KOLYVAGIN'S CONJECTURE, BIPARTITE EULER SYSTEMS, AND HIGHER CONGRUENCES OF MODULAR FORMS

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ABSTRACT. Let $E/\mathbb{Q}$ be an elliptic curve and let K be an imaginary quadratic field. Under a certain Heegner hypothesis, Kolyvagin constructed cohomology classes for E using K-CM points and conjectured they did not all vanish. Conditional on this conjecture, he described the Selmer rank of E using his system of classes. We extend work of Wei Zhang to prove new cases of Kolyvagin's conjecture by considering congruences of modular forms modulo large powers of p. Additionally, we prove an analogous result, and give a description of the Selmer rank, in a complementary "definite" case (using certain modified L-values rather than CM points). Similar methods are also used to improve known results on the Heegner point main conjecture of Perrin-Riou. One consequence of our results is a new converse theorem, that p-Selmer rank one implies analytic rank one, when the residual representation has dihedral image.

CONTENTS

1. Introduction

Let f be a weight 2 cuspidal newform for $\Gamma_0(N)$, without complex multiplication. The Birch and Swinnerton-Dyer Conjecture predicts:

$$r(A_f/\mathbb{Q}) = [E_f : \mathbb{Q}] \operatorname{ord}_{s=1} L(f, s), \tag{1}$$

where A_f is the associated abelian variety to f, r denotes the Mordell-Weil rank, and E_f is the coefficient field of f. In pioneering works on this problem, Perrin-Riou [46] and Kolyvagin [35, 36] studied ranks of elliptic curves over an auxiliary imaginary quadratic field K through the theory of Heegner points on modular curves. We prove, in new cases, conjectures made by both authors.

Fix a quadratic imaginary field K, and a prime $\wp$ of E_f of residue characteristic p, with $\mathcal{O} = \mathcal{O}_{f,\wp}$ the completion at $\wp$ of the ring of integers of E_f. Assume the following generalized Heegner hypothesis:

$$\text{(Heeg)} \quad N = N^+N^-, \text{ where all } \ell | N^+ \text{ are split in } K, \quad \text{all } \ell | N^- \text{ are inert in } K, \text{ and } N^- \text{ is squarefree,}$$

as well as:

$$\text{(unr)} \qquad p \nmid 2N \operatorname{disc}(K).$$

The p-adic Tate module V_pA_f of A_f is equipped with an action of E_f; write $V_f \coloneqq V_pA_f \otimes_{E_f} E_{f,\wp}$ for the $\wp$-adic Galois representation attached to f, and let $T_f \subset V_f$ be a Galois-stable $\mathcal{O}$-lattice. We shall assume that $\overline{T}_f \coloneqq T_f/\wp T_f$ is absolutely irreducible as a representation of the Galois group $G_K \coloneqq \mathrm{Gal}(\overline{K}/K)$.

For purposes of exposition in this introduction, we also assume:

(sclr) The image of the G_K action on $\overline{T}_f$ contains a nonzero scalar.

To state Kolyvagin's conjecture, assume that the number of prime factors $\nu(N^-)$ is even. If m is a squarefree product of primes inert in K, one can use Heegner points of conductor m on the Shimura curve X_{N^+,N^-} to construct classes

$$c(m) \in H^1(K, T_f/I_m),$$

where I_m is the ideal of $\mathcal{O} = \mathcal{O}_{f,\wp}$ generated by $\ell + 1$ and a_ℓ for all $\ell | m$. (In the text, $c(m)$ is denoted $\overline{c}(m, 1)$.) These classes are a mild generalization of the ones constructed by Kolyvagin [36]. We are able to prove the following result towards Kolyvagin's conjecture that the system $\{c(m)\}$ is nontrivial:

Theorem A ((Theorem 8.1.1, Corollary 7.3.7)). *Assume (Heeg), (unr), and (sclr) hold for $f, \wp$, and K, and $\nu(N^-)$ is even. Suppose the following conditions hold:*

$$(\diamondsuit) \quad \begin{cases} \bullet \text{ The modulo } \wp \text{ representation } \overline{T}_f \text{ associated to } f \text{ is absolutely irreducible; if } p = 3, \text{ then } \overline{T}_f \text{ is not induced from a character of } G_{\mathbb{Q}(\sqrt{-3})}. \\ \bullet \text{ If } p \text{ is inert in } K, \text{ then there exists some prime } \ell_0 || N. \\ \bullet \text{ If } a_p \text{ is not a } \wp\text{-adic unit, then either there exists } \ell_0 || N^+ \text{ such that } \overline{T}_f|_{G_{\mathbb{Q}_{\ell_0}}} \text{ is ramified and } \overline{T}_f^{G_{\mathbb{Q}_{\ell_0}}} = 0; \text{ or there exist primes } \ell_1, \ell_2 | N^- \text{ such that } \overline{T}_f|_{G_{\mathbb{Q}_{\ell_i}}} \text{ is ramified for } i = 1, 2, \ \overline{T}_f^{G_{\mathbb{Q}_{\ell_1}}} = 0, \text{ and } \overline{T}_f^{G_{\mathbb{Q}_{\ell_2}}} \neq 0. \end{cases}$$

Then there exists a nonzero Kolyvagin class

$$0 \neq c(m) \in H^1(K, T_f/I_m).$$

As Kolyvagin observed, Theorem A can be used to give a description of the Selmer ranks $r^{\pm} = \mathrm{rk}_{\mathcal{O}} \operatorname{Sel}(K, T_f)^{\pm}$, where superscripts refer to the action of complex conjugation. Indeed, define the vanishing order of the system $\{c(m)\}$ as

$$\nu := \min \{\nu(m) \,:\, c(m) \neq 0\} \tag{2}$$

where as before ν denotes the number of prime factors. Then we have:

Corollary B ((Corollary 7.3.7)). *Under the assumptions of Theorem A,*

$$\max\left\{r^+, r^-\right\} = \nu + 1.$$

Moreover $r^+ + r^-$ is odd, and the larger eigenspace has sign $(-1)^{\nu+1}\epsilon_f$, where ϵ_f is the global root number of f.

