Two conjectures on strong embeddings and 2-isomorphism for graphs

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Abstract

We present two conjectures related to strong embeddings of a graph into a surface. The first conjecture relates equivalence of integer quadratic forms given by the Laplacians of graphs, 2-isomorphism of 2-connected graphs, and strong embeddings of graphs. We prove various special cases of this conjecture, and give evidence for it. The second conjecture, motivated by ideas from physics and number theory, gives a lower bound on the number of strong embeddings of a graph. If true, this conjecture would imply the well-known Strong Embedding Conjecture.

1 Introduction

A major outstanding problem in graph theory is the *cycle double cover conjecture* which states the following:

Cycle Double Cover Conjecture. *For any bridgeless graph G, there is a list of cycles of G such that each edge of G is in precisely two of the cycles.*

The cycle double cover conjecture was stated in the 1970's by Szekeres in [12] and by Seymour in [11]. They observe that any minimal counterexample to the conjecture must be a cubic (3-regular) graph. Seymour further points out that one way to produce a cycle double cover of a graph is via a *strong embedding* of the graph into some surface. For an embedding to be strong means that each face of the embedding is a disk and corresponds to a simple cycle in the graph. The precise definitions will be given in section 2. A strong embedding gives rise to a cycle double cover since, given a strong embedding of a graph into some surface, each edge will be incident to two faces, each face being a cycle, so the list of faces from the embedding is a cycle double cover.

Seymour cites a conjecture, attributed to Tutte, that any bridgeless cubic graph has a strong embedding into a surface of its minimal genus. This conjecture turns out to be false due to an example of Xuong from 1977 in [16], which we will present in section 2, figure 2. This example is a 3-regular graph of genus one, but which has the property that all embeddings into the torus are not strong. However, this does not preclude the existence of a strong embedding of the graph into a surface of higher genus, and indeed this graph has a strong embedding into a surface of genus two. It is thus natural to conjecture the weaker *strong embedding conjecture*:

Strong Embedding Conjecture. *Every 2-connected graph has a strong embedding into some orientable surface.*

The strong embedding conjecture is still enough to imply the cycle double cover conjecture, and indeed, in 1985, Jaeger [7] showed that the strong embedding conjecture and the cycle double cover conjecture are equivalent.

The purpose of this paper is to present two conjectures related to strong embeddings of graphs. The first conjecture relates three notions of equivalence of graphs, one motivated by an aritmetic condition, one coming from a geometric condition, and one relating a combinatorial condition. Motivated by ideas from physics and number theory, our arithmetic condition invloves equivalence of quadratic forms over the

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integers given by Laplcians of graphs. Motivated by an interesting connection between duals of planar graphs with congruent Laplacians over $\mathbb{Z}$, we believe this arithmetic condition is related to a geometric equivalence condition which involves graph duals with respect to minimal strong embeddings. These dual graphs also arise as Feynman diagrams of the matrix model for 2D quantum gravity. As an intermediate step we use a combinatorial graph equivalence relation that has been well studied, namely the concept of 2-isomorphism of graphs, which was shown to be equivalent to integer congruence of the Laplacians by Watkins in [13] and [14]. The precise statement of the conjecture is in section 2. The combinatorial implications of quadratic form equivalence are also discussed in [6].

Our second conjecture, motivated by ideas from physics and number theory, conjectures a lower bound on the total number of strong embeddings of a graph. The lower bound involves the number of ways to orient the graph such that the convex hull of the columns of the resulting incidence matrix are maximal. The precise statement of the conjecture is in section 5. This conjecture would imply the strong embedding conjecture, and hence the cycle double cover conjecture.

The remainder of this paper is organized as follows. In section 2, we give the precise definitions for strong embeddings, quadratic forms, and 2-isomorphism, and all the relevent background information, as well as state Conjecture 1. In section 3, we will show that the arimetic and combinatorial conditions are equivalent. In section 4 that Conjecture 1 holds for the case of planar graphs. We will also present further evidence for why we think this conjecture holds more generally, and give examples showing that weaker versions of the conjecture cannot hold. Section 5 describes a procedure for producing a convex polytope from an orientation on a graph, and presents Conjecture 2, which gives a lower bound on the number of strong embeddings of a graph (thus implying the strong embedding conjecture). In section 6 will try to give a physical intuition to the volume of the polytope, which we hope will eventually give a physics interpretation of Conjecture 2. Finally, in section 7, we will discuss hypergeometric exponential sums associated to graphs, which give further invariants for equivalence classes of 2-connected graphs under 2-isomorphism, and may provide tools for investigating the strong embedding conjecture.

2 Strong embeddings, quadratic forms, and 2-isomorphism

2.1 Strong embeddings

Definition 1. An embedding of a graph in a surface is called a *2-cell embedding* if each face (component of the complement of the image of the embedded graph) is homeomorphic to an *open* 2-disk. It is *strong* if the closure of each face is homeomorphic to a *closed* 2-disk. In other words, the boundary of each face of a strong embedding is a simple cycle. Figure 1 shows a strong and non-strong embedding of the complete graph K_5 into the torus.

