}(H) = \mathrm{mr}(\widetilde{H})$. We will often label matrices with a $\sim$ when they are padded with rows and columns of zeros, and in this case the $\sim$ does not constitute an operator symbol but merely emphasizes a relation between a matrix and a corresponding subgraph.

Example 1.3. Let G be the graph



.

Let $G_{\mathcal{A}}$ be the graph  and let $G_{\mathcal{B}}$ be the graph . Then $\mathrm{mr}^F(G) = 3 = 1 + 2 = \mathrm{mr}^F(G_{\mathcal{A}}) + \mathrm{mr}^F(G_{\mathcal{B}})$ for any field F. Letting

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1+1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

we have $A + B \in \mathscr{MR}^F(G)$, $A \in \mathscr{MR}^F(\widetilde{G_{\mathcal{A}}})$, $B \in \mathscr{MR}^F(\widetilde{G_{\mathcal{B}}})$. This is another instance of a decomposition of a matrix in $\mathscr{MR}^F(G)$ into minimum rank matrices of subgraphs with smaller minimum rank.

The decompositions in Examples 1.1 and 1.3 are obvious as is the case in many further such examples. What is missing in these examples is an explanation of how and under what circumstances such decompositions can be obtained. We will give explicit decompositions of matrices in $\mathscr{MR}^F(G)$ for all graphs with connectivity equal to 1 or 2 (note that the decomposition is obvious for disconnected graphs). Consequently, one can think of our results as a way to reduce the study of minimum rank matrices of a graph to the study of minimum rank matrices of 3-connected graphs.

2. Prior results

Our results can be viewed as generalizing two fundamental results giving formulae for the minimum rank of a graph with a cut vertex and the minimum rank of a graph with a 2-separation.

Definition 2.1. Let G_1 and G_2 be graphs with at least two vertices, each with a non-isolated vertex labeled v. The vertex-sum at v of G_1 and G_2 is the graph on $|G_1| + |G_2| - 1$ vertices obtained by identifying the vertex v in G_1 with the vertex v in G_2. The vertex v is called a cut vertex of G.

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Definition 2.2. Let F be a field. The *rank-spread* of a vertex v of a graph G, denoted $r_v^F(G)$, is the difference between the minimum rank of G over F and the minimum rank of $G - v$ over F. i.e.,

$$r_v^F(G) = \text{mr}^F(G) - \text{mr}^F(G - v).$$

The following theorem was originally published with proofs over the real field by Hsieh in [7], and independently by Barioli et al. (see Theorem 2.3 in [1]). Proofs given over any field can be found in [5] (see Theorem 7 where an equivalent version in terms of maximum nullity is given) or in [3] (see Appendix B).

Theorem 2.3. *Let G_1 and G_2 be graphs on at least two vertices each with a vertex labeled v and let G be the vertex-sum at v of G_1 and G_2. Let F be any field. Then*

$$\text{mr}^F(G) = \min\{\text{mr}^F(G_1) + \text{mr}^F(G_2), \text{mr}^F(G_1 - v) + \text{mr}^F(G_2 - v) + 2\}.$$

Equivalently,

$$r_v^F(G) = \min\{r_v^F(G_1) + r_v^F(G_2), 2\}.$$

A more complex result applies to graphs with connectivity 2. We first recall the following definitions from [5].

Definition 2.4. Let $G = (V, E)$ be a graph with $V = \{1, 2, \dots, n\}$ which we allow to have parallel edges. We denote by F_2 the field with only two elements. If F is a field unequal to F_2, we define $\mathcal{S}^F(G)$ as the set of all F-valued symmetric $n \times n$ matrices $A = [a_{i,j}]$ with

1. $a_{i,j} = 0$ if $i \neq j$ and i and j are not adjacent,
2. $a_{i,j} \neq 0$ if $i \neq j$ and i and j are connected by exactly one edge,
3. $a_{i,j} \in F$ if $i \neq j$ and i and j are connected by multiple edges, and
4. $a_{i,i} \in F$ for all $i \in V$.

We define $\mathcal{S}^{F_2}(G)$ as the set of all F_2-valued symmetric $n \times n$ matrices $A = [a_{i,j}]$ with

1. $a_{i,j} \neq 0$ if $i \neq j$ and i and j are connected by an odd number of edges,
2. $a_{i,j} = 0$ if $i \neq j$ and i and j are connected by an even number of edges, and
3. $a_{i,i} \in F_2$ for all $i \in V$.

Definition 2.5. A 2-separation of a graph $G = (V, E)$ is a pair of subgraphs $(G_\mathcal{A}, G_\mathcal{B})$ satisfying the following: $V(G_\mathcal{A}) \cup V(G_\mathcal{B}) = V$, $| V(G_\mathcal{A}) \cap V(G_\mathcal{B}) | = 2$, $E(G_\mathcal{A}) \cup E(G_\mathcal{B}) = E$, and $E(G_\mathcal{A}) \cap E(G_\mathcal{B}) = \emptyset$.

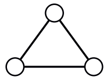
The main result of [5] (see Theorem 14 and Corollary 15) is:

Theorem 2.6. *Let $(G_\mathcal{A}, G_\mathcal{B})$ be a 2-separation of G with $R = \{r_1, r_2\} = V(G_\mathcal{A}) \cap V(G_\mathcal{B})$. Let $H_\mathcal{A}$ and $H_\mathcal{B}$ be obtained from $G_\mathcal{A}$ and $G_\mathcal{B}$, respectively, by inserting an edge between r_1 and r_2. Let $\overline{G_\mathcal{A}}$ and $\overline{G_\mathcal{B}}$ be obtained from $G_\mathcal{A}$ and $G_\mathcal{B}$, respectively, by identifying r_1 and r_2, or in other words, by inserting edges between one vertex and the neighbors of the other and then deleting the latter.*

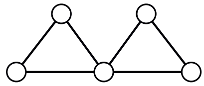
Then
$$\begin{aligned} \text{mr}^F(G) = \min\{&\text{mr}^F(G_\mathcal{A}) + \text{mr}^F(G_\mathcal{B}), \\ &\text{mr}^F(H_\mathcal{A}) + \text{mr}^F(H_\mathcal{B}), \\ &\text{mr}^F(\overline{G_\mathcal{A}}) + \text{mr}^F(\overline{G_\mathcal{B}}) + 2, \\ &\text{mr}^F(G_\mathcal{A} - r_1) + \text{mr}^F(G_\mathcal{B} - r_1) + 2, \\ &\text{mr}^F(G_\mathcal{A} - r_2) + \text{mr}^F(G_\mathcal{B} - r_2) + 2, \\ &\text{mr}^F(G_\mathcal{A} - R) + \text{mr}^F(G_\mathcal{B} - R) + 4\}. \end{aligned}$$

Example 2.7. Continuing with Example 1.3, the graphs appearing in Theorem 2.6 are:

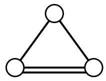
$G_{\mathcal{A}}$:



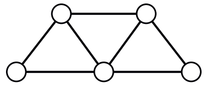
$G_{\mathcal{B}}$:



$H_{\mathcal{A}}$:

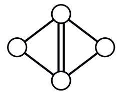


$H_{\mathcal{B}}$:

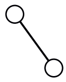


$\overline{G_{\mathcal{A}}}$:

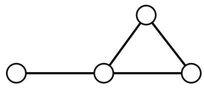
$\overline{G_{\mathcal{B}}}$:



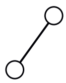
$G_{\mathcal{A}} - r_1$:



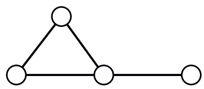
$G_{\mathcal{B}} - r_1$:



$G_{\mathcal{A}} - r_2$:

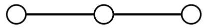


$G_{\mathcal{B}} - r_2$:



$G_{\mathcal{A}} - R$:

$G_{\mathcal{B}} - R$:



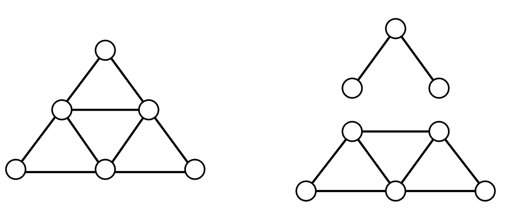
Simple calculations show that

$$\mathrm{mr}(G_{\mathcal{A}}) + \mathrm{mr}(G_{\mathcal{B}}) = 1 + 2 = 3,$$
$$\mathrm{mr}(H_{\mathcal{A}}) + \mathrm{mr}(H_{\mathcal{B}}) = 1 + 3 = 4,$$
$$\mathrm{mr}(\overline{G_{\mathcal{A}}}) + \mathrm{mr}(\overline{G_{\mathcal{B}}}) + 2 = 0 + 2 + 2 = 4,$$
$$\mathrm{mr}(G_{\mathcal{A}} - r_1) + \mathrm{mr}(G_{\mathcal{B}} - r_1) + 2 = 1 + 2 + 2 = 5,$$
$$\mathrm{mr}(G_{\mathcal{A}} - r_2) + \mathrm{mr}(G_{\mathcal{B}} - r_2) + 2 = 1 + 2 + 2 = 5,$$
$$\mathrm{mr}(G_{\mathcal{A}} - R) + \mathrm{mr}(G_{\mathcal{B}} - R) + 4 = 0 + 2 + 4 = 6.$$

So in this example, the minimum is attained uniquely by the first term.

Example 2.8. The following graphs with accompanying 2-separations show that each term in Theorem 2.6 is necessary. To the left of each we show the term that attains the minimum uniquely.

$\mathrm{mr}(H_{\mathcal{A}}) + \mathrm{mr}(H_{\mathcal{B}})$:



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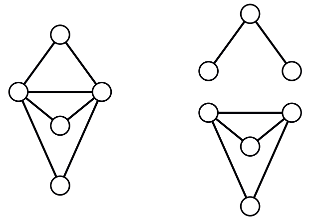
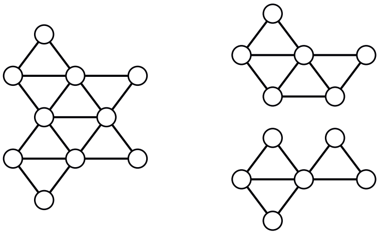
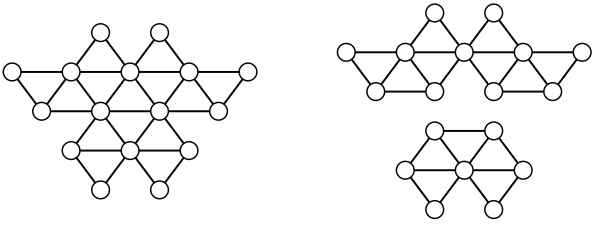
$\mathrm{mr}(\overline{G_{\mathcal{A}}}) + \mathrm{mr}(\overline{G_{\mathcal{B}}}) + 2:$



$\mathrm{mr}(G_{\mathcal{A}} - r_1) + \mathrm{mr}(G_{\mathcal{B}} - r_1) + 2:$



$\mathrm{mr}(G_{\mathcal{A}} - R) + \mathrm{mr}(G_{\mathcal{B}} - R) + 4:$



Note that a graph for the term we omitted, $\mathrm{mr}(G_{\mathcal{A}} - r_2) + \mathrm{mr}(G_{\mathcal{B}} - r_2) + 2$, can be obtained by switching the labeling of the vertices r_1 and r_2 in the graph used for the term $\mathrm{mr}(G_{\mathcal{A}} - r_1) + \mathrm{mr}(G_{\mathcal{B}} - r_1) + 2$. Also, since it is not obvious, we note that the minimum rank for the connected 16 vertex graph is 10.

We will also make use of the following lemma from [5] (see Lemma 10 where it is stated in terms of maximum nullity) and two propositions from [8].

Lemma 2.9. *Let F be a field, let $G = (V, E)$ be a graph, and let $R = \{r_1, r_2\} \subseteq V$. Let $\overline{G}$ be obtained from G by identifying the vertices of R. Then*

$$\mathrm{mr}^F(G) \leqslant \mathrm{mr}^F(\overline{G}) + 2.$$

Proposition 2.10 (Nylen)**.** *Let F be a field, let G be a graph, and let v a vertex of G. Then*

$$\mathrm{mr}^F(G - v) \leqslant \mathrm{mr}^F(G) \leqslant \mathrm{mr}^F(G - v) + 2.$$

Equivalently,

$$0 \leqslant r_v^F(G) \leqslant 2.$$

Proposition 2.11 (Nylen)**.** *Let F be a field, let G be a graph on n vertices, and let $A \in \mathscr{MR}^F(G)$. Then for any $i \in \{1, 2, \ldots, n\}$, $\operatorname{rank} A(i) = \operatorname{rank} A$ or $\operatorname{rank} A(i) = \operatorname{rank} A - 2$; i.e., $\operatorname{rank} A(i) = \operatorname{rank} A - 1$ is impossible.*

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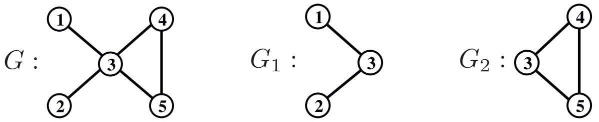
3. Decompositions for graphs with a cut vertex

In this section we generalize Theorem 2.3. We show that minimum rank matrices of a graph with a cut vertex decompose in two different ways and that these correspond to the two different possible minima in Theorem 2.3.

Theorem 3.1. *Let G be the vertex-sum at v of G_1 and G_2, and let S_{k+1} be the star subgraph of G formed by the degree k vertex v and all of its neighbors. Let F be any field and let $A \in \mathscr{MR}^F(G)$.*

1. *If* $\operatorname{rank} A = \operatorname{rank} A(v)$, *then* $A \in \mathscr{MR}^F(\widetilde{G_1}) + \mathscr{MR}^F(\widetilde{G_2})$ *and* $\operatorname{mr}^F(G) = \operatorname{mr}^F(G_1) + \operatorname{mr}^F(G_2)$.
2. *If* $\operatorname{rank} A = \operatorname{rank} A(v) + 2$, *then* $A \in \mathscr{MR}^F(\widetilde{G_1 - v}) + \mathscr{MR}^F(\widetilde{G_2 - v}) + \mathscr{MR}^F(\widetilde{S_{k+1}})$ *and* $\operatorname{mr}^F(G) = \operatorname{mr}^F(G_1 - v) + \operatorname{mr}^F(G_2 - v) + 2$.

Example 3.2. As an illustration of the above theorem, consider the graphs



noting that G is the vertex-sum at vertex 3 of G_1 and G_2. Now consider the matrices in $\mathscr{MR}(G)$

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 1 & -1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Since $\operatorname{rank} A = \operatorname{rank} A(3)$, statement 1 of the theorem applies, and

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

where the two matrices are respectively in $\mathscr{MR}(\widetilde{G_1})$ and $\mathscr{MR}(\widetilde{G_2})$. Also note $\operatorname{mr}(G) = 3 = 2 + 1 = \operatorname{mr}(G_1) + \operatorname{mr}(G_2)$.

Since $\operatorname{rank} B = \operatorname{rank} B(3) + 2$, statement 2 of the theorem applies, and

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

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where the three matrices are respectively in $\mathcal{MR}(\widetilde{G_1 - v})$, $\mathcal{MR}(\widetilde{G_2 - v})$ and $\mathcal{MR}(S_5)$ where v is vertex 3. Again note $\text{mr}(G) = 3 = 0 + 1 + 2 = \text{mr}(G_1 - v) + \text{mr}(G_2 - v) + 2$.

Proof. Let G be as stated in the theorem with $A \in \mathcal{MR}(G)$ given. Labeling the vertices of G appropriately $A = \begin{bmatrix} a & x_1^T & x_2^T \\ x_1 & C_1 & 0 \\ x_2 & 0 & C_2 \end{bmatrix}$ where each $C_i \in \mathcal{S}(G_i - v)$ and the first row and column of A correspond to the vertex v. By Proposition 2.11, either $\text{rank}\, A(v) = \text{rank}\, A$ or $\text{rank}\, A(v) = \text{rank}\, A - 2$.

Consider the case where $\text{rank}\, A(v) = \text{rank}\, A$. It follows that

$$\text{rank}\, A \geqslant \text{rank} \begin{bmatrix} x_1 & C_1 & 0 \\ x_2 & 0 & C_2 \end{bmatrix} \geqslant \text{rank}\, A(v) = \text{rank}\, C_1 + \text{rank}\, C_2.$$

Since $\text{rank}\, A(v) = \text{rank}\, A$, we actually have equality throughout. Thus $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is in the column space of $A(v)$. Further since $A(v)$ is a block diagonal matrix, each x_i is in the column space of C_i. Thus there exist vectors y_i such that $x_i = C_i y_i$. Now rewrite the matrix as $A = \begin{bmatrix} a & y_1^T C_1 & y_2^T C_2 \\ C_1 y_1 & C_1 & 0 \\ C_2 y_2 & 0 & C_2 \end{bmatrix}$. Note that the first row of A is a linear combination of the other rows of A. Thus using block Gaussian elimination we see that $a = y_1^T C_1 y_1 + y_2^T C_2 y_2$. Let

$$\widetilde{B_1} = \begin{bmatrix} y_1^T C_1 y_1 & y_1^T C_1 & 0 \\ C_1 y_1 & C_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \widetilde{B_2} = \begin{bmatrix} y_2^T C_2 y_2 & 0 & y_2^T C_2 \\ 0 & 0 & 0 \\ C_2 y_2 & 0 & C_2 \end{bmatrix}.$$

Then $A = \widetilde{B_1} + \widetilde{B_2}$, where $\text{rank}\, \widetilde{B_i} = \text{rank}\, C_i$ and $\widetilde{B_i} \in \mathcal{S}(\widetilde{G_i})$, $i = 1, 2$. By Theorem 2.3,

$$\begin{aligned} \text{mr}(G) &\leqslant \text{mr}(G_1) + \text{mr}(G_2) = \text{mr}(\widetilde{G_1}) + \text{mr}(\widetilde{G_2}) \\ &\leqslant \text{rank}\, \widetilde{B_1} + \text{rank}\, \widetilde{B_2} = \text{rank}\, C_1 + \text{rank}\, C_2 = \text{rank}\, A = \text{mr}(G). \end{aligned}$$

Then we have equality throughout, so $\text{mr}(G) = \text{mr}(G_1) + \text{mr}(G_2)$ and $\text{rank}\, \widetilde{B_i} = \text{mr}(\widetilde{G_i})$, $i = 1, 2$. Then $\widetilde{B_i} \in \mathcal{MR}(\widetilde{G_i})$, $i = 1, 2$, which completes the proof of the first statement of the theorem.