Of course, the latter two assertions follow from the parity conjecture for f, already proven by Nekovář [42].

Since $c(1) \in \operatorname{Sel}(K, T_f)$ is the Kummer image of the classical Heegner point, the Gross-Zagier formula implies that $L'(f/K, 1) \neq 0$ if and only if $c(1) \neq 0$. Hence Corollary B yields a so-called p-converse theorem (in fact, under a slightly weaker hypothesis):

Corollary C ((Theorem 7.3.6)). *Assume that (Heeg), (unr), and Condition $\diamondsuit$ hold for f, $\wp$, and K, and $\nu(N^-)$ is even. Then $L'(f/K, 1) \neq 0$ if and only if $\mathrm{rk}_{\mathcal{O}} \operatorname{Sel}(K, T_f) = 1$, in which case A_f has Mordell-Weil rank 1.*

Now suppose instead that $\nu(N^-)$ is odd; it turns out that Kolyvagin's construction, suitably modified, may still be used to relate Selmer ranks and CM points. The Jacquet-Langlands correspondence associates to f a quaternionic modular form

$$\phi_f : X_{N^+,N^-} \to \mathcal{O}_f, \tag{3}$$

where X_{N^+,N^-} is a double coset space for a definite quaternion algebra, usually called a Shimura set. If m is a squarefree product of primes inert in K, there exist analogues of CM points of conductor m on the Shimura set. Using the values of ϕ_f at these points, we construct certain special elements (well-defined up to units)

$$\lambda(m) \in \mathcal{O}/I_m \tag{4}$$

($\lambda(m, 1)$ in the text). Here the ideal $I_m \subset \mathcal{O}$ is as before. The analogues of the elements $\lambda(m)$ for p-power conductor have long been used in anticyclotomic Iwasawa theory, e.g. [2]. However, for squarefree m, a novel observation of this work is that the elements $\lambda(m)$ encode the same information about the Selmer ranks of A_f/K as Kolyvagin's classes $c(m)$.

Theorem D ((Theorem 8.1.1, Corollary 7.3.7)). *Suppose that (Heeg), (unr), (sclr), and Condition $\diamondsuit$ hold for $f, \wp$, and K, and that $\nu(N^-)$ is odd. Then the vanishing order*

$$\nu := \min \{\nu(m) \,:\, \lambda(m) \neq 0\}$$

is finite and

$$\nu = \max\left\{r^+, r^-\right\}.$$

Moreover $(-1)^\nu = \epsilon_f$ *and* $r^+ + r^-$ *is even.*

As before, the final statement is a consequence of the parity conjecture; we include it only to emphasize that it follows from the non-vanishing of some $\lambda(m)$, in analogy to the indefinite case.

1.1. **Comparison to previous results.** In the indefinite case, the first results towards Kolyvagin's conjecture were obtained by Zhang [69], under a number of additional assumptions: that $p \geq 5$, that the Galois representation associated to $\overline{T}_f$ is surjective, and additional hypotheses on the residual ramification. In particular, under the hypotheses of [69], there exists a class $c(m)$ whose reduction in $H^1(K, \overline{T}_f)$ is nonzero; this is not the case in general. In the definite case, the classes $\lambda(m)$ are a novel feature of this work and were not considered in [69].

The converse theorem we obtain (Corollary C) is new in several cases, most notably when the image of the Galois action on $\overline{T}_f$ is dihedral, or when $p = 3$. Previous results, under various additional hypotheses, were obtained by Zhang as a corollary of his work on Kolyvagin's conjecture, and by Skinner [59] by a purely Iwasawa-theoretic method. For converse theorems in other settings, see Burungale [7] for the CM case, Castella-Grossi-Lee-Skinner [9] for the residually reducible case, Castella-Wan [10] for the supersingular case, and Skinner-Zhang [61] for the case of multiplicative reduction.

1.2. **Iwasawa theory.** Now suppose again that $\nu(N^-)$ is even. While the Kolyvagin classes are constructed by varying the conductor of CM points on X_{N^+,N^-} over squarefree integers, one may instead p-adically interpolate CM points of p-power conductor to obtain a class:

$$\boldsymbol{\kappa}_\infty \in H^1(K, T_f \otimes \Lambda(\Psi)), \tag{5}$$

where $\Lambda = \mathcal{O}[\![\mathrm{Gal}(K_\infty/K)]\!]$ is the anticyclotomic Iwasawa algebra, given G_K-action by the tautological character Ψ. (Note that the specialization of $\boldsymbol{\kappa}_\infty$ at the trivial character is a multiple of $c(1)$.) The methods used to prove Theorem A also yield the following result towards Perrin-Riou's Heegner point main conjecture.

Theorem E ((Corollary 7.2.2))**.** *Suppose that (Heeg), (unr), and Condition* $\diamondsuit$ *hold for* $f, \wp$*, and* K*, and that* $\nu(N^-)$ *is even. Suppose further that* a_p *is a* $\wp$*-adic unit and* p *splits in* K*. Then there is a pseudo-isomorphism of* Λ*-modules:*

$$\mathrm{Sel}(K_\infty, W_f)^\vee \approx \Lambda \oplus M \oplus M$$

for some torsion Λ*-module* M*, and*

$$\mathrm{char}_\Lambda\left(\frac{\mathrm{Sel}(K, T_f \otimes \Lambda)}{\Lambda \cdot \boldsymbol{\kappa}_\infty}\right) = \mathrm{char}_\Lambda(M)$$

as ideals of $\Lambda \otimes \mathbb{Q}_p$*. If (sclr) holds, then the equality is true in* Λ*.*

Here, W_f is the divisible Galois module V_f/T_f. For precise definitions of the above Selmer groups and of $\boldsymbol{\kappa}_\infty$, which is denoted $\boldsymbol{\kappa}(1)$ in the text, see §5.2.

