TATE CLASSES AND ENDOSCOPY FOR GSp_4 OVER TOTALLY REAL FIELDS

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Abstract. The theory of endoscopy predicts the existence of large families of Tate classes on certain products of Shimura varieties, and it is natural to ask in what cases one can construct algebraic cycles giving rise to these Tate classes. This paper takes up the case of Tate classes arising from the Yoshida lift: these are Tate cycles in middle degree on the Shimura variety for the group $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_2 \times \mathrm{GSp}_4)$, where F is a totally real field. A special case is the family of Tate classes which reflect the appearance of two-dimensional Galois representations in the middle cohomology of both a modular curve and a Siegel modular threefold. We show that a natural algebraic cycle generates exactly the Tate classes which are associated to *generic* members of the endoscopic L-packets on $\mathrm{GSp}_{4,F}$. In the non-generic case, we give an alternate construction, which shows that the predicted Tate classes arise from Hodge cycles.

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1. Introduction

Let F be a totally real number field of degree d, and let $G = \mathrm{GSp}_{4,F}$. The unique elliptic endoscopic group for G is $M = (\mathrm{GL}_2 \times \mathrm{GL}_2 / \mathbb{G}_m)_F$, where $\mathbb{G}_m$ is embedded anti-diagonally and the L-embedding is induced by

$$\widehat{M} = \mathrm{GL}_2(\mathbb{C}) \times_{\mathbb{C}^\times} \mathrm{GL}_2(\mathbb{C}) \hookrightarrow \mathrm{GSp}_4(\mathbb{C}) = \widehat{G}. \tag{1}$$

The functorial transfer of cuspidal automorphic forms from M to G has been studied by Roberts [28] and Weissauer [35]. For any (unordered) pair of distinct cuspidal automorphic representations π_1, π_2 of $\mathrm{GL}_2(\mathbb{A}_F)$ with the same central character, one obtains an L-packet $\Pi(\pi_1, \pi_2)$ of cuspidal automorphic representations of $\mathrm{GSp}_4(\mathbb{A}_F)$. The members $\Pi_S(\pi_1, \pi_2)$ of this L-packet are indexed by finite sets S of places of F at which both π_i are discrete series, such that $|S|$ is even. The unique generic member of the L-packet $\Pi(\pi_1, \pi_2)$ is $\Pi_\emptyset(\pi_1, \pi_2)$.

Let $\mathbf{GSp}_4 = \mathrm{Res}_{F/\mathbb{Q}} G$ be the restriction of scalars, with the Shimura datum induced from that of $\mathrm{GSp}_{4,\mathbb{Q}}$. Let

$$S(\mathbf{GSp}_4) := \varprojlim_K S_K(\mathbf{GSp}_4)$$

be the resulting pro-algebraic Shimura variety over $\mathbb{Q}$, where K ranges over compact open subgroups of $\mathbf{GSp}_4(\mathbb{A}_f)$. (For the rest of the introduction, the same notation will apply when GSp_4 is replaced by any $\mathbb{Q}$-group H with a Shimura datum.) If π_1 and π_2 are unitary and of parallel weights 4 and 2, respectively, then

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the automorphic representations $\Pi_S(\pi_1, \pi_2)$ contribute to the cohomology of S, and Kottwitz's conjecture [16] predicts the associated Galois representations in étale cohomology.

To state the expected formula, choose a finite extension $E/\mathbb{Q}$ over which π_1, π_2, and $\Pi_S(\pi_1, \pi_2)$ are all defined. For a finite place λ of E, set $\rho_\Pi = \rho_{\pi_1} \oplus \rho_{\pi_2}(-1)$, where ρ_{π_i} are the usual λ-adic Galois representations associated to Hilbert modular forms, cf. [5]. Then let $(\widetilde{\rho}_\Pi, V)$ and $(\widetilde{\rho}_2, V_0)$ be the tensor induction of ρ_Π and ρ_{π_2}, respectively, from $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. There is a natural Galois-equivariant inclusion $V_0(-d) \hookrightarrow V$. Consider the involution $s \in \mathrm{End}(\widetilde{\rho}_\Pi, V)$ such that, in each factor of the decomposition

$$V = \otimes_{v|\infty} \rho_\Pi, \tag{2}$$

s acts as -1 on ρ_{π_1} and 1 on $\rho_{\pi_2}(-1)$. Taking s-eigenspaces induces a decomposition

$$V = V^+ \oplus V^-, \tag{3}$$

and $V_0(-d)$ lies inside V^+.

Kottwitz's conjectures imply that the $\Pi_S(\pi_1, \pi_2)_f$-isotypic part of étale cohomology should be:

$$H^{3d}_{\text{ét,c}}(S(\mathbf{GSp}_4)_{\overline{\mathbb{Q}}}, E_\lambda)_{\Pi_S(\pi_1,\pi_2)_f} = \begin{cases} V^+, & |S_f| \text{ even}, \\ V^-, & |S_f| \text{ odd}, \end{cases} \tag{4}$$

where $S_f \subset S$ is the subset of finite places. This expectation has been fully verified in the case $F = \mathbb{Q}$ [35]. Meanwhile, V_0 is also expected to appear in the étale cohomology of the Shimura variety $S(\mathbf{GL}_2)$:

$$H^d_{\text{ét}}(S(\mathbf{GL}_2)_{\overline{\mathbb{Q}}}, E_\lambda)_{\pi_{2,f}} = V_0. \tag{5}$$

The Tate conjecture then suggests that, for all S with $|S_f|$ even, there should be a $2d$-dimensional algebraic cycle on $S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2)$ whose étale realization induces a nontrivial map

$$H^{3d}_{\text{ét,c}}\left(S(\mathbf{GSp}_4)_{\overline{\mathbb{Q}}}, E_\lambda(d)\right)\left[\Pi_S(\pi_1,\pi_2)_f\right] \to H^d_{\text{ét}}\left(S(\mathbf{GL}_2)_{\overline{\mathbb{Q}}}, E_\lambda\right)[\pi_{2,f}]. \tag{6}$$

The natural candidate for this algebraic cycle is the sub-Shimura variety $S(\boldsymbol{H}) \subset S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2)$, where:

$$H \coloneqq \mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2 \xhookrightarrow{\iota, p} \mathrm{GSp}_4 \times \mathrm{GL}_2\,. \tag{7}$$

Here $\iota : H \hookrightarrow \mathrm{GSp}_4$ is the standard inclusion and $p : H \to \mathrm{GL}_2$ is the first projection.

Our first main theorem characterizes when the correspondence $S(\boldsymbol{H})$ does indeed induce a nontrivial map (6).

Theorem A (Theorem 7.2.5). *Let π_1 and π_2 be cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$ of parallel weights 4 and 2, respectively, and with the same unitary central character. Then the composite map*

$$H^{3d}_c(S(\mathbf{GSp}_4), E(d))[\Pi_S(\pi_1,\pi_2)_f] \xrightarrow{[S(\boldsymbol{H})]} H^d(S(\mathbf{GL}_2), E) \twoheadrightarrow H^d(S(\mathbf{GL}_2), E)[\pi_{2,f}]$$

is nontrivial if and only if $S_f = \emptyset$. In this case its image generates the $\mathrm{GL}_2(\mathbb{A}_{F,f})$-module $H^d(S(\mathbf{GL}_2), E)[\pi_{2,f}]$.

One could instead project to the π_f-isotypic component for any cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$; however, if $\pi \neq \pi_2$, we prove that the resulting map is always trivial. In the non-generic case $S_f \neq \emptyset$, we are not able to produce an algebraic cycle which induces a nontrivial map (6). However, we are able to give an alternative realization of (6) as the map induced by a nontrivial Hodge cycle:

Theorem B (Theorem 10.2.3). *Let π_1 and π_2 be cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$ of parallel weights 4 and 2, respectively, with the same unitary central character. Let S be a set of places of F at which both π_i are discrete series, such that $|S_f| \geq 2$ is even. Then there exists a Hodge class*

$$\xi \in H^{4d}(S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2), E(2d))$$

such that:

1. *For all finite places λ of E, the image of ξ in λ-adic étale cohomology is $\mathrm{Gal}(\overline{\mathbb{Q}}/F^c)$-invariant.*
2. *The composite map*

$$H^{3d}_c(S(\mathbf{GSp}_4), E(d))[\Pi_S(\pi_1,\pi_2)_f] \xrightarrow{\xi_*} H^d(S(\mathbf{GL}_2), E) \twoheadrightarrow H^d(S(\mathbf{GL}_2), E)[\pi_{2,f}]$$

is nontrivial, and its image generates the $\mathrm{GL}_2(\mathbb{A}_{F,f})$-module $H^d(S(\mathbf{GL}_2), E)[\pi_{2,f}]$.

Remark. (1) Assuming Kottwitz's conjectures, one could show that ξ is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant. However, our result is unconditional, and in particular does not rely on (4) and (5).
(2) In the text, we also prove higher-weight analogues of Theorems A and B; the results are described later in the introduction.

Overview of the proofs. Both Theorem A and Theorem B rely on the explicit realization of $\Pi_S(\pi_1,\pi_2)$ as a theta lift from a four-dimensional orthogonal group, cf. [28, 35]. Indeed, if $|S|$ is even, then there is a quaternion algebra B over F ramified exactly at the places in S, and the orthogonal group $\mathrm{GSO}(B) \simeq B^\times \times B^\times/\mathbb{G}_m$ is an inner form of M. The automorphic representation $\Pi_S(\pi_1,\pi_2)$ is the theta lift of $\pi_1^B \boxtimes \pi_2^B$ from $\mathrm{GSO}(B)$ to $\mathrm{GSp}_{4,F}$, where π^B is the Jacquet-Langlands transfer of π_i to $B^\times$. This is crucial because it allows for the calculation of period integrals involving $\Pi_S(\pi_1,\pi_2)$.

Proof of Theorem A. Since the non-vanishing of $[S(\boldsymbol{H})]$ may be detected in L^2 cohomology, the theorem is essentially a statement about periods of $\Pi_S(\pi_1,\pi_2)\boxtimes\pi_2^\vee$ along the subgroup $H \subset \mathrm{GSp}_4\times\mathrm{GL}_2$. That is, we must compute integrals of the form

$$\mathcal{P}_S(\gamma,\beta) := \int_{Z_H(\mathbb{A}_F)H(F)\backslash H(\mathbb{A}_F)} \gamma(\iota(h))\beta(p(h))\,\mathrm{d}h,\ \ \gamma\in\Pi_S(\pi_1,\pi_2),\ \beta\in\pi. \tag{8}$$

Because $\Pi_S(\pi_1,\pi_2)$ is a theta lift from $\mathrm{GSO}(B)$, we can compute (8) using the seesaw diagram:

$$\begin{tikzcd} \mathrm{GSp}_4 \arrow[d, dash] \arrow[drr, dash] & & \mathrm{GSO}(B)\times_{\mathbb{G}_m}\mathrm{GSO}(B) \arrow[d, dash] \arrow[dll, dash] \\ \mathrm{GL}_2\times_{\mathbb{G}_m}\mathrm{GL}_2 & & \mathrm{GSO}(B) \end{tikzcd}$$

Here the vertical lines are inclusions and the diagonals are dual reductive pairs inside GSp_{16}. Formally, the seesaw identity would read:

$$\mathcal{P}_S(\theta(\alpha),\beta) = \int_{[\mathrm{PGSO}(B)]} \theta(\beta)(g)\theta(\mathbb{1})(g)\alpha(g)\,\mathrm{d}g,\ \ \alpha\in\pi_1\otimes\pi_2,\ \ \beta\in\pi_2^\vee, \tag{9}$$

where the theta lifts on the right are from GL_2 to $\mathrm{GSO}(B)$, and the theta lifts on both sides depend on choices of Schwartz functions which must be made compatibly. The integral defining $\theta(\mathbb{1})$ is divergent, so a regularization step is necessary to interpret (9). However, after regularization, $\theta(\mathbb{1})$ can be recognized as 0 if B is not split (i.e. if $S\neq\emptyset$), and as a certain Eisenstein series on $\mathrm{GSO}(B)$ if B is split. The integral (9) then unfolds to an Euler product which allows us to evaluate it explicitly. The result of the calculation is:

Theorem C (Theorems 6.2.2, 6.5.2). *Let π_1, π_2, and π be cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$ such that π_i and $\pi^\vee$ have the same central character, and let S be a finite set of (possibly archimedean) places of F at which both π_i are discrete series, such that $|S|$ is even. Consider the period pairing*

$$\mathcal{P}_S(\gamma,\beta) := \int_{Z_H(\mathbb{A}_F)H(F)\backslash H(\mathbb{A}_F)} \gamma(\iota(h))\beta(p(h))\,\mathrm{d}h,\ \ \gamma\in\Pi_S(\pi_1,\pi_2),\ \beta\in\pi, \tag{10}$$

where $\mathrm{d}h$ is normalized as in (6.1.1).

(1) If $\mathcal{P}_S(\gamma,\beta)\neq 0$, then $S=\emptyset$, i.e. $\Pi_S(\pi_1,\pi_2)$ is generic, and $\pi\cong\pi_2^\vee$.
(2) Suppose given factorizable Schwartz functions

$$\phi_i = \otimes_v\phi_{i,v}\in\mathcal{S}(M_2(\mathbb{A}_F)),\ \ i=1,2$$

and factorizable vectors

$$\alpha=\otimes_v\alpha_v\in\pi_1\otimes\pi_2,\ \ \beta=\otimes_v\beta_v\in\pi_2^\vee.$$

Then the theta lift $\theta_{\phi_1\otimes\phi_2}(\alpha)$ lies in $\Pi_\emptyset(\pi_1,\pi_2)$ and, for a sufficiently large finite set S of places of F,

$$\mathcal{P}_\emptyset(\theta_{\phi_1\otimes\phi_2}(\alpha),\beta) = 2|D_F|^{1/2}\cdot\pi^{-d}\frac{L^S(1,\pi_1\times\pi_2^\vee)L^S(1,\mathrm{Ad}\,\pi_2)}{\zeta_F^S(2)}\prod_{v\in S}\frac{\mathcal{Z}_v(\phi_{1,v},\phi_{2,v},\alpha_v,\beta_v)}{1-q_v^{-1}}.$$

Here $\mathcal{Z}_v(\phi_{1,v},\phi_{2,v},\alpha_v,\beta_v)$ is an explicit local zeta integral which is nonzero for appropriate choices of test data; $\phi_1\otimes\phi_2$ is the tensor product Schwartz function in $\mathcal{S}(M_2(\mathbb{A}_F)^2)$; the theta lift $\theta_{\phi_1\otimes\phi_2}(\alpha)$ is defined in §4; and the other notations are introduced in (2.1.1).

Remark. The L-values appearing in Theorem C are nonzero by the classical result of Shahidi [30].

In fact, Theorem C amounts to a special case of the non-tempered Gan-Gross-Prasad conjectures in [8]: if π_1 and π_2 have trivial central character, then $\Pi_S(\pi_1, \pi_2)$ descends to $\mathrm{PGSp}_4 = \mathrm{SO}_5$, and the period (8) reduces to a period for the split GGP pair $\mathrm{SO}_4 \subset \mathrm{SO}_5$. Although $\Pi_S(\pi_1, \pi_2)$ is tempered, the automorphic representation of SO_4 corresponding to the forms $\beta(p(h))$ on H is not, and so this period falls outside the scope of the usual GGP conjecture.

To deduce Theorem A from Theorem C, one additional ingredient is needed. In the period integrals (8), one really wants to consider only vectors γ and β that contribute to cohomology, which in our case is equivalent to generating a minimal K-type at archimedean places. The most delicate part is to write such a vector γ as a theta lift $\theta_\phi(\alpha)$, which requires a particular choice of archimedean component for the Schwartz function ϕ; the correct choice is calculated using local Howe duality. Once we know which ϕ to consider, we can evaluate the relevant archimedean zeta integrals to show that the periods (8) are nontrivial.

Proof of Theorem B. The main difficulty in the proof of Theorem B is to find a nontrivial family of Hodge classes on $S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2)$ (besides the ones coming from the algebraic cycle $S(\boldsymbol{H})$). Once we have a good supply of Hodge classes, the proof that they induce nontrivial maps (6) uses similar methods to the proof of Theorem A.

This family of Hodge classes is constructed using nontempered, cohomological automorphic representations of $\mathrm{GSp}_6(\mathbb{A}_F)$ which contribute to cohomology in degree $4d$, and whose contribution consists entirely of Hodge cycles. More precisely, let $S = S_f \cup S_\infty$ with $|S_f|$ even, and let B be the quaternion algebra over F which is ramified exactly at S_f. Assume $S_f \neq \emptyset$, i.e. B is nonsplit. Then for any auxiliary automorphic representation π of $PB(\mathbb{A}_F)^\times$ of parallel weight 6, we consider $\Theta(\pi \boxtimes \mathbb{1})$, the theta lift from $\mathrm{GSO}(B)$ to GSp_6 of the automorphic representation $\pi \boxtimes \mathbb{1}$ of $\mathrm{GSO}(B) \simeq B^\times \times B^\times/\mathbb{G}_m$. We do not prove that $\Theta(\pi \boxtimes \mathbb{1})$ is irreducible, but for any constituent $\widetilde{\Pi}$ of $\Theta(\pi \boxtimes \mathbb{1})$, we have:

$$H^{4d}_{(2)}(S(\mathbf{GSp_6}), \mathbb{C})[\widetilde{\Pi}_f] = H^{2d,2d}_{(2)}(S(\mathbf{GSp_6}), \mathbb{C})[\widetilde{\Pi}_f] \tag{11}$$

and

$$\mathrm{Gal}(\overline{\mathbb{Q}}/F^c) \text{ acts trivially on } IH^{4d}(S(\mathbf{GSp_6}), \overline{\mathbb{Q}}_\ell(2d))[\widetilde{\Pi}_f], \tag{12}$$

where we identify $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$ to make sense of (12). In fact, (11) and (12) remain true for any $\widetilde{\Pi}$ which is only *nearly equivalent* to a constituent of $\Theta(\pi \boxtimes \mathbb{1})$.

To prove (11), it suffices (by Matsushima's formula) to understand the Lie algebra cohomology of any $\widetilde{\Pi}_\infty$ such that $\widetilde{\Pi}_f \otimes \widetilde{\Pi}_\infty$ is automorphic. For this it is better to restrict to Sp_6, where we may use the endoscopic classication of Arthur [4]. In particular, any irreducible constituent $\widetilde{\Pi}'$ of $\widetilde{\Pi}_f \otimes \widetilde{\Pi}_\infty|_{\mathrm{Sp}_6(\mathbb{A}_F)}$ has a global Arthur parameter which depends only on π, and each component $\widetilde{\Pi}'_v$ of $\widetilde{\Pi}'$ lies in the corresponding local Arthur packet for $\mathrm{Sp}_6(F_v)$. Since Lie algebra cohomology is issentially insensitive to restriction to Sp_6, it suffices to understand the cohomology of all $\widetilde{\Pi}'_v$ in the local Arthur packet, for each $v|\infty$. This is accomplished using the construction of archimedean packets in [2] and the classification of unitary representations with nonzero cohomology [32].

To prove (12), suppose that $p \neq \ell$ splits completely in F and that $\widetilde{\Pi}_v$ is spherical for all $v|p$. Then the generalized Eichler-Shimura relation proven by Lee [19, 20] provides a polynomial $P(X)$ such that $P(\mathrm{Frob}_p) = 0$ on $IH^*(S(\mathbf{GSp_6}), \overline{\mathbb{Q}}_\ell)[\widetilde{\Pi}_f]$. The coefficients of $P(X)$ depend on the Satake parameters of $\widetilde{\Pi}_v$ for $v|p$, which in turn are determined by those of π_v via the spherical theta correspondence for orthogonal-symplectic similitude pairs (Proposition 4.3.3). It turns out that $P(X)$ has a unique root of weight $4d$, which is p^{-2d}. (The other roots correspond to appearances of $\widetilde{\Pi}_f$ in higher cohomological degrees.) Thus $\mathrm{Frob}_p = p^{-2d}$ on $IH^{4d}(S(\mathbf{GSp_6}), \overline{\mathbb{Q}}_\ell)[\widetilde{\Pi}_f]$ for all such p, which shows (12) by the Chebotarev density theorem.

Let us now return to the main construction. For $\widetilde{H} \coloneqq \mathrm{GSp}_4 \times_{\mathbb{G}_m} \mathrm{GL}_2 \subset \mathrm{GSp}_6$, we have inclusions of Shimura varieties

$$S(\mathbf{GSp_6}) \xleftarrow{\iota_1} S(\widetilde{\boldsymbol{H}}) \xrightarrow{\iota_2} S(\mathbf{GSp_4}) \times S(\mathbf{GL}_2)$$

such that ι_2 is open and closed. Thus we obtain a well-defined map

$$\iota_{2,*} \circ \iota_1^* : IH^*(S(\mathbf{GSp_6}), E(d)) \to IH^*(S(\mathbf{GSp_4}) \times S(\mathbf{GL}_2), E(d)). \tag{13}$$

If $\widetilde{\Pi}$ is defined over E, then the subspace $IH^*(S(\mathbf{GSp}_6), E(d))[\widetilde{\Pi}_f]$ makes sense, and we obtain from (11), (12), and (13) a space of Galois-invariant Hodge classes

$$\mathrm{Hdg}(\widetilde{\Pi}) \subset IH^{4d}(S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2), E(d)).$$

However, we will see in a moment that, to obtain the nonvanishing of a Hodge class constructed in this way, we will have to allow π – hence also $\widetilde{\Pi}$ – to be defined over an arbitrary number field. In general, let $S(\pi)$ denote the set of automorphic representations of $\mathrm{GSp}_6(\mathbb{A}_F)$ which are nearly equivalent to a constituent of $\Theta(\pi \boxtimes \mathbb{1})$. Then the subspace

$$\sum_{\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})} \sum_{\widetilde{\Pi} \in S(\pi^\sigma)} IH^{4d}(S(\mathbf{GSp}_6), \mathbb{C})[\widetilde{\Pi}_f] \subset IH^{4d}(S(\mathbf{GSp}_6), \mathbb{C})$$

is stable under the $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ action on the coefficients. After descending to E, its image under (13) defines a subspace

$$\mathrm{Hdg}(\pi) \subset IH^{4d}(S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2), E(d))$$

which again consists of Galois-invariant Hodge classes.

It remains to show that some element $\xi \in \mathrm{Hdg}(\pi)$ induces a nonzero map as claimed in Theorem B. Similarly to the proof of Theorem A, we reduce this question to showing that the triple product period integral

$$(14) \qquad \int_{[Z_{\widetilde{H}} \backslash \widetilde{H}]} \theta(\alpha)(h,h')\beta(h)\gamma(h')\,\mathrm{d}(h,h'), \quad \alpha \in \pi \boxtimes \mathbb{1},\ \beta \in \Pi_S(\pi_1,\pi_2),\ \gamma \in \pi_2^\vee$$

is nonzero for some choice of π and some choice of test vectors α, β, and γ. Here $\widetilde{H}$ is parametrized by pairs $(h,h') \in \mathrm{GSp}_4 \times \mathrm{GL}_2$, and the theta lift, which again depends on a choice of Schwartz function, is from $\mathrm{GSO}(B)$ to GSp_6. The relevant seesaw diagram for this period is:

$$\begin{array}{ccc} \mathrm{GSp}_6 & & \mathrm{GSO}(B) \times_{\mathbb{G}_m} \mathrm{GSO}(B) \\ | & \times & | \\ \mathrm{GSp}_4 \times_{\mathbb{G}_m} \mathrm{GL}_2 & & \mathrm{GSO}(B) \end{array}$$

The seesaw identity reduces (14) to

$$(15) \qquad \int_{[\mathrm{PGSO}(B)]} \alpha(g)\theta(\beta)(g)\theta(\gamma)(g)\,\mathrm{d}g,$$

where the theta lifts are now from GSp_4 and GL_2 to $\mathrm{GSO}(B)$. (Under the assumption that B is nonsplit, all the integrals involved in the seesaw identity converge absolutely.) The theta lift $\theta(\gamma)$ runs over $(\pi_2^B)^\vee \boxtimes (\pi_2^B)^\vee$ as γ varies, and the image of the theta lift $\theta(\beta)$ includes $\pi_1^B \boxtimes \pi_2^B$ as β varies. We choose α to be a Hilbert modular eigenform on $PB^\times(\mathbb{A}_F)$ such that $\langle f_1^B \cdot f_2^B, \alpha \rangle_{\mathrm{Pet}} \neq 0$, where $f_1^B \in \pi_1^B$ and $f_2^B \in (\pi_2^B)^\vee$ are holomorphic newforms, and let π be the automorphic representation generated by α. (Note that α may be defined over a larger number field than $f_1^B \cdot f_2^B$.) Having made this choice of π and α, it follows that (15) is nonzero for appropriate choices of β and γ.

Overview of the higher-weight case. To simplify the notation, assume for now that $F = \mathbb{Q}$. The representations $\Pi_S(\pi_1,\pi_2)$ are cohomological whenever π_1 and π_2 have weights $m_1 \geq m_2 + 2 \geq 4$. In this situation one again finds nontrivial Galois-invariant classes in the étale cohomology of $S(\mathrm{GSp}_4) \times S(\mathrm{GL}_2)$, where now we take cohomology with coefficients in a local system depending on m_1 and m_2. However, it is only for special choices of weights that we are able to formulate an analogue of Theorem A.

So suppose that π_1, π_2 have weights $m+2$ and m respectively, for an integer $m \geq 2$. Let $V_{(m-2,0)}$ be the representation of $\mathrm{GSp}_4(\mathbb{Q})$ with highest weight $(m-2,0)$ and central character $t \mapsto t^{m-2}$, and let V_{m-2} be the representation $\mathrm{Sym}^{m-2} V_{\mathrm{std}}$ of $\mathrm{GL}_2(\mathbb{Q})$. We write $\mathcal{V}_{(m-2,0)}$ and $\mathcal{V}_{m-2}$ for the corresponding local systems of E-vector spaces on $S(\mathrm{GSp}_4)$ and $S(\mathrm{GL}_2)$. (The normalization of these local systems is different in the text.) With the corresponding normalization of the central characters of π_i, we have [35]:

$$(16) \qquad H^3_{\text{ét,c}}(S(\mathrm{GSp}_4)_{\overline{\mathbb{Q}}}, \mathcal{V}_{(m-2,0),\lambda})_{\Pi_S(\pi_1,\pi_2)_f} = \begin{cases} \rho_{\pi_2}(1), & \infty \notin S \\ \rho_{\pi_1}, & \infty \in S \end{cases}$$

In particular, when $\infty \notin S$ one finds a nontrivial Galois-invariant map

$$H^3_{\text{ét},c}(S(\mathrm{GSp}_4)_{\overline{\mathbb{Q}}}, \mathcal{V}_{(m-2,0),\lambda})[\Pi_S(\pi_1,\pi_2)_f] \to H^1_{\text{ét}}(S(\mathrm{GL}_2)_{\overline{\mathbb{Q}}}, \mathcal{V}_{m-2,\lambda})[\pi_{2,f}]. \tag{17}$$

The significance of our special choice of weights is that the local system

$$\iota^*\mathcal{V}^\vee_{(m-2,0)} \otimes p^*\mathcal{V}_{m-2}$$

on $S(H)$ has the constant local system $\underline{\mathbb{Q}}$ as a direct factor with multiplicity one. We may therefore define

$$[S(H)] \in H^4\left(S(\mathrm{GSp}_4) \times S(\mathrm{GL}_2), \mathcal{V}^\vee_{(m-2,0)} \boxtimes \mathcal{V}_{m-2}(2)\right) \tag{18}$$

using the pushforward of the fundamental class on $S(H)$.

Theorem D. *Let π_1 and π_2 be cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ of weights $m+2$ and m, respectively, and the same central character (suitably normalized). Then the composite map*

$$H^3_c(S(\mathrm{GSp}_4), \mathcal{V}_{(m-2,0)}(1))[\Pi_S(\pi_1,\pi_2)_f] \xrightarrow{[S(H)]_*} H^1(S(\mathrm{GL}_2), \mathcal{V}_{m-2}) \twoheadrightarrow H^1(S(\mathrm{GL}_2), \mathcal{V}_{m-2})[\pi_{2,f}]$$

is nonzero if and only if $S = \emptyset$. In this case its image generates the $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})$-module $H^1(S(\mathrm{GL}_2), \mathcal{V}_{m-2})[\pi_{2,f}]$.

It is likely that Theorem B could be generalized to the weights $m_1 \geq m_2 + 2 \geq 4$; however, we have restricted our attention to the context of Theorem D.

Theorem E. *Let π_1 and π_2 be cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ of weights $m+2$ and m, respectively, with the same central character (suitably normalized). Let S be a set of places of $\mathbb{Q}$ at which both π_i are discrete series, such that $|S| \geq 2$ is even and $\infty \notin S$. Then there exists a Hodge class*

$$\xi \in H^4\left(S(\mathrm{GSp}_4) \times S(\mathrm{GL}_2), \mathcal{V}^\vee_{(m-2,0)} \boxtimes \mathcal{V}_{m-2}(2)\right)$$

such that:

1. *For all finite places λ of E, the image of ξ in λ-adic étale cohomology is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant.*
2. *The composite map*

$$H^3_c(S(\mathrm{GSp}_4), \mathcal{V}_{(m-2,0)})[\Pi_S(\pi_1,\pi_2)_f] \to H^1(S(\mathrm{GL}_2), \mathcal{V}_{m-2}) \twoheadrightarrow H^1(S(\mathrm{GL}_2), \mathcal{V}_{m-2})[\pi_{2,f}]$$

is nontrivial, and its image generates the $\mathrm{GL}_2(\mathbb{A}_{F,f})$-module $H^d(S(\mathbf{GL}_2), \mathcal{V}_{m-2,\lambda})[\pi_{2,f}]$.

The proofs of Theorems D and E follow the same lines as the overview given above, with only minor modifications. When $F \neq \mathbb{Q}$, we also have similar results assuming π_1 and π_2 have vector-valued weights $(m_v+2)_{v|\infty}$ and $(m_v)_{v|\infty}$. However, in this case it is more cumbersome to write down the definitions of the appropriate local systems. The precise results are included in Theorems 7.2.5 and 10.2.3 below.

One could also ask for an analogue of Theorem E that uses π_1, the higher-weight representation, in the place of π_2. Unfortunately, our construction does not appear to yield any results in this direction.

Comparison with previous work. Previous results in this direction were concerned with the Jacquet-Langlands correspondence for cohomological representations of inner forms of $\mathrm{GL}_{2,F}$. For the transfer between quaternion algebras B_1 and B_2 which are split at exactly one archimedean place, the Shimura varieties associated to $B_1^\times$ and $B_2^\times$ are curves. The resulting Tate classes are known to arise from cycles by Faltings's isogeny theorem [6], but no more explicit construction of these algebraic cycles is known. When the relevant Shimura varieties have higher dimension, Ichino and Prasanna [14] have shown that the Jacquet-Langlands transfers (for general weights) are induced by Hodge cycles. Their construction is similar to the one used to prove Theorem B. However, the results of this paper demonstrate two new phenomena that did not appear in [14]: the first is the presence of the candidate algebraic cycle $S(\boldsymbol{H})$, and the second is the qualitatively different behavior of generic and non-generic members of the L-packet.