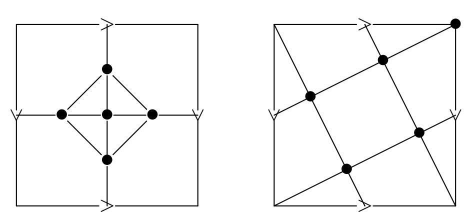
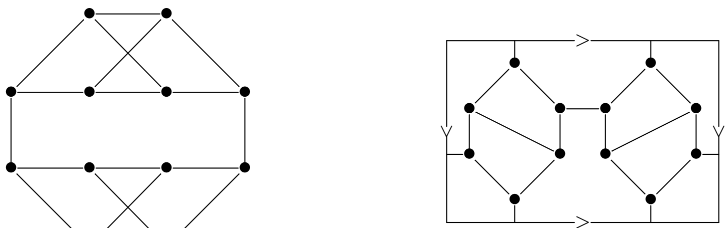


Figure 1: Two embeddings of K_5 into a torus. The first is not strong, the second is.

Note that for a strong embedding to exist G must be 2-connected (i.e., removal of any single vertex cannot disconnect the graph), and the following discussion will be limited to 2-connected graphs. Denote by $g(G)$ the minimum genus of a surface into which there exists a 2-cell embedding of G, and by $\bar{g}(G)$ the minimum genus of a surface into which there exists a strong embedding of G. For planar graphs $\bar{g}(G) = g(G) = 0$, but there exist graphs for which $\bar{g}(G) > g(G)$. An example due to Xuong [16] has $g(G) = 1$, but $\bar{g}(G) = 2$ (see Figure 2). Furthermore, Mohar in [8] gives more examples of cubic graphs with $\bar{g}(G) > g(G)$, and in fact gives an infinite family of graphs where $g(G)$ and $\bar{g}(G)$ become arbitrarily far apart.



(a) G

(b) An embedding of G into a torus that is not strong.

Figure 2: An example due to Xuong [16] with $g(G) < \bar{g}(G)$.

While it is straightforward to see that $g(G) \leq |E| < \infty$ for any graph G, the situation for $\bar{g}(G)$ is much more difficult. The strong embedding conjecture discussed above says that $\bar{g}(G)$ is finite for all 2-connected graphs.

Given a 2-cell embedding i of a graph G into an orientable surface, we define the *dual graph*, G_i^*, to be the (multi-)graph whose vertex set is the faces of the embedding, and where two vertices in G_i^* have an edge for every edge of G incident to each of the corresponding faces. When the embedding i of G is strong, the condition that the closure of each face is a simple cycle is equivalent to saying that the dual graph G_i^* has no loops on a single vertex.

2.2 Quadratic forms over $\mathbb{Z}$

Given a graph G on n vertices, consider its *combinatorial Laplacian*, $L = D - A$, where D is the diagonal degree matrix, and A is the adjacency matrix of G. It is well-known that the combinatorial Laplacian is positive semi-definite, and can be decomposed as $L = BB^t$, where B is the incidence matrix of G with respect to some arbitrary orientation of the edges. As a square matrix, L can be interpreted as a quadratic form in n variables.

Definition 2. We say $L(G) \cong L(H)$ over $\mathbb{Z}$ if there exists $M \in GL(n, \mathbb{Z})$ such that

$$ML(G)M^t = L(H).$$

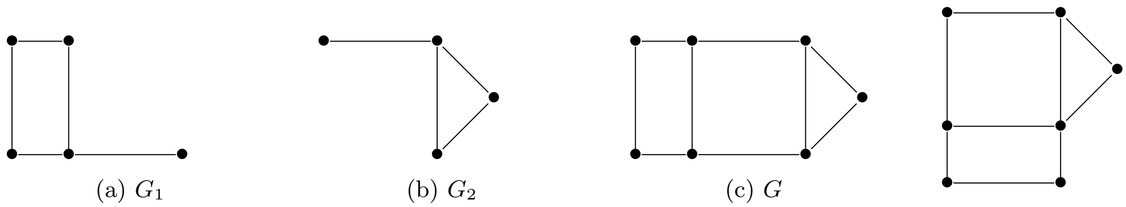
Note that if two graph G and H are isomorphic, then G and H have congruent Laplacians via a permutation matrix M. Another related equivalence is *similarity*, namely $ML(G)M^{-1} = L(H)$ for some invertible M. If $L(G)$ and $L(H)$ are similar, then the graphs G and H are *cospectral*, that is, they have the same Laplacian spectrum. Cospectral graphs have been extensively studied in spectral graph theory, but there is no known combinatorial characterization of cospectrality.

Congruence over the integers, also called *unimodular congruence* for Laplacian matrices was considered by Watkins in the 1990s. In [13] and [14], Watkins gives a combinatorial characterization of congruence of Laplacians over the integers. We will present this characterization in section 3, and before doing so, we will need some definitions in the following section.

2.3 2-isomorphism

Definition 3. For $i = 1, 2$, let G_i be a connected graph with at least 3 vertices, two of which are x_i and y_i. We say the graph obtained from the disjoint union $G_1 \cup G_2$ by identifying $x_1 \equiv x_2, y_1 \equiv y_2$ is related by a *2-switch* to the graph obtained from $G_1 \cup G_2$ by identifying $x_1 \equiv y_2, y_1 \equiv x_2$. A 2-switch is illustrated in Figure 3.