Finally, we have the following result on the anticyclotomic main conjecture for f when $\nu(N^-)$ is odd. Evaluating the quaternionic modular form ϕ_f on CM points of p-power conductor on the Shimura set X_{N^+,N^-}, one constructs the algebraic p-adic L-function

$$\boldsymbol{\lambda}_\infty \in \Lambda, \tag{6}$$

denoted $\boldsymbol{\lambda}(1)$ in the text. The square of $\boldsymbol{\lambda}_\infty$ has an interpolation property for twisted L-values of f.

Theorem F ((Corollary 7.2.3))**.** *Suppose that (Heeg), (unr), and Condition* $\diamondsuit$ *hold for* $f, \wp$*, and* K*, and that* $\nu(N^-)$ *is odd. Suppose further that* a_p *is a* $\wp$*-adic unit and* p *splits in* K*. Then there is a pseudo-isomorphism of* Λ*-modules:*

$$\mathrm{Sel}(K_\infty, W_f)^\vee \approx M \oplus M$$

for some torsion Λ*-module* M*, and*

$$(\boldsymbol{\lambda}_\infty) = \mathrm{char}_\Lambda(M)$$

as ideals of $\Lambda \otimes \mathbb{Q}_p$*. If additionally (sclr) holds, then the equality is true in* Λ*.*

One direction of this equality is due to Skinner-Urban's work on the three-variable main conjecture [60]; indeed, this is an essential ingredient in all of our results, as explained below.

1.3. **Comparison to previous results.** The hypotheses in Zhang's proof of Kolyvagin's conjecture were carried over to Burungale, Castella, and Kim's proof [6] of the lower bound on the Selmer group in the Heegner point main conjecture, where it is also assumed that p is not anomalous. While the methods used in this paper build on those of [6], Castella and Wan [11] have given a completely independent proof of a three-variable main conjecture when $\nu(N^-)$ is even. Their result also requires some hypotheses on residual ramification avoided here, and that N be squarefree.

For upper bounds on the Selmer group in Theorem E and Theorem F, various technical assumptions on the residual representation and on the image of the Galois action were used in prior works by Howard [27, 28] and Chida-Hsieh [12]. The lower bound on the Selmer group in Theorem F is contained in [60].

1.4. **Overview of the proofs.** To prove Theorems A and D, we extend Kolyvagin's construction to a larger system of classes

$$c(m, Q_1) \in H^1(K, T_f/\wp^M), \quad \lambda(m, Q_2) \in \mathcal{O}/\wp^M, \tag{7}$$

where M is a fixed integer, and m, Q_1, Q_2 are squarefree products of auxiliary primes satisfying certain congruence conditions, such that $\nu(N^- Q_1)$ is even and $\nu(N^- Q_2)$ is odd. The classes (7) form a bipartite Euler system in the sense of Howard [28] for each fixed m and a Kolyvagin system for each fixed Q_1. If $\nu(N^-)$ itself is even, then the classes $c(m, 1)$ agree with Kolyvagin's original construction. The Euler system relations are of the form:

$$\mathrm{loc}_q\, c(m, Q_1) \sim \lambda(m, Q_1 q) \sim \partial_{q'} c(m, Q_1 q q'), \tag{8}$$

where q, q' are two additional auxiliary primes not dividing Q_1; and

$$\mathrm{loc}_\ell^{\pm}\, c(m, Q_1) \sim \partial_\ell^{\mp} c(m\ell, Q_1), \tag{9}$$

where ℓ is an additional auxiliary prime not dividing m. (Here loc_q, $\partial_{q'}$, $\mathrm{loc}_\ell^{\pm}$, $\partial_\ell^{\pm}$ are certain localization maps landing in subspaces of the local cohomology free of rank one over $\mathcal{O}/\wp^M$.) The classes $c(m, Q_1)$ were introduced by Zhang, although the $\lambda(m, Q_2)$ are only implicit in [69].

If $c(m, Q_1) \neq 0$, then one can use (8) and (9) to find an auxiliary ℓ — either prime or equal to 1 — such that $\partial_q c(m\ell, Q_1) \neq 0$ for some $q | Q_1$. By (8), this implies $\lambda(m\ell, Q_1/q) \neq 0$. On the other hand, if $\lambda(m, Q_2) \neq 0$ and $q | Q_2$, then by (8) $c(m, Q_2/q) \neq 0$. Combining these two observations, we reduce the non-vanishing of some class $c(m, 1)$ or $\lambda(m, 1)$ — depending on the parity of $\nu(N^-)$ — to exhibiting a single Q_2 such that $\lambda(1, Q_2) \neq 0$.

Now, if there exists a newform g of level NQ_2 with a congruence to f modulo $\wp^M$, then $\lambda(1, Q_2)$ is essentially the reduction of the algebraic L-value $L^{\mathrm{alg}}\,(g/K, 1)$ modulo $\wp^M$, which is related to the length of the Selmer group of g by the Iwasawa main conjecture [60, 65]. To complete the proof, it therefore suffices to choose a suitable Q_2 and construct such a g with a small Selmer group. We remark that our results can only be obtained by working modulo $\wp^M$ for a large M, since in general it will not be possible to choose g such that $L^{\mathrm{alg}}(g/K, 1)$ is a $\wp$-adic unit; in [69], $M = 1$ is fixed throughout, and the need to show that the L-value is a unit is responsible for most of the additional hypotheses.

To construct g, we use the deformation-theoretic techniques developed by Ramakrishna [49]. Standard level-raising methods work by producing a modulo $\wp$ eigenform of the desired level, and then using that all modulo $\wp$ eigenforms lift to characteristic zero, but this is not the case modulo $\wp^M$. Instead, we deform the representation $T_f/\wp^M$ to a $\wp$-adic Galois representation of a suitable auxiliary level, and then apply modularity lifting to ensure the resulting representation is modular. The auxiliary level Q_2 must be chosen to control two Selmer groups: the adjoint Selmer group governing the deformation problem, and the Selmer group $\mathrm{Sel}(K, W_g)$ that is related to the L-value. (Here, W_g is the divisible $\mathcal{O}$-Galois module constructed analogously to W_f.)