Arithmetic implications. This work was originally motivated by a question of Weissauer in [34], which can be paraphrased as follows: if $F = \mathbb{Q}$ and π_2 is the automorphic representation associated to an elliptic curve $E/\mathbb{Q}$, then the motive associated to E appears attached to members of the L-packet $\Pi(\pi_1, \pi_2)$ in the cohomology of $S(\mathrm{GSp}_4)$. Can we then use Shimura curves on $S(\mathrm{GSp}_4)$ to construct interesting Selmer classes for E in the spirit of Heegner points? Theorem A implies that, when applied to quaternionic Shimura curves and a generic representation $\Pi_\emptyset(\pi_1, \pi_2)$, this construction would simply recover the Heegner points on E. Indeed, all appearances of the motive of E attached to generic representations $\Pi_\emptyset(\pi_1, \pi_2)$ are fully accounted for by Hecke translates of the correspondence from $S(\mathrm{GSp}_4)$ to the modular curve $S(\mathrm{GL}_2)$ induced by (7), and nonsplit quaternionic Shimura curves on $S(\mathrm{GSp}_4)$ are necessarily sent to CM divisors on $S(\mathrm{GL}_2)$ under this correspondence. It is an intriguing question whether Weissauer's construction yields new Selmer classes when applied to the non-generic members of the L-packets $\Pi(\pi_1, \pi_2)$.

Organization of the paper. In §2, we give some basic notations and conventions. In §3, we recall the plectic version of Matsushima's formula and its relation to vector-valued automorphic forms. In §4, we give notations and conventions for similitude theta lifts. This section also contains a proof of the L-functoriality for similitude theta lifts of spherical representations from orthogonal to symplectic groups (Proposition 4.3.3); this is presumably well-known to experts. In §5, we recall the construction of the Yoshida lift L-packets via theta lifts, and compute the plectic Hodge structures associated to $\Pi_S(\pi_1, \pi_2)_f$. The material up to this point is necessary for all the main results. However, the proofs of Theorems C and A, which are given in §6 and §7, respectively, are logically independent of the proof of Theorem B. The only exceptions are some results on the archimedean theta correspondence in §7.1. In §8, we study the nontempered representations used for the construction of Hodge classes. In §9, we compute the vector-valued triple product periods that are necessary for the nonvanishing of the Hodge classes. The proof of Theorem B is completed in §10.

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2. Preliminaries

2.1. **Basic notations.**

2.1.1. Throughout this article, F is a fixed totally real number field of degree d and discriminant D_F, $\mathcal{O}_F$ is its ring of integers, and $\mathbb{A}_F$ is its ring of adeles. For each place v of F, denote by F_v the completion; if v is non-archimedean, $\mathcal{O}_v$ is the valuation ring of F_v, $\varpi_v \in \mathcal{O}_v$ is the uniformizer, and $q_v = \#\mathcal{O}_v/\varpi_v$. For archimedean v, $q_v = 1$. The Haar measure on the additive group $\mathbb{A}_F$ is the product measure $\mathrm{d}a = \mathrm{d}a_f \prod_{v|\infty} \mathrm{d}a_v$, where $\mathrm{d}a_f$ is the Haar measure on $\mathbb{A}_{F,f}$ such that $\widehat{\mathcal{O}}_F$ has volume 1 and $\mathrm{d}a_v$ is the standard measure on $F_v \cong \mathbb{R}$.

2.1.2. If G is an algebraic group over F, $[G]$ denotes the adelic quotient $G(F)\backslash G(\mathbb{A}_F)$. If $\mathrm{d}g$ denotes a Haar measure on $G(\mathbb{A}_F)$, then we write $\mathrm{d}g$ as well for the quotient Haar measure on $[G]$ (where $G(F)$ is given the counting measure).

2.1.3. We fix the additive character $\psi = \psi_0 \circ \mathrm{tr}$ of $F\backslash\mathbb{A}_F$, where $\psi_0 : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}$ is the unique unramified character such that $\psi_0(x) = e^{2\pi i x}$ for $x \in \mathbb{R}$.

2.1.4. For any m, let $\omega_m : \mathbb{R}^\times \to \mathbb{R}^\times$ be the character

$$t \mapsto t^{m-2\cdot\lfloor\frac{m}{2}\rfloor}.$$

If $\boldsymbol{m} = (m_v)_{v|\infty}$, let $\omega_{\boldsymbol{m}} : (F \otimes \mathbb{R})^\times \to \mathbb{R}^\times$ be the character $\otimes_{v|\infty}\omega_{m_v}$. These characters will be used as the central characters for "nearly unitary" normalizations of automorphic forms appearing in cohomology.

2.1.5. If V is a vector space over a local field k (either Archimedean or non-Archimedean), then $\mathcal{S}_k(V)$ is the Schwartz space of functions on V. If V is a vector space over F and v is a place of F, then $\mathcal{S}_{F_v}(V)$ denotes the space of Schwartz functions on $V \otimes_F F_v$. Likewise, we write $\mathcal{S}_{F\otimes\mathbb{R}}(V)$ for the tensor product of the Schwartz spaces $\mathcal{S}_{F_v}(V)$ as v ranges over archimedean places of F.

2.2. **Conventions for GL_2 and SL_2.**

2.2.1. The standard Borel and unipotent subgroups of GL_2 are denoted B and N, respectively; $\overline{B}$ denotes the image of B in PGL_2 . We shall abbreviate by $c \mapsto h_c$ the section of $\det : \mathrm{GL}_2 \to \mathbb{G}_m$ given by $h_c = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}$.

2.2.2. For each non-archimedean place v of F, we normalize the Haar measure $\mathrm{d}g_v$ on $\mathrm{PGL}_2(F_v)$ to assign volume 1 to $\mathrm{PGL}_2(\mathcal{O}_v)$, and likewise for $\mathrm{SL}_2(F_v)$. For non-archimedean v, we choose the Haar measure $\mathrm{d}g_v$ on $\mathrm{PGL}_2(F_v) \cong \mathrm{PGL}_2(\mathbb{R})$ given by:

$$\mathrm{d}g_v = \frac{\mathrm{d}a\,\mathrm{d}t\,\mathrm{d}\theta}{\pi t^2}, \quad g_v = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad a \in \mathbb{R}, t \in \mathbb{R}^\times, \theta \in [0, \pi). \tag{19}$$

On $\mathrm{SL}_2(F_v) \cong \mathrm{SL}_2(\mathbb{R})$, we choose the Haar measure $\mathrm{d}g_v$ given by:

$$\mathrm{d}g_v = \frac{\mathrm{d}a\,\mathrm{d}t\,\mathrm{d}\theta}{2\pi t^2}, \quad g_v = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad a \in \mathbb{R}, t \in \mathbb{R}_{>0}, \theta \in [0, 2\pi). \tag{20}$$

2.2.3. For the standard compact subgroup $\mathrm{SO}(2)$ of $\mathrm{SL}_2(\mathbb{R})$, we denote by $\chi_m : \mathrm{SO}(2) \mapsto \mathbb{C}^\times$ the character

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mapsto (\cos\theta + i\sin\theta)^m.$$

2.3. **Conventions for symplectic groups.**

2.3.1. Let J be the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then, for any field k, the block-diagonal matrix $\begin{pmatrix} J & & \\ & \ddots & \\ & & J \end{pmatrix}$ defines a symplectic pairing on the k-space $W_{2n,k} = \langle e_1, \cdots, e_{2n} \rangle$ such that

$$W_{2n,k} = \langle e_1, e_3, \cdots, e_{2n-1} \rangle \oplus \langle e_2, e_4, \cdots, e_{2n} \rangle$$

is a decomposition into maximal isotropic subspaces; we refer to $W_{2n,k}$ as the standard symplectic space of dimension $2n$. The symplectic group $\mathrm{Sp}_{2n,k}$ and the general symplectic group $\mathrm{GSp}_{2n,k}$ are the isometry and similitude groups, respectively, of $W_{2n,k}$. When not otherwise specified, $k = F$.

2.3.2. The maximal compact-modulo-center subgroup of the symplectic group $\mathrm{GSp}_{2n,\mathbb{R}}$ is $K_n \simeq (U(n) \times \mathbb{R}^\times)/\{\pm 1\}$, consisting of the matrices whose 2×2 blocks commute with J. When K_n is viewed as a subgroup of $\mathrm{GSp}_{2n}(F_v)$ we write it $K_{n,v}$. There is a maximal compact torus $T \subset U(n)$ such that

$$\mathfrak{t} = \begin{pmatrix} \alpha_1 J & & \\ & \ddots & \\ & & \alpha_n J \end{pmatrix}, \quad \alpha_i \in \mathbb{R}.$$

We parameterize the weights of $U(n)$ by tuples of integers $(m_1, \cdots, m_n)$, corresponding to the character

$$\begin{pmatrix} \alpha_1 J & & \\ & \ddots & \\ & & \alpha_n J \end{pmatrix} \mapsto m_1\alpha_1 + \cdots + m_n\alpha_n.$$

When $n = 1$, the character $\chi_m \boxtimes \omega_m^{-1}$ on $U(1) \times Z_{\mathrm{GL}_2}$ descends to a character of K_1, which we will also denote by χ_m; we hope that this will cause no confusion.

3. Cohomology of Shimura varieties

3.1. **Plectic Hodge structures.**

3.1.1. Let G be a reductive group over F, and let $\boldsymbol{G} = \mathrm{Res}_{F/\mathbb{Q}} G_F$. Since

$$\boldsymbol{G}(\mathbb{R}) = \prod_{v|\infty} G(F_v) = \prod_{v|\infty} G_v(\mathbb{R}), \tag{21}$$

a Shimura datum $(\boldsymbol{G}, \boldsymbol{X})$ is necessarily a product $\boldsymbol{X} \simeq \prod_{v|\infty} X_v$. If $K_v \subset G_v(\mathbb{R})$ denotes the stabilizer of a distinguished point $h_v \in X_v$, then the stabilizer of the corresponding point $\boldsymbol{h} \in \boldsymbol{X}$ is

$$\boldsymbol{K}_\infty = \prod_{v|\infty} K_v.$$

Given a neat compact open subgroup

$$K \subset \boldsymbol{G}(\mathbb{A}_{\mathbb{Q},f}) = G(\mathbb{A}_{F,f}),$$

one has a smooth algebraic Shimura variety $S_K(\boldsymbol{G}, \boldsymbol{X})$ such that

$$S_K(\boldsymbol{G}, \boldsymbol{X})(\mathbb{C}) = G(F)\backslash G(\mathbb{A}_{F,f}) \times \boldsymbol{X}/K;$$

the inverse limit over K defines the pro-algebraic Shimura variety $S(\boldsymbol{G})$. (We usually drop $\boldsymbol{X}$ since it will be clear from context.) Finally, given an algebraic representation ρ of $\boldsymbol{G}$ on an E-vector space V, we have for each level subgroup K the local system $\mathcal{V}_K$ on $S_K(\boldsymbol{G})$ whose total space over $S_K(\boldsymbol{G})(\mathbb{C})$ is

$$G(F)\backslash G(\mathbb{A}_{F,f}) \times \boldsymbol{X} \times V/K. \tag{22}$$

The local systems $\mathcal{V}_K$ are compatible as K varies, and we write $\mathcal{V}$ for this compatible collection of local systems on $S(\boldsymbol{G})$.

3.1.2. Assuming $E \subset \mathbb{C}$, Matsushima's formula for the L^2 cohomology of $S(\boldsymbol{G})$ is:

$$H^*_{(2)}(S(\boldsymbol{G}), \mathcal{V}_{\mathbb{C}}) \cong \bigoplus_{\pi=\pi_f\otimes\pi_\infty} m_{\mathrm{disc}}(\pi) \cdot \pi_f \otimes H^*(\mathrm{Lie}\, \boldsymbol{G}; \boldsymbol{K}_\infty, \pi_\infty^{\mathrm{sm}} \otimes V_{\mathbb{C}})). \tag{23}$$

Here π runs over cuspidal automorphic representations of $G(\mathbb{A})$, $m_{\mathrm{disc}}(\pi)$ refers to the multiplicity in the discrete spectrum, and π_∞^{sm} is the dense subspace of smooth vectors. Moreover (23) is equivariant for the natural actions of $G(\mathbb{A}_{F,f})$ on both sides. Suppose $V_{\mathbb{C}} = \otimes_v V_v$, where V_v are $\mathbb{C}$-vector spaces equipped with algebraic representations ρ_v of $G_v(\mathbb{R})$, such that ρ factors as

$$\rho : G(F) \hookrightarrow G(F \otimes \mathbb{R}) \simeq \prod_v G_v(\mathbb{R}) \xrightarrow{\otimes\rho_v} \prod_v \mathrm{Aut}(V_v). \tag{24}$$

Since the Lie algebra of $\boldsymbol{G}$ is $\prod_{v|\infty} \mathfrak{g}_v$, the right hand side of (23) has a decomposition (cf. [23]):

$$\bigoplus_{\boldsymbol{p},\boldsymbol{q}} \left(\bigoplus_{\pi_f\otimes\pi_\infty} m_{\mathrm{disc}}(\pi) \cdot \pi_f \otimes \bigotimes_{v|\infty} H^{p_v,q_v}(\mathfrak{g}_v, K_v, \pi_v^{\mathrm{sm}} \otimes V_v) \right). \tag{25}$$

Here $\boldsymbol{p}$ and $\boldsymbol{q}$ are plectic Hodge types, i.e. tuples of positive integers $(p_v)_{v|\infty}$ and $(q_v)_{v|\infty}$. Then (23) induces a plectic Hodge decomposition on $H^*_{(2)}(S(\boldsymbol{G}), \mathcal{V}_{\mathbb{C}})$, written:

$$H^*_{(2)}(S(\boldsymbol{G}), \mathcal{V}_{\mathbb{C}}) = \bigoplus_{\boldsymbol{p},\boldsymbol{q}} H^{\boldsymbol{p},\boldsymbol{q}}_{(2)}(S(\boldsymbol{G}), \mathcal{V}_{\mathbb{C}}). \tag{26}$$

Remark 3.1.3. Because this decomposition does not take into account the variation of Hodge structures on $\mathcal{V}_{\mathbb{C}}$, it does not compare directly with the canonical mixed Hodge structure on $H^*(S(\boldsymbol{G}), \mathcal{V}_{\mathbb{C}})$. For this reason, (26) should be viewed more as a computational tool then as a suitable definition of "the" plectic Hodge structure on $H^*_{(2)}(S(\boldsymbol{G}), \mathcal{V}_{\mathbb{C}})$.

3.2. **Realizing automorphic forms in cohomology.**

3.2.1. The complex structure on X_v induces a decomposition

$$\mathfrak{g}_{v,\mathbb{C}} = \mathfrak{k}_\infty \oplus \mathfrak{p}_{v,+} \oplus \mathfrak{p}_{v,-}.$$

We define

$$\wedge^{\boldsymbol{p},\boldsymbol{q}}\mathfrak{p}_G^* := \otimes_{v|\infty}(\wedge^{p_v}\mathfrak{p}_{v,+}^* \otimes \wedge^{q_v}\mathfrak{p}_{v,-}^*), \tag{27}$$

and let $(\sigma^{\boldsymbol{p},\boldsymbol{q}}, \wedge^{\boldsymbol{p},\boldsymbol{q}})$ be the corresponding natural representation of $\boldsymbol{K}_\infty$. The vector bundle Ω^* of differential forms on $S(\boldsymbol{G})$ has a decomposition

$$\Omega^* = \oplus_{\boldsymbol{p},\boldsymbol{q}}\Omega^{\boldsymbol{p},\boldsymbol{q}},$$

where the vector bundle $\Omega^{\boldsymbol{p},\boldsymbol{q}}$ of $(\boldsymbol{p},\boldsymbol{q})$-forms on $S(\boldsymbol{G})$ corresponds to the local system whose complex points are:

$$G(F)\backslash G(\mathbb{A}_{F,f}) \times \boldsymbol{G}(\mathbb{R}) \times \wedge^{\boldsymbol{p},\boldsymbol{q}}\mathfrak{p}_G^*/\boldsymbol{K}_\infty.$$

In particular, the space $\Gamma_{(2)}(\Omega^{\boldsymbol{p},\boldsymbol{q}} \otimes \mathcal{V}_\mathbb{C})$ of L^2 global $(\boldsymbol{p},\boldsymbol{q})$-forms with coefficients in $\mathcal{V}_\mathbb{C}$ is identified with:

$$\left\{f \in C^\infty_{(2)}(G(\mathbb{A}_F)) \otimes V_\mathbb{C} \otimes \wedge^{\boldsymbol{p},\boldsymbol{q}}\mathfrak{p}_G^* \,:\, f(\gamma g k) = \rho(\gamma)\sigma^{\boldsymbol{p},\boldsymbol{q}}(k^{-1})f(g),\ \forall \gamma \in G(F),\ k \in \boldsymbol{K}_\infty\right\}. \tag{28}$$

Here $C^\infty_{(2)}(G(\mathbb{A}_F))$ is the space of smooth L^2 functions on $G(\mathbb{A}_F)$; by definition, we have:

$$\Gamma_{(2)}(\Omega^{\boldsymbol{p},\boldsymbol{q}} \otimes \mathcal{V}_\mathbb{C}) \twoheadrightarrow H^{\boldsymbol{p},\boldsymbol{q}}_{(2)}(S(\boldsymbol{G}), \mathcal{V}_\mathbb{C}). \tag{29}$$

Finally, we remark that there is a canonical isomorphism:

$$\begin{aligned}(\mathcal{A}_{(2)}(G(\mathbb{A}_F)) \otimes V_\mathbb{C}|_{\boldsymbol{K}_\infty} \otimes \wedge^{\boldsymbol{p},\boldsymbol{q}}\mathfrak{p}_G^*)^{\boldsymbol{K}_\infty} &\xrightarrow{\sim} \Gamma_{(2)}(\Omega^{\boldsymbol{p},\boldsymbol{q}} \otimes \mathcal{V}_\mathbb{C}) \\ \phi &\mapsto f_\phi, \quad f_\phi(g) = \rho(g_\infty)\phi(g).\end{aligned} \tag{30}$$

Here $\rho(g_\infty)$ is defined via the decomposition (24). By composing with (29), we obtain a realization of vector-valued automorphic forms in cohomology:

$$(\mathcal{A}_{(2)}(G(\mathbb{A}_F)) \otimes V_\mathbb{C}|_{\boldsymbol{K}_\infty} \otimes \wedge^{\boldsymbol{p},\boldsymbol{q}}\mathfrak{p}_G^*)^{\boldsymbol{K}_\infty} \twoheadrightarrow H^{\boldsymbol{p},\boldsymbol{q}}_{(2)}(S(\boldsymbol{G}), \mathcal{V}_\mathbb{C}). \tag{31}$$

3.3. **Comparison with Betti cohomology.**

3.3.1. For any local system $\mathcal{V}$ associated to a *complex* algebraic representation V of $G(F)$, recall the canonical commutative diagram of $G(\mathbb{A}_{F,f})$-modules (cf. [31, p. 293]):

$$\begin{tikzcd}
H^*_{\mathrm{cusp}}(S(\boldsymbol{G}), \mathcal{V}) \arrow[r] \arrow[d] & H^*_{(2)}(S(\boldsymbol{G}), \mathcal{V}) \arrow[d, "\sim"] \arrow[dr] & \\
H^*_c(S(\boldsymbol{G}), \mathcal{V}) \arrow[r] & IH^*(S(\boldsymbol{G})^*, \mathcal{V}) \arrow[r] & H^*(S(\boldsymbol{G}), \mathcal{V})
\end{tikzcd}$$

Here $IH^*(S(\boldsymbol{G})^*, \mathcal{V})$ is the intersection cohomology of the minimal compactification, and the indicated map is an isomorphism by the proof of Zucker's conjecture [21, 29]. Moreover it is known that the composite $H^*_{\mathrm{cusp}}(S(\boldsymbol{G}), \mathcal{V}) \to H^*(S(\boldsymbol{G}), \mathcal{V})$ is injective. Thus L^2 cohomology is related to inner cohomology by the inclusions:

$$H^*_{\mathrm{cusp}}(S(\boldsymbol{G}), \mathcal{V}) \subset H^*_!(S(\boldsymbol{G}), \mathcal{V}) \subset \operatorname{im} H^*_{(2)}(S(\boldsymbol{G}), \mathcal{V}). \tag{32}$$

3.3.2. If Π_f is a $\mathbb{C}[G(\mathbb{A}_{F,f})]$-module, Π_f is *defined* over a subfield $E \subset \mathbb{C}$ if there exists a $E[G(\mathbb{A}_{F,f})]$-module Π_f^E such that $\Pi_f^E \otimes_E \mathbb{C} \simeq \Pi_f$. In this case, if $\mathcal{V}$ is the E-local system associated to an E-linear representation V of $G(F)$, then we write:

$$\begin{aligned}H^*_?(S(\boldsymbol{G}), \mathcal{V})_{\Pi_{S_f}} &:= \operatorname{Hom}_{E[G(\mathbb{A}_{F,f})]}(\Pi_f^E, H^*_?(S(\boldsymbol{G}), \mathcal{V}), \\ H^*_?(S(\boldsymbol{G}), \mathcal{V})[\Pi_{S_f}] &:= \Pi_f^E \otimes_E H^*_?(S(\boldsymbol{G}), \mathcal{V})_{\Pi_{S_f}},\end{aligned} \tag{33}$$

where $H^*_?$ denotes compactly supported, inner, or singular cohomology as $? = c$, $!$, or $\emptyset$.

3.3.3. We say an irreducible admissible complex $G(\mathbb{A}_{F,f})$-representation Π is *Eisenstein* if Π is a subquotient of a parabolic induction $\mathrm{Ind}_{P(\mathbb{A}_{F,f})}^{G(\mathbb{A}_{F,f})}\pi_f$, for a parabolic subgroup $P = MN$ of G and a discrete automorphic representation π of $M(\mathbb{A}_F)$. For any admissible $E[G(\mathbb{A}_{F,f})]$-module H, we say H is Eisenstein if all irreducible constituents of $H \otimes \mathbb{C}$ are.

Lemma 3.3.4. *Let $\mathcal{V}$ be the automorphic local system on $S(\boldsymbol{G})$ associated to a $G(F)$-representation (V,ρ). Then the $E[G(\mathbb{A}_{F,f})]$-module*

$$H^i(S(\boldsymbol{G}),\mathcal{V})/H^i_!(S(\boldsymbol{G}),\mathcal{V})$$

is Eisenstein.

Proof. This is well-known; a lucid exposition may be found in the preprint [10, Chapter 9]. □

3.4. Symplectic Shimura varieties.

3.4.1. When $G = \mathrm{GSp}_{2n}$, equipped with its usual Shimura datum, the subgroup $\boldsymbol{K}_\infty$ is just

$$\boldsymbol{K}_n \coloneqq \prod_v K_{n,v} \subset \mathrm{GSp}_{2n}(F\otimes\mathbb{R}). \tag{34}$$

We establish some notation for local systems on $S(\mathbf{GSp}_{2n})$. Suppose given a tuple $\boldsymbol{\lambda} = (\lambda_v)_{v|\infty}$, where $\lambda_v = (m_{1,v},\cdots,m_{n,v})$ is a dominant weight of $\mathrm{Sp}_{2n,\mathbb{R}}$. We define $(\rho_{\lambda_v}, V_{\lambda_v})$ to be the unique irreducible $\mathbb{C}$-linear representation of GSp_{2n} whose restriction to Sp_{2n} has weight λ_v and whose central character is $\omega^{-1}_{m_{1,v}+\cdots+m_{n,v}}$ in the notation of (2.1.4). This defines a representation $(\rho_{\boldsymbol{\lambda}}, V_{\boldsymbol{\lambda}})$ of $\mathrm{GSp}_{2n}(F)$ according to (24), which clearly descends to F^c.

Proposition 3.4.2. *The representation $(\rho_{\boldsymbol{\lambda}}, V_{\boldsymbol{\lambda}})$ descends to a $\mathbb{Q}(\boldsymbol{\lambda})$-linear representation of $\mathrm{GSp}_{2n}(F)$, where $\mathbb{Q}(\boldsymbol{\lambda})$ is the fixed field of*

$$\{\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}) \,:\, \lambda_{\sigma\cdot v} = \lambda_v \;\forall v|\infty\}\,.$$

Proof. The proof of [33, Proposition I.3] applies unchanged. □

3.4.3. In particular, for each such $\boldsymbol{\lambda}$, we obtain a $\mathbb{Q}(\boldsymbol{\lambda})$-local system $\mathcal{V}_{\boldsymbol{\lambda}}$ on $S(\mathbf{GSp}_{2n})$ such that $\mathcal{V}_{\boldsymbol{\lambda},\mathbb{C}}$ arises from the tuple of representations $(\rho_{\lambda_v}, V_{\lambda_v})$ of $\mathrm{GSp}_{2n}(F_v)$ according to (24).

3.5. The case $G = \mathrm{GL}_2$.

3.5.1. We recall some basic results on the cohomology of $S(\boldsymbol{G})$ in the simplest case, $G = GL_2 = \mathrm{GSp}_2$. For a tuple of integers $\boldsymbol{m} = (m_v)_{v|\infty}$ with $m_v \geq 2$, define $\mathbb{Q}(\boldsymbol{m})$ to be the fixed field of

$$\{\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}) \,:\, m_{\sigma\cdot v} = m_v \;\forall v|\infty\}\,. \tag{35}$$

We then obtain from §3.4 a $\mathbb{Q}(\boldsymbol{m})$-local system $\mathcal{V}_{\boldsymbol{m}-2}$ on $S(\mathbf{GL}_2)$, where $\boldsymbol{m}-2 = (m_v-2)_{v|\infty}$.

3.5.2. Let $(p(+),q(+)) = (1,0)$ and $(p(-),q(-)) = (0,1)$, and define $(\boldsymbol{p}(\boldsymbol{\epsilon}),\boldsymbol{q}(\boldsymbol{\epsilon}))$ to be the plectic Hodge type $(p_v(\epsilon_v),q_v(\epsilon_v))_{v|\infty}$, for any choice of signs $\boldsymbol{\epsilon} = (\epsilon_v)_{v|\infty}$. Let $\boldsymbol{\chi}_{\boldsymbol{\epsilon m}}$ be the character of $\boldsymbol{K}_1$ from (2.3.2). Then we have:

$$\begin{aligned} \dim_{\mathbb{C}}\mathrm{Hom}_{\boldsymbol{K}_1}\left(\boldsymbol{\chi}_{-\boldsymbol{\epsilon m}}, \wedge^{\boldsymbol{p}(\boldsymbol{\epsilon}),\boldsymbol{q}(\boldsymbol{\epsilon})}\mathfrak{p}^*_{\mathrm{GL}_2}\otimes V_{\boldsymbol{m}-2,\mathbb{C}}\right) &= 1,\\ \dim_{\mathbb{C}}\mathrm{Hom}_{\boldsymbol{K}_1}\left(\boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon m}}, \wedge^{1-\boldsymbol{p}(\boldsymbol{\epsilon}),1-\boldsymbol{q}(\boldsymbol{\epsilon})}\mathfrak{p}^*_{\mathrm{GL}_2}\otimes V^\vee_{\boldsymbol{m}-2,\mathbb{C}}\right) &= 1. \end{aligned} \tag{36}$$

Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ of weight $\boldsymbol{m}$, whose central character has infinity type $\omega_{\boldsymbol{m}}$. Then combining (31) with (36) yields maps, well-defined up to scalars:

$$\begin{aligned} \mathrm{cl}_{\boldsymbol{\epsilon}} &: (\pi\otimes\boldsymbol{\chi}_{-\boldsymbol{\epsilon m}})^{\boldsymbol{K}_1} \to H^{\boldsymbol{p}(\boldsymbol{\epsilon}),\boldsymbol{q}(\boldsymbol{\epsilon})}_{(2)}(S(\mathbf{GL}_2),\mathcal{V}_{\boldsymbol{m}-2,\mathbb{C}})[\pi_f]\\ \mathrm{cl}'_{\boldsymbol{\epsilon}} &: (\pi^\vee\otimes\boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon m}})^{\boldsymbol{K}_1} \to H^{1-\boldsymbol{p}(\boldsymbol{\epsilon}),1-\boldsymbol{q}(\boldsymbol{\epsilon})}_{(2)}(S(\mathbf{GL}_2),\mathcal{V}^\vee_{\boldsymbol{m}-2,\mathbb{C}})[\pi^\vee_f] \end{aligned} \tag{37}$$

Here $(1-\boldsymbol{p}(\boldsymbol{\epsilon}),1-\boldsymbol{q}(\boldsymbol{\epsilon})) = (\boldsymbol{p}(-\boldsymbol{\epsilon}),\boldsymbol{q}(-\boldsymbol{\epsilon}))$ is the plectic Hodge type $(1-p(\epsilon_v),1-q(\epsilon_v))_{v|\infty}$. The following is well-known:

Proposition 3.5.3. *For each $\boldsymbol{\epsilon}$, the maps in (37) are isomorphisms, and*

$$H^{\boldsymbol{p},\boldsymbol{q}}_{(2)}(S(\mathbf{GL}_2),\mathcal{V}_{\boldsymbol{m}-2,\mathbb{C}})[\pi_f] = H^{\boldsymbol{p},\boldsymbol{q}}_{(2)}(S(\mathbf{GL}_2),\mathcal{V}^\vee_{\boldsymbol{m}-2,\mathbb{C}})[\pi^\vee_f] = 0$$

if $(\boldsymbol{p},\boldsymbol{q})$ is not of the form $(\boldsymbol{p}(\boldsymbol{\epsilon}),\boldsymbol{q}(\boldsymbol{\epsilon}))$ for some $\boldsymbol{\epsilon}$. Moreover, if π_f is defined over $E\supset\mathbb{Q}(\boldsymbol{m})$, then there are $\mathrm{GL}_2(\mathbb{A}_{F,f})$-equivariant isomorphisms

$$H^*_!(S(\mathbf{GL}_2),\mathcal{V}_{\boldsymbol{m}-2,E})[\pi_f]\otimes_E\mathbb{C}\simeq H^*_{(2)}(S(\mathbf{GL}_2),\mathcal{V}_{\boldsymbol{m},\mathbb{C}})[\pi_f]$$

and

$$H^*_c(S(\mathbf{GL}_2),\mathcal{V}_{\boldsymbol{m},E})[\pi_f]\simeq H^*(S(\mathbf{GL}_2),\mathcal{V}_{\boldsymbol{m},E})[\pi_f]\simeq H^*_!(S(\boldsymbol{G}),\mathcal{V}_{\boldsymbol{m}})[\pi_f],$$

and similarly for $\mathcal{V}^\vee_{\boldsymbol{m},E}$ and $\pi^\vee_f$.