Now consider the case where $\text{rank}\, A(v) = \text{rank}\, A - 2$. Let

$$\widetilde{E_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & C_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \widetilde{E_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C_2 \end{bmatrix}, \quad \text{and} \quad \widetilde{E} = \begin{bmatrix} a & x_1^T & x_2^T \\ x_1 & 0 & 0 \\ x_2 & 0 & 0 \end{bmatrix}.$$

Note that $A = \widetilde{E_1} + \widetilde{E_2} + \widetilde{E}$, $\widetilde{E_i} \in \mathcal{S}(\widetilde{G_i - v})$, $i = 1, 2$, and $\widetilde{E} \in \mathcal{S}(\widetilde{S_{k+1}})$. By Theorem 2.3 and the hypothesis,

$$\begin{aligned} \text{mr}(G) &\leqslant \text{mr}(G_1 - v) + \text{mr}(G_2 - v) + 2 = \text{mr}(\widetilde{G_1 - v}) + \text{mr}(\widetilde{G_2 - v}) + 2 \\ &\leqslant \text{rank}\, \widetilde{E_1} + \text{rank}\, \widetilde{E_2} + 2 = \text{rank}\, C_1 + \text{rank}\, C_2 + 2 \\ &= \text{rank}\, A(v) + 2 = \text{rank}\, A = \text{mr}(G) \end{aligned}$$

Thus we have equality throughout, so $\operatorname{mr}(G) = \operatorname{mr}(G_1 - v) + \operatorname{mr}(G_2 - v) + 2$, and $\operatorname{mr}(\widetilde{G_i - v}) = \operatorname{rank} \widetilde{E}_i$, $i = 1, 2$, i.e., $\widetilde{E}_i \in \mathscr{MR}(\widetilde{G_i - v})$. Since v is not an isolated vertex in either G_1 or G_2, $\operatorname{rank} \widetilde{E} = 2 = \operatorname{mr}(\widetilde{S_{k+1}})$, and $\widetilde{E} \in \mathscr{MR}(\widetilde{S_{k+1}})$. Since $A = \widetilde{E_1} + \widetilde{E_2} + \widetilde{E}$, the proof of the second statement of the theorem is complete. □

The following corollary gives a method for knowing the possible decompositions given only the graph and also determines when both decompositions or only one is possible.

Corollary 3.3. *Let G, F, v, G_1, G_2 and S_{k+1} be as in Theorem 3.1.*

1. *If $r_v^F(G_1) + r_v^F(G_2) < 2$, then*
$$\mathscr{MR}^F(G) = \mathscr{MR}^F(\widetilde{G_1}) + \mathscr{MR}^F(\widetilde{G_2}).$$
2. *If $r_v^F(G_1) + r_v^F(G_2) > 2$, then*
$$\mathscr{MR}^F(G) = \mathscr{MR}^F(\widetilde{G_1 - v}) + \mathscr{MR}^F(\widetilde{G_2 - v}) + \mathscr{MR}^F(\widetilde{S_{k+1}}).$$
3. *If $r_v^F(G_1) + r_v^F(G_2) = 2$, then*
$$\mathscr{MR}^F(G) = \left(\mathscr{MR}^F(\widetilde{G_1}) + \mathscr{MR}^F(\widetilde{G_2})\right) \cup \left(\mathscr{MR}^F(\widetilde{G_1 - v}) + \mathscr{MR}^F(\widetilde{G_2 - v}) + \mathscr{MR}^F(\widetilde{S_{k+1}})\right).$$

Looking back at Example 3.2, we see that $r_3(G_1) + r_3(G_2) = 2 + 0 = 2$, so G illustrates statement 3 of the corollary.

Proof. Since

$$\begin{aligned} \operatorname{mr}(G) &= \min\{\operatorname{mr}(G_1) + \operatorname{mr}(G_2), \operatorname{mr}(G_1 - v) + \operatorname{mr}(G_2 - v) + 2\} \\ &= \operatorname{mr}(G_1 - v) + \operatorname{mr}(G_2 - v) + \min\{r_v(G_1) + r_v(G_2), 2\}, \end{aligned}$$

letting $r = r_v(G_1) + r_v(G_2)$ we have

$$\operatorname{mr}(G) = \begin{cases} \operatorname{mr}(G_1) + \operatorname{mr}(G_2) < \operatorname{mr}(G_1 - v) + \operatorname{mr}(G_2 - v) + 2 & \text{if } r < 2 \\ \operatorname{mr}(G_1) + \operatorname{mr}(G_2) = \operatorname{mr}(G_1 - v) + \operatorname{mr}(G_2 - v) + 2 & \text{if } r = 2 \\ \operatorname{mr}(G_1 - v) + \operatorname{mr}(G_2 - v) + 2 < \operatorname{mr}(G_1) + \operatorname{mr}(G_2) & \text{if } r > 2. \end{cases} \tag{3.1}$$

We first prove the forward containments. Let $A \in \mathscr{MR}(G)$. By Proposition 2.11, either $\operatorname{rank} A = \operatorname{rank} A(v) + 2$ or $\operatorname{rank} A = \operatorname{rank} A(v)$.

Case 1. Suppose $r_v(G_1) + r_v(G_2) < 2$. Since $\operatorname{mr}(G) \neq \operatorname{mr}(G_1 - v) + \operatorname{mr}(G_2 - v) + 2$, by the contrapositive of statement 2 of Theorem 3.1 $\operatorname{rank} A = \operatorname{rank} A(v)$ and $A \in \mathscr{MR}(\widetilde{G_1}) + \mathscr{MR}(\widetilde{G_2})$.

Case 2. Suppose $r_v(G_1) + r_v(G_2) > 2$. Then $\operatorname{mr}(G) \neq \operatorname{mr}(G_1) + \operatorname{mr}(G_2)$ and by the contrapositive of statement 1 of Theorem 3.1 $\operatorname{rank} A = \operatorname{rank} A(v) + 2$ and $A \in \mathscr{MR}(\widetilde{G_1 - v}) + \mathscr{MR}(\widetilde{G_2 - v}) + \mathscr{MR}(\widetilde{S_{k+1}})$.

Case 3. Suppose $r_v(G_1) + r_v(G_2) = 2$. Whether $\operatorname{rank} A = \operatorname{rank} A(v) + 2$ or $\operatorname{rank} A = \operatorname{rank} A(v)$, A is in the union on the right hand side of 3.

Now we verify the reverse containments.

Case 1. Suppose $r_v(G_1) + r_v(G_2) \leqslant 2$ and $A \in \mathcal{MR}(\widetilde{G_1}) + \mathcal{MR}(\widetilde{G_2})$. Write $A = \widetilde{A_1} + \widetilde{A_2}$ with $\widetilde{A_i} \in \mathcal{MR}(\widetilde{G_i})$, $i = 1, 2$. Then $A \in \mathcal{S}(G)$. By Eq. (3.1), $\text{mr}(G) = \text{mr}(G_1) + \text{mr}(G_2)$, so

$$\begin{aligned}\text{mr}(G) &\leqslant \text{rank}\, A = \text{rank}\,(\widetilde{A_1} + \widetilde{A_2}) \leqslant \text{rank}\, \widetilde{A_1} + \text{rank}\, \widetilde{A_2} \\ &= \text{mr}(\widetilde{G_1}) + \text{mr}(\widetilde{G_2}) = \text{mr}(G_1) + \text{mr}(G_2) \\ &= \text{mr}(G).\end{aligned}$$

Then $\text{rank}\, A = \text{mr}(G)$ and $A \in \mathcal{MR}(G)$.

Case 2. Now suppose $r_v(G_1) + r_v(G_2) \geqslant 2$ and $A \in \mathcal{MR}(\widetilde{G_1 - v}) + \mathcal{MR}(\widetilde{G_2 - v}) + \mathcal{MR}(\widetilde{S_{k+1}})$. Write $A = \widetilde{B_1} + \widetilde{B_2} + \widetilde{E}$ with $\widetilde{B_i} \in \mathcal{MR}(\widetilde{G_i - v})$, $i = 1, 2$, and $\widetilde{E} \in \mathcal{MR}(\widetilde{S_{k+1}})$. Necessarily $A \in \mathcal{S}(G)$. By Eq. (3.1),

$$\begin{aligned}\text{mr}(G) &= \text{mr}(G_1 - v) + \text{mr}(G_2 - v) + 2 \\ &= \text{mr}(\widetilde{G_1 - v}) + \text{mr}(\widetilde{G_2 - v}) + 2 \\ &= \text{rank}\, \widetilde{B_1} + \text{rank}\, \widetilde{B_2} + \text{rank}\, \widetilde{E} \\ &\geqslant \text{rank}\,(\widetilde{B_1} + \widetilde{B_2} + \widetilde{E}) = \text{rank}\, A \geqslant \text{mr}(G).\end{aligned}$$

Then $\text{rank}\, A = \text{mr}(G)$ and $A \in \mathcal{MR}(G)$. $\square$

4. Decompositions for graphs with a 2-separation

We will now give a decomposition theorem associated with Theorem 2.6. There are twelve minimum rank classes associated with the terms on the right hand side of the formula for $\text{mr}^F(G)$, namely:

$$\begin{gathered}\mathcal{MR}^F(\widetilde{G_\mathcal{A}}), \quad \mathcal{MR}^F(\widetilde{G_\mathcal{B}}), \quad \mathcal{MR}^F(\widetilde{H_\mathcal{A}}), \quad \mathcal{MR}^F(\widetilde{H_\mathcal{B}}), \\ \mathcal{MR}^F(\widetilde{\overline{G_\mathcal{A}}}), \quad \mathcal{MR}^F(\widetilde{\overline{G_\mathcal{B}}}), \quad \mathcal{MR}^F(\widetilde{G_\mathcal{A} - r_1}), \quad \mathcal{MR}^F(\widetilde{G_\mathcal{B} - r_1}), \\ \mathcal{MR}^F(\widetilde{G_\mathcal{A} - r_2}), \quad \mathcal{MR}^F(\widetilde{G_\mathcal{B} - r_2}), \quad \mathcal{MR}^F(\widetilde{G_\mathcal{A} - R}), \quad \mathcal{MR}^F(\widetilde{G_\mathcal{B} - R}).\end{gathered}$$

We will need all of these and five additional graphs and their minimum rank classes in the statement of our decomposition theorem. For ease of reference, we restate the definitions of the above graphs originally given in the statement of Theorem 2.6 as well as define the five additional graphs.

Definition 4.1. Let $(G_\mathcal{A}, G_\mathcal{B})$ be a 2-separation of a graph G and let $R = \{r_1, r_2\} = V(G_\mathcal{A}) \cap V(G_\mathcal{B})$. To avoid degenerate cases we now assume that $G_\mathcal{A}$ and $G_\mathcal{B}$ each have at least 3 vertices. We define the following graphs associated with G.

1. $\overline{G_\mathcal{A}}, \overline{G_\mathcal{B}}$ are the multigraphs obtained by identifying r_1 and r_2 in $G_\mathcal{A}$ and $G_\mathcal{B}$, respectively.
2. $H_\mathcal{A}, H_\mathcal{B}$ are the multigraphs obtained from $G_\mathcal{A}$ and $G_\mathcal{B}$, respectively, by inserting one edge $r_1 r_2$.
3. For $i = 1, 2$, $\text{Star}_i(G) = (V(G), E_i)$ where E_i is the set of all edges incident to vertex r_i. $\text{Star}_{12}(G) = \text{Star}_1(G) \cup \text{Star}_2(G)$.
4. $\text{TStar}_1(G)$ is the graph obtained from $\text{Star}_1(G)$ by inserting an edge between every neighbor of r_1 in G (excluding r_2) and r_2. $\text{TStar}_2(G)$ is the graph obtained from $\text{Star}_2(G)$ by inserting an edge between every neighbor of r_2 in G (excluding r_1) and r_1.

The T in the preceding definition refers to the fact that we are twinning vertices; we are making r_2 a twin of r_1 in forming $\text{TStar}_1(G)$ and vice versa in forming $\text{TStar}_2(G)$.

It is well known that if G is a star on 4 or more vertices and $M \in \mathcal{MR}(G)$ then $m_{ii} = 0$ for every pendant vertex i (see, e.g., [2]). In the settings we will encounter, the minimum rank classes associated

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with the latter two "star classes" will also have restricted diagonal entries, so for these we modify the definitions at the beginning of the paper appropriately.

Definition 4.2. Let

$$\mathcal{S}_0^F(\mathrm{Star}_{12}(G)) = \{M \in \mathcal{S}^F(G) \mid m_{jj} = 0; j \neq r_1, r_2\}$$

$$\mathcal{S}_0^F(\mathrm{TStar}_i(G)) = \{M \in \mathcal{S}^F(G) \mid m_{jj} = 0; j \neq r_1, r_2\}, i = 1, 2$$

$$\mathrm{mr}_0^F(\mathrm{Star}_{12}(G)) = \min\{\mathrm{rank}\, A \mid A \in \mathcal{S}_0^F(\mathrm{Star}_{12}(G))\}$$

$$\mathrm{mr}_0^F(\mathrm{TStar}_i(G)) = \min\{\mathrm{rank}\, A \mid A \in \mathcal{S}_0^F(\mathrm{TStar}_i(G))\}, i = 1, 2$$

$$\mathcal{MR}_0^F(\mathrm{Star}_{12}(G)) = \{A \in \mathcal{S}_0^F(\mathrm{Star}_{12}(G)) \mid \mathrm{rank}\, A = \mathrm{mr}_0^F(\mathrm{Star}_{12}(G))\}$$

$$\mathcal{MR}_0^F(\mathrm{TStar}_i(G)) = \{A \in \mathcal{S}_0^F(\mathrm{TStar}_i(G)) \mid \mathrm{rank}\, A = \mathrm{mr}_0^F(\mathrm{TStar}_i(G))\}, i = 1, 2.$$

Proposition 4.3. *Let F be a field. Let $(G_{\mathcal{A}}, G_{\mathcal{B}})$ be a 2-separation of a graph G, assume that $G_{\mathcal{A}}$ and $G_{\mathcal{B}}$ each have at least 3 vertices, let $R = \{r_1, r_2\} = V(G_{\mathcal{A}}) \cap V(G_{\mathcal{B}})$ and assume that neither r_1 nor r_2 is a cut vertex of G. Then*

$$\mathrm{mr}^F(\mathrm{Star}_i(G)) = 2, \ \ \mathrm{mr}_0^F(\mathrm{TStar}_i(G)) = 2, \ \text{and} \ \mathrm{mr}_0^F(\mathrm{Star}_{12}(G)) \leqslant 4.$$

Proof. Since r_2 is not a cut vertex of G, r_1 is adjacent to a vertex in $V(G_{\mathcal{A}}) \setminus R$ and a vertex in $V(G_{\mathcal{B}}) \setminus R$. Then P_3 is induced in $\mathrm{Star}_1(G)$ and in $\mathrm{TStar}_1(G)$, so each has minimum rank at least 2. Similarly $\mathrm{Star}_2(G)$ and $\mathrm{TStar}_2(G)$ each have minimum rank at least 2.

For $i = 1$ or 2, $\mathrm{Star}_i(G)$ is a star so $\mathrm{mr}(\mathrm{Star}_i(G)) \leqslant 2$. If we label the graph G so that r_1, r_2 occur first, there is a matrix of either the form

$$\begin{bmatrix} 0 & 0 & x^T \\ 0 & 0 & 0 \\ x & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 & x^T \\ 1 & 0 & 0 \\ x & 0 & 0 \end{bmatrix}$$

in $\mathcal{S}(\mathrm{Star}_1(G))$. Then depending on whether or not $r_1 r_2 \in E(G)$, either

$$\begin{bmatrix} 0 & 0 & x^T \\ 0 & 0 & x^T \\ x & x & 0 \end{bmatrix} \quad \text{or else} \quad \begin{bmatrix} 1 & 1 & x^T \\ 1 & 1 & x^T \\ x & x & 0 \end{bmatrix}$$

is in $\mathcal{S}_0(\mathrm{TStar}_1(G))$ and both have rank at most 2. So $\mathrm{mr}_0(\mathrm{TStar}_1(G)) \leqslant 2$ and similarly $\mathrm{mr}_0(\mathrm{TStar}_2(G)) \leqslant 2$.

Finally, there is a matrix of the form

$$\begin{bmatrix} a & b & x^T \\ b & c & y^T \\ x & y & 0 \end{bmatrix}$$

in $\mathcal{S}_0(\mathrm{Star}_{12}(G))$, and its rank is at most 4. $\square$

The following theorem which generalizes Theorem 2.6 is our main result. We show that minimum rank matrices of graphs with a 2-separation decompose in essentially six different ways, each corresponding to one of the six possible minima in Theorem 2.6.