We now make some remarks on the construction of the Euler system. The elements $c(m, Q_1)$ (resp. $\lambda(m, Q_2)$) are constructed from CM points of conductor m on the Shimura curve $X_{N^+, N^- Q_1}$ (resp. Shimura set $X_{N^+, N^- Q_2}$). Similar Euler system constructions have been made by many authors, e.g. in [12, 2] as well as in [69], but all have relied on certain hypotheses ensuring an integral multiplicity one property for the space of algebraic modular forms on $X_{N^+, N^- Q_i}$, which we do not impose here. Instead, we obtain a control on the failure of multiplicity one, using the work of Helm [25] on maps between Jacobians of modular curves and Shimura curves. The construction of the Euler system is intimately related to level-raising, and so our method also improves results on level-raising of f to algebraic eigenforms modulo $\wp^M$ new at multiple auxiliary primes, which had previously been restricted to the multiplicity one case. A precise statement is given in Theorem 4.5.7.

The proof of Theorem E is similar to that of Theorem A: the p-adically interpolated Heegner class $\boldsymbol{\kappa}_\infty$ is viewed as the bottom layer of an Euler system $\{\boldsymbol{\kappa}(Q_1), \boldsymbol{\lambda}(Q_2)\}$. (The squarefree conductor m no longer plays a role.) If g, as above, is a newform of level NQ_2 with a congruence to f, then $\boldsymbol{\lambda}(Q_2)$ is congruent to Bertolini and Darmon's anticyclotomic p-adic L-function of g [2]. Using this and an Euler system argument, we reduce the lower bound on the Selmer group in the Heegner point main conjecture to the lower bound on the Selmer group in the anticyclotomic main conjecture for g, which was proven in [60]. Finally, the upper bound on the Selmer group in Theorems E and F follows by standard arguments from the construction of the Euler system.

In the text, the arguments described above are phrased in the language of ultrapatching, which amounts to a formalism for letting M tend to infinity; this also forces each prime factor of m, Q_1, Q_2 to tend to infinity in order to satisfy the congruence conditions. (The number of prime factors of m, Q_1, and Q_2 remains bounded.) This method was inspired by [57], where ultrapatching was applied to the Taylor-Wiles construction. Our setting is different in that we patch Galois cohomology groups and Selmer groups rather than geometric étale cohomology groups. The benefit of ultrapatching is that it allows us to consider the Euler system classes as characteristic zero objects in patched Selmer groups, significantly streamlining the Euler system arguments. For instance, with patching, we are able to make precise the heuristic that the non-vanishing of each Euler system class $c(m, Q_1)$ or $\lambda(m, Q_2)$ is equivalent to the (m, Q_i)-transverse Selmer group being rank one or zero, respectively, cf. Lemma 7.3.4.

Structure of the paper. In §2, we review basic properties of ultrafilters and introduce patched cohomology and Selmer groups. In §3, we present a simplified version of the theory of bipartite Euler systems that appeared in [28], using patched cohomology. In §4, we establish the geometric inputs that will be used to construct bipartite Euler systems: the work of Helm on maps between modular curves and Shimura curves, the modulo $\wp^M$ level-raising result, and the behavior of Heegner points on Shimura curves under reduction and specialization. In §5, we present a general framework for constructing bipartite Euler systems out of CM points, which we then specialize for our applications. In §6, we give the deformation-theoretic input to construct the newform g (in fact a sequence g_n satisfying increasingly deep congruence conditions). Finally, we prove the main results in the split ordinary case in §7. An additional calculation in cyclotomic Iwasawa theory and a comparison of periods are required for Kolyvagin's conjecture when p is non-ordinary or inert in K; this is done in §8.

Notational conventions.

- If N is a squarefree positive integer, then $\nu(N)$ denotes its number of prime factors.
- If L is a number field, we write $G_L = \mathrm{Gal}(\overline{L}/L)$ for its absolute Galois group and $\mathbb{A}_L$ (resp. $\mathbb{A}_{f,L}$) for its ring of adèles (resp. finite adéles).
- The symbol Frob_v always denotes an *arithmetic* Frobenius element.
- If K is a number field and A is a G_K-module, then $K(A)$ is the smallest algebraic extension of K such that $G_{K(A)}$ acts trivially on A.
- We fix, for each place v of $\mathbb{Q}$, an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_v$. For any number field $L \subset \overline{\mathbb{Q}}$, and any place v of L, we denote by $G_{L_v} \hookrightarrow G_L$ the resulting embedding of the local Galois group.
- The p-adic cyclotomic character is denoted $\chi : G_{\mathbb{Q}} \to \mathbb{Z}_p^\times$.
- For a cuspidal eigenform f with trivial character, its global root number is denoted ϵ_f.

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2. Ultrafilters and patching

2.1. **Ultraproducts.** The following discussion is inspired by the unpublished notes of Manning [37, §I.1].

2.1.1. A (non-principal) ultrafilter $\mathfrak{F}$ for the natural numbers $\mathbb{N} = \{0, 1, \ldots\}$ is a collection of subsets of $\mathbb{N}$ satisfying the following properties:

(1) Every set $S \in \mathfrak{F}$ is infinite.
(2) For every $S \subset \mathbb{N}$, either $S \in \mathfrak{F}$ or $\mathbb{N} - S \in \mathfrak{F}$.
(3) If $S_1 \subset S_2 \subset \mathbb{N}$ and $S_1 \in \mathfrak{F}$, then $S_2 \in \mathfrak{F}$.
(4) If $S_1, S_2 \in \mathfrak{F}$, then $S_1 \cap S_2 \in \mathfrak{F}$.