□

4. Similitude theta lifting

4.1. Local Weil representation and local theta lift.

4.1.1. Let $\epsilon=\pm1$, and let V, W be vector spaces over a field k equipped with nondegenerate ϵ-symmetric and $(-\epsilon)$-symmetric pairings, respectively. We assume $\dim W=2n$ and $\dim V=2m$ are even, and that W is equipped with a complete polarization

$$W=W_1\oplus W_2,\ \ W_2=W_1^*. \tag{38}$$

For simplicity, assume as well that the discriminant character of V is trivial (as will be the case in our applications). Let $G_1=G_1(V)$, $G=G(V)$ be the connected isometry and similitude groups, respectively, of W, and likewise $H_1=H_1(W)$ and $H=H(W)$. Let $P=P(W_1)\subset H(W)$ be the parabolic subgroup stabilizing W_1, P_1 its intersection with H_1, and $N\subset P_1$ its unipotent radical. Also set

$$R_0=\{(h,g)\in H\times G\,:\,\nu_H(h)=\nu_G(g)\}\,, \tag{39}$$

where $\nu_G:G\to\mathbb{G}_m$ and $\nu_H:H\to\mathbb{G}_m$ are the similitude characters.

4.1.2. Assume that k is a local field. Then, for any nontrivial additive character ψ_k of k, the Weil representation $\omega=\omega_{W,V,\psi_k}$ of $H_1(k)\times G_1(k)$ is realized on the Schwartz space $\mathcal{S}_k(W_2\otimes V)$; in this model, the action of the parabolic $P_1\times G_1\subset H_1\times G_1$ stabilizing $W_1\times V$ is described as follows.

$$\begin{cases}\omega(1,g)\phi(x)=\phi(g^{-1}x), & g\in G_1(k),\\ \omega(n,1)\phi(x)=\psi\left(\frac12\langle n(x),x\rangle\right)\cdot\phi(x), & n\in N(k)\subset\mathrm{Hom}(W_2,W_1),\\ \omega(h(a),1)\phi(x)=|\det(a)|^m\phi(a^tx), & a\in\mathrm{GL}(W_1)(k)\subset P_1(k),\end{cases} \tag{40}$$

where $GL(W_1)$ is viewed as the Levi factor of P_1 by the standard embedding

$$a\mapsto h(a)=\begin{pmatrix}a&0\\0&a^{-t}\end{pmatrix}\in P_1. \tag{41}$$

Following the convention of [27], ω extends naturally to $R_0(k)$ by defining

$$\omega\left(\begin{pmatrix}1&0\\0&\nu_G(g)\end{pmatrix},g\right)\phi(x)=|\nu_G(g)|^{-mn/2}\phi(g^{-1}x) \tag{42}$$

for all $g\in G(k)$. Note that ω is trivial on the center $\{(\lambda,\lambda)\}\subset R_0$.

4.1.3. Suppose that $V=V_1\oplus V_2$ is also split; then the preceding construction also defines an action of $R_0(k)$ on $\mathcal{S}_k(W\otimes V_2)$ by interchanging the roles of V and W. These two representations are isomorphic via the partial Fourier transform. More precisely, consider the map $\mathcal{F}:\mathcal{S}_k(W_2\otimes V)\to\mathcal{S}_k(W\otimes V_2)$ defined by

$$\phi\mapsto\widehat{\phi},\ \ \widehat{\phi}(x_1,x_2)=\int_{(W_2\otimes V_1)(k)}\phi(z,x_2)\psi(\langle z,x_1\rangle)\,\mathrm{d}z, \tag{43}$$

where $x_1\in W_1\otimes V_2$, $x_2\in W_2\otimes V_2$, and $\mathrm{d}z$ is the self-dual Haar measure with respect to ψ_k. Then it is well-known that $\mathcal{F}$ intertwines the actions of $H_1(k)\times G_1(k)$ on both sides, and it is immediate to check that it intertwines the actions of

$$\left(\begin{pmatrix}1&0\\0&\lambda\end{pmatrix},\begin{pmatrix}1&0\\0&\lambda\end{pmatrix}\right)\in R_0(k)$$

according to the definition (42); so $\mathcal{F}$ is equivariant for all of $R_0(k)$.

4.1.4. If π is an irreducible admissible representation of $H(k)$, then the local theta lift $\Theta(\pi) = \Theta_{W,V}(\pi)$ is the largest semisimple representation of $G(k)$ such that there is a surjection

$$\omega_{W,V,\psi_k} \twoheadrightarrow \pi^\vee \boxtimes \Theta(\pi)$$

of admissible $R_0(k)$-representations. Symmetrically, if σ is an irreducible admissible representation of $G(k)$, then the local theta lift $\Theta(\sigma) = \Theta_{V,W}(\sigma)$ is the largest semisimple representation of $H(k)$ admitting a surjection $\omega_{W,V,\psi_k} \twoheadrightarrow \Theta(\sigma) \boxtimes \sigma^\vee$. The theta lift does not depend on ψ_k by [28, Proposition 1.9].

4.1.5. We remark that the Weil representation extends naturally to the full similitude groups of V and W, not just the neutral connected components, and so a theta lift $\Theta(\pi)$ or $\Theta(\sigma)$ may be viewed as a representation of the full similitude group of V or W. The drawback of working with neutral connected components of similitude groups is that we no longer have Howe duality, and in particular the local theta lift may be reducible. However, using connected similitude groups is more convenient for our global calculations.

4.2. **Global Weil representation and global theta lifts.**

4.2.1. Now turning to the global situation, assume $k = F$ in (4.1.1). Also suppose given, for almost every place v, self-dual lattices $\mathcal{W}_v \subset W \otimes F_v$ and $\mathcal{V}_v \subset V \otimes F_v$, such that $\mathcal{W}_v$ is compatible with the polarization $W = W_1 \oplus W_2$ in the sense that:

$$\mathcal{W}_v \cap (W_1 \otimes F_v) \oplus \mathcal{W}_v \cap (W_2 \otimes F_v) = \mathcal{W}_v.$$

The adelic Schwartz space $\mathcal{S}_{\mathbb{A}_F}(W_2 \otimes V)$ is the restricted tensor product of the local Schwartz spaces $\mathcal{S}_{F_v}(W_2 \otimes V)$ with respect to the indicator function of $(\mathcal{W}_v \cap (W_2 \otimes F_v)) \otimes \mathcal{V}_v$. The global Weil representation of $R_0(\mathbb{A}_F)$, realized on $\mathcal{S}_{\mathbb{A}_F}(W_2 \otimes V)$, is defined as the restricted tensor product of the local Weil representations (using the characters ψ_{F_v} determined by the fixed global character ψ). Recall the automorphic realization of ω, given by the theta kernel:

$$\theta(h, g; \phi) = \sum_{x \in W_2(F) \otimes V(F)} \omega(h, g)\phi(x), \quad (h, g) \in R_0(\mathbb{A}_F), \quad \phi \in \mathcal{S}_{\mathbb{A}_F}(W_2 \otimes V). \tag{44}$$

If V is also split, then we again have the alternate model $\mathcal{S}_{\mathbb{A}_F}(W \otimes V_2)$, related to $\mathcal{S}_{\mathbb{A}_F}(W_2 \otimes V)$ by the adelic partial Fourier transform. Note that

$$\theta(h, g; \phi) = \theta(h, g; \widehat{\phi}) = \sum_{x \in W \otimes V_2} \omega(h, g)\widehat{\phi}(x)$$

by Poisson summation.

4.2.2. Let $f \in \mathcal{A}_0(H(\mathbb{A}_F))$ be an automorphic cusp form and choose any $\phi \in \mathcal{S}_{\mathbb{A}_F}(W_2 \otimes V)$. Then, fixing a Haar measure $\mathrm{d}h_1$ on $H_1(\mathbb{A}_F)$, the similitude theta lift $\theta_\phi(f)$ to G is the automorphic function

$$g \mapsto \int_{[H_1]} \theta(h_1 h_0, g; \phi) f(h_1 h_0)\, \mathrm{d}h_1, \quad g \in G(\mathbb{A}_F), \tag{45}$$

where $h_0 \in H(\mathbb{A}_F)$ is any element such that $\nu_H(h_0) = \nu_G(g)$.

Likewise, if $f \in \mathcal{A}_0(G(\mathbb{A}_F))$ is an automorphic cusp form and $\mathrm{d}g_1$ is a Haar measure on $G_1(\mathbb{A})$, then the similitude theta lift $\theta_\phi(f)$ to H is the automorphic function

$$h \mapsto \int_{[G_1]} \theta(h, g_1 g_0; \phi) f(g_1 g_0)\, \mathrm{d}g_1, \quad h \in H(\mathbb{A}_F),$$

where $g_0 \in G(\mathbb{A}_F)$ is any element such that $\nu_G(g_0) = \nu_H(h)$.

If π is a cuspidal automorphic representation of $H(\mathbb{A}_F)$, then the similitude theta lift $\Theta(\pi) = \Theta_{W,V}(\pi)$ is the subspace of $\mathcal{A}(G(\mathbb{A}_F))$ spanned by the theta lifts $\theta_\phi(f)$ for $f \in \pi$ and $\phi \in \mathcal{S}_{\mathbb{A}_F}(W_2 \otimes V)$; if π is a cuspidal automorphic representation of $G(\mathbb{A}_F)$, we similarly define $\Theta(\pi) = \Theta_{V,W}(\pi)$ to be the subspace of $\mathcal{A}(H(\mathbb{A}_F))$ spanned by the theta lifts $\theta_\phi(f)$ for $f \in \pi$ and $\phi \in \mathcal{S}_{\mathbb{A}_F}(W_2 \otimes V)$. A key property of the global theta lift is its compatibility with the local theta lift. Although this is well-known, we include a proof for the reader's convenience.

Proposition 4.2.3. *Let π be a cuspidal automorphic representation of either $G(\mathbb{A}_F)$ or $H(\mathbb{A}_F)$, and suppose that $\Theta(\pi)$ lies in the L^2 subspace. Then for any automorphic representation $\sigma = \otimes'_v \sigma_v \subset \Theta(\pi)$, σ_v is a constituent of $\Theta(\pi_v)$ for all v.*

Proof. Without loss of generality, suppose π is a representation of $G(\mathbb{A}_F)$. Consider the map of $R_0(\mathbb{A}_F)$-representations:

$$\begin{aligned} &\mathcal{S}_{\mathbb{A}_F}(W_2 \otimes V) \otimes \pi \otimes \sigma^\vee \to \mathbb{C} \\ &\qquad\qquad \phi \otimes f \otimes f' \mapsto \int_{[Z_H \backslash H]} \theta_\phi(f)(h) f'(h)\, \mathrm{d}h. \end{aligned}$$

This map is well-defined and nontrivial by assumption. By duality, it also gives a nontrivial map

$$\mathcal{S}_{\mathbb{A}_F}(W_2 \otimes V) \twoheadrightarrow \pi^\vee \boxtimes \sigma,$$

which is evidently a restricted tensor product. This implies the proposition. □

4.2.4. The theta lift defined in (4.2.2) generalizes readily to vector-valued automorphic forms. Suppose $K \subset G(F \otimes \otimes \mathbb{R})$ and $L \subset H(F \otimes \mathbb{R})$ are subgroups which are compact modulo center, and let

$$(L \times K)_0 \coloneqq (L \times K) \cap R_0(F \otimes \mathbb{R}).$$

Suppose given finite-dimensional representations σ and τ of L and K, and let $f \in (\mathcal{A}_0(H(\mathbb{A}_F)) \otimes \sigma)^L$ be a vector-valued automorphic form. Then for a vector-valued Schwartz function

$$\varphi \in \left(\mathcal{S}_{F\otimes\mathbb{R}}(W_2 \otimes V) \otimes \sigma^\vee \otimes \tau\right)^{(L\times K)_0},$$

and a Schwartz function

$$\phi_f \in \mathcal{S}_{\mathbb{A}_{F,f}}(W_2 \otimes V) \coloneqq \otimes'_{v\nmid\infty} \mathcal{S}_{F_v}(W_2 \otimes V),$$

we may define

$$\theta_{\phi_f \otimes \varphi}(f) \in (\mathcal{A}(G(\mathbb{A}_F)) \otimes \tau)^K$$

by the same formula (45) as for the scalar-valued theta lift. The vector-valued theta lift from G to H is defined in the same way.

4.3. **Spherical theta correspondence for similitudes.**

4.3.1. We shall require an explicit description of the spherical similitude theta correspondence in certain cases. Continuing the notation of (4.1.1), assume k is a nonarchimedean local field, that ψ_k is unramified, and that $V = V_1 \oplus V_2$ is a split orthogonal space (so that $\epsilon = +$). For this subsection only, for the purposes of clearer comparison with the literature, we let G'_1 and G' denote the **full** isometry and similitude groups of V, so that G'_1 and G' are disconnected; likewise for

$$R'_0 \coloneqq \{(h, g) \in H \times G' \,:\, \nu_H(h) = \nu_{G'}(g)\}.$$

The Weil representation of $R_0(k)$ extends naturally to $R'_0(k)$. We assume the additive character ψ_k of k used to define ω is unramified.

Now choose bases $\{e_1, \cdots, e_m\}$ and $\{f_1, \cdots, f_n\}$ of V_1 and W_1, respectively, and let $\{e_1^*, \cdots, e_m^*\}$ and $\{f_1^*, \cdots, f_n^*\}$ be the dual bases of V_2 and W_2. Let $T_{G_1} \subset \mathrm{GL}(V_1) \subset G_1$ and $T_{H_1} \subset \mathrm{GL}(W_1) \subset H_1$ be the standard diagonal tori; then we choose the maximal tori for G, H, and R_0 given (with respect to the bases $\{e_1, \cdots, e_m, e_1^*, \cdots, e_m^*\}$ and $\{f_1, \cdots, f_n, f_1^*, \cdots, f_n^*\}$) by:

$$\begin{aligned} T_G &= T_{G_1} \times \mathbb{G}_m = \left\{\mathrm{diag}(x_1, \cdots, x_m, \lambda x_1^{-1}, \cdots, \lambda x_m^{-1})\right\} \\ T_H &= T_{H_1} \times \mathbb{G}_m = \left\{\mathrm{diag}(y_1, \cdots, y_n, \kappa y_1^{-1}, \cdots, \kappa y_n^{-1})\right\} \qquad (46) \\ T_{R_0} &= T_H \times_{\mathbb{G}_m} T_G \simeq T_{G_1} \times T_{H_1} \times \mathbb{G}_m \end{aligned}$$

4.3.2. To fix notation, we recall the unramified principal series of G and H. The unramified characters of $T_{G_1}(k)$ are parameterized by tuples $\chi_1 = (\alpha_1, \cdots, \alpha_m) \in (\mathbb{C}^\times)^m$, where

$$\chi_1(\mathrm{diag}(x_1, \cdots, x_m, x_1^{-1}, \cdots, x_n^{-1})) = \prod_{i=1}^m \alpha_i^{\mathrm{ord}\, x_i}.$$

The unramified characters of $T_G(k)$ are parameterized by $\chi = (\alpha_1, \cdots, \alpha_m, s) \in (\mathbb{C}^\times)^{m+1}$, where

$$\chi(\mathrm{diag}(x_1, \cdots, x_m, \lambda x_1^{-1}, \cdots, \lambda x_m^{-1})) = s^{\mathrm{ord}\,\lambda} \prod_{i=1}^m \alpha_i^{\mathrm{ord}\, x_i}.$$

Similarly, the unramified characters of $T_{H_1}(k)$ (resp. $T_H(k)$) are parametrized by $\mu_1 = (\beta_1, \cdots, \beta_n) \in (\mathbb{C}^\times)^n$ (resp. $\mu = (\beta_1, \cdots, \beta_n, t) \in (\mathbb{C}^\times)^{n+1}$), and the unramified characters of $T_{R_0}(k)$ are parameterized by $\eta = (\beta_1, \cdots, \beta_n, \alpha_1, \cdots, \alpha_m, u) \in (\mathbb{C}^\times)^{n+m+1}$. Note that the character $\mu \boxtimes \chi$ of $T_H(k) \times T_G(k)$ pulls back to the character

$$\mu \cdot \chi := (\beta_1, \cdots, \beta_n, \alpha_1, \cdots, \alpha_m, st)$$

of $T_{R_0}(k)$ under the inclusion $T_{R_0} \subset T_H \times T_G$.

For Borel subgroups $B_G = T_G N_G \subset G$ and $B_H = T_H N_H \subset H$, the (normalized) principal series representations $\mathrm{Ind}_{B_G(k)}^{G(k)} \chi$ and $\mathrm{Ind}_{B_H(k)}^{H(k)} \mu$ possess unique spherical subquotients denoted π_χ and σ_μ, respectively; note π_χ and σ_μ depend only on the Weyl orbits of χ and μ. Moreover $\sigma_\mu \boxtimes \pi_\chi|_{R_0}$ is the unique spherical subquotient of $\mathrm{Ind}_{B_G(k)\times B_H(k)\cap R_0(k)}^{R_0(k)} \mu \cdot \chi$.

Proposition 4.3.3. *Suppose $m \leq n$, $\epsilon = +$, and that the residue field of k has odd cardinality q. If π_χ is the spherical representation of $G(k)$ associated to $\chi = (\alpha_1, \cdots, \alpha_m, s)$, and if $\mathrm{Ind}_{G(k)}^{G'(k)} \pi_\chi$ is irreducible, then $\Theta(\pi_\chi)$ is the spherical representation σ_μ of $H(k)$ for*

$$\mu = (\alpha_1, \cdots, \alpha_m, q, q^2, \cdots, q^{n-m}, sq^{-(m^2-m)/4-(n^2+n)/4+nm/2}).$$

Proof. Since $\mathrm{Ind}_{G(k)}^{G'(k)} \pi_\chi$ is irreducible, $\Theta(\pi_\chi)$ is nonzero, irreducible [27], and unramified [28, Proposition 1.11]. Thus $\Theta(\pi_\chi) = \sigma_\mu$ for some μ, and it remains to determine μ.

As in [25, §4], let $\sigma = (\sigma_1, \cdots, \sigma_m) \in \mathbb{C}^m$, and consider for all $\Re(\sigma_i) \gg 0$ the family of integrals:

$$I(\sigma, \phi) = \int \phi\left(\sum_{i=1}^m a_{ii} f_i^* \otimes e_i + \sum_{1\leq i<j\leq m} z_{ij} f_i^* \otimes e_j\right) \prod_{i=1}^m |a_{ii}|^{\sigma_i + i - r}\, \mathrm{d}^\times a_{ii} \prod_{i<j} \mathrm{d}z_{ij}, \tag{47}$$

where $\phi \in \mathcal{S}_k(W_2 \otimes V)$.

Claim. *If $\Re(\sigma_i) \gg 0$ for all i, then there exist Borel subgroups $B_G = T_G N_G \subset G$ and $B_H = T_H N_H \subset H$ such that N_G and N_H act trivially on V_1, W_1, respectively, and such that*

$$Z_\sigma(\phi)(h, g) := I(\sigma, \omega(h, g)\phi)$$

defines an $R_0'(k)$-intertwining map from ω to the induced representation

$$\begin{aligned}
I_\sigma &:= \mathrm{Ind}_{T_{R_0}(k)\cdot(N_H\times N_G)(k)}^{R_0'(k)} \eta(\sigma),\\
\eta(\sigma) &= (q^{\sigma_1+1-m}, \cdots, q^{\sigma_i+i-m}, \cdots, q^{\sigma_m}, q, q^2, \cdots, q^{n-m},\\
&\qquad q^{m-\sigma_1-1}, \cdots, q^{m-\sigma_i-i}, \cdots, q^{-\sigma_m}, q^{-(m^2-m)/4-(n^2+n)/4+nm/2}) \in (\mathbb{C}^\times)^{n+m+1}.
\end{aligned}$$

Proof of claim: Note that

$$\begin{aligned}
I_\sigma|_{H_1\times G_1'} &\simeq \mathrm{Ind}_{T_{H_1}N_H(k)}^{H_1(k)} \mu_1(\sigma) \boxtimes \mathrm{Ind}_{T_{G_1}N_G(k)}^{G_1'(k)} \chi_1(\sigma),\\
\mu_1(\sigma) &= (q^{\sigma_1+1-m}, \cdots, q^{\sigma_i+i}, \cdots, q^{\sigma_m}, q, q^2, \cdots, q^{n-m}),\\
\chi_1(\sigma) &= (q^{m-\sigma_1-1}, \cdots, q^{m-\sigma_i-i}, \cdots, q^{-\sigma_m}).
\end{aligned}$$

Thus the fact that Z_σ is an $H_1 \times G_1'$-intertwining map, for choices of N_H and N_G as in the claim, is a restatement of [25, Lemma 4.1]. (See p. 487-489 of *loc. cit.* for the choices of N_H and N_G.) To see that Z_σ

is an R_0'-intertwining map, it therefore suffices to check that, if

$$r_\lambda = \left(\begin{pmatrix} 1 & \\ & \lambda \end{pmatrix}, \begin{pmatrix} 1 & \\ & \lambda \end{pmatrix} \right) \in R_0,$$

then

$$I(\sigma, \omega(r_\lambda)\phi) = |\lambda|^{(m^2-m)/4+(n^2+n)/4-nm/2} \cdot |\det(r_\lambda, \mathrm{Lie}\, N_H \times N_G)|^{1/2} I(\sigma, \phi).$$

By definition,

$$I(\sigma, \omega(r_\lambda)\phi) = |\lambda|^{-nm/2}.$$

On the other hand, one can calculate directly that the determinant factor on the right-hand side is

$$|\lambda|^{-(m^2-m)/4-(n^2+n)/4},$$

since r_λ acts by the scalar λ^{-1} on the rootspaces $\mathrm{Hom}(V_2, V_1) \cap \mathrm{Lie}\, N_G$ and $\mathrm{Hom}(W_2, W_1) \cap \mathrm{Lie}\, N_H$, and trivially on the rest of $\mathrm{Lie}\, N_G \times N_H$. This concludes the proof of the claim.

Now choose a hyperspecial subgroup $K_{R_0'}$ of $R_0'(k)$ (arising from maximal self-dual lattices in W and V), and let $\mathcal{H}$ be the Hecke algebra of $\mathbb{C}$-valued, $K_{R_0'}$-biinvariant functions on R_0'. For all σ as in the claim, the Hecke action on the unique spherical vector in I_σ defines an algebra morphism $z_\sigma : \mathcal{H} \to \mathbb{C}$. It follows from [25] that the support of the $\mathcal{H}$-module

$$\mathcal{S}_k(W_2 \otimes V)^{K_{R_0'}}$$

is contained in the Zariski closure of the points z_σ of $\mathrm{Spec}\, \mathcal{H}$. On the other hand, the Satake isomorphism identifies complex points of $\mathrm{Spec}\, \mathcal{H}$ with R_0'-Weyl orbits of parameters $\eta = (\beta_1, \cdots, \beta_m, \alpha_1, \cdots, \alpha_n, u)$ as above. By assumption, there is a surjection

$$\mathcal{S}_k(W_2 \otimes V) \twoheadrightarrow \pi_\chi^\vee \boxtimes \sigma_\mu,$$

and hence the character $\chi^{-1} \cdot \mu$ lies in the Zariski closure of the Weyl orbit of the parameters $\eta(\sigma)$ in the claim. However, the μ listed in the proposition is the only one (up to H-Weyl action) satisfying this property. □

5. Yoshida lifts on GSp_4

5.1. **Some four-dimensional orthogonal spaces.**

5.1.1. Let B be a quaternion algebra, possibly split, over a field k. Then B comes equipped with a norm $N : B \to k$ and an involution $b \mapsto b^*$ such that $bb^* = N(b)$ for all $b \in B$. The k-orthogonal space V_B associated to B is isomorphic to B as a vector space, with the inner product defined by

$$(b_1, b_2) := \mathrm{tr}(b_1 b_2^*) = b_1 b_2^* + b_2 b_1^*. \tag{48}$$

When B is split, we often drop the subscript and abbreviate $V = V_{M_2(k)}$.

5.1.2. One has a map of algebraic groups over k:

$$\boldsymbol{p}_Z : B^\times \times B^\times \to \mathrm{GO}(V_B) \tag{49}$$

defined by

$$\boldsymbol{p}_Z(b_1, b_2) \cdot x = b_1 x b_2^*, \ \ x \in V_B.$$

The kernel of $\boldsymbol{p}_Z$ is the antidiagonally embedded $\mathbb{G}_m$, and $\boldsymbol{p}_Z$ is a surjection onto the connected similitude group $\mathrm{GSO}(V_B)$.

If k is a local field, then irreducible admissible representations of $\mathrm{GSO}(V_B)(k)$ are all of the form $\pi_1 \boxtimes \pi_2$, where π_i are irreducible admissible representations of $B^\times$ of the same central character; if $k = F$, the same is true of automorphic representations of $\mathrm{GSO}(V_B)(\mathbb{A}_F)$.

5.2. **Elliptic endoscopic L-parameters.**

5.2.1. The unique elliptic endoscopic group of $\mathrm{GSp}_{4,F}$ is $\mathrm{GSO}(V)$, equipped with the L-embedding:

$$ {}^L\,\mathrm{GSO}(V) = (\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2)(\mathbb{C}) \times \mathrm{Gal}(\overline{F}/F) \hookrightarrow \mathrm{GSp}_4(\mathbb{C}) \times \mathrm{Gal}(\overline{F}/F) = {}^L\mathrm{GSp}_4. \tag{50}$$

The Langlands functoriality principle for the map (50) then suggests that, to an automorphic representation $\pi = \pi_1 \boxtimes \pi_2$ of $\mathrm{GSO}(V)(\mathbb{A}_F)$, one can associate an L-packet of automorphic representations $\Pi(\pi_1, \pi_2)$ of $\mathrm{GSp}_4(\mathbb{A}_F)$. These L-packets and their local analogues are constructed via similitude theta lifting in [28, 35]. More precisely, for each place v of F and each irreducible admissible representation $\pi_{1,v}\boxtimes\pi_{2,v}$ of $\mathrm{GSO}(V)(F_v)$, one associates a local L-packet

$$\left\{\Pi^+(\pi_{1,v}, \pi_{2,v}), \Pi^-(\pi_{1,v}, \pi_{2,v})\right\}, \tag{51}$$

where by convention $\Pi^-(\pi_{1,v}, \pi_{2,v}) = 0$ unless both $\pi_{i,v}$ are discrete series. For all v, $\Pi^+(\pi_{1,v}, \pi_{2,v})$ is the unique generic member of the L-packet, and is explicitly given by the (nonzero, irreducible) local similitude theta lift:

$$\Pi^+(\pi_{1,v}, \pi_{2,v}) \coloneqq \Theta_{V,W_4}(\pi_{1,v} \boxtimes \pi_{2,v}). \tag{52}$$

If $\pi_{i,v}$ are both discrete series, then they admit Jacquet-Langlands transfers $\pi^B_{i,v}$ to $B^\times$, where B is the non-split quaternion algebra over F_v. In this case, we have

$$\Pi^-(\pi_{1,v}, \pi_{2,v}) \coloneqq \Theta_{V_B,W_4}(\pi^B_{1,v}, \pi^B_{2,v}), \tag{53}$$

a nonzero irreducible representation. We remark that the central character of $\Pi^\pm(\pi_{1,v}, \pi_{2,v})$ is the common central character of $\pi_{i,v}$ (since the central character of the Weil representation is trivial). The L-packets associated to π_v and $\pi'_v = \pi_{2,v} \boxtimes \pi_{1,v}$ coincide, but otherwise are all disjoint. Globally, given a cuspidal automorphic representation $\pi_1 \boxtimes \pi_2$ of $\mathrm{GSO}(V)(\mathbb{A}_F)$ and a finite set S of places where π_i are both discrete series, we form the adelic representation

$$\Pi_S(\pi_1, \pi_2) \coloneqq \bigotimes_{v\notin S}{}' \,\Pi^+(\pi_{1,v}, \pi_{2,v}) \otimes \bigotimes_{v\in S} \Pi^-(\pi_{1,v}, \pi_{2,v}). \tag{54}$$

Theorem 5.2.2 (Weissauer). *Let $\pi_1 \boxtimes \pi_2$ be a cuspidal automorphic representation of $\mathrm{GSO}(V)(\mathbb{A}_F)$, where $\pi_1 \not\cong \pi_2$. Then the automorphic multiplicity of $\Pi_S(\pi_1, \pi_2)$ is given by:*

$$m_{\mathrm{disc}}(\Pi_S(\pi_1, \pi_2)) = m_{\mathrm{cusp}}(\Pi_S(\pi_1, \pi_2)) = \begin{cases} 1, & \text{if } |S| \text{ is even,} \\ 0, & \text{if } |S| \text{ is odd.}\end{cases}$$

The representations $\Pi_S(\pi_1, \pi_2)$ constitute a full near equivalence class in the discrete spectrum of $\mathcal{A}_{(2)}(\mathrm{GSp}_4(\mathbb{A}_F))$, and are generic if and only if $S = \emptyset$. They are tempered and not CAP. Moreoever, if $|S|$ is even,

$$\Pi_S(\pi_1, \pi_2) = \Theta_{V_B,W_4}(\pi^B_1 \boxtimes \pi^B_2),$$

where B is the unique F-quaternion algebra ramified at the set of primes S and π^B_i are the Jacquet-Langlands transfers of π_i to $B^\times(\mathbb{A}_F)$.

Proof. This is a combination of [35, Theorem 5.2] (for the multiplicity formula) and [35, Corollary 5.5] (for the nonvanishing of the global theta lift); note that, given $\Theta_{V_B,W_4}(\pi^B_1 \boxtimes \pi^B_2) \neq 0$, it is cuspidal if $\pi^B_1 \neq \pi^B_2$ by [35, Theorem 4.3], and hence abstractly isomorphic to $\Pi_S(\pi_1, \pi_2)$ by Proposition 4.2.3. □

5.3. **Yoshida lifts in cohomology.**

5.3.1. Fix a tuple $\boldsymbol{m} = (m_v)_{v|\infty}$ of integers such that $m_v \geq 2$ for all v. Let π_1, π_2 be cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$ of weights $\boldsymbol{m}+2 = (m_v+2)_{v|\infty}$ and $\boldsymbol{m}$, respectively, with equal central characters of infinity type $\omega_{\boldsymbol{m}}$. For a set S_f of *finite* places of F at which π_i are both discrete series, set

$$\Pi_{S_f} = \bigotimes_{\substack{v\notin S_f\\ v\nmid\infty}}{}' \,\Pi^+(\pi_{1,v}, \pi_{2,v}) \otimes \bigotimes_{v\in S_f} \Pi^-(\pi_{1,v}, \pi_{2,v}).$$

We consider the local system $\mathcal{V}_{(\boldsymbol{m}-2,0)}$ of $\mathbb{Q}(\boldsymbol{m})$-vector spaces on $S(\mathbf{GSp_4})$ according to the conventions of §3.4 (the field $\mathbb{Q}(\boldsymbol{m})$ is defined in (3.5.1)).