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Theorem 4.4. *Let F be a field. Given G as in Definition 4.1 and that neither r_1 nor r_2 is a cut vertex, let*

$$M = \begin{bmatrix} A & x_1 & x_2 & 0 \\ x_1^T & a & b & y_1^T \\ x_2^T & b & c & y_2^T \\ 0 & y_1 & y_2 & B \end{bmatrix} \in \mathscr{MR}^F(G)$$

where the vertices are labeled so that $A \in \mathcal{S}^F(G_{\mathcal{A}} - R)$, $B \in \mathcal{S}^F(G_{\mathcal{B}} - R)$,

$$\begin{bmatrix} 0 & x_1 & 0 & 0 \\ x_1^T & a & b & y_1^T \\ 0 & b & 0 & 0 \\ 0 & y_1 & 0 & 0 \end{bmatrix} \in \mathcal{S}^F(\mathrm{Star}_1(G)), \quad \text{and} \quad \begin{bmatrix} 0 & 0 & x_2 & 0 \\ 0 & 0 & b & 0 \\ x_2^T & b & c & y_2^T \\ 0 & 0 & y_2 & 0 \end{bmatrix} \in \mathcal{S}^F(\mathrm{Star}_2(G)).$$

I. *If*

$$\mathrm{rank}\, A + \mathrm{rank}\, B = \mathrm{rank}\, [A\, x_1\, x_2] + \mathrm{rank}\, [y_1\, y_2\, B],$$

then either $M \in \mathscr{MR}^F(\widetilde{G_{\mathcal{A}}}) + \mathscr{MR}^F(\widetilde{G_{\mathcal{B}}})$ and $\mathrm{mr}^F(G) = \mathrm{mr}^F(G_{\mathcal{A}}) + \mathrm{mr}^F(G_{\mathcal{B}})$ or $M \in \mathscr{MR}^F(\widetilde{H_{\mathcal{A}}}) + \mathscr{MR}^F(\widetilde{H_{\mathcal{B}}})$ and $\mathrm{mr}^F(G) = \mathrm{mr}^F(H_{\mathcal{A}}) + \mathrm{mr}^F(H_{\mathcal{B}})$.

II. *If*

$$\mathrm{rank}\, A + \mathrm{rank}\, B = \mathrm{rank}\, [A\, x_1] + \mathrm{rank}\, [y_1\, B] < \mathrm{rank}\, [A\, x_1\, x_2] + \mathrm{rank}\, [y_1\, y_2\, B] \qquad (EI_1)$$

then $M \in \mathscr{MR}^F(\widetilde{G_{\mathcal{A}} - r_2}) + \mathscr{MR}^F(\widetilde{G_{\mathcal{B}} - r_2}) + \mathscr{MR}^F(\mathrm{Star}_2(G))$ and $\mathrm{mr}^F(G) = \mathrm{mr}^F(G_{\mathcal{A}} - r_2) + \mathrm{mr}^F(G_{\mathcal{B}} - r_2) + 2$.

III. *If*

$$\mathrm{rank}\, A + \mathrm{rank}\, B = \mathrm{rank}\, [A\, x_2] + \mathrm{rank}\, [y_2\, B] < \mathrm{rank}\, [A\, x_1\, x_2] + \mathrm{rank}\, [y_1\, y_2\, B] \qquad (EI_2)$$

then $M \in \mathscr{MR}^F(\widetilde{G_{\mathcal{A}} - r_1}) + \mathscr{MR}^F(\widetilde{G_{\mathcal{B}} - r_1}) + \mathscr{MR}^F(\mathrm{Star}_1(G))$ and $\mathrm{mr}^F(G) = \mathrm{mr}^F(G_{\mathcal{A}} - r_1) + \mathrm{mr}^F(G_{\mathcal{B}} - r_1) + 2$.

IV. *If either*

$$\mathrm{rank}\, M = \mathrm{rank}\, A + \mathrm{rank}\, B + 4, \qquad (E_1)$$

or

$$\mathrm{rank}\, A + \mathrm{rank}\, B < \mathrm{rank}\, [A\, x_1] + \mathrm{rank}\, [y_1\, B] < \mathrm{rank}\, [A\, x_1\, x_2] + \mathrm{rank}\, [y_1\, y_2\, B], \qquad (I_1)$$

or

$$\mathrm{rank}\, A + \mathrm{rank}\, B < \mathrm{rank}\, [A\, x_2] + \mathrm{rank}\, [y_2\, B] < \mathrm{rank}\, [A\, x_1\, x_2] + \mathrm{rank}\, [y_1\, y_2\, B], \qquad (I_2)$$

then $M \in \mathscr{MR}^F(\widetilde{G_{\mathcal{A}} - R}) + \mathscr{MR}^F(\widetilde{G_{\mathcal{B}} - R}) + \mathscr{MR}_0^F(\mathrm{Star}_{12}(G))$ and $\mathrm{mr}^F(G) = \mathrm{mr}^F(G_{\mathcal{A}} - R) + \mathrm{mr}^F(G_{\mathcal{B}} - R) + 4$.

V. *If*

$$\mathrm{rank}\, A + \mathrm{rank}\, B < \mathrm{rank}\, [A\, x_1] + \mathrm{rank}\, [y_1\, B] = \mathrm{rank}\, [A\, x_1\, x_2] + \mathrm{rank}\, [y_1\, y_2\, B], \qquad (IE_1)$$

$$\mathrm{rank}\, A + \mathrm{rank}\, B < \mathrm{rank}\, [A\, x_2] + \mathrm{rank}\, [y_2\, B] = \mathrm{rank}\, [A\, x_1\, x_2] + \mathrm{rank}\, [y_1\, y_2\, B], \qquad (IE_2)$$

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and

$$\operatorname{rank} M \neq \operatorname{rank} A + \operatorname{rank} B + 4$$

then

$$M \in \mathscr{MR}^F(\widetilde{\overline{G_{\mathcal{A}}}}) + \mathscr{MR}^F(\widetilde{\overline{G_{\mathcal{B}}}}) + \mathscr{MR}_0^F(\mathrm{TStar}_1(G)),$$

$$M \in \mathscr{MR}^F(\widetilde{\overline{G_{\mathcal{A}}}}) + \mathscr{MR}^F(\widetilde{\overline{G_{\mathcal{B}}}}) + \mathscr{MR}_0^F(\mathrm{TStar}_2(G)),$$

and

$$\mathrm{mr}^F(G) = \mathrm{mr}^F(\overline{G_{\mathcal{A}}}) + \mathrm{mr}^F(\overline{G_{\mathcal{B}}}) + 2.$$

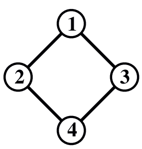
Example 4.5. Let F be a field with char $F \neq 2$. Let G be the graph in Example 1.3 with the same 2-separation $(G_{\mathcal{A}}, G_{\mathcal{B}})$. The following matrix is an example where part I applies.

$$M = \left[\begin{array}{c|cc|ccc} 1 & 1 & 1 & 0 & 0 & 0 \\ \hline 1 & 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 & 1 & 1 \\ \hline 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

Notice that $A = M[1]$, and $B = M[4, 5, 6]$. Also x_1, x_2 appear as columns in A, while y_1, y_2 appear in B. Thus M satisfies the hypothesis of part I.

In this case $M \in \mathscr{MR}^F(\widetilde{G_{\mathcal{A}}}) + \mathscr{MR}^F(\widetilde{G_{\mathcal{B}}})$ and M can be decomposed as shown in Example 1.3.

Example 4.6. The following graph and matrix provide another example where part I applies.

G: 

$$M = \left[\begin{array}{c|cc|c} 1 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ \hline 0 & -1 & -1 & -1 \end{array}\right]$$

The 2-separation is $G - \{4\}$ and $G - \{1\}$. The matrices A and B are both 1×1 matrices corresponding to $M[1]$ and $M[4]$, respectively. It is easily verified M satisfies the hypothesis of part I.

In this case $M \in \mathscr{MR}^F(\widetilde{H_{\mathcal{A}}}) + \mathscr{MR}^F(\widetilde{H_{\mathcal{B}}})$ and can be decomposed into the following matrices corresponding to $\widetilde{H_{\mathcal{A}}}$ and $\widetilde{H_{\mathcal{B}}}$, each of which is isomorphic to $K_3 \cup K_1$.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \end{bmatrix}$$

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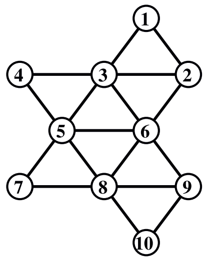
Example 4.7. Let F be a field with char $F \neq 2$. The following graph and matrix give an example where part II applies.

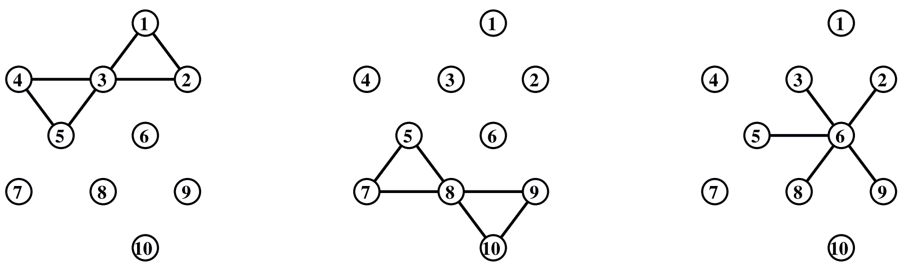
G:



$$M = \left[\begin{array}{cccc|cc|cccc}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 1 & 1 & 2 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{array}\right]$$

We have rank $M = 6 = \mathrm{mr}(G)$. The 2-separation is the graph induced by the first 6 vertices and the graph induced by the last 6 vertices excluding edge $\{56\}$ as in Example 2.8. The matrices A and B are respectively $M[1, 2, 3, 4]$ and $M[7, 8, 9, 10]$. In this case x_1 appears as a column of A and y_1 appears in B. Thus the equality in (EI_1) is satisfied. Further x_2 is not in the column space of A, which justifies the inequality in (EI_1).

The conclusion states that M can be decomposed into 3 matrices corresponding to the graphs $\widetilde{G_{\mathcal{A}}} - r_2$, $\widetilde{G_{\mathcal{B}}} - r_2$, and $\mathrm{Star}_2(G)$. The graphs and corresponding matrices are given below.



$$\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

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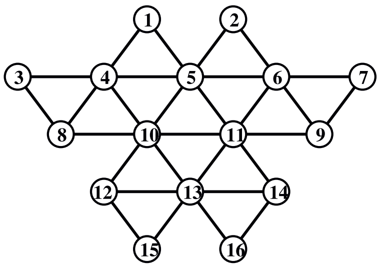
Example 4.8. Let F be a field with char $F \neq 2$. The following graph and matrix provide an example where part IV applies.

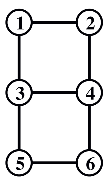
G: 

$$M = \left[\begin{array}{ccccccccc|cc|ccccc}
1&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&1&1&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&0&0&0&1&0&0&0&0&0&0&0&0\\
1&0&1&2&1&0&0&1&0&1&0&0&0&0&0&0\\
1&1&0&1&2&1&0&0&0&1&1&0&0&0&0&0\\
0&1&0&1&0&2&1&0&1&0&1&0&0&0&0&0\\
0&0&0&0&0&1&1&0&1&0&0&0&0&0&0&0\\
0&0&1&1&0&0&0&1&0&1&0&0&0&0&0&0\\
0&0&0&0&0&1&1&0&1&0&1&0&0&0&0&0\\
\hline
0&0&0&1&1&0&0&1&0&0&1&1&1&0&0&0\\
0&0&0&0&1&1&0&0&1&1&0&0&1&1&0&0\\
\hline
0&0&0&0&0&0&0&0&0&1&0&1&1&0&1&0\\
0&0&0&0&0&0&0&0&0&1&1&1&2&1&1&1\\
0&0&0&0&0&0&0&0&0&0&1&0&1&1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&1&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&1
\end{array}\right]$$

We have rank $M = 10 = \text{mr}(G)$. The 2-separation is illustrated in Example 2.8. Note that $A = M[1, 2, 3, 4, 5, 6, 7, 8, 9]$ and $B = M[12, 13, 14, 15, 16]$. The matrix B has 3 distinct columns, and the second column is the sum of the first and third columns. Thus rank $B = 2$. Since y_1, y_2, the first column of B, and the third column of B form a linearly independent set, rank $[y_1\ y_2\ B] = 4$. Therefore rank $B <$ rank $[y_1\ B] <$ rank $[y_1\ y_2\ B]$ and (I_1) follows.

The conclusion states that M can be decomposed into $\widetilde{A} = \begin{bmatrix} A&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}$, $\widetilde{B} = \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&B \end{bmatrix}$, and $S = M - \widetilde{A} - \widetilde{B}$ corresponding to $\widetilde{G_{\mathcal{A}} - R}$, $\widetilde{G_{\mathcal{B}} - R}$, and $\text{Star}_{12}(G)$, respectively.

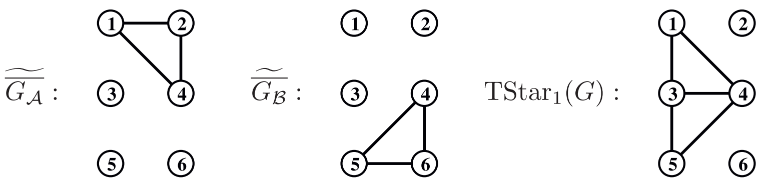
Example 4.9. Let F be a field with char $F \neq 2$. The following graph and matrix provide an example where part V applies.

G: 

$$M = \left[\begin{array}{cc|cc|cc}
1&1&1&0&0&0\\
1&1&0&1&0&0\\
\hline
1&0&2&-1&1&0\\
0&1&-1&2&0&1\\
\hline
0&0&1&0&1&1\\
0&0&0&1&1&1
\end{array}\right]$$

We have rank $M = 4 = \mathrm{mr}(G)$. The 2-separation is the four cycle induced by $\{1, 2, 3, 4\}$ and the path with edges $\{35, 56, 64\}$. Note that the matrices A and B corresponding to this 2-separation are both the 2×2 all 1's matrix. Thus rank A = rank B = 1. Neither x_1 nor x_2 is in the column space of A justifying the inequality in both (IE_1) and (IE_2). Also $x_1 + x_2$ is in the column space of A and $y_1 + y_2$ is in the column space of B justifying the equality in both (IE_1) and (IE_2). Lastly we note that rank $M = 4 \neq \mathrm{rank}\, A + \mathrm{rank}\, B + 4 = 6$.

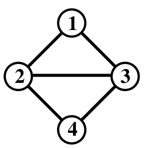
The conclusion of part V states that M can be decomposed into 3 matrices corresponding to



The matrices are

$$\begin{bmatrix} 1&1&0&1&0&0\\ 1&1&0&1&0&0\\ 0&0&0&0&0&0\\ 1&1&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&1&1\\ 0&0&0&1&1&1\\ 0&0&0&1&1&1 \end{bmatrix}, \begin{bmatrix} 0&0&1&-1&0&0\\ 0&0&0&0&0&0\\ 1&0&2&-1&1&0\\ -1&0&-1&0&-1&0\\ 0&0&1&-1&0&0\\ 0&0&0&0&0&0 \end{bmatrix}.$$

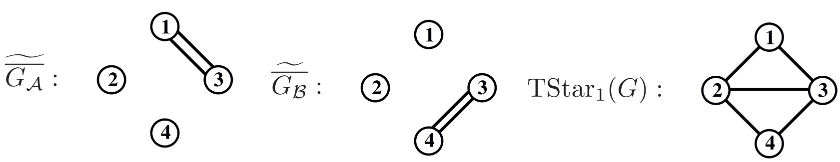
Example 4.10. We note that in rare instances the decomposition may be trivial. Consider the following graph and matrix

G: 

$$M = \left[\begin{array}{c|cc|c} 0&1&1&0\\ \hline 1&1&1&1\\ 1&1&1&1\\ \hline 0&1&1&0 \end{array}\right]$$

with 2-separation consisting of the clique induced by vertices $\{1, 2, 3\}$ and the path with edges $\{24, 34\}$. The matrices A and B are both 1×1 matrices corresponding to $M[1]$ and $M[4]$, respectively. It is easily verified M satisfies the hypothesis of part V.

The conclusion of part V states that M can be decomposed into 3 matrices corresponding to



The only matrix in either $\mathcal{MR}(\widetilde{\widetilde{G_{\mathcal{A}}}})$ or $\mathcal{MR}(\widetilde{\widetilde{G_{\mathcal{B}}}})$ is the zero matrix which leads to the trivial decomposition, $M = 0 + 0 + M$.

We require several lemmas before giving the proof of Theorem 4.4. The next one follows immediately from Theorem 2.6, but we give an independent proof that aligns with the proof we will give of Theorem 4.4.

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Lemma 4.11. *Let F be any field. Given G, $G_{\mathcal{A}}$, $G_{\mathcal{B}}$, $H_{\mathcal{A}}$, $H_{\mathcal{B}}$, $\overline{G_{\mathcal{A}}}$, $\overline{G_{\mathcal{B}}}$ as in Definition 4.1,*

$$\begin{aligned}\mathrm{mr}^F(G) \leqslant \min\{&\mathrm{mr}^F(G_{\mathcal{A}}) + \mathrm{mr}^F(G_{\mathcal{B}}),\\ &\mathrm{mr}^F(H_{\mathcal{A}}) + \mathrm{mr}^F(H_{\mathcal{B}}),\\ &\mathrm{mr}^F(\overline{G_{\mathcal{A}}}) + \mathrm{mr}^F(\overline{G_{\mathcal{B}}}) + 2,\\ &\mathrm{mr}^F(G_{\mathcal{A}} - r_1) + \mathrm{mr}^F(G_{\mathcal{B}} - r_1) + 2,\\ &\mathrm{mr}^F(G_{\mathcal{A}} - r_2) + \mathrm{mr}^F(G_{\mathcal{B}} - r_2) + 2,\\ &\mathrm{mr}^F(G_{\mathcal{A}} - R) + \mathrm{mr}^F(G_{\mathcal{B}} - R) + 4\}.\end{aligned}$$

Proof. If $r_1r_2 \in E(G)$, we assume that $r_1r_2 \in E(G_{\mathcal{A}})$.