Throughout this paper, we fix once and for all a non-principal ultrafilter $\mathfrak{F}$ on $\mathbb{N}$, which is possible assuming the axiom of choice. We will say that a statement P holds for $\mathfrak{F}$-many $n \in \mathbb{N}$ if the set S of n for which P holds lies in $\mathfrak{F}$.

Proposition 2.1.2. *Suppose that $\mathcal{C}$ is a finite set and $S \subset \mathbb{N}$ lies in $\mathfrak{F}$. Then for any function $t : S \to \mathcal{C}$, there is a unique $c \in \mathcal{C}$ such that $t(n) = c$ for $\mathfrak{F}$-many n.*

Proof. The function t defines a finite partition of $\mathbb{N}$:

$$\mathbb{N} = (\mathbb{N} - S) \sqcup \bigsqcup_{c \in \mathcal{C}} t^{-1}(c).$$

An easy induction argument shows that, for any partition of $\mathbb{N}$ into a finite number sets, exactly one of the sets lies in $\mathfrak{F}$. Since $\mathbb{N} - S \notin \mathfrak{F}$, the result follows. □

2.1.3. If $\mathcal{M} = \{M_n\}_{n \in \mathbb{N}}$ is a sequence of sets indexed by $\mathbb{N}$, then $\mathfrak{F}$ defines an equivalence relation $\sim$ on $\prod M_n$:

$$(m_n)_{n \in \mathbb{N}} \sim (m'_n)_{n \in \mathbb{N}} \iff \{n : m_n = m'_n\} \in \mathfrak{F}.$$

The quotient $\prod M_n / \sim$ is called the **ultraproduct** of the sequence $\mathcal{M}$ and is denoted $\mathcal{U}(\mathcal{M})$. The ultraproduct is functorial: let $\mathcal{M}' = \{M'_n\}$ be another sequence of sets and suppose given, for $\mathfrak{F}$-many n, maps $\varphi_n : M_n \to M'_n$. Then there is a natural map $\varphi^{\mathcal{U}} : \mathcal{U}(\mathcal{M}) \to \mathcal{U}(\mathcal{M}')$. In particular, if each M_n is endowed with the structure of an abelian group (resp. R-module for a fixed ring R), then $\mathcal{U}(\mathcal{M})$ is naturally an abelian group (resp. R-module).

Proposition 2.1.4. *(1) Let $\mathcal{M} = \{M_n\}_{n \in \mathbb{N}}$ and $\mathcal{M}' = \{M'_n\}_{n \in \mathbb{N}}$ be sequences of nonempty sets, and suppose given maps $\varphi_n : M_n \to M'_n$ for $\mathfrak{F}$-many n. If φ_n is injective (resp. surjective, bijective) for $\mathfrak{F}$-many n, then $\varphi^{\mathcal{U}}$ is injective (resp. surjective, bijective).*
(2) Suppose $\mathcal{M} = \{M_n\}_{n \in \mathbb{N}}$, where each M_n is a nonempty finite set such that $\#M_n < C$ for $\mathfrak{F}$-many n. Then $\mathcal{U}(\mathcal{M})$ is finite and $\#\mathcal{U}(\mathcal{M}) = \#M_n$ for $\mathfrak{F}$-many n.

Proof. (1) Suppose φ_n is injective for $\mathfrak{F}$-many n and let $m, m' \in \mathcal{U}(\mathcal{M})$ be the equivalence classes of sequences $(m_n)_{n \in \mathbb{N}}$ and $(m'_n)_{n \in \mathbb{N}}$. If $\varphi^{\mathcal{U}}(m) = \varphi^{\mathcal{U}}(m')$, then for $\mathfrak{F}$-many n, $\varphi_n(m_n) = \varphi_n(m'_n)$. Hence for $\mathfrak{F}$-many n, $m_n = m'_n$, so $m = m'$ in $\mathcal{U}(\mathcal{M})$. Therefore $\varphi^{\mathcal{U}}$ is injective.

Now suppose φ_n is surjective for $\mathfrak{F}$-many n, and let $m' \in \mathcal{U}(\mathcal{M}')$ be an element represented by $(m'_n)_{n \in \mathbb{N}}$. We will show that m' lies in the image of $\varphi^{\mathcal{U}}$. Let $S \in \mathfrak{F}$ be such that φ_n is surjective for $n \in S$. Define a new sequence $(m_n)_{n \in \mathbb{N}}$ by choosing $m_n \in M_n$ arbitrarily for $n \notin S$, and choosing $m_n \in M_n$ such that $\varphi_n(m_n) = m'_n$ for $n \in S$. Then the equivalence class m of this sequence satisfies $\varphi^{\mathcal{U}}(m) = m'$. Hence, $\varphi^{\mathcal{U}}$ is surjective.

(2) By Proposition 2.1.2, there exists some $c < C$ such that $\#M_n = c$ for $\mathfrak{F}$-many n. Let $[c] = \{0, \ldots, c-1\}$ and choose isomorphisms of sets

$$\varphi_n : M_n \xrightarrow{\sim} [c]$$

for $\mathfrak{F}$-many n. By (i), $\varphi^{\mathcal{U}}$ induces an isomorphism from $\mathcal{U}(\mathcal{M})$ to the ultraproduct $\mathcal{C}$ of the constant sequence $\{[c]\}_{n \in \mathbb{N}}$. However, $\mathcal{C}$ is canonically isomorphic to $[c]$ by Proposition 2.1.2.