For each $v|\infty$, let $\tau^+_{m_v}$, resp. $\tau^-_{m_v}$, be the unique irreducible representation of $K_{2,v}$ of central character $\omega^{-1}_{m_v}$ whose restriction to $U(2) \subset K_{2,v}$ has highest weight $(1, -m_v - 1)$, resp. $(m_v + 1, -1)$. Similarly, let

$\sigma^+_{m_v}$, resp. $\sigma^-_{m_v}$, be the unique irreducible representation of K_v of central character $\omega^{-1}_{m_v}$ whose restriction to $U(2)$ has highest weight $(-3,-m_v-1)$, resp. $(m_v+1,3)$. Note that $\tau^{\pm}_{m_v}$ are the duals of the minimal K_v-types of the representation $\Pi^+_v(\pi_{1,v},\pi_{2,v})$ of (5.2.1), and $\sigma^{\pm}_{m_v}$ are the duals of the minimal K_v-types of $\Pi^-_v(\pi_{1,v},\pi_{2,v})$.

For a subset $S_\infty \subset \{v|\infty\}$ and a collection of signs $\boldsymbol{\epsilon} = \{\epsilon_v\}_{v|\infty}$, define the $\boldsymbol{K}_2$-representation

$$\boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{m},S_\infty} \coloneqq \bigotimes_{v\in S_\infty} \sigma^{\epsilon_v}_{m_v} \otimes \bigotimes_{\substack{v\notin S_\infty\\ v|\infty}} \tau^{\epsilon_v}_{m_v}. \tag{55}$$

Thus $\boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{m},S_\infty}$ is a minimal $\boldsymbol{K}_2$-type of $\Pi_S(\pi_1,\pi_2)$, if S_∞ is the set of archimedean places in S.

5.3.2. Now let $(\boldsymbol{p}(\boldsymbol{\epsilon},S_\infty),\boldsymbol{q}(\boldsymbol{\epsilon},S_\infty))$ be the plectic Hodge type determined by:

$$(p_v(\boldsymbol{\epsilon},S_\infty),q_v(\boldsymbol{\epsilon},S_\infty)) = \begin{cases} (3,0), & \epsilon_v=+, v\in S_\infty,\\ (2,1), & \epsilon_v=+, v\notin S_\infty,\\ (1,2), & \epsilon_v=-, v\notin S_\infty,\\ (0,3), & \epsilon_v=-, v\in S_\infty.\end{cases} \tag{56}$$

Thus $(\boldsymbol{p}(\boldsymbol{\epsilon},\emptyset),\boldsymbol{q}(\boldsymbol{\epsilon},\emptyset)) = (\boldsymbol{p}(\boldsymbol{\epsilon})+1,\boldsymbol{q}(\boldsymbol{\epsilon})+1)$ in the notation of (3.5.1). An easy calculation shows that

$$\dim \operatorname{Hom}_{\boldsymbol{K}_2}\left(\boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{m},S_\infty}, V_{(\boldsymbol{m}-2,0),\mathbb{C}} \otimes \wedge^{\boldsymbol{p}(\boldsymbol{\epsilon},S_\infty),\boldsymbol{q}(\boldsymbol{\epsilon},S_\infty)}\mathfrak{p}^*_{\mathrm{GSp}_4}\right) = 1. \tag{57}$$

Hence, if $S = S_f \sqcup S_\infty$ is a finite set of places of F with $|S|$ even, combining (31) and (57) yields a map (well-defined up to a scalar):

$$\mathrm{cl}^{\boldsymbol{\epsilon}}_S : \left(\Pi_S(\pi_1,\pi_2)\otimes \boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{m},S_\infty}\right)^{\boldsymbol{K}_2} \to H^{\boldsymbol{p}(\boldsymbol{\epsilon},S_\infty),\boldsymbol{q}(\boldsymbol{\epsilon},S_\infty)}_{(2)}(S(\mathbf{GSp}_4),\mathcal{V}_{(\boldsymbol{m}-2,0),\mathbb{C}})[\Pi_{S_f}]. \tag{58}$$

Proposition 5.3.3. *The map* $\mathrm{cl}^{\boldsymbol{\epsilon}}_S$ *is an isomorphism of* $G(\mathbb{A}_{F,f})$*-representations, and moreover*

$$H^{\boldsymbol{p},\boldsymbol{q}}_{(2)}(S(\mathbf{GSp}_4),\mathcal{V}_{(\boldsymbol{m}-2,0),\mathbb{C}})[\Pi_{S_f}] = 0$$

if $(\boldsymbol{p},\boldsymbol{q})$ *is not of the form* $(\boldsymbol{p}(\boldsymbol{\epsilon},S_\infty),\boldsymbol{q}(\boldsymbol{\epsilon},S_\infty))$ *for some* S_∞ *such that* $|S_f\cup S_\infty|$ *is even.*

Proof. That $\mathrm{cl}^{\boldsymbol{\epsilon}}_S$ is an injection follows from [32] and the calculation of Casimir operators for G, cf. [11]. The surjectivity and the vanishing of other plectic Hodge types follows from (25), Theorem 5.2.2, and the calculation of the nonvanishing $(\mathfrak{g},K_2)$ cohomology groups:

$$\dim H^{3,0}(\mathfrak{g},K_2;\Pi^-_v(\pi_{1,v},\pi_{2,v})\otimes V_{m_v,\mathbb{C}}) = \dim H^{0,3}(\mathfrak{g},K_2;\Pi^-_v(\pi_{1,v},\pi_{2,v})\otimes V_{m_v,\mathbb{C}}) = 1,$$
$$\dim H^{2,1}(\mathfrak{g},K_2;\Pi^+_v(\pi_{1,v},\pi_{2,v})\otimes V_{m_v,\mathbb{C}}) = \dim H^{1,2}(\mathfrak{g},K_2;\Pi^+_v(\pi_{1,v},\pi_{2,v})\otimes V_{m_v,\mathbb{C}}) = 1.$$

The dimensions of these cohomology groups are calculated in [32]; the result is also recalled in [31]. □

5.3.4. Finally, we relate the Π_{S_f}-isotypic parts of the L^2 and singular cohomology.

Proposition 5.3.5. *Assume* Π *is defined over* E*, where* $\mathbb{Q}(\boldsymbol{m})\subset E\subset\mathbb{C}$*. Then there exist* $\mathrm{GSp}_4(\mathbb{A}_{F,f})$*-equivariant isomorphisms*

$$H^*_!(S(\mathbf{GSp}_4),\mathcal{V}_{(\boldsymbol{m}-2,0),E})[\Pi_{S_f}]\otimes_E\mathbb{C} \simeq H^*_{(2)}(S(\mathbf{GSp}_4),\mathcal{V}_{(\boldsymbol{m}-2,0),\mathbb{C}})[\Pi_{S_f}]$$

and

$$H^*_c(S(\mathbf{GSp}_4),\mathcal{V}_{(\boldsymbol{m}-2,0),E})[\Pi_{S_f}] \simeq H^*(S(\mathbf{GSp}_4),\mathcal{V}_{(\boldsymbol{m}-2,0),E})[\Pi_{S_f}] \simeq H^*_!(S(\mathbf{GSp}_4),\mathcal{V}_{(\boldsymbol{m}-2,0),E})[\Pi_{S_f}].$$

Proof. By Theorem 5.2.2,

$$H_{\mathrm{cusp}}(S(\mathbf{GSp}_4),\mathcal{V}_{(\boldsymbol{m}-2,0)})[\Pi_{S_f}] \simeq H_{(2)}(S(\mathbf{GSp}_4),\mathcal{V}_{(\boldsymbol{m}-2,0)})[\Pi_{S_f}],$$

and the first statement follows by the discussion in (3.3.1). The second assertion is an immediate consequence of Lemma 3.3.4 (and Poincaré duality), since Π_{S_f} is not Eisenstein. □

6. Periods of Yoshida lifts

6.1. **The period problem.**

6.1.1. Let $\pi_1 \boxtimes \pi_2$ be a cuspidal automorphic representation of $\mathrm{GSO}(V)(\mathbb{A}_F)$, and let π be an auxiliary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ such that $\pi^\vee$ and π_i have the same central character. Consider the subgroup

$$H = \mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2 \subset \mathrm{GSp}_4$$

and the period integral $\mathcal{P}_{S,\pi_1,\pi_2,\pi} : \Pi_S(\pi_1, \pi_2) \otimes \pi \to \mathbb{C}$ defined by

$$\mathcal{P}_{S,\pi_1,\pi_2,\pi}(\alpha, \beta) = \int_{[Z_H\backslash H]} \alpha(h, h') \cdot \beta(h) \,\mathrm{d}(h, h'), \tag{59}$$

where $H(\mathbb{A}_F) \subset \mathrm{GSp}_4(\mathbb{A}_F)$ is parameterized by pairs $(h, h') \in \mathrm{GL}_2(\mathbb{A}_F) \times \mathrm{GL}_2(\mathbb{A}_F)$ such that $\det(h) = \det(h')$. When π_1, π_2, and π are clear from context, we drop them from the notation $\mathcal{P}_{S,\pi_1,\pi_2,\pi}$. The goal of this section is to calculate $\mathcal{P}_{S,\pi_1,\pi_2,\pi}$ explicitly (Theorems 6.2.2 and 6.5.2). The result is applied to the cohomology of Shimura varieties in the next section.

6.1.2. Of course, we must specify a Haar measure on $[Z_H\backslash H]$ for (59) to be well-defined. Let $\mathcal{C} = \mathbb{A}_F^{\times,2} F^\times \backslash \mathbb{A}_F^\times$, and let $\mathrm{d}c$ be the Haar measure on $\mathcal{C}$ assigning volume 1 to the image of $\widehat{\mathcal{O}}_F$. As the measure on $[Z_H\backslash H]$, we take the measure induced by pullback from the surjection $[\mathrm{SL}_2] \times [\mathrm{SL}_2] \times \mathcal{C} \twoheadrightarrow [Z_H\backslash H]$. The Haar measure on SL_2 is described in (2.2.2).

6.1.3. Before we begin the calculation of (59), we explain the seesaw diagram that lies behind it:

$$\begin{tikzcd} \mathrm{GSp}_4 \arrow[d, dash] \arrow[dr, dash] & \mathrm{GSO}(V_B) \times_{\mathbb{G}_m} \mathrm{GSO}(V_B) \arrow[d, dash] \arrow[dl, dash] \\ H & \mathrm{GSO}(V_B) \end{tikzcd}$$

Here B is the quaternion algebra ramified at S, the vertical lines are inclusions, and the diagonals are similitude dual pairs inside GSp_8; the diagram corresponds to the two decompositions

$$W_4 \otimes V_B = W_2 \otimes V_B \oplus W_2 \otimes V_B$$

of W_{16}. Since $\Pi_S(\pi_1, \pi_2)$ is spanned by theta lifts $\theta_\phi(f_1 \otimes f_2)$ for $f_i \in \pi_i^B$, we wish to apply the formal seesaw identity:

$$\langle \theta_\phi(f_1 \otimes f_2)|_H, \beta \otimes \mathbb{1} \rangle_H = \langle f_1 \otimes f_2, \theta_\phi(\beta \otimes \mathbb{1})|_{\mathrm{GSO}(V_B)} \rangle_{\mathrm{GSO}(V_B)}, \tag{60}$$

Here $\beta \otimes \mathbb{1}$ is the automorphic form $(h, h') \mapsto \beta(h)$ on H. Now, the theta lift from H to $\mathrm{GSO}(V_B) \times_{\mathbb{G}_m} \mathrm{GSO}(V_B)$ is simply two copies of the theta lift from GL_2 to $\mathrm{GSO}(V_B)$; restriction to the diagonal amounts to multiplying the theta lifts of β and $\mathbb{1}$ on $\mathrm{GSO}(V_B)$. The theta lift of β to $\mathrm{GSO}(V_B)$ will be a vector in $\pi^B \boxtimes \pi^B$, where π^B is the Jacquet-Langlands transfer. However, the theta lift of the constant function is formally divergent; to regularize it, we need a certain second-term Siegel-Weil formula. Ignoring this technicality, the theta lift $\theta_\phi(\beta \otimes \mathbb{1})$ restricted to the diagonal $\mathrm{GSO}(V_B)$ should be the product of a vector in $\pi^B \boxtimes \pi^B$ and an Eisenstein series on $\mathrm{GSO}(V_B)$. Of course, the Eisenstein series can only exist when B is split, so (59) should vanish identically unless $S = \emptyset$. But when $S = \emptyset$, integrating $\theta_\phi(\beta \otimes 1)$ against the form $f_1 \otimes f_2$ gives a Rankin-Selberg integral that unfolds to an Euler product and ultimately an L-function.

Thus to compute $\mathcal{P}_{S,\pi_1,\pi_2,\pi}$, we first must dispatch the trivial case $S \neq \emptyset$, and then study the theta lift of both cusp forms and constant functions from GL_2 to $\mathrm{GSO}(V)$. This is the content of the next three subsections.

6.2. **Calculation of period integral: trivial case.**

6.2.1. The trivial case $S \neq \emptyset$ can be handled easily:

Theorem 6.2.2. *If $S \neq \emptyset$, then $\mathcal{P}_S$ is identically zero.*

Proof. Let B be the quaternion algebra over F ramified exactly at S (recall $|S|$ is even). By Theorem 5.2.2, it suffices to show the vanishing of all integrals of the form

$$I(\phi, g, f) = \int_{[Z_H\backslash H]} \theta_\phi(g)(h, h') \cdot f(h) \,\mathrm{d}(h, h'),$$

where $\phi \in \mathcal{S}_{\mathbb{A}}(W_2 \otimes B)$, $W_2 \subset W$ a maximal isotropic subspace of the standard four-dimensional symplectic space, and $g \in \pi_1^B \boxtimes \pi_2^B$. Let us fix a place v at which B ramifies, and a Schwartz function $\phi^{(v)} \in$

$\mathcal{S}_{\mathbb{A}_F^{(v)}}(W_2 \otimes B)$. Then, holding the other data f, g fixed as well, consider the linear map $I_v : \mathcal{S}_{F_v}(W_2 \otimes B) \to \mathbb{C}$ defined by

$$\phi_v \mapsto I(\phi^v \otimes \phi_v, f, g). \tag{61}$$

Now I_v clearly factors through the maximal quotient Q of $\mathcal{S}_{F_v}(W_2 \otimes B) = \mathcal{S}_{F_v}(B \oplus B)$ on which $\{1\} \times \mathrm{SL}_2(F_v) \subset H(F_v) \subset \mathrm{GSp}_4(F_v)$ acts trivially. We claim this quotient is trivial. Indeed, the action of the Borel subgroup of $\{1\} \times \mathrm{SL}_2(F_v)$ is explicitly described by:

$$\begin{aligned} \omega\left(1 \times \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, 1\right) \phi_v(b_1, b_2) &= \psi\left(\frac{1}{2} n N(b_2)\right) \phi_v(b_1, b_2) \\ \omega\left(1 \times \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, 1\right) \phi_v &= |a|^2 \phi_v(b_1, a b_2). \end{aligned} \tag{62}$$

Since B_v is anisotropic, it follows from the first equation that $\mathcal{S}_{F_v}(W_2 \otimes B) \to Q$ factors through $\phi_v \mapsto \phi_v(b_1, 0)$; then the second equation implies $Q = 0$. Therefore I_v is identically zero for all choices of (ϕ^v, f, g), and in particular (since the adelic Schwartz space is generated by factorizable Schwartz functions) all the period integrals $I(\phi, f, g)$ vanish as well. $\square$

6.3. Lifts of cuspidal representations from GL_2 to $\mathrm{GSO}(V)$.

6.3.1. Since V is split, the Weil representation for the pair (W_2, V) has the alternate model given by the complete polarization $V = V_1 \oplus V_2$, where

$$V_1 = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}, \ \ V_2 = \begin{pmatrix} 0 & 0 \\ z & w \end{pmatrix}. \tag{63}$$

6.3.2. Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. It is well-known that the theta lift $\Theta(\pi) \subset \mathcal{A}_0(\mathrm{GSO}(V)(\mathbb{A}_F))$ is isomorphic to the automorphic representation $\pi \boxtimes \pi$ of $\mathrm{GSO}(V)$. To obtain our ultimate period formula, we will require the following calculation:

Lemma 6.3.3. *Let $\phi = \otimes_v \phi \in \mathcal{S}_{\mathbb{A}}(\langle e_2 \rangle \otimes W)$ and $f = \otimes_v f_v \in \pi$ be factorizable vectors, and choose a factorization*

$$W_{\psi, f}(h) = \prod_v W_{f,v}(h_v), \ \ h = (h_v) \in \mathrm{GL}_2(\mathbb{A}_F)$$

of the global Whittaker function of f (so that $W_f(h_v)(1) = 1$ for almost all v). Then the Whittaker coefficient of $\theta_\phi(f)$ along the standard unipotent subgroup $N \times N \subset \boldsymbol{p}_Z(\mathrm{GL}_2 \times \mathrm{GL}_2)$ is given by:

$$\theta_\phi(f)(g)_{N \times N, \psi^{-1} \times \psi^{-1}} = \prod_v \left(\int_{\mathrm{SL}_2(F_v)} W_{f,v}(h_v h_{c_v}) \omega(h_v h_{c_v}, g) \widehat{\phi}(1, 0, 0, -1) \, \mathrm{d}h_v \right), \ \ c = (c_v) = \det(g).$$

Proof. We compute in two steps. First, for $(h, g) \in R_0(\mathbb{A}_F)$,

$$\begin{aligned} \theta(h, g; \phi)_{N \times 1, \psi^{-1} \times 1} &= \int_{[N]} \sum_{x \in W \otimes V_2} \omega(h, ng) \widehat{\phi}(x) \psi(n) \, \mathrm{d}n \\ &= \int_{[\mathbb{G}_a]} \sum_{(z_1, w_1, z_2, w_2)} \psi(a(w_2 z_1 - z_2 w_1)) \omega(h, g) \widehat{\phi}(z_1, w_1, z_2, w_2) \psi(a) \, \mathrm{d}a \\ &= \sum_{\substack{(z_1, w_1, z_2, w_2) \\ z_1 w_2 - w_1 z_2 = 1}} \omega(h, g) \widehat{\phi}(x) \\ &= \sum_{\gamma \in \mathrm{SL}_2(F)} \omega(\gamma h, g) \widehat{\phi}(1, 0, 0, -1). \end{aligned}$$

Here $\mathrm{d}n$ is the Haar measure on N such that $[N]$ has volume 1. Now, using the identity

$$\omega(nh, g) \widehat{\phi}(1, 0, 0, -1) = \omega(h, \boldsymbol{p}_Z(1, n) g) \widehat{\phi}(1, 0, 0, -1), \ \ (g, h) \in R_0, \ n \in N(\mathbb{A}),$$

we obtain:

$$\begin{aligned}
\theta_\phi(f)(g)_{N\times N,\psi^{-1}\times\psi^{-1}} &= \int_{[N]}\int_{[\mathrm{SL}_2]} \theta(hh_c, \boldsymbol{p}_Z(1,n)g;\phi)_{N\times 1,\psi^{-1}\times 1}\psi(n)f(hh_c)\,\mathrm{d}h\,\mathrm{d}n \\
&= \int_{[N]}\int_{\mathrm{SL}_2(\mathbb{A})} \omega(hh_c, \boldsymbol{p}_Z(1,n)g)\widehat{\phi}(1,0,0,-1)\psi(n)f(hh_c)\,\mathrm{d}h\,\mathrm{d}n \\
&= \int_{[N]}\int_{\mathrm{SL}_2(\mathbb{A})} \omega(hh_c, g)\widehat{\phi}(1,0,0,-1)\psi(n)f(n^{-1}hh_c)\,\mathrm{d}n\,\mathrm{d}h \\
&= \int_{\mathrm{SL}_2(\mathbb{A})} \omega(hh_c, g)\widehat{\phi}(1,0,0,-1)W_{\psi,f}(hh_c)\,\mathrm{d}h,
\end{aligned}$$

which gives the lemma. □

6.4. A Siegel-Weil identity for $\mathrm{GSO}(V)$.

6.4.1. *Degenerate principal series for* $\mathrm{GSO}(V)$. The maximal isotropic subspace $V_1 \subset V$ of (63) has stabilizer

$$P = \boldsymbol{p}_Z(B \times \mathrm{GL}_2) \subset \mathrm{GSO}(V). \tag{64}$$

Let v be a place of F, and consider the (normalized) induced representation

$$\boldsymbol{I}_v(s) = \mathrm{Ind}_{P(F_v)}^{\mathrm{GSO}(V)(F_v)} \delta_{P_1}^s.$$

We also consider the induced representations $I_v(s) = \mathrm{Ind}_{B(F_v)}^{\mathrm{GL}_2(F_v)} \delta_B^s$. The representations $I_v(s)$ and $\boldsymbol{I}_v(s)$ are related by the following observation.

Proposition 6.4.2. *The map*

$$M : \boldsymbol{I}_v(s) \to I_v(s)$$

defined by

$$M(\varphi)(g) = \varphi(\boldsymbol{p}_Z(g,1))$$

is a linear isomorphism and an intertwining map of $\mathrm{GL}_2(F_v)\times\mathrm{GL}_2(F_v)$ *representations, if* $\mathrm{GL}_2(F_v)\times\mathrm{GL}_2(F_v)$ *acts on the left through the quotient* $\mathrm{GL}_2(F_v)\times\mathrm{GL}_2(F_v) \twoheadrightarrow GSO(V)(F_v)$ *and on the right through the quotient* $\mathrm{GL}_2(F_v)\times\mathrm{GL}_2(F_v) \twoheadrightarrow \mathrm{GL}_2(F_v)\times\{1\}$.

Then by the well-known theory of principal series for GL_2, we deduce:

Corollary 6.4.3. *For all places v, the representation $\boldsymbol{I}_v(1/2)$ has a unique irreducible subrepresentation, and the corresponding quotient is the trivial character of* $\mathrm{GSO}(V)(F_v)$.

6.4.4. Consider the map

$$[\cdot]_v : S_{F_v}(\langle e_2\otimes\rangle V) \to \boldsymbol{I}_v(1/2)$$

defined by

$$[\phi](g) = \omega(h_{\nu(g)}, g)\widehat{\phi}(0).$$

A standard calculation shows that $[\cdot]$ is equivariant for the action of $R_0(F_v) \subset \mathrm{GL}_2(F_v)\times\mathrm{GSO}(V)(F_v)$ on both sides, where $R_0(F_v)$ acts on $\boldsymbol{I}_v(1/2)$ through the projection $R_0 \twoheadrightarrow \mathrm{GSO}(V)$. We may then extend $[\phi]_v$ to a holomorphic section $[\phi]_v(s) \in \boldsymbol{I}_v(s)$ by requiring the restriction of $[\phi]$ to the maximal compact subgroup $K_0 \subset \mathrm{GSO}(V)(F_v)$ to be independent of s.

Lemma 6.4.5. *For any place v, the map $\phi \mapsto [\phi]$ identifies $\boldsymbol{I}_v(1/2)$ with the maximal quotient of $S_{F_v}(\langle e_2\rangle \otimes V)$ on which $\mathrm{SL}_2(F_v) \subset R_0$ acts trivially.*

Proof. By [26, Theorem II.1.1] (cf. [17] in the Archimedean case), the map $\phi \mapsto [\phi]$ realizes its image as the maximal quotient of $S_{F_v}(\langle e_2\rangle\otimes V)$ on which $\mathrm{SL}_2(F_v)$ acts trivially. In light of Corollary 6.4.3 it suffices to show that there exists a nontrivial map $\mathcal{S}_{F_v}(\langle e_2\rangle\otimes V) \to \mathbb{C}$ which is equivariant for $\mathrm{SL}_2(F_v)\times SO(V)(F_v) \subset R_0(F_v)$. The L^2-norm

$$\phi \mapsto \int_{\langle e_2\rangle\otimes V} |\phi(z)|^2\,\mathrm{d}z$$

is such a map (by the Plancherel identity). □

6.4.6. *Eisenstein series on $GSO(V)$.* Let $\boldsymbol{I}(s) = \mathrm{Ind}_{P(\mathbb{A}_F)}^{\mathrm{GSO}(V)(\mathbb{A}_F)} \delta_P^s$ be the global parabolic induction, and for holomorphic sections $\varphi_s \in \boldsymbol{I}(s)$ consider the Eisenstein series:

$$\boldsymbol{E}(g, s; \varphi) = \sum_{\gamma \in P(F)\backslash \mathrm{GSO}(V)(F)} \varphi_s(\gamma g), \quad g \in \mathrm{GSO}(V)(\mathbb{A}_F), \tag{65}$$

which converges for $\Re(s) \gg 0$. We also consider $I(s) = \mathrm{Ind}_{B(\mathbb{A}_F)}^{GL_2(\mathbb{A}_F)} \delta_B^s$ and, for holomorphic sections $\varphi_s \in I(s)$, the corresponding family of Eisenstein series:

$$E(g, s; \varphi) = \sum_{\gamma \in B(F)\backslash \mathrm{GL}_2(F)} \varphi_s(\gamma g), \quad g \in \mathrm{GL}_2(\mathbb{A}_F). \tag{66}$$

Proposition 6.4.7. *Let $M = \otimes_v M_v : \boldsymbol{I}(s) \to I(s)$ be the global intertwining map. Then*

$$E(g_1; s, M(\varphi)) = \boldsymbol{E}(\boldsymbol{p}_Z(g_1, g_2); s, \varphi)$$

as functions on $\mathbb{C} \times GL_2(\mathbb{A}_F) \times GL_2(\mathbb{A}_F)$ for $\Re(s) \gg 0$ and holomorphic sections $\varphi \in \boldsymbol{I}$. □

By Proposition 6.4.7 and the well-known theory of Eisenstein series for GL_2, $\boldsymbol{E}(g, s; \varphi)$ has a meromorphic continuation to $s \in \mathbb{C}$, with at most a simple pole at $s = \frac{1}{2}$. Let

$$[\cdot]_s : \mathcal{S}_{\mathbb{A}_F}(\langle e_2 \rangle \otimes V) \to \boldsymbol{I}(s) \tag{67}$$

be the tensor product of the local maps $[\cdot]_{v,s}$. For each $\phi \in \mathcal{S}_{\mathbb{A}_F}(\langle e_2 \rangle \otimes V)$, we consider the Laurent series expansion:

$$\boldsymbol{E}(g, s; [\phi]) = \frac{A_{-1}(g; \phi)}{s - \frac{1}{2}} + A_0(g; \phi) + \cdots . \tag{68}$$

Lemma 6.4.8. *For each ϕ, $A_{-1}(g; \phi)$ is a constant function of g. Moreover, the linear map*

$$A_0 : \mathcal{S}_{\mathbb{A}_F}(\langle e_2 \rangle \otimes V) \to \mathcal{A}(\mathrm{GSO}(V)(\mathbb{A}_F))$$

is an $R_0(\mathbb{A}_F)$-intertwining operator modulo constant functions.

Proof. The first claim is immediate from Proposition 6.4.7. For the second, the proof of [9, Proposition 6.4] applies almost verbatim, taking into account Lemma 6.4.5. □

6.4.9. *The spherical Eisenstein series.* Let $\varphi_s^0 \in I(s)$ be the unique $\mathrm{GL}_2(\widehat{\mathcal{O}}_F) \cdot \mathrm{SO}(2)$-spherical section such that $\varphi_s^0(1) = 1$, and let

$$E_0(g, s) := E(g, s; \varphi_s^0) \tag{69}$$

be the resulting Eisenstein series on $\mathrm{GL}_2(\mathbb{A}_F)$. We record the following:

Proposition 6.4.10. *The residue of $E_0(h, s)$ at $s = \frac{1}{2}$ is given by:*

$$\kappa = \frac{\pi^d \operatorname{Res}_{s=1} \zeta_F(s)}{2|D_F|^{\frac{1}{2}} \zeta_F(2)}.$$

Proof. Although this is standard, we give a sketch for the reader's convenience. In the Fourier expansion of $E_0(h, s)$, the non-constant Fourier coefficients are holomorphic. We therefore wish to calculate

$$\operatorname{Res}_{s=\frac{1}{2}} \frac{1}{\mathrm{Vol}([N])} \int_{[N]} E_0(n, s)\, \mathrm{d}n,$$

where $\mathrm{d}n$ is the Haar measure on $N(\mathbb{A}_F)$ induced by the identification $N(\mathbb{A}_F) \simeq \mathbb{A}_F$ and (2.1.1). Unfolding, we obtain (using the Bruhat decomposition of GL_2):

$$\begin{aligned}
\frac{1}{\mathrm{Vol}([N])} \int_{[N]} E_0(n, s)\, \mathrm{d}n &= \frac{1}{\mathrm{Vol}([N])} \int_{[N]} \sum_{\gamma \in B(F)\backslash \mathrm{GL}_2(F)} \varphi_s^0(\gamma n)\, \mathrm{d}n \\
&= \frac{1}{\mathrm{Vol}([N])} \int_{[N]} \varphi_s^0(n)\, \mathrm{d}n + \frac{1}{\mathrm{Vol}([N])} \int_{[N]} \sum_{a \in F} \varphi_s^0(w_0 a n)\, \mathrm{d}n \\
&= \frac{1}{\mathrm{Vol}([N])} \int_{[N]} \varphi_s^0(n)\, \mathrm{d}n + \frac{1}{\mathrm{Vol}([N])} \int_{N(\mathbb{A}_F)} \varphi_s^0(w_0 n)\, \mathrm{d}n,
\end{aligned}$$

where
$$w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
is the Weyl element. The first term is holomorphic in s, so we may discard it and compute:
$$\frac{1}{\mathrm{Vol}([N])} \prod_v \int_{N(F_v)} \varphi_s^0(w_0 n_v)\, \mathrm{d}n_v,$$
where $\mathrm{d}n_v$ is the standard Haar measure assigning volume one to $\mathcal{O}_v$. By the Gindikin-Karpelevich formula (e.g. [7]), this product is
$$\frac{1}{\mathrm{Vol}([N])} \left(\sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\right)^d \prod_{v\nmid\infty} \frac{1-q_v^{-2s-1}}{1-q_v^{-2s}} = \frac{1}{\mathrm{Vol}([N])} \left(\sqrt{\pi}\frac{\Gamma(s)}{\Gamma(s+1/2)}\right)^d \frac{\zeta_F(2s)}{\zeta_F(2s+1)}.$$
Taking residue at $s = \frac{1}{2}$, we obtain
$$\kappa = \frac{\pi^d \operatorname{Res}_{s=1} \zeta_F(s)}{2\,\mathrm{Vol}([N])\zeta_F(2)}.$$
Finally, we may calculate
$$\mathrm{Vol}([N]) = \mathrm{Vol}(F\backslash \mathbb{A}_F/\widehat{O}_F) = \mathrm{Vol}(\mathbb{R}^d/\mathcal{O}_F) = |D_F|^{\frac{1}{2}}$$
by strong approximation. □

6.4.11. *Regularized theta integrals.* We now recall the regularization, due to Kudla and Rallis [18], of the (non-convergent) theta integral
$$g \mapsto \int_{[\mathrm{SL}_2]} \theta(h_1 h_{\nu(g)}, g; \phi)\, \mathrm{d}h_1, \quad g \in \mathrm{GSO}(V)(\mathbb{A}_F),$$
where $\phi \in \mathcal{S}_{\mathbb{A}_F}(\langle e_2\rangle \otimes V)$. The first step of the regularization is to define a certain element z of the universal enveloping algebra of $\mathfrak{sl}_2$; for the precise definition, see [18, §5.1]. Kudla-Rallis' regularized theta integral (adapted to the similitude case) is then:

$$I(g,s;\phi) \coloneqq \frac{1}{\kappa\cdot(4s^2-1)} \int_{[\mathrm{SL}_2]} \theta(g, h_1 h_{\nu(g)}; \omega(z)\phi) E_0(h_1, s)\, \mathrm{d}h_1, \quad g \in \mathrm{GSO}(V)(\mathbb{A}_F). \tag{70}$$

(The factor of $4s^2-1$ is designed to cancel the effect of $\omega(z)$, cf. [18, §5.5]. Our normalization of s differs from *loc. cit.* by a factor of two.) The regularized integral $I(g,s;\phi)$ is a meromorphic function of s whose poles coincide with the poles of $E_0(h_1,s)$. The Laurent expansion about $s=\frac{1}{2}$ has the form:

$$I(g,s;\phi) = \frac{B_{-2}(g,\phi)}{\left(s-\frac{1}{2}\right)^2} + \frac{B_{-1}(g,\phi)}{s-\frac{1}{2}} + B_0(g,\phi) + \cdots \tag{71}$$

By definition, the linear maps

$$B_d : \mathcal{S}_{\mathbb{A}_F}(\langle e_2\rangle \otimes V) \to \mathcal{A}(GSO(V)) \tag{72}$$

are $\mathrm{GSO}(V)(\mathbb{A}_F)$-equivariant, where $g \in \mathrm{GSO}(V)(\mathbb{A}_F)$ acts on the left by $\phi \mapsto \omega(h_{\nu(g)}, g)\phi$.