Two graphs are called *2-switch equivalent* if one can be obtained from the other via a sequence of 2-switches. Two graphs G and H are called *cycle equivalent* or *cycle isomorphic* if there is a bijection between the edge sets $E(G)$ and $E(H)$ that maps cycles to cycles. In 1933, Whitney [15] proved the well-known Whitney 2-isomorphism theorem which states that if G and H are 2-connected graphs, then G and H are



(a) G_1 (b) G_2 (c) G (d) H

Figure 3: The graph G and H are each formed by identifying two vertices of G_1 wtih two vertices of G_2, but in different orders. Hence G and H are related by a 2-switch.

cycle isomorphic if and only if they are 2-switch equivalent. Therefore for 2-connected graphs, both these equivalences can then be referred to as *2-isomorphism.* Note in particular that two 3-connected graphs are 2-isomporphic if and only if they are isomorphic.

Remark. Notice also that two graphs are 2-isomorphic if the associated cycle matroids of the two graphs are isomorphic as matroids. In [10], it is shown that the computational complexity of the matroid isomorphism problem is polynomial time equivalent to the graph isomorphism problem. Due to recent work of Babai [1], graph isomorphism can be solved in quasipolynomial time. Hence 2-isomorphism can also be solved in quasipolynomial time.

2.4 Conjecture 1

The first goal of this paper is to present a conjecture which relates congruence over the integers (an arithmetic condition we will call (A)), 2-isomorphism (a combinatorial condition we will call (C)), and a geometric condition (G) that equates duals of minimal strong embeddings.

Denote by G_i^* the dual of G with respect to some 2-cell embedding i of G into a surface. Denote by I_G the set of all minimal strong embeddings of G, that is, embeddings of G into a surface of genus $\bar{g}(G)$.

Conjecture 1. *The following are equivalent for 2-connected graphs G and H:*

(A) $L(G) \cong L(H)$ *over* $\mathbb{Z}$.

(C) G *and* H *are 2-isomorphic.*

(G) There exists a bijection $\phi : I_G \to I_H$ *such that* G_i^* *is isomorphic to* $H^*_{\phi(i)}$, *for every* $i \in I_G$.

In section 3, we will prove that (A) and (C) are equivalent, so the conjecture is that the dual graph condition (G) is equivalent to each of these. In section 4, we will show that (C) is equivalent to (G) for the case of planar graphs, and we will give evidence for why we think the conjecture holds more generally, including examples that show that weaker forms of the conjecture do not hold in general.

3 Quadratic form equivalence and 2-isomorphism

In this section, we prove that conditions (A) and (C) are equivalent. That is, if G and H are 2-connected graphs, then they have equivalent quadratic forms over the integers if and only if they are 2-isomorphic. We will present the proof given by Watkins in [13] and [14], with some modifications to put more emphasis on the incidence matrix of the graph.

Theorem 1 (Watkins)**.** *Two 2-connected graphs G and H on n vertices, with Laplacians L_1 and L_2 respectively, are 2-isomorphic if and only if there exists some* $M \in GL(n, \mathbb{Z})$ *such that* $ML_1M^t = L_2$.

Lemma 1. *Let G be a graph on n vertices with a 2-separation, and H be the graph obtained after performing a 2-switch on that separation. Then there is a matrix* $M \in GL(n, \mathbb{Z})$ *such that* $ML_1M^t = L_2$.

Proof. Let x and y denote the vertices of the 2-separation, and define

$$M = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 1 & 0 & \mathbf{1}^T \\ 0 & 0 & 1 & \mathbf{1}^T \\ 0 & 0 & 0 & -I \end{bmatrix}$$

where the middle two rows are indexed by x and y, and $\mathbf{1}$ denotes the vector of all ones. It is clear that $M \in GL(n, \mathbb{Z})$, and direct computation shows that $ML_1M^t = L_2$. See [14] for details. □

Lemma 2. *Suppose $ML_1M^t = L_2$ for some $M \in GL(n, \mathbb{Z})$, then $\mathbf{1} = (1, ..., 1)^t$ is an eigenvector of M^t, with eigenvalue 1 or -1.*

Proof. By basic properties of graph Laplacians, $L_2\mathbf{1} = 0$. Thus $ML_1M^t\mathbf{1} = 0$, which implies $L_1M^t\mathbf{1} = 0$ since M is invertible. As G is connected, the multiplicity of 0 as an eigenvalue of L_1 is 1. Thus we must have $M^t\mathbf{1} = \lambda\mathbf{1}$ for some complex number λ. As $M^t \in GL(n, \mathbb{Z})$, it is then clear that $\lambda \in \mathbb{Z}$. We have also $(M^t)^{-1}\mathbf{1} = \lambda^{-1}\mathbf{1}$. As $(M^t)^{-1} \in GL(n, \mathbb{Z})$, we have $\lambda^{-1} \in \mathbb{Z}$ as well. Thus $\lambda = 1$ or $\lambda = -1$. □

Lemma 3. *If B_1, B_2 are incidence matrices for G, H respectively, then MB_1 is an incidence matrix for H as well.*