□

Proposition 2.1.5. *Let $\mathcal{S}$ be the category of sequences of abelian groups indexed by $\mathbb{N}$. Then $\mathcal{U}$ is exact as a functor from $\mathcal{S}$ to the category of abelian groups.*

Proof. Let $\mathcal{A} = (A_n)_{n\in\mathbb{N}}$, $\mathcal{B} = (B_n)_{n\in\mathbb{N}}$, and $\mathcal{C} = (C_n)_{n\in\mathbb{N}}$ be three sequences of abelian groups, and suppose given exact sequences

$$0 \to A_n \xrightarrow{\varphi_n} B_n \xrightarrow{\psi_n} C_n \to 0$$

for all $n \in \mathbb{N}$. We wish to show that

$$0 \to \mathcal{U}(\mathcal{A}) \xrightarrow{\varphi^{\mathcal{U}}} \mathcal{U}(\mathcal{B}) \xrightarrow{\psi^{\mathcal{U}}} \mathcal{U}(\mathcal{C}) \to 0$$

is exact. By Proposition 2.1.4(i), it suffices to show that the kernel of $\psi^{\mathcal{U}}$ is the image of $\varphi^{\mathcal{U}}$. Suppose $(b_n)_{n\in\mathbb{N}}$ represents an element $b \in \ker \psi^{\mathcal{U}}$. Then, by definition, $\psi_n(b_n) = 0$ for $\mathfrak{F}$-many n, so for $\mathfrak{F}$-many n there exists $a_n \in A_n$ with $\varphi_n(a_n) = b_n$. Hence, there exists a sequence $(a_n)_{n\in\mathbb{N}}$ representing an element $a \in \mathcal{U}(\mathcal{A})$ with $\varphi^{\mathcal{U}}(a) = b$. We have shown that $\ker \psi^{\mathcal{U}} \subset \operatorname{im} \varphi^{\mathcal{U}}$. Since the opposite inclusion is clear, this completes the proof. □

2.2. **Ultraprimes.**

2.2.1. Fix a number field $L \subset \overline{\mathbb{Q}}$ and let M_L be its set of places. If $\mathcal{M}_L$ is the constant sequence of sets $\{M_L\}_{n\in\mathbb{N}}$, then we define the set of **ultraprimes** of L as

$$\mathsf{M}_L = \mathcal{U}(\mathcal{M}_L).$$

By definition, an ultraprime $\mathsf{v} \in \mathsf{M}_L$ is an equivalence class of sequences $(v_n)_{n\in\mathbb{N}}$, where each v_n is a place of L. The map $v \mapsto (v, v, \dots)$ induces an embedding $M_L \hookrightarrow \mathsf{M}_L$, written $v \mapsto \underline{v}$, and we say an ultraprime is constant if it lies in the image of this embedding.

Proposition 2.2.2. *Let* v *be a non-constant ultraprime. Then there exists a unique Frobenius element* $\mathrm{Frob}_{\mathsf{v}} \in \mathrm{Gal}(\overline{L}/L)$ *with the following property: for each finite Galois extension* $L \subset E \subset \overline{L}$*, and for any representative* (v_n) *of* v*, there are* $\mathfrak{F}$*-many* n *such that* v_n *is unramified in* E/L *and the Frobenius of* v_n *in* $\mathrm{Gal}(E/L)$ *is the natural image of* $\mathrm{Frob}_{\mathsf{v}}$*.*

Proof. Let $(v_n)_{n\in\mathbb{N}}$ be a representative of v, and fix for the time being a finite Galois extension E/L inside $\overline{\mathbb{Q}}$. If v_n is archimedian or ramified in E for $\mathfrak{F}$-many n, then Proposition 2.1.2 implies that v is constant. Thus for $\mathfrak{F}$-many n, the Frobenius of v_n is a well-defined element of $\mathrm{Gal}(E/L)$ (determined exactly, and not only up to conjugacy, by the fixed embeddings $E \hookrightarrow \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_v$). By Proposition 2.1.2, the map $n \mapsto \mathrm{Frob}_{v_n} \in \mathrm{Gal}(E/L)$ sends $\mathfrak{F}$-many n to a (unique) common value $g_E \in \mathrm{Gal}(E/L)$. Note that g_E does not depend on the representative (v_n). By the uniqueness of g_E, the association $E \mapsto g_E$ is compatible with restriction to subextensions $E' \subset E$, hence defines an element of the absolute Galois group. □

2.2.3. Let v be an ultraprime. We define its abstract Galois group G_{v} as $\mathrm{Gal}(\overline{L_v}/L_v)$ if $\mathsf{v} = \underline{v}$ is constant, and as the semi direct product

$$\widehat{\mathbb{Z}}(1) \rtimes \langle \mathrm{Frob}_{\mathsf{v}} \rangle$$

otherwise. Here, $\langle \mathrm{Frob}_{\mathsf{v}} \rangle$ denotes the free profinite group on one generator, acting on $\widehat{\mathbb{Z}}(1)$ by $\mathrm{Frob}_{\mathsf{v}}$. We define the inertia group $\mathsf{I}_{\mathsf{v}} \subset \mathsf{G}_{\mathsf{v}}$ of v to be the usual inertia group if v is constant, and the normal subgroup $\widehat{\mathbb{Z}}(1) \subset \mathsf{G}_{\mathsf{v}}$ otherwise.

2.3. **Local cohomology.**

2.3.1. For any (continuous) Galois module A defined over L and unramified at almost all places, and for any $\mathsf{v} \in \mathsf{M}_L$, there is a natural action of G_{v} on A (factoring through the quotient $\mathsf{G}_{\mathsf{v}} \to \langle \mathrm{Frob}_{\mathsf{v}} \rangle$ if v is nonconstant). We define local cohomology groups by:

$$\begin{aligned} \mathsf{H}^i(L_{\mathsf{v}}, A) &\coloneqq H^i_{cts}(\mathsf{G}_{\mathsf{v}}, A), \\ \mathsf{H}^i(L^{nr}_{\mathsf{v}}, A) &\coloneqq H^i_{cts}(\mathsf{I}_{\mathsf{v}}, A), \quad i \geq 0. \end{aligned}$$

Note that the local cohomology commutes with direct limits and countable inverse limits of finite, discrete Galois modules; the former is essentially by definition of continuous cohomology and the latter is by [44, Corollary 2.6.7] applied to $\mathsf{G}_{\mathsf{v}}, \mathsf{I}_{\mathsf{v}}$.