1. Let

$$M_1 = \begin{bmatrix} A & x_1 & x_2 \\ x_1^T & a & b \\ x_2^T & b & c \end{bmatrix} \in \mathcal{S}(G_{\mathcal{A}}) \text{ with rank } M_1 = \mathrm{mr}(G_{\mathcal{A}})$$

and

$$M_2 = \begin{bmatrix} h & 0 & y_1^T \\ 0 & k & y_1^T \\ y_1 & y_2 & B \end{bmatrix} \in \mathcal{S}(G_{\mathcal{B}}) \text{ with rank } M_2 = \mathrm{mr}(G_{\mathcal{B}}).$$

Then

$$M = \begin{bmatrix} A & x_1 & x_2 & 0 \\ x_1^T & a+h & b & y_1^T \\ x_2^T & b & c+k & y_2^T \\ 0 & y_1 & y_2 & B \end{bmatrix} = \begin{bmatrix} A & x_1 & x_2 & 0 \\ x_1^T & a & b & 0 \\ x_2^T & b & c & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h & 0 & y_1^T \\ 0 & 0 & k & y_2^T \\ 0 & y_1 & y_2 & B \end{bmatrix} \in \mathcal{S}(G)$$

whether or not $r_1r_2 \in E(G)$. Then

$$\mathrm{mr}(G) \leqslant \mathrm{rank}\, M \leqslant \mathrm{rank}\, M_1 + \mathrm{rank}\, M_2 = \mathrm{mr}(G_{\mathcal{A}}) + \mathrm{mr}(G_{\mathcal{B}}).$$

2. Let

$$M_1 = \begin{bmatrix} A & x_1 & x_2 \\ x_1^T & a & b \\ x_2^T & b & c \end{bmatrix} \in \mathcal{S}(H_{\mathcal{A}}) \text{ with rank } M_1 = \mathrm{mr}(H_{\mathcal{A}})$$

and

$$M_2 = \begin{bmatrix} r & s & y_1^T \\ s & t & y_2^T \\ y_1 & y_2 & B \end{bmatrix} \in \mathcal{S}(H_{\mathcal{B}}) \text{ with rank } M_2 = \mathrm{mr}(H_{\mathcal{B}})$$

(so $s \neq 0$).

Subcase 1. $r_1r_2 \notin E(G)$.
Then $r_1r_2 \in E(H_{\mathcal{A}})$ so $b \neq 0$.
Let

$$M = s\begin{bmatrix} A & x_1 & x_2 & 0 \\ x_1^T & a & b & 0 \\ x_2^T & b & c & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - b\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & r & s & y_1^T \\ 0 & s & t & y_2^T \\ 0 & y_1 & y_2 & B \end{bmatrix}.$$

Then $M \in \mathcal{S}(G)$ and $\mathrm{mr}(G) \leqslant \mathrm{rank}\, M \leqslant \mathrm{rank}\, M_1 + \mathrm{rank}\, M_2 = \mathrm{mr}(H_{\mathcal{A}}) + \mathrm{mr}(H_{\mathcal{B}})$.

Subcase 2. $r_1r_2 \in E(G)$.
Then there is a double edge between r_1 and r_2 in $H_{\mathcal{A}}$. For $F \neq F_2$, b may or may not be 0, while for $F = F_2$, $b = 0$. Since $s \neq 0$, there is a nonzero k such that $kb + s \neq 0$. Let

$$M = k\begin{bmatrix} A & x_1 & x_2 & 0 \\ x_1^T & a & b & 0 \\ x_2^T & b & c & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & r & s & y_1^T \\ 0 & s & t & y_2^T \\ 0 & y_1 & y_2 & B \end{bmatrix}.$$

Then $M \in \mathcal{S}(G)$ and $\mathrm{mr}(G) \leqslant \mathrm{rank}\, M \leqslant \mathrm{rank}\, M_1 + \mathrm{rank}\, M_2 = \mathrm{mr}(H_{\mathcal{A}}) + \mathrm{mr}(H_{\mathcal{B}})$.

3. The multigraph $\overline{G}$, obtained from G by identifying r_1 and r_2, is the vertex sum at v of $\overline{G_{\mathcal{A}}}$ and $\overline{G_{\mathcal{B}}}$. By Lemma 2.9 and Theorem 2.3, $\mathrm{mr}(G) \leqslant \mathrm{mr}(\overline{G}) + 2 \leqslant \mathrm{mr}(\overline{G_{\mathcal{A}}}) + \mathrm{mr}(\overline{G_{\mathcal{B}}}) + 2$.

4. Let

$$M_1 = \begin{bmatrix} A & x_2 \\ x_2^T & c_1 \end{bmatrix} \in \mathcal{S}(G_{\mathcal{A}} - r_1) \text{ with } \mathrm{rank}\, M_1 = \mathrm{mr}(G_{\mathcal{A}} - r_1)$$

and

$$M_2 = \begin{bmatrix} c_2 & y_2^T \\ y_2 & B \end{bmatrix} \in \mathcal{S}(G_{\mathcal{B}} - r_1) \text{ with } \mathrm{rank}\, M_2 = \mathrm{mr}(G_{\mathcal{B}} - r_1).$$

Then

$$M = \begin{bmatrix} A & x_2 & 0 \\ x_2^T & c_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & c_2 & y_2^T \\ 0 & y_2 & B \end{bmatrix} \in \mathcal{S}(G - r_1)$$

and by Proposition 2.10, $\mathrm{mr}(G) \leqslant \mathrm{mr}(G - r_1) + 2 \leqslant \mathrm{rank}\, M + 2 \leqslant \mathrm{rank}\, M_1 + \mathrm{rank}\, M_2 + 2 = \mathrm{mr}(G_{\mathcal{A}} - r_1) + \mathrm{mr}(G_{\mathcal{B}} - r_1) + 2$.

By the same argument, $\mathrm{mr}(G) \leqslant \mathrm{mr}(G_{\mathcal{A}} - r_2) + \mathrm{mr}(G_{\mathcal{B}} - r_2) + 2$.

5. We have $\mathrm{mr}(G) \leqslant \mathrm{mr}(G - r_1) + 2 \leqslant \mathrm{mr}(G - r_1 - r_2) + 4 = \mathrm{mr}(G - R) + 4 = \mathrm{mr}((G_{\mathcal{A}} - R) \cup (G_{\mathcal{B}} - R)) + 4 = \mathrm{mr}(G_{\mathcal{A}} - R) + \mathrm{mr}(G_{\mathcal{B}} - R) + 4$. $\square$

Lemma 4.12. *Let F be a field, let G be a graph on 3 or more vertices, let u, v be vertices of G and let $\overline{G}$ be the multigraph obtained by identifying u and v. Let*

$$M = \begin{bmatrix} A & x & y \\ x^T & a & b \\ y^T & b & c \end{bmatrix} \in \mathcal{S}^F(G)$$

where the last 2 rows and columns are associated with u and v. Then for any nonzero h,k and any scalar a',

$$\overline{M} = \begin{bmatrix} A & hx + ky \\ hx^T + ky^T & a' \end{bmatrix} \in \mathcal{S}^F(\overline{G})$$

Proof. Let $i \neq u, v$ be a vertex of G. Then

$$hx_i + ky_i \text{ is } \begin{cases} 0 & \text{if neither } iu \text{ nor } iv \text{ is an edge of } G \\ \text{nonzero} & \text{if exactly one of } iu, iv \text{ is an edge of } G \\ 0 \text{ or nonzero} & \text{if } iu, iv \text{ are both edges of } G \text{ and } F \neq F_2 \\ 0 & \text{if } iu, iv \text{ are both edges of } G \text{ and } F = F_2. \end{cases}$$

It follows that $\overline{M} \in \mathcal{S}^F(\overline{G})$. □

Lemma 4.13. *Let F be a field. Given G as in Definition 4.1, let*

$$M = \begin{bmatrix} A & x_1 & x_2 & 0 \\ x_1^T & a & b & y_1^T \\ x_2^T & b & c & y_2^T \\ 0 & y_1 & y_2 & B \end{bmatrix} \in \mathcal{MR}^F(G),$$

$$K_1 = \begin{bmatrix} A & x_1 & 0 \\ x_1^T & a & y_1^T \\ 0 & y_1 & B \end{bmatrix}, \quad \text{and} \quad K_2 = \begin{bmatrix} A & x_2 & 0 \\ x_2^T & c & y_2^T \\ 0 & y_2 & B \end{bmatrix}.$$

If $\operatorname{rank}[A\,x_1] + \operatorname{rank}[y_1\,B] < \operatorname{rank}[A\,x_1\,x_2] + \operatorname{rank}[y_1\,y_2\,B]$,
then $K_1 \in \mathcal{MR}^F(G - r_2)$, $\operatorname{rank} K_1 = \operatorname{rank} M - 2$, and $\operatorname{mr}^F(G) = \operatorname{mr}^F(G - r_2) + 2$,
while if $\operatorname{rank}[A\,x_2] + \operatorname{rank}[y_2\,B] < \operatorname{rank}[A\,x_1\,x_2] + \operatorname{rank}[y_1\,y_2\,B]$,
then $K_2 \in \mathcal{MR}^F(G - r_1)$, $\operatorname{rank} K_2 = \operatorname{rank} M - 2$, and $\operatorname{mr}^F(G) = \operatorname{mr}^F(G - r_1) + 2$.

Proof. It suffices to prove the first claim. Then either $\operatorname{rank}[A\,x_1] < \operatorname{rank}[A\,x_1\,x_2]$ or else $\operatorname{rank}[y_1\,B] < \operatorname{rank}[y_1\,y_2\,B]$. Without loss of generality assume $\operatorname{rank}[A\,x_1] < \operatorname{rank}[A\,x_1\,x_2]$. Since $x_2 \notin C([A\,x_1])$, $\operatorname{rank} K_1 < \operatorname{rank} M$. By Propositions 2.11 and 2.10,

$$\operatorname{mr}(G - r_2) \leqslant \operatorname{rank} K_1 = \operatorname{rank} M - 2 = \operatorname{mr}(G) - 2 \leqslant \operatorname{mr}(G - r_2).$$

Therefore $\operatorname{mr}(G - r_2) = \operatorname{rank} K_1 = \operatorname{rank} M - 2 = \operatorname{mr}(G) - 2$ and $K_1 \in \mathcal{MR}(G - r_2)$. □

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Lemma 4.14. *Let F be a field. Given G as in Definition 4.1, let*

$$M = \begin{bmatrix} A & x_1 & x_2 & 0 \\ x_1^T & a & b & y_1^T \\ x_2^T & b & c & y_2^T \\ 0 & y_1 & y_2 & B \end{bmatrix} \in \mathscr{MR}^F(G).$$

Then $\operatorname{rank} M - \operatorname{rank} A - \operatorname{rank} B$ *equals* $4, 2, 1,$ *or* 0.

Proof. Let K_1 be as in Lemma 4.13. By Proposition 2.11, $\operatorname{rank} K_1$ is either $\operatorname{rank} M$ or $\operatorname{rank} M - 2$.

Case 1. $\operatorname{rank} K_1 = \operatorname{rank} M - 2$

Then $K_1 \in \mathscr{MR}(G - r_2)$ and again by Proposition 2.11 the $\operatorname{rank} K_1 - \operatorname{rank} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is either 0 or 2. Then

$$\operatorname{rank} M - \operatorname{rank} A - \operatorname{rank} B = (\operatorname{rank} M - \operatorname{rank} K_1) + \left(\operatorname{rank} K_1 - \operatorname{rank} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right)$$

which is either $2 + 0$ or $2 + 2$; i.e., 2 or 4.

Case 2. $\operatorname{rank} K_1 = \operatorname{rank} M$

Then $\operatorname{rank} K_1 - \operatorname{rank} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is 0,1, or 2 and

$$\operatorname{rank} M - \operatorname{rank} A - \operatorname{rank} B = \operatorname{rank} K_1 - \operatorname{rank} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

which is 0,1, or 2. □

Proof. (Of Theorem 4.4)
The five hypotheses of I, II, III, IV, V are mutually exclusive and exhaustive.
We adopt the same convention as in the proof of Lemma 4.11 that if $r_1r_2 \in E(G)$ then $r_1r_2 \in E(G_{\mathcal{A}})$.

I. Assume that $\operatorname{rank} A + \operatorname{rank} B = \operatorname{rank}\,[A\, x_1\, x_2] + \operatorname{rank}\,[y_1\, y_2\, B]$. It follows that $x_1, x_2 \in C(A)$ and $y_1, y_2 \in C(B)$. Then there are vectors u_1, v_1, u_2, v_2 such that $x_1 = Au_1, y_1 = Bv_1, x_2 = Au_2, y_2 = Bv_2$ and hence $M = \begin{bmatrix} A & Au_1 & Au_2 & 0 \\ u_1^TA & a & b & v_1^TB \\ u_2^TA & b & c & v_2^TB \\ 0 & Bv_1 & Bv_2 & B \end{bmatrix}$. It is straightforward that M is row and column equivalent to

$$R = \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & a - u_1^TAu_1 - v_1^TBv_1 & b - u_1^TAu_2 - v_1^TBv_2 & 0 \\ 0 & b - u_2^TAu_1 - v_2^TBv_1 & c - u_2^TAu_2 - v_2^TBv_2 & 0 \\ 0 & 0 & 0 & B \end{bmatrix}.$$

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So rank M = rank R = rank A + rank B + rank T where

$$T = \begin{bmatrix} a - u_1^T A u_1 - v_1^T B v_1 & b - u_1^T A u_2 - v_1^T B v_2 \\ b - u_2^T A u_1 - v_2^T B v_1 & c - u_2^T A u_2 - v_2^T B v_2 \end{bmatrix}.$$

We now explain why rank T is 0 or 1. If the off-diagonal entry of T is 0 then since $M \in \mathscr{MR}(G)$, the two diagonal entries of T must be 0. And if the off-diagonal entry of T is nonzero then since a and c could be chosen to make $\det T = 0$ and since $M \in \mathscr{MR}(G)$ it must be the case that they were so chosen.

Case 1. T is the zero matrix. Then rank M = rank A + rank B and $a = u_1^T A u_1 + v_1^T B v_1$, $b = u_1^T A u_2 + v_1^T B v_2$ $(= u_2^T A u_1 + v_2^T B v_1)$ and $c = u_2^T A u_2 + v_2^T B v_2$.

$$\text{So } M = \begin{bmatrix} A & Au_1 & Au_2 & 0 \\ u_1^T A & u_1^T A u_1 + v_1^T B v_1 & u_1^T A u_2 + v_1^T B v_2 & v_1^T B \\ u_2^T A & u_2^T A u_1 + v_2^T B v_1 & u_2^T A u_2 + v_2^T B v_2 & v_2^T B \\ 0 & Bv_1 & Bv_2 & B \end{bmatrix}.$$

$$\text{Let } \widetilde{M_\mathcal{A}} = \begin{bmatrix} A & Au_1 & Au_2 & 0 \\ u_1^T A & u_1^T A u_1 & u_1^T A u_2 & 0 \\ u_2^T A & u_2^T A u_1 & u_2^T A u_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \widetilde{M_\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & v_1^T B v_1 & v_1^T B v_2 & v_1^T B \\ 0 & v_2^T B v_1 & v_2^T B v_2 & v_2^T B \\ 0 & Bv_1 & Bv_2 & B \end{bmatrix}.$$

Then $M = \widetilde{M_\mathcal{A}} + \widetilde{M_\mathcal{B}}$, rank $\widetilde{M_\mathcal{A}}$ = rank A and rank $\widetilde{M_\mathcal{B}}$ = rank B.

Subcase 1. $v_1^T B v_2 = 0$.

Then $u_1^T A u_2 = b \neq 0$ if and only if $r_1 r_2 \in E(G)$ if and only if $r_1 r_2 \in E(\widetilde{G_\mathcal{A}})$. Then $\widetilde{M_\mathcal{A}} \in \mathcal{S}(\widetilde{G_\mathcal{A}})$ and $\widetilde{M_\mathcal{B}} \in \mathcal{S}(\widetilde{G_\mathcal{B}})$. By Lemma 4.11,

$$\begin{aligned} \text{mr}(G_\mathcal{A}) + \text{mr}(G_\mathcal{B}) &\geqslant \text{mr}(G) = \text{rank}\, M = \text{rank}\, A + \text{rank}\, B \\ &= \text{rank}\, \widetilde{M_\mathcal{A}} + \text{rank}\, \widetilde{M_\mathcal{B}} \geqslant \text{mr}(\widetilde{G_\mathcal{A}}) + \text{mr}(\widetilde{G_\mathcal{B}}) = \text{mr}(G_\mathcal{A}) + \text{mr}(G_\mathcal{B}). \end{aligned}$$

It follows that rank $\widetilde{M_\mathcal{A}} = \text{mr}(\widetilde{G_\mathcal{A}})$ and rank $\widetilde{M_\mathcal{B}} = \text{mr}(\widetilde{G_\mathcal{B}})$ so $\widetilde{M_\mathcal{A}} \in \mathscr{MR}(\widetilde{G_\mathcal{A}})$, $\widetilde{M_\mathcal{B}} \in \mathscr{MR}(\widetilde{G_\mathcal{B}})$ and $M \in \mathscr{MR}(\widetilde{G_\mathcal{A}}) + \mathscr{MR}(\widetilde{G_\mathcal{B}})$. Furthermore $\text{mr}(G) = \text{mr}(G_\mathcal{A}) + \text{mr}(G_\mathcal{B})$.

Subcase 2. $v_1^T B v_2 \neq 0$.

Then $\widetilde{M_\mathcal{B}} \in \mathcal{S}(\widetilde{H_\mathcal{B}})$. We claim that $\widetilde{M_\mathcal{A}} \in \mathcal{S}(\widetilde{H_\mathcal{A}})$ also. For if $r_1 r_2 \notin E(G)$, $r_1 r_2 \in E(\widetilde{H_\mathcal{A}})$ and $u_1^T A u_2 = -v_1^T B v_2 \neq 0$. If $r_1 r_2 \in E(G)$, there is a double edge between r_1 and r_2 in $\widetilde{H_\mathcal{A}}$. Here $u_1^T A u_2$ may be zero or nonzero for any $F \neq F_2$. But if $F = F_2$,

$$1 = b = u_1^T A u_2 + v_1^T A v_2 = u_1^T A u_2 + 1$$

and $u_1^T A u_2 = 0$. So in either case $\widetilde{M_\mathcal{A}} \in \mathcal{S}(\widetilde{H_\mathcal{A}})$. By Lemma 4.11,

$$\text{mr}(H_\mathcal{A}) + \text{mr}(H_\mathcal{B}) \geqslant \text{mr}(G) = \text{rank}\, M = \text{rank}\, A + \text{rank}\, B$$

$$= \text{rank}\, \widetilde{M_\mathcal{A}} + \text{rank}\, \widetilde{M_\mathcal{B}} \geqslant \text{mr}(\widetilde{H_\mathcal{A}}) + \text{mr}(\widetilde{H_\mathcal{B}}) = \text{mr}(H_\mathcal{A}) + \text{mr}(H_\mathcal{B}).$$

It follows that rank $\widetilde{M_\mathcal{A}} = \text{mr}(\widetilde{H_\mathcal{A}})$ and rank $\widetilde{M_\mathcal{B}} = \text{mr}(\widetilde{H_\mathcal{B}})$ so $\widetilde{M_\mathcal{A}} \in \mathscr{MR}(\widetilde{H_\mathcal{A}})$, $\widetilde{M_\mathcal{B}} \in \mathscr{MR}(\widetilde{H_\mathcal{B}})$, and $M \in \mathscr{MR}(\widetilde{H_\mathcal{A}}) + \mathscr{MR}(\widetilde{H_\mathcal{B}})$. Furthermore $\text{mr}(G) = \text{mr}(H_\mathcal{A}) + \text{mr}(H_\mathcal{B})$.