Proof. Let E_1, E_2 denote the number of edges of G and H, respectively. By assumption, there exists $M \in GL(n, \mathbb{Z})$, such that $ML_1M^t = L_2$. We can multiply M by $-I$ if necessary, to make sure that $\mathbf{1}$ is an eigenvector of M^t, with eigenvalue 1. Choose arbitrary labeling and orientations of all edges of G and H, and consider the corresponding vertex-edge incidence matrices B_1, B_2 of size $n \times E_1$ and $n \times E_2$, respectively. Let $F := MB_1$. We have $B_iB_i^t = L_i, i = 1, 2$, and $FF^t = L_2$. Note that any column of B_1 is of the form $\pm(0, ..., 1, 0, ..., 0, -1, 0, ...0)$, so $\mathbf{1}^tF = \mathbf{1}^tMB_1 = \mathbf{1}^tB_1 = 0$, so F is an integral matrix all of whose column sums are equal to 0. Suppose some column of F is a zero vector, say, without loss of generality, the first column of F is the zero vector. Then we have $\sum_{k=1}^{n} M_{ik}B_{1k1} = 0$, for all $1 \le i \le n$. Suppose the u, v-th places of the first column of B_1 are nonzero, then the identity gives $M_{iu} = M_{iv}$, for all $1 \le i \le n$. Namely, M has two different columns which are equal. This is not possible since M is invertible. Thus, any column of F is nonzero, and we therefore have for any $1 \le k \le E_1$,

$$\sum_{i=1}^{n} F_{ik}^2 \ge 2 \tag{1}$$

which takes the equal sign if and only if the k-th column of F is of the form $\pm(0, ..., 1, 0, ..., 0, -1, 0, ...0)$.

Indeed, since L_1 and L_2 are equivalent as integral quadratic forms, then (1) must take the equal sign for all $1 \le k \le E_1$. Thus,

$$E_2 = \frac{1}{2}trL_2 = \frac{1}{2}\sum_{k=1}^{E_1}\sum_{i=1}^{n} F_{ik}^2 = E_1 \tag{2}$$

So G and H have the same number of edges, which we will denote now as E, and furthermore, each column of F has the form $\pm(0, ..., 1, 0, ..., 0, -1, 0, ...0)$.

For any $1 \le u \ne v \le n$, as $L_{2uv} = \sum_{k=1}^{E} B_{2uk}B_{2vk}$, $L_{2uv} = -1$ if and only if there exists some (necessarily unique) k, such that the k-th column of B_2 is nonzero precisely at the u, v-th places. As $FF^t = L_2$, by the previous lemma, the same applies to F. Thus, for any column of B_2, there exists a column of F, which is equal to the column up to a sign. As F and B_2 have the same number of columns, then F and B_2 are identical up to re-labeling and re-orienting the edges. Thus the lemma follows

□

Proof of Theorem 1. First, assume that G and H are 2-isomorphic. Then we know that H can be obtained from G by a sequence of 2-switches. Applying Lemma 1 to each successive 2-switch, we obtain the desired matrix.

For the other direction, let B_1 be the incidence matrix of G, and by Lemma 3, we have $F = MB_1$ is an incidence matrix for H. Now a cycle in G corresponds to a vector $c \in \{0, \pm1\}^{|E(G)|}$ in the kernel of B, and hence in the kernel of F, so it is a cycle in H too. This means that the bijection between $E(G)$ and $E(H)$ arising from pairing columns of B_1 and of F preserves cycles, and hence is a cycle equivalence, which we have remarked is equivalent to 2-isomphism for 2-connected graphs.

□

Remark. The hypothesis that G and H be 2-connected is not really necessary, but we are only interested in the 2-connected case for the purposes of our conjecture. In fact, [14] has a slightly more general definition of 2-switch equivalence, and gives a version for Theorem 1 for all pairs of connected finite simple graphs.

Remark. S. Friedland [5] showed that the quadratic form $Q_\alpha(G)$ represented by $L(G) + \alpha I$ determines the graph G (up to isomorphism) for sufficiently large $\alpha > 0$. On the other hand, it is easy to see that $Q_\alpha(G)$ is far from sufficient when $\alpha < 0$. Theorem 1 shows that the untranslated Laplacian ($\alpha = 0$) exhibits a unique interesting behavior, by giving a complete combinatorial characterization of this arithmetic invariant of graphs.

4 Graph embeddings and 2-isomorphism

We have seen that the arithmetic (A) and combinatorial (C) conditions are equivalent, so the conjecture relates these to a geometric condition (G). We can show that the conjecture holds for planar graphs.

Theorem 2. *If G and H are 2-connected planar graphs, then G and H are 2-isomorphic if and only if there is a bijection between the set of embeddings of G and the set of embeddings of H that preserves the dual graph.*

Proof. Assume that we have such a bijection that preserves duals. Pick a planar embedding of G and a matching planar embedding of H such that the duals G^* and H^* are two embeddings of the same graph. It is well-known that two embeddings of a planar graph have 2-isomorphic duals, hence G and H are 2-isomorphic.

For the other direction, it suffices to produce a bijection for two planar graphs G and H related by a single 2-switch, say at x and y. We do this as follows.

Given an embedding of G into the sphere, using the Jordan closed curve theorem we can find a canonical curve γ through x and y that separates the sphere to two regions. We then cut the sphere along γ and glue one region back after a 180° rotation, yielding an embedding of H (see Figure 4).

One can verify that this operation does not change whether two regions neighbor each other, thus the dual graph remains unchanged.