Theorem 6.4.12 (Gan-Qiu-Takeda)**.** *For all $\phi \in \mathcal{S}_{\mathbb{A}_F}(\langle e_2\rangle \otimes V)$ and all $g \in \mathrm{GSO}(V)(\mathbb{A}_F)$, we have:*
$$B_{-2}(g,\phi) = \mathrm{Vol}([\mathrm{SL}_2])A_{-1}(g,\phi)$$
$$B_{-1}(g,\phi) = \mathrm{Vol}([\mathrm{SL}_2])A_0(g,\phi) + C(\nu(g),\phi),$$
where the volume of $[\mathrm{SL}_2]$ is taken with respect to $\mathrm{d}h_1$.

Proof. Fix ϕ; it follows immediately from [9] that the identities hold for all $g \in \mathrm{SO}(V)(\mathbb{A}_F)$, and for some $C(1,\phi)$. On the other hand, the map $\phi \mapsto B_{-2}(\cdot,\phi)$ is SL_2-invariant and $\mathrm{GSO}(V)(\mathbb{A}_F)$-invariant, in particular $R_0(\mathbb{A}_F)$-invariant; thus it factors through the maximal $\mathrm{GSO}(V)(\mathbb{A}_F)$-quotient of $\boldsymbol{I}(1/2)$ on which $\mathrm{SO}(V)(\mathbb{A}_F)$ acts trivially, i.e. the trivial character of $\mathrm{GSO}(V)(\mathbb{A}_F)$, and $B_{-2}(\cdot,\phi)$ is constant. Since $A_{-1}(\cdot,\phi)$ is also constant by Lemma 6.4.8, the first identity follows.

For the second identity, for all $g = g_1 g_0 \in \mathrm{GSO}(V)(\mathbb{A}_F)$ with $g_0 \in \mathrm{SO}(V)(\mathbb{A}_F)$, Lemma 6.4.8 implies:
$$A_0(g_1, \omega(h_{\nu(g_0)}, g_0)\phi) = A_0(g_1 g_0, \phi) + C(g_0,\phi);$$

since this applies to all decompositions $g = g_1 g_0$, $C(g_0, \phi)$ depends only on $\nu(g_0) = \nu(g)$. Combining this with the identity for isometry groups,

$$\begin{aligned} B_{-1}(g, \phi) &= B_{-1}(g_1, \omega(h_{\nu(g_0)}, g_0)\phi) \\ &= \mathrm{Vol}([\mathrm{SL}_2])A_0(g_1, \omega(h_{\nu(g_0)}, g_0)\phi) + C(\phi) \\ &= \mathrm{Vol}([SL_2])A_0(g_1 g_0, \phi) + C'(\nu(g), \phi). \end{aligned}$$

□

6.5. **Calculation of the period: nontrivial case.**

6.5.1. We now assume that $S = \emptyset$, so that $\Pi = \Pi_\emptyset(\pi_1, \pi_2)$ is generic, and compute $\mathcal{P}_{\emptyset,\pi_1,\pi_2,\pi}$.

Theorem 6.5.2. *(1) Choose vectors $\phi_1 \in \mathcal{S}_{\mathbb{A}_F}(\langle e_2\rangle \otimes V)$, $\phi_2 \in \mathcal{S}_{\mathbb{A}_F}(\langle e_4\rangle \otimes V)$, $\alpha \in \pi_1 \boxtimes \pi_2$, and $\beta \in \pi$. Then:*

$$\mathcal{P}_{\emptyset,\pi_1,\pi_2,\pi}(\theta_{\phi_1\otimes\phi_2}(\alpha), \beta) = \mathrm{Val}_{s=\frac{1}{2}} \int_{[\mathrm{PGSO}(V)]} \boldsymbol{E}(g, s; [\phi_2])\alpha(g)\theta_{\phi_1}(\beta)(g)\, \mathrm{d}g,$$

where $\mathrm{PGSO}(V)(\mathbb{A}_F) = \mathrm{PGL}_2(\mathbb{A}_F) \times \mathrm{PGL}_2(\mathbb{A}_F)$ is given the product Haar measure.

(2) $\mathcal{P}_{\emptyset,\pi_1,\pi_2,\pi}$ is identically zero unless $\pi \cong \pi_2^\vee$.

(3) Suppose we are given factorizations:

$$\phi_1 = \otimes_v \phi_{1,v} \in \mathcal{S}_{\mathbb{A}_F}(V), \quad \phi_2 = \otimes_v \phi_{2,v} \in \mathcal{S}_{\mathbb{A}_F}(V),$$

$$\alpha = \otimes_v \alpha_v \in \pi_1 \boxtimes \pi_2, \quad \beta = \otimes_v \beta_v \in \pi_2^\vee,$$

along with decompositions of the global Whittaker functions:

$$\alpha_{N\times N,\psi\times\psi}(g) = \prod_v W_{\alpha,v}(g_v), \quad g = (g_v) \in \mathrm{GSO}(V)(\mathbb{A}_F),$$

$$\beta_{N,\psi^{-1}}(h) = \prod_v W_{\beta,v}(h_v), \quad h = (h_v) \in \mathrm{GL}_2(\mathbb{A}).$$

Then for a sufficiently large finite set of primes S, we have:

$$\mathcal{P}_\emptyset(\theta_{\phi_1\otimes\phi_2}(\alpha), \beta) = 2|D_F|^{\frac{1}{2}} \cdot \pi^{-d} \frac{L^S(1, \pi_1 \times \pi_2^\vee)L^S(1, \mathrm{Ad}\, \pi_2)}{\zeta_F^S(2)} \prod_{v\in S} \frac{\mathcal{Z}_v(\phi_{1,v}, \phi_{2,v}, \alpha_v, \beta_v)}{1 - q_v^{-1}}$$

where $\mathcal{Z}_v(\phi_{1,v}, \phi_{2,v}, \alpha_v, \beta_v)$ is the local zeta integral:

$$\int_{(N\times N\backslash \mathrm{PGSO}(V))(F_v)} \int_{SL_2(F_v)} W_{\alpha,v}(g)W_{\beta,v}(h_1 h_c)\omega(h_1 h_c, g)\widehat{\phi}_1(1, 0, 0, -1)\varphi^0(g_2)[\phi_2](g_1)\, \mathrm{d}h_1\, \mathrm{d}g \tag{73}$$

$$c = \det(g_1 g_2), \quad g = \boldsymbol{p}_Z(g_1, g_2).$$

Here $\varphi^0(g_2)$ is the standard spherical section of $I(1)$.

(4) The L-values $L^S(1, \pi_1 \times \pi_2^\vee)$ and $L^S(1, \mathrm{Ad}\, \pi_2)$ are nonzero. Moreover, for each place v, there exist choices of $\phi_{i,v}, \alpha_v$, and β_v such that

$$\mathcal{Z}_v(\phi_{1,v}, \phi_{2,v}, \alpha_v, \beta_v) \neq 0.$$

Proof. First, fix the Haar measure dg on $\mathrm{SO}(V)(\mathbb{A}_F)$ such that, under the surjective natural map $[\mathrm{SO}(V)] \times \mathcal{C} \to [\mathrm{PGSO}(V)] = [\mathrm{PGL}_2] \times [\mathrm{PGL}_2]$, the Haar measure on $[\mathrm{PGL}_2] \times [\mathrm{PGL}_2]$ induced from (2.2.2) pulls back to dg dc. We expand:

$$\mathcal{P}_{\emptyset,\pi_1,\pi_2,\pi}(\theta_{\phi_1\otimes\phi_2}(\alpha), \beta) = \int_{[Z_H\backslash H]} \theta_{\phi_1\otimes\phi_2}(\alpha)(h, h')\beta(h), \tag{$*$}$$

which by definition is:

$$\begin{aligned}(*) &= \int_{[Z_H\backslash H]}\int_{[\mathrm{GSO}(V)^{\nu(h)}]}\theta(h,g;\phi_1)\theta(h',g;\phi_2)\alpha(g)\beta(h)\\ &= \int_{\mathcal{C}}\int_{[\mathrm{SL}_2]}\int_{[\mathrm{SO}(V)]}\int_{[\mathrm{SL}_2]}\theta(hh_c,gg_c;\phi_1)\theta(h'h_c,gg_c;\phi_2)\alpha(gg_c)\beta(h)\\ &= \int_{\mathcal{C}}\int_{[\mathrm{SL}_2]}\int_{[\mathrm{SO}(V)]}\theta(h'h_c,gg_c;\phi_2)\theta_{\phi_1}(\beta)(gg_c)\alpha(gg_c).\end{aligned}$$

Now, by the reasoning of [18, §5.5], the latter integral is equal to the residue at $s=\frac{1}{2}$ of:

$$\frac{1}{\kappa\cdot(4s^2-1)}\int_{\mathcal{C}}\int_{[\mathrm{SL}_2]}\int_{[\mathrm{SO}(V)]}\theta(hh_c,gg_c;\omega(z)\phi_2)E_0(h,s)\alpha(g)\theta_{\phi_1}(\beta)(gg_c)\,\mathrm{d}g\,\mathrm{d}h\,\mathrm{d}c,$$

which is meromorphic for $\Re(s)\gg 0$. Here κ is as in Proposition 6.4.10. Now, by the principle of meromorphic continuation, we may interchange the integrals over SL_2 and $\mathrm{SO}(V)$, and obtain:

$$\begin{aligned}(*) &= \int_{[\mathrm{PGSO}(V)]}B_{-1}(g,\phi_2)\alpha(g)\theta_{\phi_1}(\beta)(g)\,\mathrm{d}g\\ &= \mathrm{Val}_{s=\frac{1}{2}}\int_{[\mathrm{PGSO}(V)]}\boldsymbol{E}(g,s;[\phi_2]_s)\alpha(g)\theta_{\phi_1}(\beta)(g)\,\mathrm{d}g,\end{aligned}$$

by Theorem 6.4.12 and the cuspidality of A. This is (1). For (2), since $\theta_{\phi_1}(\beta)(g)$ lies in the automorphic representation $\pi'\boxtimes\pi'$ of $\mathrm{GSO}(V)(\mathbb{A}_F)$, it is a linear combination of functions of the form

$$\boldsymbol{p}_Z(g_1,g_2)\mapsto f_1(g_1)f_2(g_2).$$

Combining this observation with Proposition 6.4.7, it follows that $(*)$ is a linear combination of integrals of the form

$$\mathrm{Val}_{s=\frac{1}{2}}\int_{[\mathrm{PGL}_2\times\mathrm{PGL}_2]}E(g_1,s;M[\phi_2])\alpha(\boldsymbol{p}_Z(g_1,g_2))f_1(g_1)f_2(g_2)\,\mathrm{d}g_1\,\mathrm{d}g_2,$$

which clearly vanish unless $f_2\in\pi_2^\vee$, i.e., unless $\pi'\cong\pi_2$. This proves (2). In order to prove (3), we replace $(*)$ with an equivalent integral that can be unfolded:

$$\begin{aligned}(*) &= \frac{1}{\kappa}\mathrm{Res}_{s=\frac{1}{2}}\int_{[\mathrm{PGSO}(V)]}E(g_1,s;M[\phi_2])E_0(g_2,s)\alpha(g)\theta_{\varphi_1}(\beta)(g)\,\mathrm{d}g,\quad g=\boldsymbol{p}_Z(g_1,g_2)\\ &= \frac{1}{\kappa}\mathrm{Res}_{s=\frac{1}{2}}\int_{N(\mathbb{A})\times N(\mathbb{A})\backslash\,\mathrm{PGSO}(V)(\mathbb{A})}M[\phi_2]_s(g_1)\varphi_s^0(g_2)\alpha_{N\times N,\psi\times\psi}(g)\theta_{\varphi_1}(\beta)_{N\times N,\psi^{-1}\times\psi^{-1}}(g)\,\mathrm{d}g.\end{aligned}$$

This factors into an Euler product

$$(*) = \frac{1}{\kappa}\mathrm{Res}_{s=\frac{1}{2}}\prod_v\mathcal{Z}_v(s,\phi_{1,v},\phi_{2,v},\alpha_v,\beta_v),$$

where the local zeta integrals are (applying Lemma 6.3.3):

$$\int_{(N\times N\backslash\,\mathrm{PGSO}(V))(F_v)}\int_{SL_2(F_v)}W_{\alpha,v}(g)W_{\beta,v}(h_1h_c)\omega(h_1h_c,g)\widehat{\phi}_1(1,0,0,-1)\varphi^0(g_2)[\phi_2](g_1)\,\mathrm{d}h_1\,\mathrm{d}g \qquad c=\det(g_1g_2),\quad g=\boldsymbol{p}_Z(g_1,g_2). \tag{74}$$

At an unramified place v such that $W_{\alpha,v}$, $W_{\beta,v}$, $\phi_{i,v}$ are all the standard spherical vectors, the inner integral

$$\int_{\mathrm{SL}_2(F_v)}W_{\beta,v}(h_1h_c)\omega(h_1h_c,g)\widehat{\phi}_1(1,0,0,-1)\,\mathrm{d}h_1$$

is exactly the standard spherical Whittaker function for $\pi_2^\vee\boxtimes\pi_2^\vee$, by the unramified theta correspondence. Then, the standard Rankin-Selberg calculations show that we have the Euler factor

$$\mathcal{Z}_v(s,\phi_{1,v},\phi_{2,v},\alpha_v,\beta_v)=\frac{L_v(s+\frac{1}{2},\pi_1\boxtimes\pi_2^\vee)L_v(s+\frac{1}{2},\pi_2\boxtimes\pi_2^\vee)}{1-q_v^{-2}}.$$

Now the formula (3) follows by comparing with Proposition 6.4.10. The non-vanishing of the L-values in (4) is well-known; see for instance [30] and [12]. The non-vanishing of the local zeta integrals at ramified places also follows from the non-vanishing for Rankin-Selberg local zeta integrals, cf. [15]. □

7. Proof of main result: special cycles in the generic case

In this section, we apply the results of §6 to the cohomology of Shimura varieties. Since the Schwartz functions at the Archimedean places must be chosen rather carefully to obtain automorphic forms that contribute to cohomology, we must begin with several local calculations.

7.1. Archimedean calculations.

7.1.1. We first establish some general conventions for the local Weil representation for the pair $(V, W_{2n,\mathbb{R}})$, where $V = V_{M_2(\mathbb{R})}$. Fix coordinates on $W_{2n} \otimes V$ by:

$$(75)\qquad \begin{aligned} (\underline{x}_1, \cdots, \underline{x}_{2n}) &\longleftrightarrow \sum e_i \otimes \underline{x}_i, \\ \underline{x}_i = (x_i, y_i, z_i, w_i) &\longleftrightarrow \begin{pmatrix} x_i & y_i \\ z_i & w_i \end{pmatrix}. \end{aligned}$$

Let $K_n \subset G = \mathrm{GSp}_{2n,\mathbb{R}}$ be as in (2.3.2), let $H = \mathrm{GSO}(V)$, let R_0 be as in (4.1.1), and let $L = Z_H \cdot \boldsymbol{p}_Z(\mathrm{SO}(2) \times \mathrm{SO}(2)) \subset H(\mathbb{R})$. Also let $L_1 \subset L$ be the kernel of ν_H restricted to L, so that $L_1 = \boldsymbol{p}_Z(\mathrm{SO}(2) \times \mathrm{SO}(2))$. For any integers m_1, m_2 with $m_1 \equiv m_2 \pmod 2$, let χ_{m_1,m_2} be the character of L which is given by $\omega_{m_1}^{-1} = \omega_{m_2}^{-1}$ on Z_H and by $\chi_{m_1} \boxtimes \chi_{m_2}$ on L_1. Finally let $(K \times L)_0 = (K \times L) \cap R_0$.

7.1.2. Let $S^0(n) \subset \mathcal{S}_{F_v}(\langle e_2, \cdots, e_{2n} \rangle \otimes V)$ be the subspace of Schwartz functions of the form

$$\phi(\underline{x}_2, \cdots, \underline{x}_{2n}) = p(\underline{x}_2, \cdots, \underline{x}_{2n}) \exp(-\pi(|\underline{x}_2|^2 + \cdots + |\underline{x}_{2n}|^2),$$

where p is a polynomial, and let $S^0_d(n) \subset S^0(n)$ be the subset such that p is homogeneous of degree d. As a $(\mathfrak{r}_0, (K_n \times L)_0)$-module, S^0 is isomorphic to the Fock space $\mathcal{F}_n$ of complex polynomials in $4n$ variables, cf. [13]; the isomorphism does not preserve degrees, but it does carry $S^0_{\leq d}(n) = \oplus_{i \leq d} S^0_i$ isomorphically onto $\mathcal{F}_{\leq d}$, the subspace of polynomials of degree less than or equal to d. The following proposition is the key fact we will need about the structure of the $(K_n \times L)_0$-module $S^0_{\leq d}(n)$.

Proposition 7.1.3. *(1) If the $U(n)$-representation of highest weight $(a_1, \cdots, a_n)$ appears in $S^0(n)_{\leq d}$, then $|a_1| + \cdots + |a_n| \leq d$.*

(2) If $m_1 \equiv m_2 \pmod 2$ are integers such that $m_1 = \pm m_2$ if $n = 1$, define

$$a = \frac{|m_1 + m_2|}{2}, \quad b = \frac{|m_1 - m_2|}{2},$$

and let τ be the unique representation of K_n whose restriction to $\mathbb{R}^\times$ is ω_m^{-1} and which has weight $(a, 0, \cdots, 0, -b)$ when restricted to $U(n)$. Then

$$\dim \left(S^0_{\leq a+b}(n) \otimes \tau \otimes \chi^\vee_{m_1,m_2}\right)^{(K_n \times L)_0} = 1.$$

Proof. This follows from [11, Proposition 4.2.1]; see Remark 3.2.2 of *loc. cit.* to translate the $O(2) \times O(2)$ parameters into $\boldsymbol{p}_Z(\mathrm{SO}(2) \times \mathrm{SO}(2))$-parameters. □

In practice, we supplement this proposition with an explicit calculation:

Proposition 7.1.4. *Suppose $n = 1$ and $m \geq 0$. Then for $\epsilon = \pm 1$, a generator for the one-dimensional space*

$$(S^0_{\leq m}(1) \otimes \chi_{\epsilon m} \otimes \chi^\vee_{m,\epsilon m})^{(K_1 \times L)_0}$$

is given by

$$\phi^\epsilon_m(x, y, z, w) := (x + \epsilon i y + i z - \epsilon w)^m \exp(-\pi |\underline{x}|^2).$$

Proof. It suffices to show that for all

$$(k, \boldsymbol{p}_Z(k_1, k_2)) \in U(1) \times \boldsymbol{p}_Z(\mathrm{SO}(2) \times \mathrm{SO}(2)) \subset \mathrm{SL}_2(F_v) \times \mathrm{SO}(V)(F_v),$$

we have:

$$\omega(k, \boldsymbol{p}_Z(k_1, k_2))\phi^\epsilon_m = \chi_{-\epsilon m}(k)\chi_m(k_1)\chi_{\epsilon m}(k_2)\phi^\epsilon_m.$$

For the action of $U(1) \subset \mathrm{SL}_2$, we calculate on the Lie algebra level using the following formulas for differential $\mathrm{d}\omega$ of the Weil representation:

$$\mathrm{d}\omega\left(\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, 0\right) = \frac{1}{2\pi i}\left(\frac{\partial^2}{\partial z \partial y} - \frac{\partial^2}{\partial w \partial x}\right),$$
$$\mathrm{d}\omega\left(\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, 0\right) = 2\pi i(xw - yz).$$

Since

$$\mathrm{d}\omega\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, 0\right)\phi_m^\epsilon = -im\epsilon\phi_m^\epsilon,$$

the lemma follows. □

7.1.5. For the remainder of this subsection, $n = 2$. We now define the vector-valued Schwartz functions adapted to constructing cohomology classes on Shimura varieties as in §3, and compute two related local zeta integrals. Fix a choice of sign $\epsilon = \pm$ and an integer $m \geq 2$, and let τ_m^ϵ be the representation of K_2 defined in (5.3.1).

Remark 7.1.6. In practice, it would suffice to perform the calculations below for a single choice of ϵ; we have included both for maximum clarity and for the convenience of the reader.

Proposition 7.1.7. *Let φ_m^ϵ generate the one-dimensional space*

$$(S^0_{\leq m+2}(2) \otimes \tau_m^\epsilon \otimes \chi^\vee_{(m+2),-\epsilon m})^{(K_2 \times L)_0}.$$

If $\ell \in \tau_m^{\epsilon,\vee}$ is a highest (resp. lowest) weight vector if $\epsilon = +$ (resp. $\epsilon = -$), then $\ell(\varphi)$ is proportional to

$$\phi_{m+1}^{-\epsilon}(\underline{x}_2) \cdot \phi_1^\epsilon(\underline{x}_4).$$

Proof. The one-dimensionality follows from Proposition 7.1.3(2). For the rest, $\ell(\varphi)$ is a vector of weight $\epsilon(m+1,-1)$ for $U(2)$. By Proposition 7.1.3(1), $\tau^\vee$ is the only $U(2)$-type containing the highest weight $(\epsilon(m+1), -\epsilon)$ to appear in $S^0_{\leq m+2}(2)$, so $\ell(\varphi)$ spans the weight $\epsilon(m+1,-1)$ subspace of $(S^0_{\leq m+2} \otimes \chi^\vee_{(m+2,-\epsilon m)})^{L_1}$; on the other hand $\phi_{m+1}^{-\epsilon}(\underline{x}_2) \cdot \phi_1^\epsilon(\underline{x}_4)$ lies in this subspace by Proposition 7.1.4, and the proposition follows. □

Proposition 7.1.8. *Let $\ell \in \tau_m^{\epsilon,\vee}$ be a vector of weight $(\epsilon m, 0)$. Then $\ell(\varphi_m^\epsilon)$ is proportional to*

$$\varphi_m^\epsilon := \left((m+1)\left((x_4 + iz_4)^2 + (y_4 + iw_4)^2\right) - (x_2 + iz_2)^2 - (y_2 + iw_2)^2\right)\phi_m^{-\epsilon}(\underline{x}_2)\exp(-\pi|x_4|^2).$$

Proof. In light of Proposition 7.1.7, we must apply a lowering (resp. raising) operator to $\phi_{m+1}^{-\epsilon}(\underline{x}_2) \cdot \phi_{1,v}^\epsilon(\underline{x}_4)$ in the case $\epsilon = +$ (resp. $\epsilon = -$). First, on the Lie algebra level, the lowering operator for K_2 is:

$$L = \begin{pmatrix} 0 & 0 & 1 & i\\ 0 & 0 & -i & 1\\ -1 & i & 0 & 0\\ -i & -1 & 0 & 0\end{pmatrix} \in \mathfrak{k} \otimes \mathbb{C} \subset \mathfrak{g} \otimes \mathbb{C},$$

and the raising operator R is its complex conjugate. For compactness of notation, set $L^+ := L$ and $L^- := R$. Recall the partial Fourier transform of 4.1.3:

$$\widehat{\phi}(z_1, z_2, z_3, z_4, w_1, w_2, w_3, w_4) = \int \phi(\underline{x}_2, \underline{x}_4)\psi(y_2 z_1 - x_2 w_1 + y_4 z_3 - x_4 w_3)\,\mathrm{d}x_2\,\mathrm{d}y_2\,\mathrm{d}x_4\,\mathrm{d}y_4.$$

Using the identity

$$\widehat{\omega(g,1)}\phi(z_1, z_2, z_3, z_4, w_1, w_2, w_3, w_4) = \widehat{\phi}\left(g^{-1}\begin{pmatrix}z_1\\ z_2\\ z_3\\ z_4\end{pmatrix}, g^{-1}\begin{pmatrix}w_1\\ w_2\\ w_3\\ w_4\end{pmatrix}\right),$$

we calculate that

(76)
$$\omega(L^\epsilon)\left[p(\underline{x}_2, \underline{x}_4)\phi_0(\underline{x}_2)\phi_0(\underline{x}_4)\right] = \big[-(z_2+\epsilon y_2)(\partial_{z_4}p+\epsilon\partial_{y_4}p)+(z_4-\epsilon y_4)(\partial_{z_2}p-\epsilon\partial_{y_2}p)-(w_2-\epsilon x_2)(\partial_{w_4}p-\epsilon\partial_{x_4}p)$$
$$+ (w_4 + \epsilon x_4)(\partial_{w_2}p + \epsilon\partial_{x_2}p) + \frac{\epsilon}{2\pi}(\partial^2_{y_2,z_4}p + \partial^2_{z_2,y_4}p - \partial^2_{x_2,w_4}p - \partial^2_{w_2,x_4}p)\big]\phi_0(\underline{x}_2)\phi_0(\underline{x}_4),$$

where we have abbreviated

$$\phi_0(\underline{x}) \coloneqq \exp(-\pi|\underline{x}|^2).$$

One can then check that

$$\omega(L^\epsilon)\left[\phi_{m+1}^{-\epsilon}(\underline{x}_2)\cdot\phi_{1,v}^{\epsilon}(\underline{x}_4)\right] = 2\overline{\varphi}_m^\epsilon,$$

which proves the proposition. $\square$

7.1.9. For later use, we calculate two archimedean local zeta integrals related to $\overline{\varphi}_m^\epsilon$. Write:

$$\begin{aligned}
\phi_1 &= \phi_m^{-\epsilon}\\
\phi_2 &= \left((x+iz)^2+(y+iw)^2\right)\exp(-\pi|\underline{x}|^2),\\
\phi_1' &= \left((x+iz)^2+(y+iw)^2\right)\phi_m^\epsilon\\
\phi_2' &= \exp(-\pi|\underline{x}|^2),
\end{aligned}$$

so that

$$\overline{\varphi}_m^\epsilon = (m+1)\phi_1\otimes\phi_2 - \phi_1'\otimes\phi_2'. \tag{77}$$

For each integer $n\geq 2$ and pair of signs $\epsilon,\delta\in\{\pm\}$, let $W_{n,\psi^\delta}^\epsilon$ be the normalized weight ϵn vector in the ψ^δ-Whitaker model of the discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ of weight n; thus

$$W_{n,\psi^\delta}^\epsilon\begin{pmatrix}\epsilon\delta t^{1/2} & 0\\ 0 & t^{1/2}\end{pmatrix} = t^{n/2}e^{-2\pi t},\ \ W_{n,\psi^\delta}^\epsilon\begin{pmatrix}-\epsilon\delta t^{1/2} & 0\\ 0 & t^{-1/2}\end{pmatrix} = 0,\ \ \forall t>0. \tag{78}$$

Proposition 7.1.10. *With notation as above,*

$$\mathcal{Z}_v\left(\phi_1,\phi_2,W_{m+2,\psi}^-\otimes W_{m,\psi}^\epsilon, W_{m,\psi}^{-\epsilon}\right) = \frac{\epsilon^m\cdot m!(m-1)!}{\pi^{2m+2}2^{3m+2}}$$

and

$$\mathcal{Z}_v\left(\phi_1',\phi_2',W_{m+2,\psi}^-\otimes W_{m,\psi}^\epsilon, W_{m,\psi}^{-\epsilon}\right) = 0.$$

Proof. First, note that:

$$\begin{aligned}
\omega(k,\boldsymbol{p}_Z(k_1,k_2))\phi_1 &= \chi_{\epsilon m}(k)\chi_m(k_1)\chi_{-\epsilon m}(k_2)\phi_1,\\
\omega(k,\boldsymbol{p}_Z(k_1,k_2))\phi_2 &= \chi_2(k_1)\phi_2,\\
\omega(k,\boldsymbol{p}_Z(k_1,k_2))\phi_1' &= \chi_{\epsilon m}(k)\chi_{m+2}(k_1)\chi_{-\epsilon m}(k_2)\phi_1',\\
\omega(k,\boldsymbol{p}_Z(k_1,k_2))\phi_2' &= \phi_2',\quad \forall(k,\boldsymbol{p}_Z(k_1,k_2))\in U(1)\times\boldsymbol{p}_Z(\mathrm{SO}(2)\times\mathrm{SO}(2))\subset\mathrm{SL}_2\times\mathrm{SO}(V).
\end{aligned}$$

The first identity is just Proposition 7.1.4, and the latter three may be proved similarly. (Alternatively, the action of the first factor $U(1)\subset\mathrm{SL}_2$ can be deduced from Proposition 7.1.8.) Next, we compute the inner integral for $\mathcal{Z}_v\left(\phi_1,\phi_2,W_{m+2,\psi}^-\otimes W_{m,\psi}^\epsilon,W_{m,\psi}^{-\epsilon}\right)$:

$$I(g_1,g_2) = \int_{\mathrm{SL}_2(F_v)}\omega(h_1h_c,\boldsymbol{p}_Z(g_1,g_2))\widehat{\phi}_1(1,0,0,-1)W_{m,\psi}^{-\epsilon}(h_1h_c)\,\mathrm{d}h_1,\quad c=\det(g_1)\det(g_2). \tag{79}$$

By the archimedean local theta correspondence for $\mathrm{GL}_2\times\mathrm{GSO}(V)$ and Proposition 7.1.4, we have

$$I(g_1,g_2) = \lambda W_{m,\psi^{-1}}^+(g_1)W_{m,\psi^{-1}}^{-\epsilon}(g_2) \tag{80}$$

for a scalar λ. To pin down the scalar, it suffices to calculate:

$$I(g_0),\quad g_0\coloneqq\boldsymbol{p}_Z\left(\begin{pmatrix}-1&0\\0&1\end{pmatrix},\begin{pmatrix}\epsilon&0\\0&1\end{pmatrix}\right). \tag{$*$}$$

By definition, we have:

$$\begin{aligned}
(*) &= \int\omega\left(h_1\begin{pmatrix}1&0\\0&-\epsilon\end{pmatrix},g_0\right)\widehat{\phi}_m^{-\epsilon}(1,0,0,-1)W_{m,\psi}^{-\epsilon}\left(h_1\begin{pmatrix}1&0\\0&-\epsilon\end{pmatrix}\right)\mathrm{d}h_1\\
&= \int\omega\left(h_1\begin{pmatrix}1&0\\0&-\epsilon\end{pmatrix},g_0\right)\phi_m^\epsilon(x,y,0,-1)\psi(y)(-\epsilon)^mW_{m,\psi}^{-\epsilon}\left(h_1\begin{pmatrix}-\epsilon&0\\0&1\end{pmatrix}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}h_1.
\end{aligned}$$

Recall that the Haar measure on SL_2 is given by

$$\mathrm{d}h_1 = \frac{\mathrm{d}a\,\mathrm{d}t\,\mathrm{d}\theta}{2\pi t^2}, \quad h_1 = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{-1/2} \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Adopting these coordinates, the integral becomes

$$\begin{aligned}(*) &= (-\epsilon)^m \int \psi(a) t^{m/2-2} e^{-2\pi t}\psi(-ax)\omega\left(\begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{-1/2}\end{pmatrix}, 1\right)\phi_m^{-\epsilon}(-x, -\epsilon y, 0, \epsilon)\psi(y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}a\,\mathrm{d}t \\ &= (-\epsilon)^m \int t^{m-1}(-2+iy)^m \exp(-\pi t(4+y^2))\psi(y)\,\mathrm{d}t\,\mathrm{d}y \\ &= (-i\epsilon)^m \frac{(m-1)!}{\pi^m}\int \frac{e^{2\pi i y}\,\mathrm{d}y}{(y-2i)^m} \\ &= (-i\epsilon)^m (2\pi i)^m \frac{(m-1)!}{(m-1)!\pi^m} e^{-4\pi} = (2\epsilon)^m e^{-4\pi};\end{aligned}$$

hence $\lambda = (2\epsilon)^m$.