Case 2. Now assume rank $T = 1$. Then rank M = rank A + rank B + 1. Let

$$\widetilde{M_{\mathcal{A}}} = \begin{bmatrix} A & Au_1 & Au_2 & 0 \\ u_1^T A & a - v_1^T Bv_1 & b - v_1^T Bv_2 & 0 \\ u_2^T A & b - v_2^T Bv_1 & c - v_2^T Bv_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \widetilde{M_{\mathcal{B}}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & v_1^T Bv_1 & v_1^T Bv_2 & v_1^T B \\ 0 & v_2^T Bv_1 & v_2^T Bv_2 & v_2^T B \\ 0 & Bv_1 & Bv_2 & B \end{bmatrix}.$$

Then $M = \widetilde{M_{\mathcal{A}}} + \widetilde{M_{\mathcal{B}}}$, rank $\widetilde{M_{\mathcal{B}}} =$ rank B and rank $\widetilde{M_{\mathcal{A}}} =$ rank $A + 1$ since $\widetilde{M_{\mathcal{A}}}$ is row and column equivalent to $A \oplus T$. So

$$\text{mr}(G) = \text{rank}\, M = \text{rank}\, A + 1 + \text{rank}\, B = \text{rank}\, \widetilde{M_{\mathcal{A}}} + \text{rank}\, \widetilde{M_{\mathcal{B}}}, \tag{E_2}$$

Subcase 1. $v_1^T Bv_2 = 0$.
Then $b - v_1^T Bv_2 = b$ and whether or not $b = 0$, $\widetilde{M_{\mathcal{A}}} \in \mathcal{S}(\widetilde{G_{\mathcal{A}}})$ and $\widetilde{M_{\mathcal{B}}} \in \mathcal{S}(\widetilde{G_{\mathcal{B}}})$. Then by ($E_2$) and Lemma 4.11,

$$\text{mr}(G) = \text{rank}\, \widetilde{M_{\mathcal{A}}} + \text{rank}\, \widetilde{M_{\mathcal{B}}} \geqslant \text{mr}(\widetilde{G_{\mathcal{A}}}) + \text{mr}(\widetilde{G_{\mathcal{B}}}) = \text{mr}(G_{\mathcal{A}}) + \text{mr}(G_{\mathcal{B}}) \geqslant \text{mr}(G).$$

Then we have $\widetilde{M_{\mathcal{A}}} \in \mathcal{MR}(\widetilde{G_{\mathcal{A}}})$, $\widetilde{M_{\mathcal{B}}} \in \mathcal{MR}(\widetilde{G_{\mathcal{B}}})$ and $\text{mr}(G) = \text{mr}(G_{\mathcal{A}}) + \text{mr}(G_{\mathcal{B}})$.

Subcase 2. $v_1^T Bv_2 \neq 0$.
Then $\widetilde{M_{\mathcal{B}}} \in \mathcal{MR}(\widetilde{H_{\mathcal{B}}})$. If $r_1 r_2 \notin E(G)$, $r_1 r_2 \in E(\widetilde{H_{\mathcal{A}}})$; also $b = 0 \Rightarrow b - v_1^T Bv_2 \neq 0$ and $\widetilde{M_{\mathcal{A}}} \in \mathcal{S}(\widetilde{H_{\mathcal{A}}})$. If $r_1 r_2 \in E(G)$, there is a double edge between r_1 and r_2 in $\widetilde{H_{\mathcal{A}}}$. We only need to check the case in which $F = F_2$ and in that case $b = 1 = v_1^T Bv_2 \Rightarrow b - v_1^T Bv_2 = 0$. So $M_{\mathcal{A}} \in \mathcal{S}(\widetilde{H_{\mathcal{A}}})$. Again by ($E_2$) and Lemma 4.11,

$$\text{mr}(G) = \text{rank}\, \widetilde{M_{\mathcal{A}}} + \text{rank}\, \widetilde{M_{\mathcal{B}}} \geqslant \text{mr}(\widetilde{H_{\mathcal{A}}}) + \text{mr}(\widetilde{H_{\mathcal{B}}}) = \text{mr}(H_{\mathcal{A}}) + \text{mr}(H_{\mathcal{B}}) \geqslant \text{mr}(G).$$

Now $\widetilde{M_{\mathcal{A}}} \in \mathcal{MR}(\widetilde{H_{\mathcal{A}}})$, $\widetilde{M_{\mathcal{B}}} \in \mathcal{MR}(\widetilde{H_{\mathcal{B}}})$ and $\text{mr}(G) = \text{mr}(H_{\mathcal{A}}) + \text{mr}(H_{\mathcal{B}})$.

II. Now assume (EI_1):

$$\text{rank}\, A + \text{rank}\, B = \text{rank}\, [A\, x_1] + \text{rank}\, [y_1\, B] < \text{rank}\, [A\, x_1\, x_2] + \text{rank}\, [y_1\, y_2\, B].$$

Because of the inequality, we can apply Lemma 4.13 to conclude

$$K_1 = \begin{bmatrix} A & x_1 & 0 \\ x_1^T & a & y_1^T \\ 0 & y_1 & B \end{bmatrix} \in \mathcal{MR}(G - r_2), \quad \text{rank}\, K_1 = \text{rank}\, M - 2,$$

and $\text{mr}(G) = \text{mr}(G - r_2) + 2$.

The equality implies that $x_1 \in C(A)$ and $y_1 \in C(B)$. Write $x_1 = Au_1$ and $y_1 = Bv_1$ so that

$$K_1 = \begin{bmatrix} A & Au_1 & 0 \\ u_1^T A & a & v_1^T B \\ 0 & Bv_1 & B \end{bmatrix}.$$

Since K_1 is row and column equivalent to

$$R_1 = \begin{bmatrix} A & 0 & 0 \\ 0 & a - u_1^T Au_1 + v_1^T Bv_1 & 0 \\ 0 & 0 & B \end{bmatrix}, \text{rank}\, K_1 = \text{rank}\, R_1.$$

Also, since K_1 is a minimum rank matrix, we must have $a - u_1^T Au_1 - v_1^T Bv_1 = 0$. Then rank $M =$ rank $K_1 + 2 =$ rank $R_1 + 2 =$ rank $A +$ rank $B + 2$. Now let

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$$\widetilde{M_{\mathcal{A}}} = \begin{bmatrix} A & Au_1 & 0 & 0 \\ u_1^TA & u_1^TAu_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \widetilde{M_{\mathcal{B}}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & v_1^TBv_1 & 0 & v_1^TB \\ 0 & 0 & 0 & 0 \\ 0 & Bv_1 & 0 & B \end{bmatrix},$$

$$\text{and } S_2 = \begin{bmatrix} 0 & 0 & x_2 & 0 \\ 0 & 0 & b & 0 \\ x_2^T & b & c & y_2^T \\ 0 & 0 & y_2 & 0 \end{bmatrix}.$$

Then $\widetilde{M_{\mathcal{A}}} \in \mathcal{S}(\widetilde{G_{\mathcal{A}} - r_2})$, $\operatorname{rank} \widetilde{M_{\mathcal{A}}} = \operatorname{rank} A$, $\widetilde{M_{\mathcal{B}}} \in \mathcal{S}(\widetilde{G_{\mathcal{B}} - r_2})$, $\operatorname{rank} \widetilde{M_{\mathcal{B}}} = \operatorname{rank} B$, $S_2 \in \mathcal{S}(\text{Star}_2(G))$, and $M = \widetilde{M_{\mathcal{A}}} + \widetilde{M_{\mathcal{B}}} + S_2$. Also, since r_1 is a cut vertex for $G - r_2$ and $G - r_2$ is the vertex sum at r_1 of $G_{\mathcal{A}} - r_2$ and $G_{\mathcal{B}} - r_2$, we have by Theorem 2.3, $\operatorname{mr}(G - r_2) \leqslant \operatorname{mr}(G_{\mathcal{A}} - r_2) + \operatorname{mr}(G_{\mathcal{B}} - r_2)$. Therefore,

$$\begin{aligned} \operatorname{mr}(G) &= \operatorname{mr}(G - r_2) + 2 \leqslant \operatorname{mr}(G_{\mathcal{A}} - r_2) + \operatorname{mr}(G_{\mathcal{B}} - r_2) + 2 \\ &= \operatorname{mr}(\widetilde{G_{\mathcal{A}} - r_2}) + \operatorname{mr}(\widetilde{G_{\mathcal{B}} - r_2}) + 2 \leqslant \operatorname{rank} \widetilde{M_{\mathcal{A}}} + \operatorname{rank} \widetilde{M_{\mathcal{B}}} + 2 \\ &= \operatorname{rank} A + \operatorname{rank} B + 2 = \operatorname{rank} M = \operatorname{mr}(G). \end{aligned}$$

So $\operatorname{mr}(G) = \operatorname{mr}(G_{\mathcal{A}} - r_2) + \operatorname{mr}(G_{\mathcal{B}} - r_2) + 2$, $\operatorname{rank} \widetilde{M_{\mathcal{A}}} = \operatorname{mr}(\widetilde{G_{\mathcal{A}} - r_2})$, and $\operatorname{rank} \widetilde{M_{\mathcal{B}}} = \operatorname{mr}(\widetilde{G_{\mathcal{B}} - r_2})$. Thus $\widetilde{M_{\mathcal{A}}} \in \mathcal{MR}(\widetilde{G_{\mathcal{A}} - r_2})$ and $\widetilde{M_{\mathcal{B}}} \in \mathcal{MR}(\widetilde{G_{\mathcal{B}} - r_2})$.

Finally, by Proposition 4.3, $\operatorname{mr}(\text{Star}_2(G)) = 2$ so $\operatorname{rank} S_2 = 2$. Then $S_2 \in \mathcal{MR}(\text{Star}_2(G))$. Since $M = \widetilde{M_{\mathcal{A}}} + \widetilde{M_{\mathcal{B}}} + S_2$, the proof is complete.

III. The only difference between (EI_1) and (EI_2) is that the roles of x_1, x_2 and of y_1, y_2 are both reversed. So the result in *III* follows from that of *II*.

For convenience we let $G_A = G_{\mathcal{A}} - R$ and $G_B = G_{\mathcal{B}} - R$.

IV. We first show that any of (E_1), (I_1), (I_2) imply that

$\operatorname{mr}(G) = \operatorname{mr}(G_A) + \operatorname{mr}(G_B) + 4$, $A \in \mathcal{MR}(G_A)$, and $B \in \mathcal{MR}(G_B)$.

First, assume (E_1), that $\operatorname{rank} M = \operatorname{rank} A + \operatorname{rank} B + 4$. Since $A \in \mathcal{S}(G_A)$ and $B \in \mathcal{S}(G_B)$, $\operatorname{mr}(G) = \operatorname{rank} M = \operatorname{rank} A + \operatorname{rank} B + 4 \geqslant \operatorname{mr}(G_A) + \operatorname{mr}(G_B) + 4 \geqslant \operatorname{mr}(G)$, where the last inequality follows from Lemma 4.11. So equality holds, and $A \in \mathcal{MR}(G_A)$, and $B \in \mathcal{MR}(G_B)$.

Next, assume that (I_1) holds:

$$\operatorname{rank} A + \operatorname{rank} B < \operatorname{rank}\,[A\,x_1] + \operatorname{rank}\,[y_1\,B] < \operatorname{rank}\,[A\,x_1\,x_2] + \operatorname{rank}\,[y_1\,y_2\,B].$$

By Lemma 4.13, $K_1 \in \mathcal{MR}(G - r_2)$, $\operatorname{rank} K_1 = \operatorname{rank} M - 2$, and $\operatorname{mr}(G) = \operatorname{mr}(G - r_2) + 2$. Now either $\operatorname{rank}\,[A\,x_1] > \operatorname{rank} A$ or else $\operatorname{rank}\,[y_1\,B] > \operatorname{rank} B$; equivalently, $x_1 \notin C(A)$ or $y_1 \notin C(B)$. Applying Proposition 2.11, $\operatorname{rank} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \operatorname{rank} K_1(r_1) = \operatorname{rank} K_1 - 2$. Then

$$\operatorname{mr}(G) = \operatorname{rank} M = \operatorname{rank} K_1 + 2 = \operatorname{rank} A + \operatorname{rank} B + 4 \geqslant \operatorname{mr}(G_A) + \operatorname{mr}(G_B) + 4 \geqslant \operatorname{mr}(G),$$

and again equality holds, $A \in \mathcal{MR}(G_A)$ and $B \in \mathcal{MR}(G_B)$.

The case in which (I_2) holds is similar, so any of (E_1), (I_1), (I_2) imply $\operatorname{mr}(G) = \operatorname{mr}(G_A) + \operatorname{mr}(G_B) + 4$, $A \in \mathcal{MR}(G_A)$, and $B \in \mathcal{MR}(G_B)$.

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Now write $M=\widetilde{A}+\widetilde{B}+S$ where $\widetilde{A}=\begin{bmatrix} A&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$, $\widetilde{B}=\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&B\end{bmatrix}$, and $S=\begin{bmatrix} 0&x_1&x_2&0\\x_1^T&a&b&y_1^T\\x_2^T&b&c&y_2^T\\0&y_1&y_2&0\end{bmatrix}$.
Since $A\in\mathscr{MR}(G_A)$ and $B\in\mathscr{MR}(G_B)$, we have $\widetilde{A}\in\mathscr{MR}(\widetilde{G_A})$ and $\widetilde{B}\in\mathscr{MR}(\widetilde{G_B})$. Now $S\in\mathcal{S}_0(\mathrm{Star}_{12}(G))$ and clearly rank $S\leqslant 4$. Let S' be any matrix in $\mathscr{MR}_0(\mathrm{Star}_{12}(G))$, and let $M'=\widetilde{A}+\widetilde{B}+S'\in\mathcal{S}(G)$. Then

$$\mathrm{mr}(G)\leqslant \mathrm{rank}\,M'\leqslant \mathrm{rank}\,\widetilde{A}+\mathrm{rank}\,\widetilde{B}+\mathrm{rank}\,S'\leqslant \mathrm{rank}\,A+\mathrm{rank}\,B+4=\mathrm{mr}(G)$$

which implies rank $S'=4$. Therefore $\mathrm{mr}_0(\mathrm{Star}_{12}(G))=4$ and $S\in\mathscr{MR}_0(\mathrm{Star}_{12}(G))$. Then $M\in\mathscr{MR}(\widetilde{G_A})+\mathscr{MR}(\widetilde{G_B})+\mathscr{MR}_0(\mathrm{Star}_{12}(G))$. We arrive at the same conclusion if (I_2) holds, so this concludes the proof of IV.

V. By Lemma 4.14, rank $M\leqslant$ rank $A+$ rank $B+2$. By the inequality of (IE_1), either $x_1\notin C(A)$ or else $y_1\notin C(B)$. Without loss of generality say $x_1\notin C(A)$. The equality of (IE_1) implies that $x_2\in C([A\,x_1])$ and $y_2\in C([y_1\,B])$. Therefore there are vectors u and v and scalars h and k such that $x_2=Au+hx_1$ and $y_2=Bv+ky_1$. By the inequality of (IE_2), h and k cannot both be 0. Now

$$M=\begin{bmatrix} A & x_1 & Au+hx_1 & 0\\ x_1^T & a & b & y_1^T\\ u^TA+hx_1^T & b & c & v^TB+ky_1^T\\ 0 & y_1 & Bv+ky_1 & B\end{bmatrix}$$

and it is straightforward to show that M is row and column equivalent to

$$M'=\begin{bmatrix} A & x_1 & 0 & 0\\ x_1^T & a & b-x_1^Tu-ah & y_1^T\\ 0 & b-u^Tx_1-ha & c-u^TAu+h^2a-2hb & v^TB+(k-h)y_1^T\\ 0 & y_1 & Bv+(k-h)y_1 & B\end{bmatrix}.$$

Since $x_1\notin C(A)$ we know that the rank of the matrix

$$M'(2)=\begin{bmatrix} A & 0 & 0\\ 0 & c-u^TAu+h^2a-2hb & v^TB+(k-h)y_1^T\\ 0 & Bv+(k-h)y_1 & B\end{bmatrix}$$

is rank $M-2$. Since

$$\mathrm{rank}\,A+\mathrm{rank}\,B\leqslant \mathrm{rank}\,M'(2)=\mathrm{rank}\,M-2\leqslant \mathrm{rank}\,A+\mathrm{rank}\,B,$$
$$\mathrm{rank}\,M=\mathrm{rank}\,A+\mathrm{rank}\,B+2 \text{ and } \mathrm{rank}\,M'(2)=\mathrm{rank}\,A+\mathrm{rank}\,B.$$

It follows that rank $B=$ rank $\begin{bmatrix} c-u^TAu+h^2a-2hb & v^TB+(k-h)y_1^T\\ Bv+(k-h)y_1 & B\end{bmatrix}$. Thus $Bv+(k-h)y_1\in C(B)$ which implies $(k-h)y_1\in C(B)$. Then either $k=h$ or $y_1\in C(B)$, so we consider these two cases.

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Case 1. $k = h$.