□

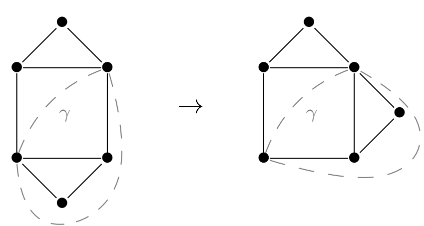


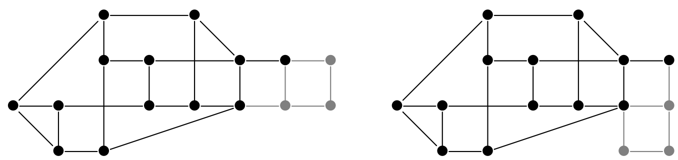
Figure 4: The canonical cut curve γ giving a region whose rotation accomplishes a 2-switch.

Intuitively, it seems that our proof that (C) $\implies$ (G) for planar graphs should be generalizable to 2-connected graphs of higher genus. This may involve some more involved topological arguments to construct the cononical cut curve γ for minimal strong embeddings on a higher genus surface. This is a work in progress.

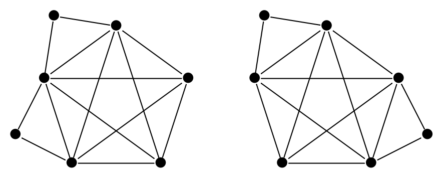
4.1 Rigidity of the conjecture

In this section, we will discuss the rigitiy of our conjecture. Namely, we will discuss why each of the conditions we have placed in each equivalence is necessary, and discuss why several weaker conjectures do not hold.

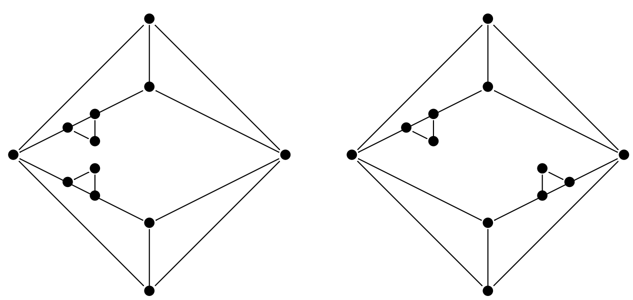
- Congruence in (A) must be over $\mathbb{Z}$: e.g. the conjecture holds for neither congruence over $\mathbb{R}$, $\mathbb{Q}$, nor over all p-adic integers $\mathbb{Z}_p$.
- Embeddings in (G) must be strong; the conjecture fails to hold when using all embeddings of G into a surface of genus $g(G)$. In the graphs pictured below, the graph on the right has more embeddings into the torus, but both have the same number of strong ones.



- We need to consider all minimal strong embeddings in (G); one pair of minimal strong embeddings with isomorphic dual graphs does not suffice. In the pair of graphs pictured below, there is a pair of embeddings such that the dual graphs are isomorphic, but there is no bijection between sets of embeddings for which all dual graphs are isomorphic. Observe also that these graphs are not 2-isomorphic since any 2-switch on either of them leaves the graph unchanged.



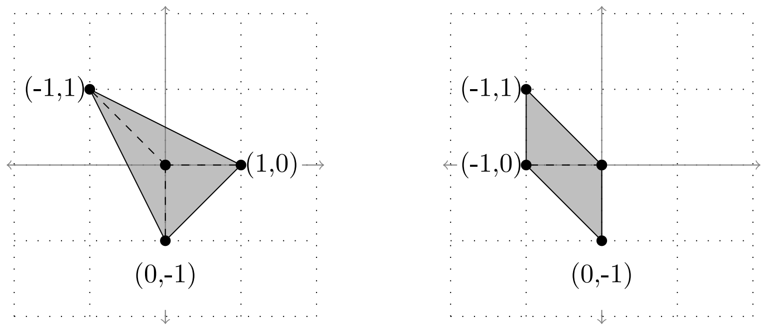
- We must consider only *minimal* strong embeddings. The conjecture fails to hold if we consider all strong embeddings of the graph. The two graphs below are 2-switch equivalent, and the conjectured bijection between minimal strong embeddings exists for this pair of graphs, however, such a bijection does not exist between the sets of all strong embeddings of the graphs.



5 A partial quantification of the strong embedding conjecture

Partially inspired by Conjecture 1, the aim of the following is to present a conjecture regarding a lower bound of the number on minimal strong embeddings, using the incidence matrix B.

Let $G = (V, E)$ be a 2-connected finite simple graph. For each choice of its edge orientations, we consider the Newton polytope (convex hull) of the columns of B, together with the zero vector, and denote it by P_B. Let $n = |V|$ and $m = |E|$. Clearly P_B is a full rank polytope on the hyperplane $x_1 + ... + x_n = 0$ in $\mathbb{R}^n$. We are interested in the Euclidean volume of this convex polytope. For convenience, we take the following normalization for the volume: pick any row of B and delete it from B. Then we are left with column vectors of the reduced matrix. The Newton polytope of these column vectors and the zero vector is a full rank polytope in $\mathbb{R}^{n-1}$, that is the projection of P_B onto one of the coordinate hyperplanes. We define the normalized volume $Vol(P_B)$ to be the Euclidean volume of this projected polytope, multiplied by $(n-1)!$. Clearly, this normalization is independent of the choice of projection, and the resulting volume is always a positive integer. See Figure 5 for an example involving two orientations of a cycle on three vertices, C_3.