Proposition 2.3.2. *Let $\mathsf{v} \in \mathsf{M}_L$ be an ultraprime represented by a sequence $(v_n)_{n\in\mathbb{N}}$. If A is a finite, discrete Galois module over L, then for $\mathfrak{F}$-many n there are canonical isomorphisms (functorial in A, compatible with cup products, and compatible with the natural restriction maps):*

$$H^i(L_{v_n}, A) \simeq \mathsf{H}^i(L_\mathsf{v}, A),$$
$$H^i(L_{v_n}^{nr}, A) \simeq \mathsf{H}^i(L_\mathsf{v}^{nr}, A), \quad i \geq 0.$$

Proof. If v is the constant ultraprime $\underline{v}$, then $v_n = v$ for $\mathfrak{F}$-many n, and the desired isomorphisms are given by the identity maps; so suppose v is nonconstant. For $\mathfrak{F}$-many n, the action of the decomposition group G_{v_n} at v_n on A is unramified and the Frobenius of v_n acts by Frob_v. Let ℓ_n be the prime of $\mathbb{Q}$ lying under v_n; since $L/\mathbb{Q}$ is a finite extension and A is a finite Galois module, for $\mathfrak{F}$-many n we have $\ell_n \nmid |A|$. Restricting to these n, the inflation map induces isomorphisms:

$$H^i(G_{v_n}^t, A) \simeq H^i(L_{v_n}, A), \quad H^i(I_{v_n}^t, A) \simeq H^i(L_{v_n}^{nr}, A),$$

where $G_{v_n}^t$ and $I_{v_n}^t$ denote the tame quotients. The tame Galois group $G_{v_n}^t$ is identified with the semi direct product:

$$I_{v_n}^t \rtimes \langle \mathrm{Frob}_{v_n} \rangle \simeq \widehat{\mathbb{Z}}^{(\ell_n)}(1) \rtimes \langle \mathrm{Frob}_{v_n} \rangle.$$

Since Frob_{v_n} and Frob_v may act differently on the Tate twist, $G_{v_n}^t$ and G_v cannot be compared directly; however, for $\mathfrak{F}$-many n, Frob_{v_n} and Frob_v act the same way on any finite quotient of $\widehat{\mathbb{Z}}(1)$. The cohomology groups are therefore canonically isomorphic for $\mathfrak{F}$-many n by the following easy lemma in group cohomology (applied to both $G_{v_n}^t$ and G_v). □

Lemma 2.3.3. *Let $G = I \rtimes \langle F \rangle$ be a group, where I is abelian and profinite of cohomological dimension at most one, and $\langle F \rangle$ denotes the free profinite group on one generator, acting on I by an automorphism. If A is a finite $\mathbb{Z}[F]$-module, viewed as a G-module via $G \twoheadrightarrow \langle F \rangle$, then there are canonical isomorphisms:*

$$\begin{aligned} H^i_{cts}(I, A) &= H^i(I/|A|, A), && i = 0, 1, \\ H^i_{cts}(G, A) &= H^i_{cts}(I/|A| \rtimes \langle F \rangle, A), && i \neq 2, \\ H^2_{cts}(G, A) &= H^1_{cts}(\langle F \rangle, \mathrm{Hom}(I/|A|, A)). \end{aligned}$$

Proof. The isomorphisms $H^i_{cts}(I, A) \simeq H^i_{cts}(I/|A|, G)$ are immediate for $i = 0, 1$. For the cohomology groups $H^i(G, A)$, note first that G has cohomological dimension at most 2. The Hochschild-Serre spectral sequence gives a canonical isomorphism

$$H^2_{cts}(G, A) = H^1_{cts}(\langle F \rangle, H^1_{cts}(I, A)) = H^1_{cts}(\langle F \rangle, \mathrm{Hom}(I/|A|, A)),$$

since both $\langle F \rangle$ and I have cohomological dimension at most 1.

It remains to show that the inflation map induces an isomorphism

$$H^1_{cts}(I/|A| \rtimes \langle F \rangle, A) \xrightarrow{\sim} H^1_{cts}(G, A).$$

Equivalently, if $H \subset G$ is the subgroup $|A|I \subset I$, we wish to show that the restriction map $H^1_{cts}(G, A) \to H^1_{cts}(H, A)$ is trivial. Indeed, the restriction map factors through $H^1_{cts}(I, A) \to H^1_{cts}(H, A)$, which is the zero map since I acts trivially on A.

□

2.4. **Patched cohomology.**

2.4.1. Let $\mathsf{S} \subset \mathsf{M}_L$ be a finite set of ultraprimes $\{\mathsf{s}_1, \mathsf{s}_2, \ldots, \mathsf{s}_r\}$. A **representative** of S is a sequence of sets $S^n \subset M_L$ such that $S^n = \{s_1^n, \cdots, s_r^n\}$ for some sequences $(s_i^n)_{n\in\mathbb{N}}$ representing s_i. If A is a $\mathrm{Gal}(\overline{L}/L)$ module, we say A is unramified outside $\mathsf{S} \subset \mathsf{M}_L$ if it is unramified outside $\mathsf{S} \cap M_L$.

Definition 2.4.2. Let A be a topological $\mathrm{Gal}(\overline{L}/L)$-module unramified outside a finite set $\mathsf{S} \subset \mathsf{M}_L$, represented by a sequence $S^n \subset M_L$. If A is profinite, then we define the ith unramified-outside-S patched cohomology, for all $i \geq 0$, by:

$$\mathsf{H}^i(L^\mathsf{S}/S, A) = \varprojlim_{A \twoheadrightarrow A'} \mathcal{U}\left(\left\{H^i(L^{S^n}/L, A')\right\}\right)_{n\in\mathbb{N}},$$

where the inverse limit runs over continuous finite quotients of A. If A is ind-finite, then its unramified-outside-S patched cohomology is defined as:

$$\mathsf{H}^i(L^{\mathsf{S}}/L, A) = \varinjlim_{A' \subset A} \mathcal{U}\left(\left\{H^i(L^{S^n}/L, A')\right\}_{n\in\mathbb{N}}\right),$$

where the direct limit runs over finite submodules. If A is either profinite or ind-finite, then the totally patched cohomology is defined as

$$\mathsf{H}^i(L, A) = \varinjlim_{\mathsf{S}\subset \mathsf{M}_L} \mathsf{H}^i(L^{\mathsf{S}}/L, A),$$

where the direct limit is over finite subsets such that A is unramified outside S and the transition maps are induced by the functoriality of the ultraproduct.