By the equivariance properties of ϕ_2,

$$M[\phi_2] \in \mathrm{Ind}_{\overline{B}(\mathbb{R})}^{\mathrm{PGL}_2(\mathbb{R})} |\cdot|^{1/2}$$

is a section of weight two for $\mathrm{SO}(2)$. Thus it is determined by:

$$M[\phi_2](1) = \int \phi_2(x, y, 0, 0)\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{\pi}. \tag{81}$$

Now, our local zeta integral is given by

$$(2\epsilon)^m \cdot \left(\int_{N\backslash \mathrm{PGL}_2(F_v)} W^-_{m+2,\psi}(g_1)[\phi_2](g_1) W^+_{m,\psi^{-1}}(g_1)\,\mathrm{d}g_1\right)\cdot\left(\int_{N\backslash \mathrm{PGL}_2(F_v)} W^{\epsilon}_{m,\psi}(g_2) W^{-\epsilon}_{m,\psi^{-1}}(g_2)\varphi^0(g_2)\,\mathrm{d}g_2\right).$$

Since both integrands are right $\mathrm{SO}(2)$-invariant, and since the Haar measure on $\mathrm{PGL}_2(\mathbb{R})$ is given by

$$\mathrm{d}g = \frac{\mathrm{d}a\,\mathrm{d}t\,\mathrm{d}\theta}{\pi t^2}, \quad g = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad t \in \mathbb{R}^\times, \theta \in [0, \pi),$$

we obtain

$$\frac{(2\epsilon)^m}{\pi}\left(\int_0^\infty t^m e^{-4\pi t}\,\mathrm{d}t\right)\cdot\left(\int_0^\infty t^{m-1} e^{-4\pi t}\,\mathrm{d}t\right) = \frac{\epsilon^m \cdot m!(m-1)!}{\pi^{2m+2} 2^{3m+2}}, \tag{82}$$

as claimed.

To show that

$$\mathscr{Z}_v\left(\phi_1', \phi_2', W^-_{m+2,\psi} \otimes W^{\epsilon}_{m,\psi}, W^{-\epsilon}_{m,\psi}\right) = 0,$$

we may ignore scalar factors. The inner integral

$$I'(g_1, g_2) = \int_{\mathrm{SL}_2(F_v)} \omega(h_1 h_c, \boldsymbol{p}_Z(g_1, g_2))\widehat{\phi}_1'(1, 0, 0, -1) W^{-\epsilon}_{m,\psi}(h_1 h_c)\,\mathrm{d}h_1, \quad c = \det(g_1)\det(g_2) \tag{83}$$

must be of the form

$$I'(g_1, g_2) = W'(g_1) W^{-\epsilon}_{m,\psi^{-1}}(g_2), \tag{84}$$

where W' is a weight $m+2$ vector in the ψ^{-1}-Whittaker model of the discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ of weight m. To calculate W', let

$$X \coloneqq \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix} \in \mathfrak{gl}_2 \tag{85}$$

be the $\mathrm{SO}(2)$-raising operator. Then, up to scalar,

$$W' = X \cdot W^+_{m,\psi^{-1}} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} W^+_{m,\psi^{-1}} + 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\cdot W^+_{m,\psi^{-1}} - im W^+_{m,\psi^{-1}}. \tag{86}$$

In particular, $W'\begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{-1/2}\end{pmatrix} = 0$ and

$$W'\begin{pmatrix} -t^{1/2} & 0 \\ 0 & t^{-1/2}\end{pmatrix} = -2imt^{m/2}e^{-2\pi t} + 8\pi i t^{m/2+1}e^{-2\pi t} \tag{87}$$

(up to scalar) for $t > 0$. To complete the proof, we will show the integral

$$\int_{N\backslash \mathrm{PGL}_2(F_v)} W'(g_1)[\phi_2](g_1)W^+_{m,\psi^{-1}}(g_1)\,\mathrm{d}g_1 \tag{$*$}$$

vanishes. Since $[\phi_2]$ is a section of weight zero, $(*)$ is proportional to

$$\begin{aligned}(**) &= \int_0^\infty (-2imt^{m/2} + 8\pi i t^{m/2+1})\cdot t \cdot t^{m/2}e^{-4\pi t}\frac{\mathrm{d}t}{t^2}\\ &= -2im\int_0^\infty t^{m-1}e^{-4\pi t}\,\mathrm{d}t + 8\pi i\int_0^\infty e^{-4\pi t}t^m\,\mathrm{d}t\\ &= -2im\frac{(m-1)!}{(4\pi)^m} + 8\pi i\frac{m!}{(4\pi)^{m+1}}\\ &= 0.\end{aligned}$$

□

7.2. **Cohomological span of special cycle.**

7.2.1. Let $H = \mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2 \subset \mathrm{GSp}_4$, viewed as an algebraic group over F. Then $\boldsymbol{H}$ possesses a Shimura datum, and we have a natural embedding of pro-algebraic varieties

$$\iota : S(\boldsymbol{H}) \hookrightarrow S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2),$$

induced from the map on the level of groups: $(h_1, h_2) \mapsto ((h_1, h_2), h_1)$. For all weights $\boldsymbol{m}$ as in (5.3.1) above, abbreviate by $\mathcal{W}_{\boldsymbol{m}}$ the local system $\mathcal{V}^\vee_{(\boldsymbol{m}-2,0)} \boxtimes \mathcal{V}_{\boldsymbol{m}-2}$ on $S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2)$. Note that the constant local system $\underline{\mathbb{Q}(\boldsymbol{m})}$ on $S(\boldsymbol{H})$ is a direct factor with multiplicity one of the pullback $\iota^*(\mathcal{W}_{\boldsymbol{m}})$, and in particular, we have a composite map (well-defined up to a scalar):

$$H^{4d}_c(S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2), \mathcal{W}^\vee_{\boldsymbol{m}}) \to H^{4d}_c(S(\boldsymbol{H}), \iota^*(\mathcal{W}^\vee_{\boldsymbol{m}})) \to H^{4d}_c(S(\boldsymbol{H}), \mathbb{Q}(\boldsymbol{m})). \tag{88}$$

Definition 7.2.2. The cycle class $[\mathcal{Z}] \in H^{4d}(S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2), \mathcal{W}_{\boldsymbol{m}})(2d)$ is the image of the fundamental class of $S(\boldsymbol{H})$ under the map

$$H^0(S(\boldsymbol{H}), \mathbb{Q}(\boldsymbol{m})) \to H^{2d}(S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2), \mathcal{W}_{\boldsymbol{m}})(2d)$$

induced by the dual of (88). We write

$$[\mathcal{Z}]_* : H^{3d}_c(S(\mathbf{GSp}_4), \mathcal{V}_{(\boldsymbol{m}-2,0)})(d) \to H^d(S(\mathbf{GL}_2), \mathcal{V}_{\boldsymbol{m}-2})$$

for the induced map.

7.2.3. Let π be an automorphic cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_F)$ of weight $\boldsymbol{m}$ whose central character has infinity type $\omega_{\boldsymbol{m}}$. If π is defined over E, recall that the trace map induces a perfect pairing:

$$\langle\cdot,\cdot\rangle : H^d_c(S(\mathbf{GL}_2), \mathcal{V}_{\boldsymbol{m}-2,E})[\pi_f] \times H^d(S(\mathbf{GL}_2), \mathcal{V}^\vee_{\boldsymbol{m}-2,E})[\pi^\vee_f] \to H^{2d}_c(S(\mathbf{GL}_2), E) \to E(d).$$

Proposition 7.2.4. *Let π be as above, and let π_1, π_2 be as in (5.3.1), with $\Pi = \Pi_S(\pi_1, \pi_2)$, for some $S = S_f \sqcup S_\infty$ such that $|S|$ is even.*

(1) *For choices of signs $\boldsymbol{\epsilon}, \boldsymbol{\epsilon}'$, let $\sigma_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}'} : \boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{m},S_\infty} \to \mathbb{C}$ be the projection onto the weight $(-\boldsymbol{\epsilon}'\boldsymbol{m}, 0)$-component (hence σ is trivial unless $S_\infty = \emptyset$ and $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}'$). Then the following diagram commutes up to a nonzero scalar:*

$$
\begin{tikzcd}
\left(\Pi\otimes\boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{m},S_\infty}\right)^{\boldsymbol{K}_2}\otimes\left(\pi^\vee\otimes\boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon}'\boldsymbol{m}}\right)^{\boldsymbol{K}_1} \arrow[r, "\sigma_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}'}\otimes\mathrm{id}"] \arrow[d, "\mathrm{cl}^{\boldsymbol{\epsilon}}_S\otimes\mathrm{cl}'_{\boldsymbol{\epsilon}'}"] & \Pi\otimes\pi \arrow[dddd, "\mathcal{P}_S"] \\
H^*_{(2)}(S(\mathbf{GSp}_4),\mathcal{V}_{(\boldsymbol{m}-2,0),\mathbb{C}})[\Pi_{S_f}]\otimes H^*_{(2)}(S(\mathbf{GL}_2),\mathcal{V}^\vee_{\boldsymbol{m}-2,\mathbb{C}})[\pi^\vee_f] \arrow[d, "\sim"] & \\
H^{3d}_c(S(\mathbf{GSp}_4),\mathcal{V}_{(\boldsymbol{m}-2,0),\mathbb{C}})[\Pi_{S_f}]\otimes H^d_c(S(\mathbf{GL}_2),\mathcal{V}^\vee_{\boldsymbol{m}-2,\mathbb{C}})[\pi_f] \arrow[d, "{[\mathcal{Z}]_*\otimes\mathrm{id}}"] & \\
H^d(S(\mathbf{GL}_2),\mathcal{V}_{\boldsymbol{m}-2,\mathbb{C}})\otimes H^d_c(S(\mathbf{GL}_2),\mathcal{V}^\vee_{\boldsymbol{m}-2,\mathbb{C}}) \arrow[d, "\langle\cdot,\cdot\rangle"] & \\
\mathbb{C} \arrow[r, equal] & \mathbb{C}
\end{tikzcd}
$$

(2) *Suppose $S_\infty=\emptyset$ and $\boldsymbol{\epsilon}=\boldsymbol{\epsilon}$. After fixing isomorphisms*

$$\Pi_f\simeq\left(\Pi\otimes\boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{m},\emptyset}\right)^{\boldsymbol{K}_2},\ \ \pi^\vee_f\simeq\left(\pi^\vee\otimes\boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon}\boldsymbol{m}}\right)^{\boldsymbol{K}_1},$$

the composites with the map from (1):

$$\Pi_f\otimes\pi^\vee_f\xrightarrow{\sim}\left(\Pi\otimes\boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{m},\emptyset}\right)^{\boldsymbol{K}_2}\otimes\left(\pi^\vee\otimes\boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon}\boldsymbol{m}}\right)^{\boldsymbol{K}_1}\longrightarrow\mathbb{C}$$

are independent of $\boldsymbol{\epsilon}$ up to a nonzero scalar.

Proof. Let $\ell:V_{(\boldsymbol{m}-2,0)}\otimes V^\vee_{\boldsymbol{m}-2}\to\mathbb{Q}(\boldsymbol{m})$ be an $H(F)$-invariant projection. There exists a basis $\mathbb{1}_{\boldsymbol{H}}$ of $\wedge^{4d}\mathfrak{p}_H$ such that

$$\int_{S(\boldsymbol{H})}\omega=\int_{[Z_H\backslash H]}h^*\omega(\mathbb{1}_{\boldsymbol{H}})\,\mathrm{d}h$$

for all top-degree forms ω on $S(\boldsymbol{H})$. Then by definition,

$$\langle[\mathcal{Z}]_*\alpha,\beta\rangle=\int_{[Z_H\backslash H]}\ell\left[h^*\iota^*(\alpha\wedge\beta)(\mathbb{1}_{\boldsymbol{H}})\right]\mathrm{d}h.$$

The composite map:

$$
\begin{aligned}
\boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{m},S_\infty}\otimes\boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon}'\boldsymbol{m}}\to\wedge^{\boldsymbol{p}(\boldsymbol{\epsilon},S_\infty),\boldsymbol{q}(\boldsymbol{\epsilon},S_\infty)}\mathfrak{p}^*_{\mathrm{GSp}_4}\otimes\wedge^{1-\boldsymbol{p}(\boldsymbol{\epsilon}'),1-\boldsymbol{q}(\boldsymbol{\epsilon}')}\mathfrak{p}^*_{\mathrm{GL}_2}\otimes V_{(\boldsymbol{m}-2,0)}\otimes V_{\boldsymbol{m}-2}\\
\xrightarrow{\iota^*\otimes\ell}\wedge^{\boldsymbol{p}(\boldsymbol{\epsilon},S_\infty)+1-\boldsymbol{p}(\boldsymbol{\epsilon}'),\boldsymbol{q}(\boldsymbol{\epsilon},S_\infty)+1-\boldsymbol{q}(\boldsymbol{\epsilon}')}\mathfrak{p}_H\xrightarrow{\mathbb{1}_{\boldsymbol{H}}}\mathbb{C}
\end{aligned}
\tag{89}
$$

is a map of $U(1)^d\times U(1)^d$-modules, where the action on $\boldsymbol{\chi}_{-\boldsymbol{\epsilon}'\boldsymbol{m}}$, $\wedge^{\boldsymbol{p}(\boldsymbol{\epsilon}'),\boldsymbol{q}(\boldsymbol{\epsilon}')}\mathfrak{p}_{\mathrm{GL}_2}$, and $V_{\boldsymbol{m}-2}$ is through projection to the first factor. In particular, (89) is trivial unless $S_\infty=\emptyset$ and $\boldsymbol{\epsilon}=\boldsymbol{\epsilon}'$, in which case it is proportional to the projection onto the weight $(-\boldsymbol{\epsilon}\boldsymbol{m},0)$-component of $\boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{m},\emptyset}$; moreover, a direct calculation shows it is nonzero, and (1) follows.

For (2), let

$$g_{\boldsymbol{\epsilon}}=\begin{pmatrix}\epsilon_v&&&\\&\epsilon_v&&\\&&1&\\&&&1\end{pmatrix}_{v|\infty}\in G(F\otimes\mathbb{R})\subset G(\mathbb{A}_F),\quad g'_{\boldsymbol{\epsilon}}=\begin{pmatrix}\epsilon_v&\\&1\end{pmatrix}_{v|\infty}\in G'(F\otimes\mathbb{R})\subset G'(\mathbb{A}_F).$$

We have an obvious commutative diagram

$$
\begin{tikzcd}
\left(\Pi\otimes\boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{m},\emptyset}\right)^{\boldsymbol{K}_2}\otimes\left(\pi^\vee\otimes\boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon}\boldsymbol{m}}\right)^{\boldsymbol{K}_1} \arrow[r, "\sigma_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}\otimes\mathrm{id}"] \arrow[d] & \Pi\otimes\pi^\vee \arrow[d] \\
\left(\Pi\otimes\boldsymbol{\tau}^{+}_{\boldsymbol{m},\emptyset}\right)^{\boldsymbol{K}_2}\otimes\left(\pi^\vee\otimes\boldsymbol{\chi}^\vee_{-\boldsymbol{m}}\right)^{\boldsymbol{K}_1} \arrow[r, "\sigma_{+,-}\otimes\mathrm{id}"] & \Pi\otimes\pi^\vee
\end{tikzcd}
$$

in which the vertical arrows are translation by $(g_{\boldsymbol{\epsilon}}, g'_{\boldsymbol{\epsilon}})$ and $\pm$ stands for the constant sign $(\pm)_v|\infty$. However, since $(g_{\boldsymbol{\epsilon}}, g'_{\boldsymbol{\epsilon}})$ lies in $H(\mathbb{A}_F) \subset G(\mathbb{A}_F) \times G'(\mathbb{A}_F)$, this translation has no effect on the period integral $\mathcal{P}_S$, and (2) follows. $\square$

Theorem 7.2.5. *Let π_1, π_2, π be cuspidal automorphic representations of $G'(\mathbb{A})$ of weights $\boldsymbol{m}+2$, $\boldsymbol{m}$, and $\boldsymbol{m}$, respectively, where $\boldsymbol{m} = (m_v)_{v|\infty}$ for positive integers m_v. Assume that the central characters of π_1 and π_2 agree, and that the the central characters of π_1, π_2, and π all have infinity type $\omega_{\boldsymbol{m}}$. Let Π_{S_f} be as in (5.3.1) for set S_f of finite places of F. Then, for any coefficient field $E \supset F^c$ such that Π, π_i, and π are defined over E, the induced map*

$$[\mathcal{Z}]_* : H^{3d}_!(S(\mathbf{GSp}_4), \mathcal{V}_{(\boldsymbol{m}-2,0)})(d)[\Pi_{S_f}] \to H^d_!(S(\mathbf{GL}_2), \mathcal{V}_{\boldsymbol{m}-2})[\pi_f]$$

is trivial unless $\pi = \pi_2$ and $S_f = \emptyset$. In the latter case, $[\mathcal{Z}]_*$ takes the form:

$$\Pi_\emptyset \otimes H^{3d}_!(S(\mathbf{GSp}_4), \mathcal{V}_{(\boldsymbol{m}-2,0)})_{\Pi_\emptyset}(d) \xrightarrow{\ell\otimes s} \pi_{2,f} \otimes H^d_!(S(\mathbf{GL}_2), \mathcal{V}_{\boldsymbol{m}-2})_{\pi_{2,f}},$$

where s is an surjection and ℓ is a nontrivial E-linear map.

Proof. Without loss of generality, suppose $E = \mathbb{C}$. By Proposition 7.2.4, Theorem 6.2.2, and Theorem 6.5.2, we immediately reduce to the case $S_f = \emptyset$ and $\pi = \pi_2$. In this case, write $\Pi = \Pi_\emptyset(\pi_1, \pi_2)$. Under the decomposition

$$H^{3d}_!(S(\mathbf{GSp}_4), \mathcal{V}_{(\boldsymbol{m}-2,0),\mathbb{C}})[\Pi_f] = \bigoplus_{S_\infty} \bigoplus_{\boldsymbol{\epsilon}} H^{\boldsymbol{p}(\boldsymbol{\epsilon},S_\infty),\boldsymbol{q}(\boldsymbol{\epsilon},S_\infty)}_{(2)}(S(\mathbf{GSp}_4), \mathcal{V}_{(\boldsymbol{m}-2,0),\mathbb{C}})[\Pi_f]$$

provided by 5.3.3 and 5.3.5, Proposition 7.2.4(1) implies that $[\mathcal{Z}]_*$ is trivial on components with $S_\infty \neq \emptyset$, and maps $H^{\boldsymbol{p}(\boldsymbol{\epsilon},\emptyset),\boldsymbol{q}(\boldsymbol{\epsilon},\emptyset)}_{(2)}(S(\mathbf{GSp}_4), \mathcal{V}_{(\boldsymbol{n}-2,0),\mathbb{C}})(d)[\Pi_f]$ to $H^{\boldsymbol{p}(\boldsymbol{\epsilon}),\boldsymbol{q}(\boldsymbol{\epsilon})}_{(2)}(S(\mathbf{GL}_2), \mathcal{V}_{\boldsymbol{m}-2,\mathbb{C}})[\pi_{2,f}]$. Moreover, by Proposition 7.2.4(2) and Proposition 3.5.3, $[\mathcal{Z}]_*$ is a pure tensor $\ell \otimes s$, and s is surjective provided it is nontrivial. Thus, for any single choice of $\boldsymbol{\epsilon}$, it suffices to show that

$$H^{\boldsymbol{p}(\boldsymbol{\epsilon},\emptyset),\boldsymbol{q}(\boldsymbol{\epsilon},\emptyset)}_{(2)}(S(\mathbf{GSp}_4), \mathcal{V}_{(\boldsymbol{m}-2,0),\mathbb{C}})[\Pi_f] \otimes H^{\boldsymbol{1}-p(\boldsymbol{\epsilon}),\boldsymbol{1}-q(\boldsymbol{\epsilon})}_{(2)}(S(\mathbf{GL}_2), \mathcal{V}_{\boldsymbol{m}-2,\mathbb{C}})[\pi^\vee_{2,f}] \xrightarrow{\langle [\mathcal{Z}]_* \cdot, \cdot \rangle} \mathbb{C} \tag{90}$$

is nontrivial.

Indeed, let

$$\boldsymbol{\varphi}^{\boldsymbol{\epsilon}}_\infty = \otimes_{v|\infty} \varphi^{\epsilon_v}_{m_v} \in \mathcal{S}_{F\otimes\mathbb{R}}(\langle e_2, e_4\rangle \otimes V) \otimes \boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{m},\emptyset} \otimes \boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon}},$$

where $\varphi^{\epsilon_v}_{m_v}$ is the vector-valued Schwartz function of Proposition 7.1.7. Also let

$$\theta_{\boldsymbol{\epsilon}} : \mathcal{S}_{\mathbb{A}_{F,f}}(\langle e_2, e_4\rangle \otimes V) \twoheadrightarrow \left(\Pi \otimes \boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{m},\emptyset}\right)^{\boldsymbol{K}_2} \tag{91}$$

be the $\mathbb{C}$-linear map

$$\phi_f \mapsto \theta_{\phi_f \otimes \varphi^{\boldsymbol{\epsilon}}_\infty}(f_1 \otimes f_2),$$

where $f_1 \in \pi_1$ and $f_2 \in \pi_2$ are nonzero newforms of weights $-(\boldsymbol{m}+2)$ and $\boldsymbol{\epsilon m}$, respectively.

Now Proposition 7.2.4 and Proposition 7.1.8 imply that the composite map

$$\mathcal{S}_{\mathbb{A}_{F,f}}(\langle e_2, e_4\rangle \otimes V) \otimes (\pi^\vee_2 \otimes \boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon m}})^{\boldsymbol{K}_1} \xrightarrow{\theta_{\boldsymbol{\epsilon}}\otimes \mathrm{id}} \left(\Pi \otimes \boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{m},\emptyset}\right)^{\boldsymbol{K}_2} \otimes (\pi^\vee_2 \otimes \boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon m}})^{\boldsymbol{K}_1} \xrightarrow{\langle [\mathcal{Z}]_* \circ \mathrm{cl}^{\boldsymbol{\epsilon}}_\emptyset, \mathrm{cl}'_{\boldsymbol{\epsilon}}\rangle} \mathbb{C} \tag{92}$$

is given by

$$\phi_f \otimes \beta \mapsto \mathcal{P}_\emptyset(\theta_{\phi_f \otimes \overline{\varphi}^{\boldsymbol{\epsilon}}_\infty}(f_1 \otimes f_2), \beta) \tag{93}$$

up to a nonzero scalar, where

$$\overline{\varphi}^{\boldsymbol{\epsilon}}_\infty = \otimes_v \overline{\varphi}^{\epsilon_v}_{m_v}$$

for $\overline{\varphi}^{\epsilon_v}_{m_v} \in \mathcal{S}_{F_v}(\langle e_2, e_4\rangle \otimes V)$ as in Proposition 7.1.8. Suppose given factorizable test data

$$\phi_{i,f} = \otimes_{v\nmid\infty} \phi_{i,v} \in \mathcal{S}_{\mathbb{A}_{F,f}}(V), \quad i = 1, 2$$

and

$$\beta = \otimes_v \beta_v \in (\pi^\vee_2 \otimes \boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon m}})^{\boldsymbol{K}_1};$$

we also fix decompositions of the global Whittaker functionals as in Theorem 6.5.2. Then, from Theorem 6.5.2 and Proposition 7.1.10, we conclude the formula

(94)
$$\langle [\mathcal{Z}]_* \circ \mathrm{cl}^{\boldsymbol{\epsilon}}_{\emptyset}(\theta_{\boldsymbol{\epsilon}}(\phi_{1,f} \otimes \phi_{2,f})), \mathrm{cl}'_{\boldsymbol{\epsilon}}(\beta)\rangle \doteq \frac{L^S(1, \pi_1 \times \pi_2^\vee) L^S(1, \mathrm{Ad}\,\pi_2)}{\zeta_F^S(2)} \prod_{\substack{v \in S \\ v \nmid \infty}} \frac{\mathcal{Z}_v(\phi_{1,v}, \phi_{2,v}, f_{1,v} \otimes f_{2,v}, \beta_v)}{1 - q_v^{-1}} \cdot \prod_{v \mid \infty} \frac{W_{\beta_v}(1)}{e^{-2\pi}},$$

where $\doteq$ denotes equality up to a nonzero constant. The nonvanishing of the right-hand side, which follows from Theorem 6.5.2, concludes the proof. □

Remark 7.2.6. In fact, since the formula (94) is uniform in $\boldsymbol{\epsilon}$, Proposition 7.2.4(2) was not strictly necessary for the proof of the theorem.

8. Non-tempered theta lifts on GSp_6

8.1. Arthur parameters.

8.1.1. Let G be a split reductive group over F, and recall that a local Arthur parameter for G is a $\widehat{G}$-conjugacy class of homomorphisms

$$\psi_v : WD_{F_v} \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G},$$

such that the restriction to WD_{F_v} is a bounded Langlands parameter. If v is non-archimedean and ψ_v is unramified, then it determines a local Langlands parameter φ_v by the rule

$$\varphi_v(w) = \psi_v \left(w, \begin{pmatrix} |w|_v^{1/2} & 0 \\ 0 & |w|_v^{-1/2} \end{pmatrix} \right). \tag{95}$$

An unramified representation of $G(F_v)$ is said to have Arthur parameter ψ_v if its Langlands parameter is determined by the rule (95).

8.1.2. In [4], Arthur defines discrete global parameters for Sp_{2n} to be formal (unordered) sums

$$\oplus \pi_i[d_i], \tag{96}$$

where:

- π_i are cuspidal automorphic representations of PGL_{n_i};
- $d_i \geq 0$ are integers such that $\sum n_i d_i = 2n+1$;
- π_i is conjugate symplectic if d_i is even, and conjugate orthogonal if d_i is odd;
- the pairs (π_i, d_i) are distinct.

The integers d_i are to be interpreted as the dimensions of irreducible representations of $\mathrm{SL}_2(\mathbb{C})$. Moreover, a global Arthur parameter ψ induces a local Arthur parameter ψ_v for each place v of F via the local Langlands classification for GL_m. The reason we must take this rather circuitous definition of global Arthur parameters is the lack of a global Langlands group.

8.1.3. In [4], Arthur classified discrete automorphic representations of Sp_{2n} by constructing local and global packets for these parameters, denoted Π_{ψ_v} and Π_ψ, respectively. An automorphic representation $\pi \subset \mathcal{A}_{(2)}(\mathrm{Sp}_{2n}(\mathbb{A}_F))$ belongs to Π_ψ if and only if, for almost all places v, π_v is the spherical representation determined by φ_v. Although a full endoscopic classification for GSp_{2n} is not available, one can obtain partial results using the pullback along the natural map

$$\iota : \mathrm{Sp}_{2n} \to \mathrm{GSp}_{2n}.$$

Indeed, for our purposes the following rather weak result is sufficient.

Proposition 8.1.4. *Let $\pi_1, \pi_2 \subset \mathcal{A}_{(2)}(\mathrm{GSp}_{2n}(\mathbb{A}_F))$ be nearly equivalent discrete automorphic representations of $\mathrm{GSp}_{2n}(\mathbb{A}_F)$, and suppose that some irreducible constituent of $\iota^*\pi_1$ has parameter ψ. Then for all places v of F, any irreducible constituent of the admissible $\mathrm{Sp}_{2n}(F_v)$-module $\iota^*\pi_2$ belongs to the Arthur packet Π_{ψ_v}.*

Proof. Let $\pi_{0,v}$ be an irreducible constituent of $\iota^*\pi_{2,v}$. There exists a vector $f \in \pi_2$ generating an irreducible $\mathrm{Sp}_{2n}(\mathbb{A}_F))$-module $\pi_0 \subset \mathcal{A}_{(2)}(\mathbb{A}_F)$ whose component at v is $\pi_{0,v}$. By construction, π_0 has Arthur parameter ψ, and so the result follows from [4]. □

8.2. Theta lift from $\mathrm{GSO}(V_B)$ to GSp_6.

8.2.1. For the remainder of this section, fix a *non-split* quaternion algebra B over F, and let π be a tempered automorphic representation of $PB^\times$. We consider the representation $\pi \boxtimes \mathbb{1}$ of $\mathrm{GSO}(V_B) \simeq B^\times \times B^\times/\mathbb{G}_m$ and its theta lift $\Theta(\pi\boxtimes\mathbb{1})$ to $\mathrm{GSp}_6(\mathbb{A}_F)$; this is well-defined because V_B is anisotropic, and descends to $\mathrm{PGSp}_6(\mathbb{A}_F)$ because the similitude theta lift preserves central characters. Although this is not strictly necessary for our argument, it follows from [9] that $\Theta(\pi \boxtimes \mathbb{1}) \neq 0$ for any such π. However, $\Theta(\pi \boxtimes \mathbb{1})$ need not be irreducible.