(a) $B = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ (b) $B = \begin{bmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

Figure 5: Two orientations of C_3 give two different incidence matrices and two polytopes obtained by deleting the last row and using the columns as points. The (normalized) volume of the first is 3, and of the second is 2.

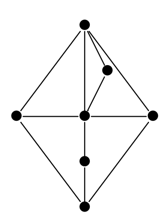
Next, we pick out all the edge orientations of G that maximize the resulting volume $Vol(P_B)$, and denote the collection of these orientations by the set O_m. For any two orientations X and Y in O_m, we say $X \cong Y$ if and only if there exists a sequence of elements in O_m, $X_1 = X, X_2, ..., X_s = Y$, such that any consecutive elements X_i, X_{i+1} differ by a graph automorphism, a possible reversal of orientations on all the edges, or a possible reversal of all edge orientations on an isolated directed path, where by isolated directed path, we mean a subgraph which is itself a directed path, and is connected to the rest of the graph only through its starting vertex and its ending vertex. Clearly, this is an equivalence relation, and we consider the equivalence classes $O_m/\sim$.

On the other hand, given two minimal strong embeddings of G into Σ_g with chosen surface orientation, we say they are equivalent if and only if there is an orientation preserving homeomorphism of Σ_g to itself, such that the homeomorphism maps the image of G under one embedding to that under the other embedding, through an automorphism of G. Now we state

Conjecture 2. *The number of equivalent classes in O_m, is a lower bound of the number of equivalence classes of minimal strong embeddings.*

Remark. Clearly, the above conjecture would imply the strong embedding conjecture.

We have gathered numerous numerical evidence in support of this conjecture. For instance, for any cylce graph C_k, there is only one embedding of C_k into the plane (or sphere), and the only orientation for which the volume of the polytope is maximal is the cyclic orientation. (Either direction achieves the maximum, but these are equivalent under our equivalence relation.) Thus, in this case the number of equivalence classes of orientations in O_m and the number of equivalence classes of strong embeddings are equal. Further examples that we have checked include the complete graph K_5, the complete bipartite graph $K_{3,3}$, the Xuong graph of Example 2, and numerous others. The graph pictured below is an example where all of the conditions in our equivalence relation are necessary for the conjecture to hold. We give more details in the appendix.



6 Free fermions on a graph

In this section, we give a physics interpretation of $Vol(P_B)$ in Conjecture 2. For a general background on the physics, one can see [9].

Since $L = BB^t$ for any chosen orientation of edges, B has been called the "Dirac matrix" in graph theory. In the following, we shall make sense of a theory of free fermions on a graph, and show that its partition function computes precisely the volume of P_B.

Let ψ_E, ψ_V, respectively, denote functions $E \to \mathbb{R}^+$, $V \to \mathbb{R}$.[1] Note that B maps ψ_E to ψ_V. In the conventional continuous case, the Dirac operator is local, in the sense that it acts on germs of sections. However, in the discrete situation of a graph, B is no longer local in this sense, but instead, it maps functions on edges to functions on vertexes.

We first formally write down the Langrangian $\mathcal{L} = \psi_V B \psi_E$. In order to match the physical degrees of freedom[2] on edges and on vertices, we should regard two field configurations (values of ψ_V and ψ_E) as equivalent if $\psi_V B \psi_E$ become equal. For ψ_V, this amounts to saying that any constant configuration is equivalent to the zero configuration. For ψ_E, this is slightly more complicated, as each loop of G gives rise to a nontrivial element in the kernel of B. Therefore, we adopt the following instruction on performing the path integral: For each spanning tree T of G, components of $\psi_E|_T$ are all independent degrees of freedom on the edges. We integrate over all possible configurations of $\psi_E|_T$, and ψ_V up to the constant shift. As it is a fermion integral, by standard Grassmann integration we have

$$\int e^{\mathcal{L}} d\psi_E|_T d\psi_V = det(B|_T) \tag{3}$$

where it is easy to show that $det(B|_T) = 1$ or -1.

For each T, $B|_T$ maps the configuration space of $\psi_E|_T$[3] to a cone in $\mathbb{R}^{n-1}$, over the face formed by the corresponding column vectors. Note that the sign of $det(B|_T)$ depends on how we label the columns (and the vertices), whereas the Dirac operator should be independent of this choice. Thus, we use the convention that the labeling of columns in the integration measure is always in a way such that this determinant is positive. Note that $1 = det(B|_T)$ is the normalized volume of the polytope spanned by the column vectors of T and the zero vector.