Let $\widetilde{\widetilde{M_{\mathcal{A}}}} = \begin{bmatrix} A & 0 & Au & 0 \\ 0 & 0 & 0 & 0 \\ u^TA & 0 & u^TAu & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\widetilde{\widetilde{M_{\mathcal{B}}}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & v^TBv & v^TB \\ 0 & 0 & Bv & B \end{bmatrix}$, and

$$N = M - \widetilde{\widetilde{M_{\mathcal{A}}}} - \widetilde{\widetilde{M_{\mathcal{B}}}} = \begin{bmatrix} 0 & x_1 & hx_1 & 0 \\ x_1^T & a & b & y_1^T \\ hx_1^T & b & c - u^TAu - v^TBv & hy_1^T \\ 0 & y_1 & hy_1 & 0 \end{bmatrix}.$$

Then since $h = k$, $M'(2) = \begin{bmatrix} A & 0 & 0 \\ 0 & c - u^TAu + h^2a - 2hb & v^TB \\ 0 & Bv & B \end{bmatrix}$. Since $\operatorname{rank} M'(2) = \operatorname{rank} A + \operatorname{rank} B$, it follows that $v^TBv = c - u^TAu + h^2a - 2hb \Rightarrow c - u^TAu - v^TBv = 2hb - h^2a$. Thus $N = \begin{bmatrix} 0 & x_1 & hx_1 & 0 \\ x_1^T & a & b & y_1^T \\ hx_1^T & b & 2hb - h^2a & hy_1^T \\ 0 & y_1 & hy_1 & 0 \end{bmatrix}$ which is row and column equivalent to $\begin{bmatrix} 0 & x_1 & 0 & 0 \\ x_1^T & a & b - ha & y_1^T \\ 0 & b - ha & 0 & 0 \\ 0 & y_1 & 0 & 0 \end{bmatrix}$, which has rank 2.

Since $Au = -hx_1 + x_2$, $Bv = -ky_1 + y_2 = -hy_1 + y_2$, and h is nonzero, by Lemma 4.12, $\widetilde{\widetilde{M_{\mathcal{A}}}} \in \mathcal{S}(\widetilde{\widetilde{G_{\mathcal{A}}}})$ and $\widetilde{\widetilde{M_{\mathcal{B}}}} \in \mathcal{S}(\widetilde{\widetilde{G_{\mathcal{B}}}})$. Also $h \neq 0$ gives that $N \in \mathcal{S}_0(\mathrm{TStar}_1(G))$. Moreover

$$\begin{aligned} \operatorname{mr}(G) &= \operatorname{rank} M = \operatorname{rank} A + \operatorname{rank} B + 2 = \operatorname{rank} \widetilde{\widetilde{M_{\mathcal{A}}}} + \operatorname{rank} \widetilde{\widetilde{M_{\mathcal{B}}}} + \operatorname{rank} N \\ &\geqslant \operatorname{mr}(\widetilde{\widetilde{G_{\mathcal{A}}}}) + \operatorname{mr}(\widetilde{\widetilde{G_{\mathcal{B}}}}) + 2 = \operatorname{mr}(\overline{G_{\mathcal{A}}}) + \operatorname{mr}(\overline{G_{\mathcal{B}}}) + 2 \geqslant \operatorname{mr}(G) \end{aligned}$$

where the last inequality follows by Lemma 4.11. Therefore $\operatorname{mr}(\overline{G_{\mathcal{A}}}) + \operatorname{mr}(\overline{G_{\mathcal{B}}}) + 2 = \operatorname{mr}(G)$, and rank $\widetilde{\widetilde{M_{\mathcal{A}}}} = \operatorname{mr}(\widetilde{\widetilde{G_{\mathcal{A}}}})$, rank $\widetilde{\widetilde{M_{\mathcal{B}}}} = \operatorname{mr}(\widetilde{\widetilde{G_{\mathcal{B}}}})$, and by Proposition 4.3
$\operatorname{rank} N = \operatorname{mr}_0(\mathrm{TStar}_1(G))$. In other words,

$$\widetilde{\widetilde{M_{\mathcal{A}}}} \in \mathscr{MR}(\widetilde{\widetilde{G_{\mathcal{A}}}}), \quad \widetilde{\widetilde{M_{\mathcal{B}}}} \in \mathscr{MR}(\widetilde{\widetilde{G_{\mathcal{B}}}}), \quad \text{and} \quad N \in \mathscr{MR}_0(\mathrm{TStar}_1(G)).$$

Case 2: $y_1 \in C(B)$.
Thus there is a vector z such that $y_1 = Bz$. Since $y_2 = Bv + ky_1 = Bv + kBz$, $y_2 \in C(B)$ as well and there exists a vector w such that $y_2 = Bw$. Then we have

$$M = \begin{bmatrix} A & x_1 & Au + hx_1 & 0 \\ x_1^T & a & b & z^TB \\ u^TA + hx_1^T & b & c & w^TB \\ 0 & Bz & Bw & B \end{bmatrix}.$$

Since $y_2 \in C(B)$, the inequality of (IE_2) implies that $x_2 \notin C(A)$. Therefore $h \neq 0$.

Let $\widetilde{\widetilde{M_{\mathcal{A}}}} = \begin{bmatrix} A & 0 & Au & 0 \\ 0 & 0 & 0 & 0 \\ u^TA & 0 & u^TAu & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

$\widetilde{\widetilde{M_{\mathcal{B}}}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (w-hz)^TB(w-hz) & (w-hz)^TB \\ 0 & 0 & B(w-hz) & B \end{bmatrix}$, and

$$N = M - \widetilde{\widetilde{M_{\mathcal{A}}}} - \widetilde{\widetilde{M_{\mathcal{B}}}} = \begin{bmatrix} 0 & x_1 & hx_1 & 0 \\ x_1^T & a & b & z^TB \\ hx_1^T & b & c - u^TAu - (w-hz)^TB(w-hz) & hz^TB \\ 0 & Bz & hBz & 0 \end{bmatrix}.$$

Since $Au = -hx_1 + x_2$ and $B(w-hz) = -hy_1 + y_2$, by Lemma 4.12, $\widetilde{\widetilde{M_{\mathcal{A}}}} \in \mathcal{S}(\widetilde{\widetilde{G_{\mathcal{A}}}})$, and $\widetilde{\widetilde{M_{\mathcal{B}}}} \in \mathcal{S}(\widetilde{\widetilde{G_{\mathcal{B}}}})$. Also note $N \in \mathcal{S}_0(\mathrm{TStar}_1(G))$. Substituting $y_1 = Bz$ and $Bw = Bv + kBz$ into $M'(2)$, we find

$$M'(2) = \begin{bmatrix} A & 0 & 0 \\ 0 & c - u^TAu - 2hb + h^2a & (w^T - hz^T)B \\ 0 & B(w-hz) & B \end{bmatrix}.$$

Since $M'(2)$ has rank equal to rank $A+$rank B, it follows that $c - u^Tu - 2hb + h^2a = (w-hz)^TB(w-hz)$. Thus $c - u^TAu - (w-hz)^TB(w-hz) = 2hb - h^2a$.

Substituting this into N we obtain $N = \begin{bmatrix} 0 & x_1 & hx_1 & 0 \\ x_1^T & a & b & z^TB \\ hx_1^T & b & 2hb - h^2a & hz^TB \\ 0 & Bz & hBz & 0 \end{bmatrix}$. Row and column reducing N we obtain $\begin{bmatrix} 0 & x_1 & 0 & 0 \\ x_1^T & a & b - ha & z^TB \\ 0 & b - ha & 0 & 0 \\ 0 & Bz & 0 & 0 \end{bmatrix}$, which has rank 2.

The remainder of the proof of Case 2 is the same as the end of the proof of Case 1.

If we had begun the proof by considering the second given inequality to conclude that either $x_2 \notin C(A)$ or $y_2 \notin C(B)$, an entirely similar proof yields the conclusion

$$M \in \mathscr{MR}(\widetilde{\widetilde{G_{\mathcal{A}}}}) + \mathscr{MR}(\widetilde{\widetilde{G_{\mathcal{B}}}}) + \mathscr{MR}_0(\mathrm{TStar}_2(G)).$$

This concludes the proof. $\square$

Given any minimum rank matrix M corresponding to a graph G as in Definition 4.1, Theorem 4.4 explains how M can be decomposed (except for rare exceptions as in Example 4.10) as a sum of minimum rank matrices of simpler graphs related to G. We now ask the question, is it possible to build the class of minimum rank matrices of G ($\mathscr{MR}(G)$) by summing the classes of minimum rank matrices of simpler graphs related to G? Again, except for rare exceptions, the answer is yes and is given by the next theorem.

Definition 4.15. Let F be a field. Let $\mathcal{G}_2$ be the class of all connected graphs with a 2-separation $(G_{\mathcal{A}}, G_{\mathcal{B}})$ as in Definition 4.1, and assume neither r_1 nor r_2 is a cut vertex as in the hypothesis of Theorem 4.4. A particular graph may appear more than once in $\mathcal{G}_2$ because it may have several such two separations. So properly each graph in $\mathcal{G}_2$ is a graph with two labeled vertices r_1, r_2, all other vertices unlabeled, and a specified 2-separation $(G_{\mathcal{A}}, G_{\mathcal{B}})$. Let

$$\mathscr{C}_{1.1} = \{G \in \mathcal{G}_2 \mid \operatorname{mr}^F(G) = \operatorname{mr}^F(G_{\mathcal{A}}) + \operatorname{mr}^F(G_{\mathcal{B}})\}$$

$$\mathscr{C}_{1.2} = \{G \in \mathcal{G}_2 \mid \operatorname{mr}^F(G) = \operatorname{mr}^F(H_{\mathcal{A}}) + \operatorname{mr}^F(H_{\mathcal{B}})\}$$

$$\mathscr{C}_{2} = \{G \in \mathcal{G}_2 \mid \operatorname{mr}^F(G) = \operatorname{mr}^F(G_{\mathcal{A}} - r_2) + \operatorname{mr}^F(G_{\mathcal{B}} - r_2) + 2\}$$

$$\mathscr{C}_{3} = \{G \in \mathcal{G}_2 \mid \operatorname{mr}^F(G) = \operatorname{mr}^F(G_{\mathcal{A}} - r_1) + \operatorname{mr}^F(G_{\mathcal{B}} - r_1) + 2\}$$

$$\mathscr{C}_{4} = \{G \in \mathcal{G}_2 \mid \operatorname{mr}^F(G) = \operatorname{mr}^F(G_{\mathcal{A}} - R) + \operatorname{mr}^F(G_{\mathcal{B}} - R) + 4\}$$

$$\mathscr{C}_{5} = \{G \in \mathcal{G}_2 \mid \operatorname{mr}^F(G) = \operatorname{mr}^F(\overline{G_{\mathcal{A}}}) + \operatorname{mr}^F(\overline{G_{\mathcal{B}}}) + 2\}$$

For each $G \in \mathcal{G}_2$ let

$$J(G) = \{i \in \{1.1, 1.2, 2, 3, 4, 5\} \mid G \in \mathscr{C}_i\}.$$

Now define the following matrix classes for $G \in \mathcal{G}_2$,

$$\mathcal{D}_{1.1}(G) = \mathscr{MR}^F(\widetilde{G_{\mathcal{A}}}) + \mathscr{MR}^F(\widetilde{G_{\mathcal{B}}})$$

$$\mathcal{D}_{1.2}(G) = \mathscr{MR}^F(\widetilde{H_{\mathcal{A}}}) + \mathscr{MR}^F(\widetilde{H_{\mathcal{B}}})$$

$$\mathcal{D}_{2}(G) = \mathscr{MR}^F(\widetilde{G_{\mathcal{A}} - r_2}) + \mathscr{MR}^F(\widetilde{G_{\mathcal{B}} - r_2}) + \mathscr{MR}^F(\operatorname{Star}_2(G))$$

$$\mathcal{D}_{3}(G) = \mathscr{MR}^F(\widetilde{G_{\mathcal{A}} - r_1}) + \mathscr{MR}^F(\widetilde{G_{\mathcal{B}} - r_1}) + \mathscr{MR}^F(\operatorname{Star}_1(G))$$

$$\mathcal{D}_{4}(G) = \mathscr{MR}^F(\widetilde{G_{\mathcal{A}} - R}) + \mathscr{MR}^F(\widetilde{G_{\mathcal{B}} - R}) + \mathscr{MR}_0^F(\operatorname{Star}_{12}(G))$$

$$\begin{aligned}\mathcal{D}_{5}(G) = {} & (\mathscr{MR}^F(\widetilde{\overline{G_{\mathcal{A}}}}) + \mathscr{MR}^F(\widetilde{\overline{G_{\mathcal{B}}}}) + \mathscr{MR}_0^F(\operatorname{TStar}_1(G))) \\ & \cup (\mathscr{MR}^F(\widetilde{\overline{G_{\mathcal{A}}}}) + \mathscr{MR}^F(\widetilde{\overline{G_{\mathcal{B}}}}) + \mathscr{MR}_0^F(\operatorname{TStar}_2(G))).\end{aligned}$$

Theorem 4.16. *Let $G \in \mathcal{G}_2$. Then*

$$\mathscr{MR}^F(G) = \left(\bigcup_{i \in J(G)} \mathcal{D}_i(G)\right) \cap \mathcal{S}^F(G).$$

Proof. First note that $J(G) \neq \emptyset$ by Theorem 2.6.

Let $M \in \mathscr{MR}(G)$. We show $M \in \cup_{i \in J(G)} D_i$.

Since the hypotheses in the five statements of Theorem 4.4 are mutually exclusive and exhaustive, M satisfies the hypothesis of exactly one of the statements. Call this statement R.
Case 1. Suppose $R \in \{\text{II,III,IV,V}\}$. For notational convenience we define

$$I(R) = \begin{cases} 2 & \text{if } R = \text{II} \\ 3 & \text{if } R = \text{III} \\ 4 & \text{if } R = \text{IV} \\ 5 & \text{if } R = \text{V}. \end{cases}$$

By Theorem 4.4, $G \in \mathscr{C}_{I(R)}$ and $M \in \mathcal{D}_{I(R)}$. Thus $I(R) \in J(G)$ and $M \in \cup_{i\in J(G)} D_i$.
Case 2. Suppose $R = \text{I}$. By Theorem 4.4 $G \in \mathscr{C}_{1.1} \cup \mathscr{C}_{1.2}$ and $M \in \mathcal{D}_{1.1} \cup \mathcal{D}_{1.2}$. If $\{1.1, 1.2\} \subseteq J(G)$, $M \in \cup_{i\in J(G)} D_i$. If $1.2 \notin J(G)$, $\text{mr}(G) < \text{mr}(H_\mathcal{A}) + \text{mr}(H_\mathcal{B})$. Since $R = \text{I}$ we must have $\text{mr}(G) = \text{mr}(G_\mathcal{A}) + \text{mr}(G_\mathcal{B})$ and $M \in \mathcal{D}_{1.1}$. Thus $1.1 \in J(G)$ and $M \in \cup_{i\in J(G)} D_i$. Similarly, if $1.1 \notin J(G)$, $M \in \cup_{i\in J(G)} D_i$.

Therefore $\mathscr{MR}(G) \subseteq \cup_{i\in J(G)} D_i \cap \mathcal{S}(G)$, since $\mathscr{MR}(G) \subseteq \mathcal{S}(G)$.

Let $M \in (\cup_{i\in J(G)} \mathcal{D}_i(G)) \cap \mathcal{S}(G)$. Then M is in at least one of $\mathcal{D}_{1.1}(G) \cap \mathcal{S}(G)$, $\mathcal{D}_{1.2}(G) \cap \mathcal{S}(G)$, $\mathcal{D}_2(G) \cap \mathcal{S}(G)$, $\mathcal{D}_3(G) \cap \mathcal{S}(G)$, $\mathcal{D}_4(G) \cap \mathcal{S}(G)$, $\mathcal{D}_5(G) \cap \mathcal{S}(G)$.

Suppose $1.1 \in J(G)$ and $M \in \mathcal{D}_{1.1}(G) \cap \mathcal{S}(G)$.
Then $\text{mr}(G) = \text{mr}(G_\mathcal{A}) + \text{mr}(G_\mathcal{B})$ and there exist $\widetilde{M_\mathcal{A}} \in \mathscr{MR}(\widetilde{G_\mathcal{A}})$ and $\widetilde{M_\mathcal{B}} \in \mathscr{MR}(\widetilde{G_\mathcal{B}})$ such that $M = \widetilde{M_\mathcal{A}} + \widetilde{M_\mathcal{B}} \in \mathcal{S}(G)$. Then

$$\begin{aligned} \text{mr}(G) \leqslant \text{rank}\, M \leqslant \text{rank}\, \widetilde{M_\mathcal{A}} + \text{rank}\, \widetilde{M_\mathcal{B}} \\ = \text{mr}(\widetilde{G_\mathcal{A}}) + \text{mr}(\widetilde{G_\mathcal{B}}) \\ = \text{mr}(G_\mathcal{A}) + \text{mr}(G_\mathcal{B}) = \text{mr}(G). \end{aligned}$$

Then $\text{rank}\, M = \text{mr}(G)$ so $M \in \mathscr{MR}(G)$.
Suppose $1.2 \in J(G)$ and $M \in \mathcal{D}_{1.2}(G) \cap \mathcal{S}(G)$.
Then $\text{mr}(G) = \text{mr}(H_\mathcal{A}) + \text{mr}(H_\mathcal{B})$ and there exist $\widetilde{M_\mathcal{A}} \in \mathscr{MR}(\widetilde{H_\mathcal{A}})$ and $\widetilde{M_\mathcal{B}} \in \mathscr{MR}(\widetilde{H_\mathcal{B}})$ such that $M = \widetilde{M_\mathcal{A}} + \widetilde{M_\mathcal{B}} \in \mathcal{S}(G)$. Then

$$\begin{aligned} \text{mr}(G) \leqslant \text{rank}\, M \leqslant \text{rank}\, \widetilde{M_\mathcal{A}} + \widetilde{\text{rank}\, M_\mathcal{B}} \\ = \text{mr}(\widetilde{H_\mathcal{A}}) + \text{mr}(\widetilde{H_\mathcal{B}}) \\ = \text{mr}(H_\mathcal{A}) + \text{mr}(H_\mathcal{B}) = \text{mr}(G). \end{aligned}$$

Then $\text{rank}\, M = \text{mr}(G)$ so $M \in \mathscr{MR}(G)$.