Proposition 8.2.2. *The theta lift $\Theta(\pi \boxtimes \mathbb{1})$ lies in the L^2 subspace $\mathcal{A}_{(2)}(\mathrm{PGSp}_6(\mathbb{A}_F))$.*

By the usual criterion for square-integrability [22, Lemma I.4.11], we must check that, for each standard parabolic subgroup $P = MN$ of GSp_6, the characters of $Z(M)$ appearing in the cuspidal component of the normalized Jacquet module

$$\Theta(\pi \boxtimes \mathbb{1})_N \otimes \delta_P^{-1/2}$$

all lie in the interior of the cone spanned by the *negatives* of the characters appearing in the action of $Z(M)$ on N. Since $\pi\boxtimes\mathbb{1}$ is not a theta lift from $\mathrm{GSp}_2 = \mathrm{GL}_2$, [26, Theorem I.1.1] implies that the Jacquet modules are given by:

$$\Theta(\pi \boxtimes \mathbb{1})_N = \begin{cases} |\cdot|^2\Theta'(\pi \boxtimes \mathbb{1}), & M = \mathrm{GSp}_4 \times \mathrm{GL}_1 \\ 0, & \text{otherwise.} \end{cases} \tag{97}$$

Here $\Theta'(\pi\boxtimes\mathbb{1})$ denotes the theta lift to GSp_4, and $|\cdot|$ is the norm character of GL_1 . On the other hand, the action of $Z(M)$ on N is through positive powers of $|\cdot|$, and $\delta_P = |\cdot|^6$; thus the criterion for square-integrability is satisfied.

Proposition 8.2.3. *Suppose Π is an irreducible constituent of $\Theta(\pi \boxtimes \mathbb{1})$.*

(1) For all non-archimedean v, if π_v is unramified with local Langlands parameter ϕ_v, then Π_v is unramified with an Arthur parameter ψ_v such that the composite

$$W_{F_v} \xrightarrow{\psi_v} \widehat{\mathrm{PGSp}_6} = \mathrm{Spin}_7 \xrightarrow{r_{spin}} \mathrm{GL}_8$$

is given by

$$\phi_v \otimes S_2 \oplus S_3 \oplus S_1,$$

where S_i is the i-dimensional irreducible representation of SL_2.

(2) Any irreducible constituent of $\iota^*\Pi$ has global Arthur parameter

$$\mathrm{JL}(\pi)[2] \oplus \mathbb{1}[3].$$

Proof. For (1), Propositions 8.2.2 and 4.2.3 imply that Π_v is an irreducible constituent of $\Theta_v(\pi_v\boxtimes\mathbb{1})$ for all v. Since π_v is tempered at all unramified v, $\mathrm{Ind}_{\mathrm{GSO}(V)(F_v)}^{\mathrm{GO}(V)(F_v)} \pi_v \boxtimes \mathbb{1}$ is irreducible. Adopting the notation of (4.3.2) with $G = \mathrm{GSO}(V)$ and $H = \mathrm{GSp}_6$, $\pi_v \boxtimes \mathbb{1}$ is the spherical representation π_χ for the unramified character of $T_G(F_v)$ defined by $(-\log_q \alpha_v - 1/2, 1/2 - \log_q \alpha_v, \log_q \alpha_v)$, where $\{\alpha_v, \alpha_v^{-1}\}$ are the Sataka parameters of π_v. Then Proposition 4.3.3 implies that $\Theta_v(\pi_v \boxtimes \mathbb{1})$ is the irreducible representation σ_μ with $\mu(\alpha_v) = (-\log_q \alpha_v - 1/2, 1/2 - \log_q \alpha_v, -1, 1/2 + \log_q \alpha_v)$. Recall that any μ determines an unramified Langlands parameter for H: the characters $x_i, \lambda \in \mathrm{Hom}_{F_v}(T_G, \mathbb{G}_m)$ correspond to cocharacters in $\mathrm{Hom}_{\mathbb{C}}(\mathbb{G}_m, \widehat{T}_G)$, and any unramified character $\mu = (\beta_1, \cdots, \beta_3, t)$ may be viewed as the element

$$\lambda(q^{-t}) \prod_i x_i(q^{-\beta_i}) \in \widehat{T}_H(\mathbb{C}).$$

Then the Langlands parameter of σ_μ is the conjugacy class of the unramified map

$$\phi_\mu : W_{F_v} \to \widehat{T}_H(\mathbb{C}) \hookrightarrow \widehat{H}(\mathbb{C})$$

such that $\phi(\mathrm{Frob}_v)$ is the element corresponding to μ. Now, the eigenvalues of $r_{\mathrm{spin}} \circ \phi_\mu$ on $\mathbb{C}^8$ are given by $q^{-t}\prod_{i\in S} q^{-\beta_i}$ as S ranges over subsets of $\{1, 2, 3\}$. Hence in our case, the eight Frobenius eigenvalues are (with multiplicity) $q^{\pm 1/2}\alpha_v^{\pm 1}$, q, q^{-1}, 1, and 1. These Frobenius eigenvalues correspond to the unramified Arthur parameter $\phi_v \otimes S_2 \oplus S_3 \oplus S_1$ according to (95).

The second claim follows from the endoscopic classification for Sp_6, since the local Arthur parameter for any irreducible constituent of $\iota^*\Pi_v$ is the composition of ψ_v with the projection $\mathrm{Spin}_7 \to \mathrm{SO}_7$ (cf. [25]). $\square$

8.3. **Archimedean Arthur packet and cohomology.**

8.3.1. In order to construct Hodge classes from $\Theta(\pi \boxtimes \mathbb{1})$, it is essential to understand the structure of the cohomological local Arthur packet Π_{ψ_v}, where ψ is a global parameter for Sp_6 of the kind in Proposition 8.2.3(2) and v is archimedean. We now recall the construction of Π_{ψ_v} due to Adams and Johnson [2]. (This construction agrees with [4] by [1, 3].)

8.3.2. We first fix an anisotropic maximal torus T of $G = \mathrm{Sp}_6$. However, we depart from our usual coordinates for G, and choose a *complex* basis of the $2n$-dimensional symplectic space with respect to which

$$\mathfrak{g}_{\mathbb{C}} = \left\{ \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \middle| A, B, C \in \mathfrak{gl}_{3,\mathbb{C}},\ B = B^T,\ C = C^T \right\}, \tag{98}$$

$$T_{\mathbb{C}} = \left\{ \begin{pmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & & \\ & & \lambda_3 & & & \\ & & & \lambda_1^{-1} & & \\ & & & & \lambda_2^{-1} & \\ & & & & & \lambda_3^{-1} \end{pmatrix} \right\} \simeq (\mathbb{C}^\times)^3 \subset G_{\mathbb{C}}, \tag{99}$$

and complex conjugation acts by

$$\begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \mapsto \begin{pmatrix} -\overline{A}^T & \overline{C} \\ \overline{B} & \overline{A}. \end{pmatrix} \tag{100}$$

We choose the Borel subgroup B of $G_{\mathbb{C}}$ given by

$$B = \left\{ \begin{pmatrix} A & B \\ 0 & A^{-T} \end{pmatrix} \in G_{\mathbb{C}} \middle| A \text{ is upper triangular} \right\}. \tag{101}$$

Let θ be the Cartan involution of G that acts on $\mathfrak{g}_{\mathbb{C}}$ by

$$\begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \mapsto \begin{pmatrix} A & -B \\ -C & -A^T \end{pmatrix}. \tag{102}$$

The corresponding maximal compact subgroup of G is $K = U(3)$. The absolute Weyl group of G is

$$W(G,T) = \{\pm 1\}^3 \rtimes S_3, \tag{103}$$

where S_3 acts on T by permuting $(\lambda_1, \lambda_2, \lambda_3)$ and $\{\pm 1\}^3$ acts by

$$(e_1, e_2, e_3) \cdot (\lambda_1, \lambda_2, \lambda_3) = (\lambda_1^{e_1}, \lambda_2^{e_2}, \lambda_3^{e_3}).$$

The relative Weyl group $W_{\mathbb{R}}(G,T)$ of G may be identified with S_3.

8.3.3. Let $L \subset G$ be the unique $\mathbb{R}$-subgroup isomorphic to $U(2) \times SU(1,1)$ such that

$$L_{\mathbb{C}} = \left\{ \left(\begin{array}{cc|cc} A & & & \\ & a & & b \\ \hline & & A^{-T} & \\ & c & & d \end{array} \right) \middle| A \in \mathrm{GL}_2,\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2 \right\} \subset G_{\mathbb{C}}. \tag{104}$$

Then $L_{\mathbb{C}}$ is the Levi factor of a θ-stable standard parabolic $Q \subset G_{\mathbb{C}}$, which does not descend to $\mathbb{R}$. The absolute Weyl group $W(L,T)$ is given by

$$W(L,T) = \{(1,1,\pm 1)\} \times \{(1),(12)\} \subset W(G,T). \tag{105}$$

Note that the local Arthur parameter factors as

$$\psi_v : W_{F_v} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L L \to {}^L G. \tag{106}$$

If $\mathrm{JL}(\pi_v)$ is the discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ of weight $2k \geq 6$, then, in the notation of [2],

$$\Pi_{\psi_v} = \left\{ A_{\mathfrak{q}_w}(w^{-1} \cdot (k-3, k-3, 0)) \,:\, w \in S \right\}, \tag{107}$$

where $(k-3, k-3, 0)$ is viewed as a character of T,

$$S = W(L,T) \backslash W(G,T) / W_{\mathbb{R}}(G,T), \tag{108}$$

and, for $w \in S$, Q_w is the θ-stable parabolic subgroup $w^{-1}Qw$ of $G_{\mathbb{C}}$.

8.3.4. Choose representatives

$$w_0 = 1, \ \ w_1 = (-1,1,1), \ \ w_2 = (-1,-1,1) \tag{109}$$

of S in $W(G,T)$, and label the elements of Π_{ψ_v} as

$$\Pi_{\psi_v,i} = A_{\mathfrak{q}_{w_i}}(w_i^{-1} \cdot (k-3,k-3,0)), \ \ 0 \leq i \leq 2. \tag{110}$$

Proposition 8.3.5. *For any complex, irreducible algebraic representation V of G,*

$$H^*(\mathfrak{g}, K; \Pi_{\psi_v,i} \otimes V) = 0$$

unless V is the highest weight representation $V_{(k-3,k-3,0)}$. If $V = V_{(k-3,k-3,0)}$, then the dimensions of the nonzero cohomology groups of the representations $\Pi_{\psi_v,i}$ are computed as follows.

$$\dim_{\mathbb{C}} H^{5,0}(\mathfrak{g}, K; \Pi_{\psi_v,0} \otimes V) = \dim_{\mathbb{C}} H^{6,1}(\mathfrak{g}, K; \Pi_{\psi_v,0} \otimes V) = 1$$

$$\dim_{\mathbb{C}} H^{2,2}(\mathfrak{g}, K; \Pi_{\psi_v,1} \otimes V) = \dim_{\mathbb{C}} H^{4,4}(\mathfrak{g}, K; \Pi_{\psi_v,1} \otimes V) = 1$$
$$\dim_{\mathbb{C}} H^{3,3}(\mathfrak{g}, K; \Pi_{\psi_v,1} \otimes V) = 2$$

$$\dim_{\mathbb{C}} H^{0,5}(\mathfrak{g}, K; \Pi_{\psi_v,2} \otimes V) = \dim_{\mathbb{C}} H^{1,6}(\mathfrak{g}, K; \Pi_{\psi_v,2} \otimes V) = 1$$

Proof. Write $Q_{w_i} = L_i U_i$. The proposition follows from [32, Proposition 6.19], together with the calculations:

$$\mathfrak{u}_0 = \left(\begin{array}{ccc|ccc} I & & * & * & * & * \\ & & * & * & * & * \\ 0 & 0 & 1 & * & * & 0 \\ \hline & & & I & & 0 \\ & & & & & 0 \\ & & & * & * & 1 \end{array}\right), \quad W_{\mathbb{R}}(L_0, T) = \{1, (12)\},$$

$$\mathfrak{u}_1 = \left(\begin{array}{ccc|ccc} 1 & & & & & \\ * & 1 & * & & * & * \\ * & & 1 & & * & \\ \hline * & & * & & * & \\ & & & & 1 & \\ * & & & & * & 1 \end{array}\right), \quad W_{\mathbb{R}}(L_1, T) = \{1\},$$

$$\mathfrak{u}_2 = \left(\begin{array}{ccc|ccc} I & & 0 & & & \\ & & 0 & & & \\ * & * & 1 & & & \\ \hline * & * & * & I & & * \\ * & * & * & & & * \\ * & * & 0 & 0 & 0 & 1 \end{array}\right), \quad W_{\mathbb{R}}(L_2, T) = \{1, (12)\}.$$

□

8.3.6. If $V = V_{(k-3,k-3,0)}$, then V has a K-stable Hodge decomposition of weight $m = 2k-6$ defined by

$$V^{p,q} = \left\{ v \in V : z \cdot v = z^{n-2p} v, \ \forall z \in Z_K \simeq U(1) \right\}, \ p + q = m. \tag{111}$$

The decomposition (111) induces a refined decomposition of the Lie algebra cohomology, cf. [36]:

$$H^{p,q}(\mathfrak{g}, K; \Pi_{\psi_v,i} \otimes V) = \bigoplus_{\substack{r+s=m \\ r,s \geq 0}} H^{(p,q);(r,s)}(\mathfrak{g}, K; \Pi_{\psi_v,i} \otimes V).$$

For later use, we now calculate this decomposition.

Proposition 8.3.7. *Let $V = V_{(k-3,k-3,0)}$. The dimensions of the nonzero refined components of the cohomology groups of the representations $\Pi_{\psi_v,i}$ are computed as follows.*

$$\dim_{\mathbb{C}} H^{(5,0);(2k-6,0)}(\mathfrak{g}, K; \Pi_{\psi_v,0} \otimes V) = \dim_{\mathbb{C}} H^{(6,1);(2k-6,0)}(\mathfrak{g}, K; \Pi_{\psi_v,0} \otimes V = 1$$

$$\begin{aligned} &\dim_{\mathbb{C}} H^{(2,2);(k-3,k-3)}(\mathfrak{g}, K; \Pi_{\psi_v,1} \otimes V) = \dim_{\mathbb{C}} H^{(4,4);(k-3,k-3)}(\mathfrak{g}, K; \Pi_{\psi_v,1} \otimes V) = 1 \\ &\dim_{\mathbb{C}} H^{(3,3);(k-3,k-3)}(\mathfrak{g}, K; \Pi_{\psi_v,1} \otimes V) = 2 \end{aligned}$$

$$\dim_{\mathbb{C}} H^{(0,5);(0,2k-6)}(\mathfrak{g}, K; \Pi_{\psi_v,2} \otimes V) = \dim_{\mathbb{C}} H^{(1,6);(0,2k-6)}(\mathfrak{g}, K; \Pi_{\psi_v,2} \otimes V) = 1$$

Proof. The proof is similar to [14, Proposition 11.4]. Consider the set of coset representatives for $W_{\mathbb{R}}(G,T)\backslash W(G,T)$ given by

$$W_0 := \left\{ w \in W(G,T) : w^{-1}(\Delta_c^+) \subset \Delta^+ \right\}, \tag{112}$$

where Δ^+ is the set of positive roots with respect to the Borel subgroup (101) and $\Delta_c^+ \subset \Delta^+$ is the subset of compact roots (i.e. the positive roots of K). In the notation of [36], let μ and λ be the highest characters of Z_K appearing in the action of K on $\mathfrak{g}_{\mathbb{C}}$ and V, respectively; identifying characters of Z_K with $\mathbb{Z}$, $\mu = 2$ and $\lambda = 2k - 6 = m$ (the weight of V). Also let $\Lambda = (k-3, k-3, 0)$, considered as a character of T. By [36, §5], if

$$H^{(p,q);(n-p,m+p-n)}(\mathfrak{g}, K; \Pi_{\psi_v,i} \otimes V) \neq 0$$

for any i, then there exists $w \in W_0$ with $\ell(w) = p$, such that the K-representation of highest weight $w(\Lambda + \rho) - \rho$ has central character $\lambda - n\mu = 2k - 2n - 6$. Here $\rho = (3,2,1)$ is the half sum of positive roots for $\mathfrak{g}_{\mathbb{C}}$. Thus each $w \in W_0$ defines exactly one choice of p and n such that a nonzero contribution of bidegree $(p,q);(n-p, m+p-n)$ is possible. These choices are summarized in the following table.

$w \cdot (a,b,c)$	$p = \ell(w)$	$w(\Lambda+\rho)-\rho$	n	possible types
(a,b,c)	0	$(k-3,k-3,0)$	0	$(0,q);(0,2k-6)$
$(a,b,-c)$	1	$(k-3,k-3,-2)$	1	$(1,q);(0,2k-6)$
$(a,c,-b)$	2	$(k-3,-1,-k)$	$k-1$	$(2,q);(k-3,k-3)$
$(b,c,-a)$	3	$(k-4,-1,-k-1)$	k	$(3,q);(k-3,k-3)$
$(a,-c,-b)$	3	$(k-3,-3,-k)$	k	$(3,q);(k-3,k-3)$
$(b,-c,-a)$	4	$(k-4,-3,-k-1)$	$k+1$	$(4,q);(k-3,k-3)$
$(c,-b,-a)$	5	$(-2,-k-1,-k-1)$	$2k-1$	$(5,q);(2k-6,0)$
$(-c,-b,-a)$	6	$(-4,-k-1,-k-1)$	$2k$	$(6,q);(2k-6,0)$

Comparing with Proposition 8.3.5 completes the proof. □

8.4. **Contributions to the cohomology of Shimura varieties.**

8.4.1. Consider the Shimura variety for $\mathbf{GSp}_6$ as in §3.4. Following the notation of (3.1.1), we obtain a local system $\mathcal{V}_{(\boldsymbol{m}-2,\boldsymbol{m}-2,0)}$ of $\mathbb{Q}(\boldsymbol{m})$-vector spaces. Let σ_{m_v} be the unique irreducible representation of $K_{3,v}$ with trivial central character and whose restriction to $U(3)$ has highest weight $(m_v+1, 0, -m_v-1)$, and let $\boldsymbol{\sigma_m}$ be the representation $\otimes_{v|\infty}\sigma_{m_v}$ of $\boldsymbol{K}_3$. One calculates that

$$\dim \mathrm{Hom}_{\boldsymbol{K}_3}\left(\boldsymbol{\sigma_m}, V_{(\boldsymbol{m}-2,\boldsymbol{m}-2,0),\mathbb{C}} \otimes \wedge^{\mathbf{2,2}}\mathfrak{p}^*_{\mathrm{GSp}_6}\right) = 1, \tag{113}$$

where $(\mathbf{2},\mathbf{2})$ is the constant plectic Hodge type. Thus we have, from (31), a class map

$$\left(\mathcal{A}_{(2)}(\mathrm{GSp}_6(\mathbb{A}_F)) \otimes \boldsymbol{\sigma_m}\right)^{\boldsymbol{K}_3} \to H^{\mathbf{2,2}}_{(2)}(S(\mathbf{GSp}_6), \mathcal{V}_{(\boldsymbol{m}-2,\boldsymbol{m}-2,0),\mathbb{C}}). \tag{114}$$

8.4.2. We now choose a totally indefinite, non-split quaternion algebra B over F. Let π be an auxiliary automorphic representation of $PB^\times(\mathbb{A}_F)$ of weight $2\boldsymbol{m}+2 = (2m_v+2)_{v|\infty}$.

Lemma 8.4.3. *If $\widetilde{\Pi}$ is a discrete automorphic representation of $\mathrm{GSp}_6(\mathbb{A}_F)$ which is nearly equivalent to a constituent of the theta lift $\iota^*\Theta(\pi \boxtimes \mathbb{1})$, then we have:*

(1) View $\mathcal{V}_{(\boldsymbol{m}-2,\boldsymbol{m}-2,0),\mathbb{C}}$ *as a variation of structures of weight 0. Then the* L^2*-cohomology*

$$H^{4d}_{(2)}(S(\mathbf{GSp}_6), \mathcal{V}_{(\boldsymbol{m}-2,\boldsymbol{m}-2,0),\mathbb{C}})[\widetilde{\Pi}_f]$$

is purely of Hodge type $(2d, 2d)$ *for the (non-plectic) Hodge structure on* L^2 *cohomology defined in* [36].

(2) The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/F^c)$ *acts on the intersection cohomology*

$$IH^{4d}(S(\mathbf{GSp}_6)_{\overline{\mathbb{Q}}}, \mathcal{V}_{(\boldsymbol{m}-2,\boldsymbol{m}-2,0),\overline{\mathbb{Q}}_\ell})[\widetilde{\Pi}_f]$$

via χ^{-2d}*, where* χ *is the* ℓ*-adic cyclotomic character.*

Proof. For (1), it follows from Propositions 8.1.4, 8.2.3, and 8.3.5 that

$$H^{4d}_{(2)}(S(\mathbf{GSp}_6), \mathcal{V}_{(\boldsymbol{m}-2,\boldsymbol{m}-2,0),\mathbb{C}})[\widetilde{\Pi}_f] \simeq \widetilde{\Pi}_f \otimes \bigotimes_{v|\infty} H^{2,2}(\mathrm{GSp}_{6,\mathbb{R}}, K_3; \pi_v^{\mathrm{sm}} \otimes V_{(m_v-2,m_v-2,0),\mathbb{C}}).$$

Thus it follows from Proposition 8.3.7 and from [36] that the cohomology is pure of Hodge type (m, m) in Zucker's normalization, where $m = \sum m_v$. However, since $V_{\boldsymbol{m},\mathbb{C}}$ has trivial central character and hence total weight 0 in the algebraic normalization, we must twist by $(2d-m, 2d-m)$, which shows the claim. For (2), choose a compact open subgroup $K = \prod K_v \subset \mathrm{GSp}_6(\mathbb{A}_{F,f})$. It suffices to show that Frob_p acts trivially on

$$H \coloneqq IH^{4d}(S_K(\mathbf{GSp}_6)_{\overline{\mathbb{Q}}}, \mathcal{V}_{(\boldsymbol{m}-2,\boldsymbol{m}-2,0),\overline{\mathbb{Q}}_\ell})[\widetilde{\Pi}_f]$$

for almost all p such that p splits completely in F^c and K_p is hyperspecial. Assume without loss of generality that $\widetilde{\Pi}_p$ is unramified with local Langlands parameter

$$\phi_p : W_{\mathbb{Q}_p} \to {}^L\,\mathbf{GSp}_6 = \mathrm{GSpin}_7(\mathbb{C})^d \times W_{\mathbb{Q}_p},$$

and consider the $8d$-dimensional representation defined by the composite:

$$W_{\mathbb{Q}_p} \xrightarrow{\phi_p} \mathrm{GSpin}_7(\mathbb{C})^d \xrightarrow{r_{\mathrm{spin}}^{\otimes d}} \mathrm{GL}_{8d}(\mathbb{C}). \tag{115}$$

After picking an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$, (115) defines an $8d$-dimensional ℓ-adic unramified local Galois representation V_p. By [20, §2], since p splits in $F(\boldsymbol{m})$, the action of the geometric Frobenius Frob_p^{-1} on H satisfies the characteristic polynomial of $p^3 \,\mathrm{Frob}_p^{-1}$ on V_p. Now by Proposition 8.2.3, for almost every such p the representation V_p is given by

$$\bigotimes_{v|p} \left(\overline{\mathbb{Q}}_\ell(-1) \oplus \overline{\mathbb{Q}}_\ell^2 \oplus \overline{\mathbb{Q}}_\ell(1) \oplus \rho_\pi|_{F_v} \oplus \rho_\pi|_{F_v}(1) \right),$$

where ρ_π is the 2-dimensional ℓ-adic Galois representation associated to π, which we normalize to have weight one. (Recall that ρ_π is pure since π is discrete series at a finite place of F [24].) On the other hand, it is known that H is pure of weight $4d$; comparing with the weights in V_p, it follows that Frob_p acts as p^{-2d} on H. □

9. Triple product periods

9.1. **The vector-valued period problem.**

9.1.1. Let $\boldsymbol{m}$, π_1, π_2, $\boldsymbol{\epsilon}$, $\boldsymbol{\tau}_{\boldsymbol{m}}^{\boldsymbol{\epsilon}}$, and $\boldsymbol{\sigma}_{\boldsymbol{m}}$ be as in (5.3.1) and (8.4.1), and let B be a *non-split* totally indefinite quaternion algebra over F, ramified at a set S of places of F at which π_i are both discrete series.

9.1.2. For auxiliary automorphic representations π of $PB(\mathbb{A}_F)^\times$ of weight $2\boldsymbol{m}+2$, we will consider triple product period integrals of $\Theta(\pi \boxtimes \mathbb{1})$ along the subgroup

$$\widetilde{H} \coloneqq (\mathrm{GSp}_4 \times_{\mathbb{G}_m} \mathrm{GL}_2) \subset \mathrm{GSp}_6\,. \tag{116}$$

The maximal compact-modulo-center subgroup of $\widetilde{H}(F \otimes \mathbb{R})$ is

$$(\boldsymbol{K}_2 \times \boldsymbol{K}_1)_0 \coloneqq (\boldsymbol{K}_2 \times \boldsymbol{K}_1) \cap \widetilde{H}(F \otimes \mathbb{R}). \tag{117}$$

To define the vector-valued period integral, note that (by the classical branching law for unitary groups), the space

$$\mathrm{Hom}_{(\boldsymbol{K}_2 \times \boldsymbol{K}_1)_0}(\boldsymbol{\sigma}_{\boldsymbol{m}} \otimes \boldsymbol{\tau}_{\boldsymbol{m}}^{\boldsymbol{\epsilon}} \otimes \boldsymbol{\chi}_{-\boldsymbol{\epsilon}\boldsymbol{m}}^\vee, \mathbb{C}) \tag{118}$$

is one-dimensional, say with generator ℓ.

We then define, for any auxiliary representation π of $PB(\mathbb{A}_F)^\times$ of weight $2\boldsymbol{m}+2$, the triple product period:

$$\begin{aligned}&\widetilde{\mathcal{P}}^{\boldsymbol{\epsilon}}_{S,\pi_1,\pi_2,\pi}(\alpha,\beta,\gamma) = \int_{[\widetilde{H}]} \ell(\alpha(g,g')\otimes\beta(g)\otimes\gamma(g'))\,\mathrm{d}(g,g') \neq 0,\\ &\qquad \alpha\in(\Theta(\pi\boxtimes\mathbb{1})\otimes\boldsymbol{\sigma_m})^{\boldsymbol{K}_3},\ \ \beta\in(\Pi_S(\pi_1\otimes\pi_2)\otimes\boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{m}})^{\boldsymbol{K}_2},\ \ \gamma\in(\pi_2^\vee\otimes\boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon m}})^{\boldsymbol{K}_1}.\end{aligned} \tag{119}$$

Since we will not give a precise formula for $\widetilde{\mathcal{P}}^{\boldsymbol{\epsilon}}_{S,\pi_1,\pi_2,\pi}$ and are only interested in its non-vanishing, we ignore the problem of normalization. The non-vanishing of (119), for a good choice of π, is the key input to the non-vanishing of the Hodge classes we construct in the next section.

9.1.3. The strategy for calculating (119) is to use the seesaw diagram:

$$\begin{tikzcd} \mathrm{GSO}(V_B)\times_{\mathbb{G}_m}\mathrm{GSO}(V_B) \arrow[d, dash] \arrow[dr, dash] & \mathrm{GSp}_6 \arrow[d, dash] \arrow[dl, dash] \\ \mathrm{GSO}(V_B) & \widetilde{H} \end{tikzcd}$$

There are two main inputs to the non-vanishing of our period integral (for a good choice of π): the first is a vector-valued version of the usual global seesaw identity, and the second is a non-vanishing result for the vector-valued theta lifts along the "other" diagonal in the seesaw diagram, i.e. from GSp_4 and GL_2 to $\mathrm{GSO}(V_B)$.

9.2. **Vector-valued seesaw identity.**

9.2.1. Continuing with the notation from (7.1.1), let $m\geq 2$ be an integer and let σ_m be the unique representation of K_3 of trivial central character and whose restriction to $U(3)$ has highest weight $(m+1,0,-m-1)$. Let

$$\widetilde{\varphi}_m \in (S^0_{\leq 2m+2}(3)\otimes\sigma_m\otimes\chi_{-(2m+2),0})^{(K_3\times L)_0}$$

be a generator, which makes sense by Proposition 7.1.3. If $(K_2\times K_1)_0$ is the intersection of K_3 with $\mathrm{GSp}_{4,\mathbb{R}}\times_{\mathbb{G}_m}\mathrm{GL}_{2,\mathbb{R}}$ inside $\mathrm{GSp}_{6,\mathbb{R}}$, then we have, for any $\epsilon=\pm1$,

$$\dim\mathrm{Hom}_{(K_2\times K_1)_0}(\sigma_m,\tau^\epsilon_m\otimes\chi^\vee_{-\epsilon m}) = 1; \tag{120}$$

let ℓ_ϵ denote a generator. Also let $(K_1\times K_2\times L)_0 = (K_1\times K_2)_0\times L\cap(K_3\times L)_0$.