In addition, if we have two trees T_1 and T_2, and two configurations ψ_{E1} and ψ_{E2}, such that

$$B|_{T_1}(\psi_{E1}|_{T_1}) = B|_{T_2}(\psi_{E2}|_{T_2}), \tag{4}$$

then these two edge configurations couple in exactly the same way to vertex degrees of freedom, thus, we should only count for them once in the path integral. Note that equation 4 means the associated cones of T_1 and T_2 have overlap in the interior. Thus, in defining the full fermion path integral, we only sum over trees T, such that the associated cones are minimal, in the sense that the interior of the cones do not overlap with any other cone constructed in this way by a spanning tree that does not contain the original cones. To justify this procedure, we start with the following canonical triangulation of P_B into simplexes:

Lemma 4. *Connecting the origin 0 with all (other) vertices of P_B, gives a triangulation of P_B into simplexes.*

Proof. Take any face F of P_B, F is itself a convex polytope, all of whose vertices are vertices of P_B. If $0 \notin F$, then the subspace spanned by F does not contain 0, thus for any cycle C in the graph G, not all the column vectors corresponding to its edges can lie in F. Therefore, the collection of edges whose column lie in F form a forest subgraph of G. On the other hand, F has dimension $n-2$, so the forest has to be a spanning tree, thus F is itself a simplex.

Now suppose $0 \in F$. Suppose all column vectors corresponding to its edges lie in F. We denote by F_C the convex hull of these vectors and 0. Then $F_C \subset F$, and the subspace spanned by F_C intersecting F is F_C. Connecting 0 with all of these column vectors of edges of C gives a triangulation of F_C into simplexes. Then one easily sees by induction that, further connecting 0 with all other vectors on F extends this triangulation to a triangulation of F by polytopes. Next, connecting 0 with all other vectors of P_B

[1] One may think of them as the simplest spinor sections on the graph G.

[2] i.e. the number of independent variables.

[3] i.e. the space of all possible $\psi_E|_T$.

extends this triangulation of F, into a triangulation of P_B by polytopes. These new polytopes have the property that column vectors corresponding to edges of C no longer can all lie on any one polytope entirely.

Thus, we obtain a triangulation of P_B by connecting 0 with all vectors of P_B. Suppose C_1 is any cylce in G. Repeating the above argument for C_1, we see that this exact same triangulation also prohibits all edge columns of C_1 to lie on any single polytope. Therefore, any polytope of the triangulation corresponds to a forest, and thus, by the same dimension argument as in the beginning of the proof, the forest is a spanning tree, and so the polytope is a simplex. □

Remark. It is easy to see that any column vector of an edge is a vertex of P_B.

So any of these simplexes corresponds to a spanning tree, and thus has volume equal to 1. Note that all these simplexes in the triangulation are polytopes corresponding to minimal cones: for if not, there exists another cone over a polytope of column vectors of a spanning tree, such that its intersection with the original cone is a proper subcone, which has to be again a cone over a polytope of column vectors of a spanning tree, thus the proper subcone has normalized volume equal to 1, a contradiction to convexity. On the other hand, for any spanning tree, the polytope of its vertices and zero is contained in P_B, thus it is covered by the union of polytopes corresponding to minimal cones. Therefore, any spanning tree with a minimal cone gives one of the simplexes in the canonical triangulation, and conversely, any spanning tree whose associated cone is not minimal does not correspond to one of the simplexes in the canonical triangulation. This implies that our sum over spanning trees that show up in the canonical triangulation covers the configuration space with no repetition.

Therefore, we have the following:

$$\int e^{\mathcal{L}} d\psi_E d\psi_V = Vol(P_B). \tag{5}$$

7 Exponential sums

Given a graph G, let $B = (b_{ij})$ be its $|V| \times |E|$ incidence matrix with respect to a particular edge orientation (also called the Dirac matrix). Fix a finite field $\mathbb{F}_q$; B gives rise to a *hypergeometric exponential sum*

$$S_B = \sum_{x_1,\dots,x_{|V|} \in \mathbb{F}_q^*} e^{\frac{2\pi i}{q}\left(\sum_{j=1}^{|E|} t_j \prod_{i=1}^{|V|} x_i^{b_{ij}}\right)}$$

An example is given by the famous Kloosterman sums, achieved by taking B to be the Dirac matrix for a cyclically oriented cycle graph. The sum is independent of the choice of vertex and edge labelings, up to a permutation of the parameters t_j.

A direct computation shows that S_B depends on B only up to a $GL(|V|, \mathbb{Z})$ transformation. Consider the set of all possible exponential sums associated to a graph in this way. From our arithmetic characterization of 2-isomorphism, one shows that this set gives an invariant of 2-connected graphs, that are preserved under 2-switchings (or cycle equivalence). From number theory, one expects S_B to determine B up to the $GL(|V|, \mathbb{Z})$ transformation, therefore there are good reasons to expect that this set of exponential sums gives a very strong invariant of the 2-isomorphic class of graphs.

Next, as explained earlier, it is clear that 2-isomorphic graphs should have the same minimal strong embeddings as detected by the dual graphs, one can then expect that most information regarding those dual graphs can be extracted from these exponential sums. This provides a much more refined invariant of the graph than $Vol(P_B)$.

On the other hand, exponential sums like these were very fruitfully studied by many people e.g. A. Weil, Grothendieck-Deligne, using deep tools from arithmetic algebraic geometry, via *ℓ-adic sheaves*. [4] provides a general introduction. More specifically, sums of our kind can be studied through these ideas, via the additional help of *GKZ theory* [3]. In particular, by ℓ-adic GKZ theory, the rank of (the ℓ-adic sheaf corresponding to) S_B is given by volume of the Newton polytope of columns of B, together with the zero vector (normalized so that the unit simplex has volume one), the (Frobenius) weight is $|V| - 1$ [2], and therefore a sharp upper bound estimate on the absolute value of generic exponential sums is given by

$$|S_B| \leq q^{weight/2} \times \text{rank}.$$

For example, take the cycle graph C_n, clockwise or counterclockwise edge orientations correspond to a unique maximal rank exponential sum, which is exactly the Kloosterman sum in $n-1$ variables.