Suppose $2 \in J(G)$ and $M \in \mathcal{D}_2(G) \cap \mathcal{S}(G)$.
Then $\text{mr}(G) = \text{mr}(G_\mathcal{A} - r_2) + \text{mr}(G_\mathcal{B} - r_2) + 2$ and there exist $\widetilde{M_\mathcal{A}} \in \mathscr{MR}(\widetilde{G_\mathcal{A} - r_2})$, $\widetilde{M_\mathcal{B}} \in \mathscr{MR}(\widetilde{G_\mathcal{B} - r_2})$ and $M_1 \in \mathscr{MR}(\text{Star}_2(G))$ such that
$M = \widetilde{M_\mathcal{A}} + \widetilde{M_\mathcal{B}} + M_1 \in \mathcal{S}(G)$. Then similar to the previous cases and by Proposition 4.3

$$\begin{aligned} \text{mr}(G) \leqslant \text{rank}\, M \leqslant \text{rank}\, \widetilde{M_\mathcal{A}} + \text{rank}\, \widetilde{M_\mathcal{B}} + \text{rank}\, M_1 \\ = \text{mr}(\widetilde{G_\mathcal{A} - r_2}) + \text{mr}(\widetilde{G_\mathcal{B} - r_2}) + \text{mr}(\text{Star}_2(G)) \\ = \text{mr}(G_\mathcal{A} - r_2) + \text{mr}(G_\mathcal{B} - r_2) + 2 = \text{mr}(G). \end{aligned}$$

Then $\text{rank}\, M = \text{mr}(G)$ so $M \in \mathscr{MR}(G)$.

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Suppose $3 \in J(G)$ and $M \in \mathcal{D}_3(G) \cap \mathcal{S}(G)$.
This case follows by replacing 2 with 1 in the previous case.

Suppose $4 \in J(G)$ and $M \in \mathcal{D}_4(G) \cap \mathcal{S}(G)$.
Then $\mathrm{mr}(G) = \mathrm{mr}(G_\mathcal{A} - R) + \mathrm{mr}(G_\mathcal{B} - R) + 4$ and there exist $\widetilde{M_\mathcal{A}} \in \mathscr{MR}(\widetilde{G_\mathcal{A} - R})$, $\widetilde{M_\mathcal{B}} \in \mathscr{MR}(\widetilde{G_\mathcal{B} - R})$ and $M_{12} \in \mathscr{MR}_0(\mathrm{Star}_{12}(G))$ such that
$M = \widetilde{M_\mathcal{A}} + \widetilde{M_\mathcal{B}} + M_{12} \in \mathcal{S}(G)$. Then similar to the previous cases and by Proposition 4.3

$$\begin{aligned}\mathrm{mr}(G) \leqslant \mathrm{rank}\, M \leqslant \mathrm{rank}\, \widetilde{M_\mathcal{A}} + \mathrm{rank}\, \widetilde{M_\mathcal{B}} + \mathrm{rank}\, M_{12} \\ = \mathrm{mr}(\widetilde{G_\mathcal{A} - R}) + \mathrm{mr}(\widetilde{G_\mathcal{B} - R}) + \mathrm{mr}_0(\mathrm{Star}_{12}(G)) \\ \leqslant \mathrm{mr}(G_\mathcal{A} - R) + \mathrm{mr}(G_\mathcal{B} - R) + 4 = \mathrm{mr}(G).\end{aligned}$$

Then $\mathrm{rank}\, M = \mathrm{mr}(G)$ so $M \in \mathscr{MR}(G)$.

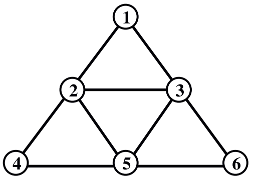
Suppose $5 \in J(G)$ and $M \in \mathcal{D}_5(G) \cap \mathcal{S}(G)$.
Then $\mathrm{mr}(G) = \mathrm{mr}(\overline{G_\mathcal{A}}) + \mathrm{mr}(\overline{G_\mathcal{B}}) + 2$ and there exist $\widetilde{M_\mathcal{A}} \in \mathscr{MR}(\widetilde{\overline{G_\mathcal{A}}})$, $\widetilde{M_\mathcal{B}} \in \mathscr{MR}(\widetilde{\overline{G_\mathcal{B}}})$ and $M_1 \in \mathscr{MR}_0(\mathrm{TStar}_1(G))$ or $\mathscr{MR}_0(\mathrm{TStar}_2(G))$ such that $M = \widetilde{M_\mathcal{A}} + \widetilde{M_\mathcal{B}} + M_1 \in \mathcal{S}(G)$. We suppose $M_1 \in \mathscr{MR}_0(\mathrm{TStar}_1(G))$ and note that the argument is similar if $M_1 \in \mathscr{MR}_0(\mathrm{TStar}_2(G))$. Then similar to the previous cases and by Proposition 4.3

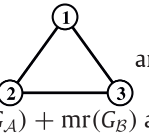
$$\begin{aligned}\mathrm{mr}(G) \leqslant \mathrm{rank}\, M \leqslant \mathrm{rank}\, \widetilde{M_\mathcal{A}} + \mathrm{rank}\, \widetilde{M_\mathcal{B}} + \mathrm{rank}\, M_1 \\ = \mathrm{mr}(\widetilde{\overline{G_\mathcal{A}}}) + \mathrm{mr}(\widetilde{\overline{G_\mathcal{B}}}) + \mathrm{mr}_0(\mathrm{TStar}_1(G)) \\ = \mathrm{mr}(\overline{G_\mathcal{A}}) + \mathrm{mr}(\overline{G_\mathcal{B}}) + 2 = \mathrm{mr}(G).\end{aligned}$$

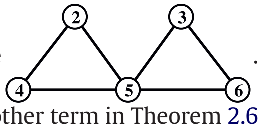
Then $\mathrm{rank}\, M = \mathrm{mr}(G)$ so $M \in \mathscr{MR}(G)$.

Thus in every case, $M \in \mathscr{MR}(G)$ and so $\mathscr{MR}(G) \supset (\cup_{i \in J(G)} \mathcal{D}_i(G)) \cap \mathcal{S}(G)$.
Therefore $\mathscr{MR}(G) = (\cup_{i \in J(G)} \mathcal{D}_i(G)) \cap \mathcal{S}(G)$. □

Example 4.17. Let G be



as in Example 1.3. We use Theorem 4.16 and Corollary 3.3 to determine the structure of every minimum rank matrix with graph G. Let $M \in \mathscr{MR}(G)$. Let $G_\mathcal{A}$ be 

and $G_\mathcal{B}$ be 

.
Then $(G_\mathcal{A}, G_\mathcal{B})$ is a 2-separation of G, $\mathrm{mr}(G) = \mathrm{mr}(G_\mathcal{A}) + \mathrm{mr}(G_\mathcal{B})$ and no other term in Theorem 2.6 achieves the minimum rank (see Example 2.7). By Theorem 4.16, $\mathscr{MR}(G) = (\mathscr{MR}(\widetilde{G_\mathcal{A}}) + \mathscr{MR}(\widetilde{G_\mathcal{B}})) \cap \mathcal{S}(G) = \mathscr{MR}(\widetilde{G_\mathcal{A}}) + \mathscr{MR}(\widetilde{G_\mathcal{B}})$. Thus there exist $\widetilde{M_\mathcal{A}} \in \mathscr{MR}(\widetilde{G_\mathcal{A}})$ and $\widetilde{M_\mathcal{B}} \in \mathscr{MR}(\widetilde{G_\mathcal{B}})$ such that $M = \widetilde{M_\mathcal{A}} + \widetilde{M_\mathcal{B}}$. Further, $\widetilde{G_\mathcal{B}}$ is a vertex sum of

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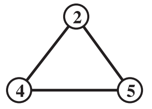
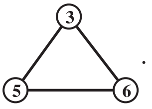
$G_1 =$  and $G_2 =$  .

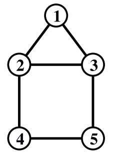
Since $r_5(G_1) = 0$ and $r_5(G_2) = 0$ by Corollary 3.3, $\mathscr{MR}(\widetilde{G_{\mathcal{B}}}) = \mathscr{MR}(\widetilde{G_1}) + \mathscr{MR}(\widetilde{G_2})$. Thus there exist $\widetilde{M_1} \in \mathscr{MR}(\widetilde{G_1})$ and $\widetilde{M_2} \in \mathscr{MR}(\widetilde{G_2})$ such that $\widetilde{M_{\mathcal{B}}} = \widetilde{M_1} + \widetilde{M_2}$. Thus $M = \widetilde{M_{\mathcal{A}}} + \widetilde{M_1} + \widetilde{M_2}$. The minimum rank of each of $\widetilde{G_{\mathcal{A}}}$, $\widetilde{G_1}$, and $\widetilde{G_2}$ is one. Since every rank one matrix has the form $\pm xx^T$,

$$M = \pm \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 & 0 & 0 & 0 \end{bmatrix} \pm \begin{bmatrix} 0 \\ b_2 \\ 0 \\ b_4 \\ b_5 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & b_2 & 0 & b_4 & b_5 & 0 \end{bmatrix}$$

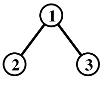
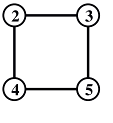
$$\pm \begin{bmatrix} 0 \\ 0 \\ c_3 \\ 0 \\ c_5 \\ c_6 \end{bmatrix} \begin{bmatrix} 0 & 0 & c_3 & 0 & c_5 & c_6 \end{bmatrix}.$$

Thus every matrix in $\mathscr{MR}(G)$ can be constructed using the form given above.

Example 4.18. Let G be the house graph



and let $M \in \mathscr{MR}(G)$. We apply Theorem 4.16 with the 2-separation

$G_{\mathcal{A}}$:  $G_{\mathcal{B}}$: 

It is then easy to check that $J(G) = \{1.2, 5\}$. By Theorem 4.16,

$$\begin{aligned}\mathscr{MR}(G) = &(\mathscr{MR}(\widetilde{H_{\mathcal{A}}}) + \mathscr{MR}(\widetilde{H_{\mathcal{B}}}) \\ &\cup (\mathscr{MR}(\widetilde{\widetilde{G_{\mathcal{A}}}}) + \mathscr{MR}(\widetilde{\widetilde{G_{\mathcal{B}}}}) + \mathscr{MR}_0(\mathrm{TStar}_1(G))) \\ &\cup (\mathscr{MR}(\widetilde{\widetilde{G_{\mathcal{A}}}}) + \mathscr{MR}(\widetilde{\widetilde{G_{\mathcal{B}}}}) + \mathscr{MR}_0(\mathrm{TStar}_2(G)))) \cap \mathcal{S}(G)\end{aligned}$$

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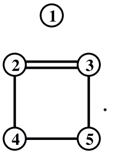
Case 1. $M = \widetilde{M_{\mathcal{A}}} + \widetilde{M_{\mathcal{B}}}$ where $\widetilde{M_{\mathcal{A}}} \in \mathscr{MR}(\widetilde{H_{\mathcal{A}}})$ and $\widetilde{M_{\mathcal{B}}} \in \mathscr{MR}(\widetilde{H_{\mathcal{B}}})$.

Since $\widetilde{H_{\mathcal{A}}}$ is the union of a clique on 3 vertices and 2 isolated vertices, any matrix in $\mathscr{MR}(\widetilde{H_{\mathcal{A}}})$ can be expressed as $\pm aa^T$ where $a^T = [a_1\ a_2\ a_3\ 0\ 0]$, all $a_i \neq 0$. To finish this case we need to decompose $\mathscr{MR}(\widetilde{H_{\mathcal{B}}})$ where $\widetilde{H_{\mathcal{B}}}$ is



To obtain a minimum rank matrix for $\widetilde{H_{\mathcal{B}}}$ the 2,3 entry must be nonzero and thus decomposing $\mathscr{MR}(\widetilde{H_{\mathcal{B}}})$ is equivalent to decomposing $\mathscr{MR}(C_4 \cup K_1)$.

Consider the 2-separation $G'_{\mathcal{A}} = \widetilde{H_{\mathcal{B}}} - \{5\}$ and $G'_{\mathcal{B}} = \widetilde{H_{\mathcal{B}}} - \{1,\ 2\}$. Then again $J(\widetilde{H_{\mathcal{B}}}) = \{1.2, 5\}$ and $\widetilde{M_{\mathcal{B}}} \in \mathcal{D}_{1.2}(\widetilde{H_{\mathcal{B}}}) \cup \mathcal{D}_5(\widetilde{H_{\mathcal{B}}})$.

1. $\widetilde{M_{\mathcal{B}}} \in \mathcal{D}_{1.2}(\widetilde{H_{\mathcal{B}}})$.

Since $\widetilde{H'_{\mathcal{A}}}$ and $\widetilde{H'_{\mathcal{B}}}$ both consist of the union of a clique on 3 vertices and two isolated vertices $\text{mr}(\widetilde{H'_{\mathcal{A}}}) = \text{mr}(\widetilde{H'_{\mathcal{B}}}) = 1$ and $\widetilde{M_{\mathcal{B}}}$ has the form $\pm bb^T \pm cc^T$ with $b^T = [0\ b_2\ b_3\ b_4\ 0]$, all $b_i \neq 0$, and $c^T = [0\ 0\ c_3\ c_4\ c_5]$, all $c_i \neq 0$, with the additional condition $\pm b_3 b_4 \pm c_3 c_4 = 0$ since $34 \notin E(\widetilde{H_{\mathcal{B}}})$.

Then $M = \pm aa^T \pm bb^T \pm cc^T$ with this same condition.

2. $\widetilde{M_{\mathcal{B}}} \in \mathcal{D}_5(G)$.

Similar to Example 4.10, $\text{mr}(\widetilde{\widetilde{G'_{\mathcal{A}}}}) = \text{mr}(\widetilde{\widetilde{G'_{\mathcal{B}}}}) = 0$ so $\widetilde{M_{\mathcal{B}}} \in \mathcal{S}_0(\text{TStar}_1(\widetilde{H_{\mathcal{B}}}))$.

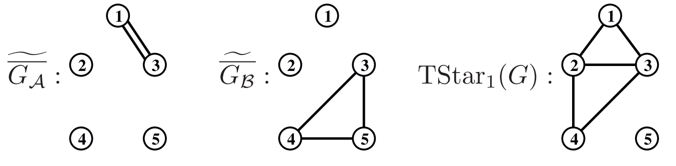
Then rank $\widetilde{M_{\mathcal{B}}} = 2$ implies that $\widetilde{M_{\mathcal{B}}}$ can be written

$$\widetilde{M_{\mathcal{B}}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & ka & 0 \\ 0 & a & b & 0 & c \\ 0 & ka & 0 & -k^2 b & kc \\ 0 & 0 & c & kc & 0 \end{bmatrix}$$

for some $a, c, k \neq 0$, and so $M = \pm aa^T \pm \widetilde{M_{\mathcal{B}}}$, where $\pm a_2 a_3 + a \neq 0$.

Case 2. $M \in \mathscr{MR}(\widetilde{\widetilde{G_{\mathcal{A}}}}) + \mathscr{MR}(\widetilde{\widetilde{G_{\mathcal{B}}}}) + \mathscr{MR}_0(\text{TStar}_1(G))$.

The graphs are:



The only matrix in $\mathscr{MR}(\widetilde{\widetilde{G_{\mathcal{A}}}})$ is the zero matrix, and any matrix in $\mathscr{MR}(\widetilde{\widetilde{G_{\mathcal{B}}}})$ has the form $\pm cc^T$ with c as in Case 1. Since any matrix $C \in \mathscr{MR}_0(\text{TStar}_1(G))$ has rank 2, it can be written

$$C = \begin{bmatrix} 0 & a & ka & 0 & 0 \\ a & b & c & d & 0 \\ ka & c & 2kc - k^2 b & kd & 0 \\ 0 & d & kd & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad a, c, d, k \neq 0.$$

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So in this case $M = C \pm cc^T$ where $kd \pm c_3c_4 = 0$.

The case in which $\mathscr{MR}_0(\text{TStar}_1(G))$ is replaced by $\mathscr{MR}_0(\text{TStar}_2(G))$ is almost the same.

5. Decompositions of positive semidefinite minimum rank matrices

In this section we establish analogues of Theorems 3.1 and 4.4 for positive semidefinite minimum rank.

We first provide some needed definitions and previous results.

Definition 5.1. Given a graph G, let $\mathcal{S}_+(G)$ be the subset of $\mathcal{S}^{\mathbb{R}}(G)$ consisting of all positive semidefinite matrices. The *minimum positive semidefinite rank* of G is

$$\text{mr}_+(G) = \min_{A \in \mathcal{S}_+(G)} \{\text{rank}\, A\}.$$

Definition 5.2. Given a graph G, let

$$\mathscr{MR}_+(G) = \{A \in \mathcal{S}_+(G) \mid \text{rank}\, A = \text{mr}_+(G)\}.$$

The following is a well known result for positive semidefinite matrices.

Lemma 5.3 (Column inclusion)**.** *If* $A = \begin{bmatrix} B & y \\ y^T & c \end{bmatrix}$ *is positive semidefinite, then* $y \in C(B)$.

The following three results appear as Proposition 1.4 in [4] and Corollaries 2.5 and 2.9 in [6].

Lemma 5.4. *Let A, B be real symmetric $n \times n$ matrices. Then*

$$\pi(A + B) \leqslant \pi(A) + \pi(B)$$

where $\pi(C)$ denotes the number of positive eigenvalues of C.

Theorem 5.5. *Let G be the vertex-sum of G_1 and G_2. Then*

$$\text{mr}_+(G) = \text{mr}_+(G_1) + \text{mr}_+(G_2).$$

Theorem 5.6. *Let $G = (G_{\mathcal{A}}, G_{\mathcal{B}})$ be a 2-separation of a graph G and let $H_{\mathcal{A}}$ and $H_{\mathcal{B}}$ be as in Definition 4.1. Then*

$$\text{mr}_+(G) = \min\{\text{mr}_+(G_{\mathcal{A}}) + \text{mr}_+(G_{\mathcal{B}}), \text{mr}_+(H_{\mathcal{A}}) + \text{mr}_+(H_{\mathcal{B}})\}.$$

We now give the analogues of Theorems 3.1 and 4.4 for positive semidefinite minimum rank.