Proposition 9.2.2. *The Schwartz function*

$$\ell_\epsilon(\widetilde{\varphi}_m)\in\big(\mathcal{S}_\mathbb{R}(\langle e_2,e_4,e_6\rangle\otimes V)\otimes\tau^\epsilon_m\otimes\chi^\vee_{-\epsilon m}\otimes\chi_{2m+2,0}\big)^{(K_1\times K_2\times L)_0}$$

is a nonzero scalar multiple of the tensor product

$$\varphi^\epsilon_m\otimes\phi^\epsilon_m\in\Big(\big(\mathcal{S}_\mathbb{R}(\langle e_2,e_4\rangle\otimes V)\otimes\tau^\epsilon_m\otimes\chi^\vee_{m+2,-\epsilon m}\big)^{(K_2\times L)_0}\otimes\big(\mathcal{S}_\mathbb{R}(\langle e_6\rangle\otimes V)\otimes\chi^\vee_{-\epsilon m}\otimes\chi_{-m,\epsilon m}\big)^{(K_1\times L)_0}\Big)\Big|_{(K_1\times K_2\times L)_0}.$$

Proof. Assume $\epsilon=+$; the other case is similar. Let $\phi\in S^0_{\leq 2m+2}(3)$ be the contraction of $\varphi^\epsilon_m\otimes\phi^\epsilon_m$ with any nonzero vector. Then ϕ generates the irreducible $U(2)\times U(1)$-representation of highest weight $(1,-m-1,m)$, and it suffices to show that $U(3)\cdot\phi$ is the irreducible representation of highest weight $(m+1,0,-m-1)$. Without loss of generality, assume $U(3)\cdot\phi$ is irreducible, say with highest weight (a,b,c). It follows (using the branching law for unitary groups) that

$$\begin{aligned} a\geq 1\geq b\geq -m-1\geq c,\\ a+b+c=0.\end{aligned} \tag{121}$$

On the other hand, considering Proposition 7.1.3(2), we have

$$|a|+|b|+|c|\leq 2m+2. \tag{122}$$

The combination of (121) and (122) force $(a,b,c)=(m+1,0,-m-1)$. $\square$

9.2.3. We now return to the global situation. Choose an isomorphism $V_B \otimes_F \mathbb{R} \simeq V \otimes_F \mathbb{R}$, which induces an isomorphism $\mathrm{GSO}(V_B)(F\otimes\mathbb{R}) \simeq \mathrm{GSO}(V)(F\otimes\mathbb{R})$. Then let $\boldsymbol{L} = \prod_{v|\infty} L \subset \mathrm{GSO}(V_B)(F\otimes\mathbb{R})$, and similarly for $(\boldsymbol{K}_n \times \boldsymbol{L})_0$, etc. We fix vector-valued Schwartz functions as follows:

$$
\begin{aligned}
\boldsymbol{\varphi}_{\boldsymbol{m}}^{-\boldsymbol{\epsilon}} = \otimes_{v|\infty}\varphi_{m_v}^{-\epsilon_v} &\in \left(\mathcal{S}_{F\otimes\mathbb{R}}(\langle e_2, e_4\rangle \otimes V) \otimes \boldsymbol{\tau}_{\boldsymbol{m}}^{-\boldsymbol{\epsilon}} \otimes \boldsymbol{\chi}^\vee_{(\boldsymbol{m}+2),\boldsymbol{\epsilon m}}\right)^{(\boldsymbol{K}_2\times\boldsymbol{L})_0} \qquad \text{(Proposition 7.1.7)}\\
&\simeq \left(\mathcal{S}_{F\otimes\mathbb{R}}(\langle e_2, e_4\rangle \otimes V_B) \otimes (\tau_{\boldsymbol{m}}^{\boldsymbol{\epsilon}})^\vee \otimes \boldsymbol{\chi}_{-(\boldsymbol{m}+2),-\boldsymbol{\epsilon m}}\right)^{(\boldsymbol{K}_2\times\boldsymbol{L})_0}\\
\boldsymbol{\phi}_{\boldsymbol{m}}^{-\boldsymbol{\epsilon}} = \otimes_{v|\infty}\phi_{m_v}^{-\epsilon_v} &\in \left(\mathcal{S}_{F\otimes\mathbb{R}}(\langle e_6\rangle \otimes V) \otimes \boldsymbol{\chi}_{-\boldsymbol{\epsilon m}} \otimes \boldsymbol{\chi}^\vee_{\boldsymbol{m},-\boldsymbol{\epsilon m}}\right)^{(\boldsymbol{K}_1\times\boldsymbol{L})_0} \qquad \text{(Proposition 7.1.4)}\\
&\simeq \left(\mathcal{S}_{F\otimes\mathbb{R}}(\langle e_6\rangle \otimes V_B) \otimes \boldsymbol{\chi}_{-\boldsymbol{\epsilon m}} \otimes \boldsymbol{\chi}^\vee_{\boldsymbol{m},-\boldsymbol{\epsilon m}}\right)^{(\boldsymbol{K}_1\times\boldsymbol{L})_0}\\
\widetilde{\boldsymbol{\varphi}}_{\boldsymbol{m}} = \otimes_{v|\infty}\widetilde{\varphi}_{m_v} &\in \left(\mathcal{S}_{\mathbb{R}}(\langle e_2, e_4, e_6\rangle V) \otimes \boldsymbol{\sigma}_{\boldsymbol{m}} \otimes \boldsymbol{\chi}_{-(2\boldsymbol{m}+2),0}\right)^{(\boldsymbol{K}_2\times\boldsymbol{L})_0}\\
&\simeq \left(\mathcal{S}_{\mathbb{R}}(\langle e_2, e_4, e_6\rangle \otimes V_B) \otimes \boldsymbol{\sigma}_{\boldsymbol{m}} \otimes \boldsymbol{\chi}_{-(2\boldsymbol{m}+2),0}\right)^{(\boldsymbol{K}_2\times\boldsymbol{L})_0}
\end{aligned} \tag{123}
$$

Proposition 9.2.4. *Let ℓ be as above. For all*

$$
\alpha \in \left(\mathcal{A}_0(\mathrm{PGSO}(V_B)(\mathbb{A}_F)) \otimes \boldsymbol{\chi}_{2\boldsymbol{m}+2,0}\right)^{\boldsymbol{L}}, \quad \beta \in \left(\Pi_S(\pi_1,\pi_2)\otimes \boldsymbol{\tau}_{\boldsymbol{m}}^{\boldsymbol{\epsilon}}\right)^{\boldsymbol{K}_2}, \quad \gamma \in \left(\pi_2^\vee \otimes \boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon m}}\right)^{\boldsymbol{K}_1},
$$
$$
\phi_{1,f} \in \mathcal{S}_{\mathbb{A}_{F,f}}(\langle e_2, e_4\rangle \otimes V_B), \quad \phi_{2,f} \in \mathcal{S}_{\mathbb{A}_{F,f}}(\langle e_6\rangle \otimes V_B),
$$

and up to a nonzero scalar depending on the normalizations, we have the identity:

$$
\begin{aligned}
&\int_{[Z_{\widetilde{H}}\backslash \widetilde{H}]} \ell\left(\theta_{\phi_{1,f}\otimes\phi_{2,f}\otimes\widetilde{\boldsymbol{\varphi}}_{\boldsymbol{m}}}(\alpha)(g,g')\otimes\beta(g)\otimes\gamma(g')\right)\mathrm{d}(g,g') =\\
&\qquad \int_{[\mathrm{PGSO}(V_B)]} \alpha(g)\theta_{\phi_{1,f}\otimes\boldsymbol{\varphi}_{\boldsymbol{m}}^{-\boldsymbol{\epsilon}}}(\beta)(g)\theta_{\phi_{2,f}\otimes\boldsymbol{\phi}_{\boldsymbol{m}}^{-\boldsymbol{\epsilon}}}(\gamma)(g)\,\mathrm{d}g.
\end{aligned} \tag{124}
$$

Proof. This is formal from Proposition 9.2.2 and the usual seesaw identity, i.e. exchanging the order of integration. □

9.3. Proof of the non-vanishing result.

Proposition 9.3.1. *Let π_i^B be the Jacquet-Langlands transfers of π_i to $B(\mathbb{A}_F)^\times$.*

(1) The map

$$
\theta_{\boldsymbol{\varphi}_{\boldsymbol{m}}^{-\boldsymbol{\epsilon}}} : \mathcal{S}_{\mathbb{A}_{F,f}}(\langle e_2, e_4\rangle\otimes V_B)\otimes\left(\Pi_S(\pi_1,\pi_2)\otimes\boldsymbol{\tau}^{\boldsymbol{\epsilon}}\right)^{\boldsymbol{K}_2} \to \left(\mathcal{A}(\mathrm{GSO}(V_B))(\mathbb{A}_F)\otimes\boldsymbol{\chi}_{-(\boldsymbol{m}+2),-\boldsymbol{\epsilon m}}\right)^{\boldsymbol{L}},
$$

defined by

$$
(\phi,\alpha)\mapsto \theta_{\phi\otimes\boldsymbol{\varphi}_{\boldsymbol{m}}^{-\boldsymbol{\epsilon}}}(\alpha),
$$

has image containing $\left((\pi_1^B\boxtimes\pi_2^B)\otimes\boldsymbol{\chi}_{-(\boldsymbol{m}+2),-\boldsymbol{\epsilon m}}\right)^{\boldsymbol{L}}$.

(2) The map

$$
\theta_{\boldsymbol{\phi}_{\boldsymbol{m}}^{-\boldsymbol{\epsilon}}} : \mathcal{S}_{\mathbb{A}_{F,f}}(\langle e_6\rangle\otimes V_B)\otimes\left(\pi_2^\vee\otimes\boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon m}}\right)^{\boldsymbol{K}_1} \to \left(\mathcal{A}(\mathrm{GSO}(V_B))(\mathbb{A}_F)\otimes\boldsymbol{\chi}^\vee_{\boldsymbol{m},-\boldsymbol{\epsilon m}}\right)^{\boldsymbol{L}},
$$

defined by

$$
(\phi,\alpha)\mapsto \theta_{\phi\otimes\boldsymbol{\phi}_{\boldsymbol{m}}^{-\boldsymbol{\epsilon}}}(\alpha),
$$

has image containing $\left(((\pi_2^B)^\vee\boxtimes(\pi_2^B)^\vee)\otimes\boldsymbol{\chi}^\vee_{\boldsymbol{m},-\boldsymbol{\epsilon m}}\right)^{\boldsymbol{L}}$.

Proof. In the general setup of §4, suppose $\Theta_{V,W}(\pi) = \Pi$ for cuspidal automorphic representations π of $G(V)(\mathbb{A}_F)$ and Π of $H(W)(\mathbb{A}_F)$. Then by definition we have a surjective composite

$$
\mathcal{S}_{\mathbb{A}_F}(W_2\otimes V) \xrightarrow{\theta} \mathcal{A}(R_0(\mathbb{A}_F)) \twoheadrightarrow \Pi\otimes\pi^\vee. \tag{125}
$$

Now, the theta kernel satisfies

$$
\theta(\phi)(g,h) = \overline{\theta(\overline{\phi})}\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} g \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, h\right)
$$

(cf. [27]), so we deduce that $\overline{\Pi\otimes\pi^\vee} = \Pi^\vee\otimes\pi$ also appears in the spectrum of the theta kernel. (Recall that the central characters of Π and π must agree since the central character of the Weil representation is trivial.)

In particular $\Theta_{W,V}(\Pi)$ contains the nonzero irreducible constituent π. For (1), take $W = W_4$, $V = V_B$, and $\Pi = \Pi_S(\pi_1, \pi_2)$. As in the proof of Proposition 4.2.3, the global theta lift gives rise to a nontrivial map:

$$\mathcal{S}_{\mathbb{A}_F}(\langle e_2, e_4\rangle \otimes V_B) \twoheadrightarrow \Pi_S(\pi_1,\pi_2)^\vee \otimes \Theta_{W_4,V_B}(\Pi_S(\pi_1,\pi_2)). \tag{126}$$

(Since $\mathrm{GSO}(V_B)$ is anisotropic, all theta lifts are square-integrable.) The map (126) is a restricted tensor product of local maps. To prove the proposition, it suffices to show that, for all $v|\infty$ and for some vector $0 \neq \ell \in \tau_{m_v}^{\epsilon_v}$, the contraction $\ell(\varphi_{m_v}^{-\epsilon_v})$ has nontrivial image under the local component

$$\mathcal{S}_{F_v}(\langle e_2, e_4\rangle \otimes V_B) \twoheadrightarrow \Pi^+(\pi_{1,v},\pi_{2,v})^\vee \otimes \left(\pi_{1,v}^B \boxtimes \pi_{2,v}^B\right) \tag{127}$$

of (126). Now, by local Howe duality for the disconnected similitude group $\mathrm{GO}(V_B)$, the local theta lift $\Theta_{W_4,V_B}(\Pi^+(\pi_{1,v},\pi_{2,v})$ is irreducible when viewed as a representation of $\mathrm{GO}(V_B)(F_v)$ [13, 27]. Hence

$$\Theta_{W_4,V_B}(\Pi^+(\pi_{1,v},\pi_{2,v}) = \pi_{1,v}^B \boxtimes \pi_{2,v}^B \oplus \pi_{2,v}^B \boxtimes \pi_{1,v}^B,$$

and the map (127) factors as

$$\mathcal{S}_{F_v}(\langle e_2, e_4\rangle \otimes V_B) \twoheadrightarrow \Pi^+(\pi_{1,v},\pi_{2,v})^\vee \otimes \left(\pi_{1,v}^B \boxtimes \pi_{2,v}^B \oplus \pi_{2,v}^B \boxtimes \pi_{1,v}^B\right) \twoheadrightarrow \Pi^+(\pi_{1,v},\pi_{2,v})^\vee \otimes \left(\pi_{1,v}^B \boxtimes \pi_{2,v}^B\right).$$

We now note that $\ell(\varphi_{m_v}^{\epsilon_v})$ is a harmonic in the sense of [13] by Proposition 7.1.3, and generates a $U(2)$-type that appears in $\Pi^+(\pi_{1,v},\pi_{2,v})^\vee$. It follows from [13] that its image under

$$\mathcal{S}_{F_v}(\langle e_2, e_4\rangle \otimes V_B) \twoheadrightarrow \Pi^+(\pi_{1,v},\pi_{2,v})^\vee \otimes \left(\pi_{1,v}^B \boxtimes \pi_{2,v}^B \oplus \pi_{2,v}^B \boxtimes \pi_{1,v}^B\right)$$

is nontrivial. However, $\ell(\varphi_{m_v}^{\epsilon_v})$ generates the L-type $\chi^\vee_{-(m_v+2),\epsilon_v m_v}$, which does not appear in $\pi_{2,v}^B \boxtimes \pi_{1,v}^B$. It follows that $\ell(\varphi_{m_v}^{\epsilon_v})$ has nonzero image under (127). This proves (1). The proof of (2) is analogous. □

Finally we come to the main result of this section:

Lemma 9.3.2. *There exists an automorphic representation π of $PB(\mathbb{A}_F)^\times$ of weight $2\boldsymbol{m}+2$, along with vectors*

$$\alpha \in (\Theta(\pi \boxtimes \mathbb{1}) \otimes \boldsymbol{\sigma_m})^{\boldsymbol{K}_3}, \quad \beta \in (\Pi_S(\pi_1 \otimes \pi_2) \otimes \boldsymbol{\tau}_{\boldsymbol{m}}^{\boldsymbol{\epsilon}})^{\boldsymbol{K}_2}, \quad \gamma \in (\pi_2^\vee \otimes \boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon m}})^{\boldsymbol{K}_1},$$

such that:

$$\widetilde{\mathcal{P}}^{\boldsymbol{\epsilon}}_{S,\pi_1,\pi_2,\pi}(\alpha,\beta,\gamma) = \int_{[\widetilde{H}]} \ell(\alpha(g,g') \otimes \beta(g) \otimes \gamma(g'))\,\mathrm{d}(g,g') \neq 0.$$

Proof. First, fix newforms

$$f_1 \in \pi_1^B, \quad f_2^{\boldsymbol{\epsilon}} \in \pi_2^B, \quad f_2^\vee \in (\pi_2^B)^\vee, \quad (f_2^{\boldsymbol{\epsilon}})^\vee \in (\pi_2^B)^\vee$$

of weights $\boldsymbol{m}+2$, $\boldsymbol{\epsilon m}$, $\boldsymbol{m}$, and $-\boldsymbol{\epsilon m}$, respectively. Then Proposition 9.3.1 implies that we may choose vectors

$$\beta \in (\Pi_S(\pi_1,\pi_2) \otimes \boldsymbol{\tau}_{\boldsymbol{m}}^{\boldsymbol{\epsilon}})^{\boldsymbol{K}_2}, \quad \gamma \in (\pi_2^\vee \otimes \boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon m}})^{\boldsymbol{K}_1}$$

and Schwartz functions

$$\phi_{1,f} \in \mathcal{S}_{\mathbb{A}_{F,f}}(\langle e_2, e_4\rangle \otimes V_B), \quad \phi_{2,f} \in \mathcal{S}_{\mathbb{A}_{F,f}}(\langle e_6\rangle \otimes V_B)$$

such that:

$$\theta_{\phi_{1,f}\otimes\boldsymbol{\varphi}_{\boldsymbol{m}}^{-\boldsymbol{\epsilon}}} = f_1 \otimes f_2^{\boldsymbol{\epsilon}}; \quad \theta_{\phi_{2,f}\otimes\boldsymbol{\varphi}_{\boldsymbol{m}}^{-\boldsymbol{\epsilon}}}(\gamma) = f_2^\vee \otimes (f_2^{\boldsymbol{\epsilon}})^\vee. \tag{128}$$

Now, the automorphic function $g \mapsto f_1(g)\cdot f_2^\vee(g)$ corresponds to a Hilbert modular form on $B^\times$ of weight $2\boldsymbol{m}+2$ and trivial central character. We may therefore choose some automorphic representation π of $PB(\mathbb{A}_F)^\times$ of weight $2\boldsymbol{m}+2$, with a vector α_0 of weight $-(2\boldsymbol{m}+2)$, such that

$$\int_{[PB^\times]} \alpha_0(g) f_1(g) f_2^\vee(g)\,\mathrm{d}g \neq 0.$$

Now, we turn α_0 into an automorphic form α on $\mathrm{PGSO}(V_B)(\mathbb{A}_F)$ by setting $\alpha(\boldsymbol{p}_Z(g_1,g_2)) = \alpha_0(g_1)$. It is clear that α is a vector in $((\pi\boxtimes\mathbb{1})\otimes\boldsymbol{\chi}_{2\boldsymbol{m}+2,0})^{\boldsymbol{L}}$. Then Proposition 9.2.4 allows us to compute:

$$\widetilde{\mathcal{P}}_{S,\pi_1,\pi_2,\pi}(\theta_{\phi_{1,f}\otimes\phi_{2,f}\otimes\widetilde{\boldsymbol{\varphi}}_{\boldsymbol{m}}}(\alpha),\beta,\gamma) = \left(\int_{[PB^\times]} \alpha_0(g) f_1(g) f_2^\vee(g)\,\mathrm{d}g\right)\cdot\left(\int_{[PB^\times]} f_2^{\boldsymbol{\epsilon}}(g)(f_2^{\boldsymbol{\epsilon}})^\vee(g)\,\mathrm{d}g\right) \neq 0.$$

□

10. Proof of main result: Hodge classes in the non-generic case

10.1. **Construction.**

10.1.1. Consider the inclusions of Shimura varieties:

$$S(\mathbf{GSp}_6) \xleftarrow{\iota_1} S(\widetilde{\boldsymbol{H}}) \xrightarrow{\iota_2} S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2), \tag{129}$$

where

$$\widetilde{H} \coloneqq \mathrm{GSp}_4 \times_{\mathbb{G}_m} \mathrm{GL}_2 \subset \mathrm{GSp}_6\,.$$

Note that $\iota_1^* \mathcal{V}_{(\boldsymbol{m}-2,\boldsymbol{m}-2,0)}$ contains $\iota_2^* \mathcal{W}_{\boldsymbol{m}}$ as a direct factor with multiplicity one. Since ι_2 is an open and closed embedding at sufficiently small level, one obtains from (129) a map

$$H^i(S(\mathbf{GSp}_6), \mathcal{V}_{(\boldsymbol{m}-2,\boldsymbol{m}-2,0)}) \to H^i(S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2), \mathcal{W}_{\boldsymbol{m}}). \tag{130}$$

The Hodge classes we construct will be the images of classes on $S(\mathbf{GSp}_6)$ under the map (130).

10.1.2. Let π_1, π_2, and Π_{S_f} be as in (5.3.1), where $|S_f| \geq 2$ is even. We let B be the unique quaternion algebra over F which is ramified at S_f and split at all archimedean places. For any finite set $\Sigma \supset S_f$ of places of F, including all infinite ones, we consider the unramified Hecke algebra with $\mathbb{Q}$-coefficients:

$$\widetilde{\mathbb{T}}^{\Sigma} = \otimes_{v \notin \Sigma} \mathcal{H}(\mathrm{GSp}_6(F_v), \mathrm{GSp}_6(\mathcal{O}_v)). \tag{131}$$

For an auxiliary automorphic representation π of $PB^\times$ which is tempered, unramified outside of Σ, and of weight $2\boldsymbol{m}+2$, the Hecke action on $\Theta(\pi \boxtimes \mathbb{1})$ defines a maximal ideal $I^{\Sigma} \subset \widetilde{\mathbb{T}}^{\Sigma}$.

Definition 10.1.3. Fix π and Σ as above, a sufficiently small compact open subgroup $K = \prod K_v \subset \mathrm{GSp}_6(\mathbb{A}_{F_f})$ such that $K_v = \mathrm{GSp}_6(\mathcal{O}_v)$ for $v \notin \Sigma$, and a coefficient field $E \supset \mathbb{Q}(\boldsymbol{m})$ over which Π_{S_f} is defined. Then we define

$$\mathrm{Hdg}_E(\pi, K, \Sigma) \subset H^{4d}(S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2), \mathcal{W}_{\boldsymbol{m},E})(2d)\left[\Pi_{S_f}^\vee \boxtimes \pi_2\right]$$

to be the image of the Tate twist of the composite map

$$\begin{aligned} IH^{4d}(S_K(\mathbf{GSp}_6)^*, \mathcal{V}_{(\boldsymbol{m}-2,\boldsymbol{m}-2,0),E})[I^{\Sigma}] \to H^{4d}(S(\mathbf{GSp}_6), \mathcal{V}_{(\boldsymbol{m}-2,\boldsymbol{m}-2,0),E}) \xrightarrow{(130)} H^{4d}(S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2), \mathcal{W}_{\boldsymbol{m},E}) \\ \twoheadrightarrow H^{4d}_!(S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2), \mathcal{W}_{\boldsymbol{m},E})\left[\Pi_{S_f}^\vee \boxtimes \pi_2\right]. \end{aligned}$$

Lemma 10.1.4. *Any $\xi \in \mathrm{Hdg}_E(\pi, K, \Sigma)$ is a Hodge class of weight $(0,0)$. Moreover, the image of ξ in*

$$H^{4d}_{\text{ét},!}\left((S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2)_{\overline{\mathbb{Q}}}, \mathcal{W}_{\boldsymbol{m},E_\lambda}\right)(2d)\left[\Pi_{S_f}^\vee \boxtimes \pi_{2,f}\right]$$

is $\mathrm{Gal}(\overline{\mathbb{Q}}/F^c)$-invariant, where λ is any finite prime of E.

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} H^{4d}_{(2)}(S_K(\mathbf{GSp}_6), \mathcal{V}_{(\boldsymbol{m}-2,\boldsymbol{m}-2,0),\mathbb{C}})(2d)[I^{\Sigma}] & \longrightarrow & H^{4d}(S_K(\mathbf{GSp}_6), \mathcal{V}_{(\boldsymbol{m}-2,\boldsymbol{m}-2,0),\mathbb{C}})(2d) \\ \downarrow & & \downarrow \\ H^{4d}_{(2)}(S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2), \mathcal{W}_{\boldsymbol{m},\mathbb{C}})(2d)\left[\Pi_{S_f}^\vee \boxtimes \pi_2\right] & \xrightarrow{\sim} & H^{4d}_!(S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2), \mathcal{W}_{\boldsymbol{m},\mathbb{C}})(2d)\left[\Pi_{S_f}^\vee \boxtimes \pi_2\right]. \end{array}$$

The left and bottom arrows in this diagram are maps of pure Hodge structures, and by construction $\mathrm{Hdg}_E(\pi, K, \Sigma) \otimes_E \mathbb{C}$ is the image of the composite from the top left to the bottom right. We wish to show that every element of $H^{4d}_{(2)}(S_K(\mathbf{GSp}_6), \mathcal{V}_{(\boldsymbol{m}-2,\boldsymbol{m}-2,0),\mathbb{C}})(2d)[I^{\Sigma}]$ has weight $(0,0)$. If $\widetilde{\Pi}$ is an automorphic representation of $\mathrm{GSp}_6(\mathbb{A}_F)$ such that $\widetilde{\Pi}_f^K \neq 0$ is annihilated by I^{Σ}, then there exists a Galois element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\widetilde{\Pi}$ is nearly equivalent to a constituent of $\Theta(\pi \boxtimes \mathbb{1})^\sigma$. Since the spherical theta correspondence is defined over $\mathbb{Q}$, it follows that $\widetilde{\Pi}$ is nearly equivalent to a constituent of $\Theta(\pi^\sigma \boxtimes \mathbb{1})$. Thus, by Lemma 8.4.3, $\mathrm{Hdg}_E(\pi, K, \Sigma)$ consists of hodge classes of weight $(0,0)$.

The Galois invariance is similar, replacing $\mathbb{C}$ with $\overline{\mathbb{Q}}_\ell$ and L^2 cohomology with absolute intersection cohomology in the diagram above. □

10.2. **Nonvanishing.**

10.2.1. To test the non-degeneracy of the subspace $\mathrm{Hdg}_(\pi, K, \Sigma)$, we will use the following proposition.

Proposition 10.2.2. *Let $\Pi = \Pi_S$, where $S = S_f \sqcup S_\infty$, and choose an auxiliary π as above. Suppose given*

$$\alpha \in (\Theta(\pi \boxtimes \mathbb{1}) \otimes \boldsymbol{\sigma_m})^{\boldsymbol{K}_3}.$$

(1) Fix choices of signs $\boldsymbol{\epsilon} = \{\epsilon_v\}_{v|\infty}$ and $\boldsymbol{\epsilon}' = \{\epsilon_v\}_{v|\infty}$. Then:

$$\langle \iota_{2,*} \circ \iota_1^*(\mathrm{cl}(\alpha)), \mathrm{cl}^{\boldsymbol{\epsilon}}_{S_\infty}(\beta) \boxtimes \mathrm{cl}'_{\boldsymbol{\epsilon}'}(\gamma) = \begin{cases} \widetilde{\mathcal{P}}_{S,\pi_1,\pi_2,\pi}(\alpha,\beta,\gamma), & \text{if } S_\infty = \emptyset \text{ and } \boldsymbol{\epsilon} = \boldsymbol{\epsilon}'; \\ 0, & \text{otherwise.} \end{cases}$$

(2) After choosing isomorphisms

$$\Pi_{S_f} \simeq (\Pi \otimes \boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{m}})^{\boldsymbol{K}_3}, \quad \pi_{2,f}^\vee \simeq (\pi_2^\vee \otimes \boldsymbol{\chi}_{-\boldsymbol{\epsilon m}})^{\boldsymbol{K}_1},$$

the composite maps

$$\Pi_{S_f} \otimes \pi_{2,f}^\vee \xrightarrow{\mathrm{cl}^{\boldsymbol{\epsilon}} \otimes \mathrm{cl}^{-\boldsymbol{\epsilon}}} H^{4d}_{(2)}(S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2), \mathcal{W}_{\boldsymbol{m},\mathbb{C}}) \xrightarrow{\langle \iota_{2,*}\iota_1^*\zeta, \cdot\rangle} \mathbb{C}$$

are independent of $\boldsymbol{\epsilon}$ up to a scalar.

Proof. The proof is essentially identical to Proposition 7.2.4. The only new ingredient is the calculation of the $(\boldsymbol{K}_2 \times \boldsymbol{K}_1)_0$-equivariant composite:

$$\begin{aligned} \boldsymbol{\sigma_m} \otimes \boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{m},S_\infty} \otimes \boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon}'\boldsymbol{m}} &\to \wedge^{\mathbf{2},\mathbf{2}} \mathfrak{p}^*_{\mathrm{GSp}_6} \otimes V_{(\boldsymbol{m}-2,\boldsymbol{m}-2,0),\mathbb{C}} \otimes \wedge^{\boldsymbol{p}(\boldsymbol{\epsilon},S_\infty),\boldsymbol{q}(\boldsymbol{\epsilon},S_\infty)} \mathfrak{p}^*_{\mathrm{GSp}_4} \otimes V_{(\boldsymbol{m}-2,0),\mathbb{C}} \\ &\otimes \wedge^{1-\boldsymbol{p}(\boldsymbol{\epsilon}'),1-\boldsymbol{q}(\boldsymbol{\epsilon}')} \mathfrak{p}^*_{\mathrm{GL}_2} \otimes V^\vee_{\boldsymbol{m}-2,\mathbb{C}} \to \wedge^{3+\boldsymbol{p}(\boldsymbol{\epsilon},S_\infty)-\boldsymbol{p}(\boldsymbol{\epsilon}'),3+\boldsymbol{q}(\boldsymbol{\epsilon},S_\infty)-\boldsymbol{q}(\boldsymbol{\epsilon}')} \mathfrak{p}^*_{\widetilde{H}} \xrightarrow{\mathbb{1}_{\widetilde{H}}} \mathbb{C}, \end{aligned} \tag{132}$$

which is automatically trivial unless $S_\infty = \emptyset$ and $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}'$, in which case one can check that it is not trivial. □

Theorem 10.2.3. *Let π_1 and π_2 be cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ of weights $\boldsymbol{m}+2$ and $\boldsymbol{m}$, respectively, where $\boldsymbol{m} = (m_v)_{v|\infty}$ for positive integers m_v. Assume that the central characters of π_1 and π_2 agree and have infinity type $\omega_{\boldsymbol{m}}$. Let Π_{S_f} be as in (5.3.1) for a set S_f of finite places of F such that $|S_f| \geq 2$ is even, and choose a coefficient field $E \supset \mathbb{Q}(\boldsymbol{m})$ over which π_i and Π_{S_f} are defined. Then there exists a triple (π, K, Σ) as in Definition 10.1.3 and a Hodge class*

$$\xi \in \mathrm{Hdg}_E(\pi, K, \Sigma) \subset H^{4d}_!(S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2), \mathcal{W}_{\boldsymbol{m},E})(2d)\left[\Pi_{S_f} \boxtimes \pi_2^\vee\right]$$

such that the induced map

$$\xi_* : H^{3d}_!(S(\mathbf{GSp}_4), \mathcal{V}_{\boldsymbol{m},E})(d)[\Pi_{S_f}] \to H^d_!(S(\mathbf{GL}_2), \mathcal{V}'_{\boldsymbol{m},E})[\pi_{2,f}]$$

is of the form

$$\Pi^E_{S_f} \otimes H^{3d}_!(S(\mathbf{GSp}_4), \mathcal{V}_{\boldsymbol{m},E})_{\Pi_{S_f}}(d) \xrightarrow{\ell \otimes s} \pi^E_{2,f} \otimes H^d_!(S(\mathbf{GL}_2), \mathcal{V}'_{\boldsymbol{m},E})_{\pi_{2,f}},$$

where s is a surjection and ℓ is a nontrivial E-linear map. Moreover, the image of ξ in ℓ-adic étale cohomology is $\mathrm{Gal}(\overline{\mathbb{Q}}/F^c)$-equivariant for all ℓ.

Proof. The same argument as for Theorem 7.2.5 implies ξ_* is always a pure tensor $\ell \otimes s$, and the map s is always either trivial or a surjection. Thus it suffices to show that there exists a *complex* Hodge class $\xi \in \mathrm{Hdg}_{\mathbb{C}}(\pi, K, \Sigma)$ with $\xi_* \neq 0$. However, this is guaranteed by Lemma 9.3.2 and Proposition 10.2.2. □

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