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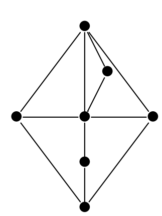
A Appendix

Denote by G the graph pictured below.



This graph is one in which we need all of the conditions in our equivalence relation on orientations on a graph in order for Conjecture 2 to hold. Figure 6 displays all of the orientations of G for which the volume of the convex polytope is maximized (fixing the orientation on one edge). Several of these are equivalent up to automorphism of the graph, e.g. (i) and (j). In other cases, one of these orientations can be obtained by reversing the orientations on an isolated (directed) path. For example, (b) can be obtained from (a) by reversing the orientation of the upper central edge (note that a single edge is an isolated path in our defintion). As a further example, (t) can be obtained from (m) simply be reversing the orientations of an isolated path of length two. Since each of these orientations achieves the maximum volume, each of these operations stays in the same equivalence class. Checking each case, it can be verified that this graph has two equivalence classes under our equivalence relation, namely $\{a, b, c, g, h, l, q, s\}$ and $\{d, e, f, i, j, k, m, n, o, p, r, t\}$. It can also be seen that G has two inequivalent embeddings in the plane.

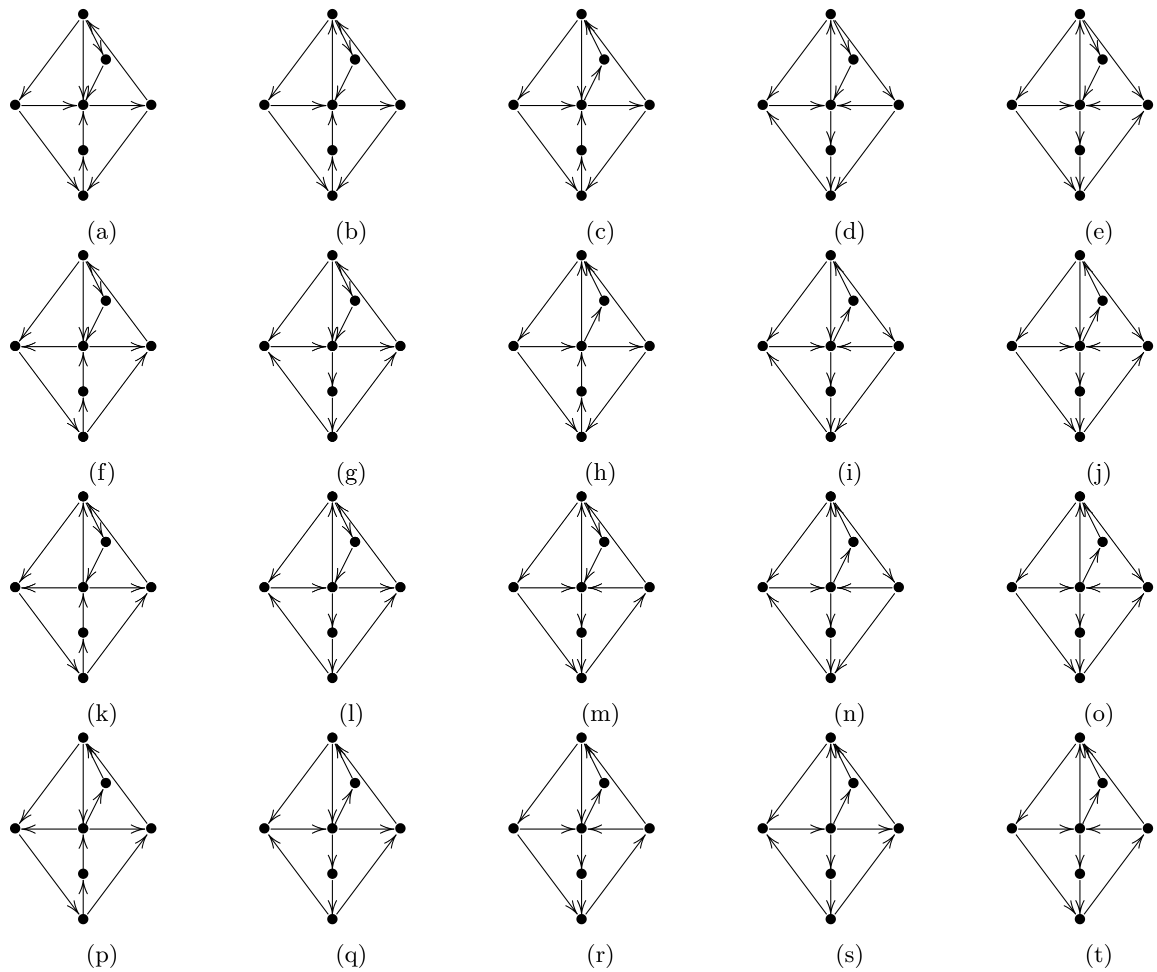


Figure 6: The orientations of G that achieve the maximum volume of the associated polytope.