Theorem 5.7. *If G is the vertex-sum at v of G_1 and G_2, then*

$$\mathscr{MR}_+(G) = \mathscr{MR}_+(\widetilde{G_1}) + \mathscr{MR}_+(\widetilde{G_2}).$$

Proof. Let G be as stated in the theorem with $M \in \mathscr{MR}_+(G)$ given. Labeling the vertices of G appropriately $M = \begin{bmatrix} a & x_1^T & x_2^T \\ x_1 & C_1 & 0 \\ x_2 & 0 & C_2 \end{bmatrix}$ where the first row and column of M correspond to the vertex v. Note that each $C_i \in \mathcal{S}_+(G_i - v)$ since they are principal submatrices of a positive semidefinite matrix. By Lemma 5.3, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is in the column space of $\begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}$. Thus there exist vectors y_1 and y_2 such that $x_1 = C_1 y_1$ and $x_2 = C_2 y_2$. Now rewrite the matrix as $M = \begin{bmatrix} a & y_1^T C_1 & y_2^T C_2 \\ C_1 y_1 & C_1 & 0 \\ C_2 y_2 & 0 & C_2 \end{bmatrix}$. Now consider

$$N = \begin{bmatrix} y_1^T & y_2^T \\ I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{bmatrix} y_1 & I & 0 \\ y_2 & 0 & I \end{bmatrix} = \begin{bmatrix} y_1^T C_1 y_1 + y_2^T C_2 y_2 & y_1^T C_1 & y_2^T C_2 \\ C_1 y_1 & C_1 & 0 \\ C_2 y_2 & 0 & C_2 \end{bmatrix}.$$

Since N is of the form $A^T BA$ where B is positive semidefinite, N is positive semidefinite. Thus $N \in \mathcal{S}_+(G)$. Furthermore $\operatorname{rank} N = \operatorname{rank} C_1 + \operatorname{rank} C_2$. Since $M \in \mathcal{MR}_+(G)$

$$\operatorname{rank} C_1 + \operatorname{rank} C_2 \leqslant \operatorname{rank} M \leqslant \operatorname{rank} N = \operatorname{rank} C_1 + \operatorname{rank} C_2.$$

Therefore $\operatorname{rank} M = \operatorname{rank} C_1 + \operatorname{rank} C_2$. We note that the first row of M is a linear combination of the other rows of M and using block Gaussian elimination we see that $a = y_1^T C_1 y_1 + y_2^T C_2 y_2$; i.e., $M = N$. Let

$$\widetilde{M_1} = \begin{bmatrix} y_1^T C_1 y_1 & y_1^T C_1 & 0 \\ C_1 y_1 & C_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \widetilde{M_2} = \begin{bmatrix} y_2^T C_2 y_2 & 0 & y_2^T C_2 \\ 0 & 0 & 0 \\ C_2 y_2 & 0 & C_2 \end{bmatrix}.$$

Then $M = \widetilde{M_1} + \widetilde{M_2}$ and $\operatorname{rank} \widetilde{M_i} = \operatorname{rank} C_i$, $i = 1, 2$. Furthermore

$$\widetilde{M_1} = \begin{bmatrix} y_1^T & 0 \\ I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \widetilde{M_2} = \begin{bmatrix} y_2^T & 0 \\ 0 & 0 \\ I & 0 \end{bmatrix} \begin{bmatrix} C_2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_2 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $\widetilde{M_1}$ and $\widetilde{M_2}$ are positive semidefinite. By Theorem 5.5

$$\begin{aligned} \operatorname{mr}_+(G) = \operatorname{mr}_+(G_1) + \operatorname{mr}_+(G_2) &= \operatorname{mr}_+(\widetilde{G_1}) + \operatorname{mr}_+(\widetilde{G_2}) \\ &\leqslant \operatorname{rank} \widetilde{M_1} + \operatorname{rank} \widetilde{M_2} = \operatorname{rank} M = \operatorname{mr}_+(G). \end{aligned}$$

Therefore $\operatorname{rank} \widetilde{M_i} = \operatorname{mr}_+(\widetilde{G_i})$, $i = 1, 2$ and $\widetilde{M_i} \in \mathcal{MR}_+(\widetilde{G_i})$, $i = 1, 2$. Thus $\mathcal{MR}_+(G) \subset \mathcal{MR}_+(\widetilde{G_1}) + \mathcal{MR}_+(\widetilde{G_2})$.

Let $M_1 \in \mathcal{MR}_+(\widetilde{G_1})$ and $M_2 \in \mathcal{MR}_+(\widetilde{G_2})$. Let $M = M_1 + M_2$. Note that M is positive semidefinite and in $\mathcal{S}_+(G)$. Then

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$$\begin{aligned}\mathrm{mr}_+(G) \leqslant \operatorname{rank} M &\leqslant \operatorname{rank} M_1 + \operatorname{rank} M_2\\ &= \mathrm{mr}_+(\widetilde{G_1}) + \mathrm{mr}_+(\widetilde{G_2})\\ &= \mathrm{mr}_+(G_1) + \mathrm{mr}_+(G_2)\\ &= \mathrm{mr}_+(G)\end{aligned}$$

where the last equality follows from Theorem 5.5. Thus $\operatorname{rank} M = \mathrm{mr}_+(G)$ so $M \in \mathscr{MR}_+(G)$ and $\mathscr{MR}_+(G) \supset \mathscr{MR}_+(\widetilde{G_1}) + \mathscr{MR}_+(\widetilde{G_2})$.

Therefore $\mathscr{MR}_+(G) = \mathscr{MR}_+(\widetilde{G_1}) + \mathscr{MR}_+(\widetilde{G_2})$. $\square$

Definition 5.8. Let $\mathcal{G}_2$ be as in Definition 4.15. Let

$$\begin{aligned}\mathscr{C}_{1.1+} &= \{G \in \mathcal{G}_2 \mid \mathrm{mr}_+(G) = \mathrm{mr}_+(G_{\mathcal{A}}) + \mathrm{mr}_+(G_{\mathcal{B}})\}\\ \mathscr{C}_{1.2+} &= \{G \in \mathcal{G}_2 \mid \mathrm{mr}_+(G) = \mathrm{mr}_+(H_{\mathcal{A}}) + \mathrm{mr}_+(H_{\mathcal{B}})\}\end{aligned}$$

For each $G \in \mathcal{G}_2$ let

$$J(G) = \{i \in \{1.1, 1.2\} \mid G \in \mathscr{C}_{i+}\}.$$

Now define the following matrix classes for $G \in \mathcal{G}_2$,

$$\begin{aligned}\mathcal{D}_{1.1+}(G) &= \mathscr{MR}_+(\widetilde{G_{\mathcal{A}}}) + \mathscr{MR}_+(\widetilde{G_{\mathcal{B}}})\\ \mathcal{D}_{1.2+}(G) &= \mathscr{MR}_+(\widetilde{H_{\mathcal{A}}}) + \mathscr{MR}_+(\widetilde{H_{\mathcal{B}}})\end{aligned}$$

Theorem 5.9. *Let $G \in \mathcal{G}_2$. Then*

$$\mathscr{MR}_+(G) = \left(\bigcup_{i \in J(G)} \mathcal{D}_{i+}(G)\right) \cap \mathcal{S}_+(G).$$

Proof. First note that $J(G) \neq \emptyset$ by Theorem 5.6.

Let $M \in \mathscr{MR}_+(G)$. Then $M \in \mathcal{S}_+(G)$. Labeling the vertices of G appropriately,

$$M = \begin{bmatrix} A & x_1 & x_2 & 0\\ x_1^T & a & b & y_1^T\\ x_2^T & b & c & y_2^T\\ 0 & y_1 & y_2 & B\end{bmatrix}.$$

Since $\begin{bmatrix} A & x_1\\ x_1^T & a\end{bmatrix}$ and $\begin{bmatrix} A & x_2\\ x_2^T & c\end{bmatrix}$ are principal submatrices of a positive semidefinite matrix, they are also positive semidefinite matrices. By Lemma 5.3, $x_1, x_2 \in C(A)$. By a similar argument $y_1, y_2 \in C(B)$. Thus

$$\operatorname{rank} A + \operatorname{rank} B = \operatorname{rank}\,[A\, x_1\, x_2] + \operatorname{rank}\,[y_1\, y_2\, B].$$

Then following the proof of I in Theorem 4.4, in either Case 1 or Case 2 we have

$$M = \widetilde{M_{\mathcal{A}}} + \widetilde{M_{\mathcal{B}}} \text{ and } \operatorname{rank} M = \operatorname{rank} \widetilde{M_{\mathcal{A}}} + \operatorname{rank} \widetilde{M_{\mathcal{B}}}.$$

Then

$$\pi(\widetilde{M_{\mathcal{A}}}) + \pi(\widetilde{M_{\mathcal{B}}}) \leqslant \operatorname{rank} \widetilde{M_{\mathcal{A}}} + \operatorname{rank} \widetilde{M_{\mathcal{B}}} = \operatorname{rank} M = \pi(M) \leqslant \pi(\widetilde{M_{\mathcal{A}}}) + \pi(\widetilde{M_{\mathcal{B}}})$$

where the last inequality follows from Lemma 5.4. It follows that we have equality throughout and therefore $\widetilde{M_{\mathcal{A}}}$ and $\widetilde{M_{\mathcal{B}}}$ are both positive semidefinite.

Continuing with the proof of I, we see if $v_1^T B v_2 = 0$ then $\widetilde{M_{\mathcal{A}}} \in \mathcal{S}_+(G_{\mathcal{A}})$ and $\widetilde{M_{\mathcal{B}}} \in \mathcal{S}_+(G_{\mathcal{B}})$. Also by Theorem 5.6

$$\begin{aligned} \operatorname{mr}_+(G_{\mathcal{A}}) + \operatorname{mr}_+(G_{\mathcal{B}}) \geqslant \operatorname{mr}_+(G) = \operatorname{rank} M &= \operatorname{rank} \widetilde{M_{\mathcal{A}}} + \operatorname{rank} \widetilde{M_{\mathcal{B}}} \\ &\geqslant \operatorname{mr}_+(\widetilde{G_{\mathcal{A}}}) + \operatorname{mr}_+(\widetilde{G_{\mathcal{B}}}) = \operatorname{mr}_+(G_{\mathcal{A}}) + \operatorname{mr}_+(G_{\mathcal{B}}). \end{aligned}$$

Thus equality holds throughout and

$$\operatorname{mr}_+(G) = \operatorname{mr}_+(G_{\mathcal{A}}) + \operatorname{mr}_+(G_{\mathcal{B}}) \text{ and } \widetilde{M_{\mathcal{A}}} \in \mathscr{MR}_+(\widetilde{G_{\mathcal{A}}}),\ \widetilde{M_{\mathcal{B}}} \in \mathscr{MR}_+(\widetilde{G_{\mathcal{B}}}).$$

If $v_1^T B v_2 \neq 0$ then a similar argument shows that

$$\operatorname{mr}_+(G) = \operatorname{mr}_+(H_{\mathcal{A}}) + \operatorname{mr}_+(H_{\mathcal{B}}) \text{ and } \widetilde{M_{\mathcal{A}}} \in \mathscr{MR}_+(\widetilde{H_{\mathcal{A}}}),\ \widetilde{M_{\mathcal{B}}} \in \mathscr{MR}_+(\widetilde{H_{\mathcal{B}}}).$$

We have shown that $\mathscr{MR}_+(G) \subset \mathcal{D}_{1.1+}(G) \cup \mathcal{D}_{1.2+}(G)$. If $1.2 \notin J(G)$, $\operatorname{mr}_+(G) < \operatorname{mr}_+(H_{\mathcal{A}}) + \operatorname{mr}_+(H_{\mathcal{B}})$. Then necessarily $v_1^T B v_2 = 0$ and $M \in \mathcal{D}_{1.1+}(G) \cap \mathcal{S}_+(G) = (\cup_{i \in J(G)} \mathcal{D}_{i+}(G)) \cap \mathcal{S}_+(G)$. If $1.1 \notin J(G)$, similarly $M \in \mathcal{D}_{1.2+}(G) \cap \mathcal{S}_+(G) = (\cup_{i \in J(G)} \mathcal{D}_{i+}(G)) \cap \mathcal{S}_+(G)$. Therefore $\mathscr{MR}_+(G) \subset (\cup_{i \in J(G)} \mathcal{D}_{i+}(G)) \cap \mathcal{S}_+(G)$.

Let $M \in (\cup_{i \in J(G)} \mathcal{D}_{i+}(G)) \cap \mathcal{S}_+(G)$. Then M is in $\mathcal{D}_{1.1+}(G) \cap \mathcal{S}_+(G)$ or $\mathcal{D}_{1.2+}(G) \cap \mathcal{S}_+(G)$.
Suppose $1.1 \in J(G)$ and $M \in \mathcal{D}_{1.1+}(G) \cap \mathcal{S}_+(G)$.
Then $\operatorname{mr}_+(G) = \operatorname{mr}_+(G_{\mathcal{A}}) + \operatorname{mr}_+(G_{\mathcal{B}})$ and there exist $\widetilde{M_{\mathcal{A}}} \in \mathscr{MR}_+(\widetilde{G_{\mathcal{A}}})$ and $\widetilde{M_{\mathcal{B}}} \in \mathscr{MR}_+(\widetilde{G_{\mathcal{B}}})$ such that
$M = \widetilde{M_{\mathcal{A}}} + \widetilde{M_{\mathcal{B}}} \in \mathcal{S}_+(G)$. Then

$$\begin{aligned} \operatorname{mr}_+(G) \leqslant \operatorname{rank} M \leqslant \operatorname{rank} \widetilde{M_{\mathcal{A}}} + \operatorname{rank} \widetilde{M_{\mathcal{B}}} &\\ = \operatorname{mr}_+(\widetilde{G_{\mathcal{A}}}) + \operatorname{mr}_+(\widetilde{G_{\mathcal{B}}}) &\\ = \operatorname{mr}_+(G_{\mathcal{A}}) + \operatorname{mr}_+(G_{\mathcal{B}}) = \operatorname{mr}_+(G). \end{aligned}$$

Then $\operatorname{rank} M = \operatorname{mr}_+(G)$ so $M \in \mathscr{MR}_+(G)$.

Suppose $1.2 \in J(G)$ and $M \in \mathcal{D}_{1.2+}(G) \cap \mathcal{S}_+(G)$.
Then $\operatorname{mr}_+(G) = \operatorname{mr}_+(H_{\mathcal{A}}) + \operatorname{mr}_+(H_{\mathcal{B}})$ and there exist $\widetilde{M_{\mathcal{A}}} \in \mathscr{MR}_+(\widetilde{H_{\mathcal{A}}})$ and $\widetilde{M_{\mathcal{B}}} \in \mathscr{MR}_+(\widetilde{H_{\mathcal{B}}})$ such that
$M = \widetilde{M_{\mathcal{A}}} + \widetilde{M_{\mathcal{B}}} \in \mathcal{S}_+(G)$. Then

$$\begin{aligned} \operatorname{mr}_+(G) \leqslant \operatorname{rank} M \leqslant \operatorname{rank} \widetilde{M_{\mathcal{A}}} + \operatorname{rank} \widetilde{M_{\mathcal{B}}} &\\ = \operatorname{mr}_+(\widetilde{H_{\mathcal{A}}}) + \operatorname{mr}_+(\widetilde{H_{\mathcal{B}}}) &\\ = \operatorname{mr}_+(H_{\mathcal{A}}) + \operatorname{mr}_+(H_{\mathcal{B}}) = \operatorname{mr}_+(G). \end{aligned}$$

Then $\operatorname{rank} M = \operatorname{mr}_+(G)$ so $M \in \mathscr{MR}_+(G)$.

Thus in both cases, $M \in \mathscr{MR}_+(G)$ and so $\mathscr{MR}_+(G) \supset (\cup_{i \in J(G)} \mathcal{D}_{i+}(G)) \cap \mathcal{S}_+(G)$.
Therefore $\mathscr{MR}_+(G) = (\cup_{i \in J(G)} \mathcal{D}_{i+}(G)) \cap \mathcal{S}_+(G)$. □

6. Conclusion

Our aim in this work was to go beyond the problem of determining the minimum rank of a specified graph G to understanding the structure of the class of matrices which attain the minimum rank of G.

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For graphs with a cut vertex, the structure is given by Theorem 3.1 and Corollary 3.3. For graphs with a 2-separation, it is given by Theorems 4.4 and 4.16. Theorems 5.7 and 5.9 give the positive semidefinite analogues. As a by-product our results have clarified some of the principal results on minimum rank. There are two terms on the right hand side of the formula in Theorem 2.3 for $\mathrm{mr}^F(G)$ because of the two different ways a matrix in $\mathscr{MR}^F(G)$ can decompose according to Theorem 3.1. Theorem 4.4 explains more clearly the reason for the six terms on the right hand side of Theorem 2.6. They arise from the five mutually exclusive and exhaustive cases in Theorem 4.4 involving equalities and inequalities on ranks of particular submatrices of a given minimum rank matrix.

More importantly, we expect our results to provide a simpler approach to the inverse eigenvalue problem for $\mathscr{MR}(G)$ (see [2]) for graphs for which Corollary 3.3 and Theorem 4.16 (or the positive semidefinite analogues) can be applied recursively to obtain a parametric representation of all matrices in $\mathscr{MR}(G)$ as in Example 4.17. This will be possible not only for trees, but for many graphs with relatively few edges.

For those graphs G with a complete characterization of $\mathscr{MR}(G)$, one can think of extending these results to other matrices in $\mathcal{S}(G)$, for example

$$\{A \in \mathcal{S}(G) \mid \operatorname{rank} A = \mathrm{mr}(G) + 1\}.$$

Such results would conceivably help in solving the inverse eigenvalue problem for $\mathcal{S}(G)$.

Finally, we expect that many of these results will extend directly to inertia classes of graphs, a line of inquiry that some of the co-authors plan to pursue.

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