Remark 2.4.3. (1) To see that these cohomology groups are well-defined, first note that they are independent of the choice of S^n since any two representatives of a finite set $\mathsf{S} \subset \mathsf{M}_L$ agree for $\mathfrak{F}$-many n. Moreover, if A is both profinite and ind-finite, then it is finite, and it is clear that either definition gives the same cohomology groups.

(2) There is a canonical isomorphism $\mathsf{H}^0(L^{\mathsf{S}}/L, A) = H^0(L, A)$ for all profinite or ind-finite A and all finite $\mathsf{S} \subset \mathsf{M}_L$ such that A is unramified outside S.

(3) The assignment $A \mapsto \mathsf{H}^i(L^{\mathsf{S}}/L, A)$ is functorial in G_L-modules A unramified outside S, hence $A \mapsto \mathsf{H}^i(L, A)$ is functorial in A. If A is an R-Galois module unramified outside S, then each patched cohomology group $\mathsf{H}^i(L^{\mathsf{S}}/L, A)$, $\mathsf{H}^i(L, A)$ has a natural R-module structure.

(4) In practice, we will want our profinite Galois modules to be **countably profinite**, i.e. to have a presentation as a countable inverse limit of finite, discrete topological Galois modules. The significance of this technical hypothesis is that countable inverse limits of finite abelian groups are exact. For example, see [44, Corollary 2.7.6].

(5) Suppose A is ind-finite or countably profinite and unramified outside S. If every ultraprime in S is constant, and $S \subset M_L$ is the corresponding finite set of places, then $\mathsf{H}^i(L^{\mathsf{S}}/L, A)$ is canonically isomorphic to $H^i(L^S/L, A)$.

(6) Suppose A is ind-finite or countably profinite. For each ultraprime v, there are natural localization maps

$$\mathrm{Res}_{\mathsf{v}} : \mathsf{H}^i(L, A) \to \mathsf{H}^i(L_{\mathsf{v}}, A)$$

deduced from Proposition 2.3.2 (and from [44, Corollary 2.7.6] applied to G_{v} in the profinite case).

(7) If the Galois action on A is the restriction of an action of G_K, where L/K is a Galois extension, then $\mathrm{Gal}(L/K)$ acts naturally on $\mathsf{H}^i(L, A)$, again by functoriality of ultraproducts; this is compatible with the localization maps in the obvious way.

Lemma 2.4.4. *Let A be a finite Galois module over L, and let $S \subset M_L$ be a finite set of primes outside which A is unramified. Then the cardinality of $H^i(L^S/L, A)$ is uniformly bounded, with a bound depending only on A, L, and $|S|$. In particular, if $\mathsf{S} \subset \mathsf{M}_L$ is finite, then the patched cohomology groups $\mathsf{H}^i(L^{\mathsf{S}}/L, A)$ are finite for each finite Galois module A and each $i \geq 0$.*

Proof. The second claim follows from the first by Proposition 2.1.4(ii). The first claim is immediate for $i = 0$, and the case $i \geq 3$ is handled by [41, Chapter 1, Theorem 4.10(c)]. For $i = 1$ and 2, let S_0 be the set of primes at which A is ramified, or with residue characteristic dividing $|A|$; without loss of generality, we may assume $S_0 \subset S$. Now the map

$$H^i(L^S/L, A) \to \prod_{v\in S} H^i(L_v, A)$$

has kernel contained in $\text{Ш}^i_{S_0}(A)$, which is finite by part (a) of loc. cit. Since $|H^i(L_v, A)| \leq |A|^2$ for $v \notin S_0$, the lemma follows. ☐

Proposition 2.4.5. *If A is either countably profinite or ind-finite, then, for all i, the natural map induces an isomorphism*

$$\mathsf{H}^i(L^{\mathsf{S}}/L, A) \simeq \ker\left(\mathsf{H}^i(L, A) \to \prod_{\mathsf{v}\in \mathsf{M}_L - \mathsf{S}} \mathsf{H}^i(L^{nr}_{\mathsf{v}}, A)\right).$$

Proof. It suffices to show that, for all finite sets $\mathsf{T} \subset \mathsf{M}_L - \mathsf{S}$,

$$\mathsf{H}^i(L^{\mathsf{S}}/L, A) \simeq \ker\left(\mathsf{H}^i(L^{\mathsf{S}\cup\mathsf{T}}, A) \to \prod_{\mathsf{t}\in\mathsf{T}} \mathsf{H}^i(L_{\mathsf{t}}^{nr}, A)\right).$$

This holds when A is finite by Lemma 2.4.4 and Proposition 2.1.5; the general case follows by taking limits. □

Lemma 2.4.6. *Let*

$$0 \to A \to B \to C \to 0$$

be an exact sequence of either countably profinite or ind-finite Galois modules unramified outside S. *Then there is an induced long exact sequence beginning:*

$$\begin{aligned} 0 \to \mathsf{H}^0(L^{\mathsf{S}}/L, A) \to \mathsf{H}^0(L^{\mathsf{S}}/L, B) \to \mathsf{H}^0(L^{\mathsf{S}}/L, C) \to \\ \to \mathsf{H}^1(L^{\mathsf{S}}/L, A) \to \mathsf{H}^1(L^{\mathsf{S}}/L, B) \to \cdots \end{aligned}$$

Proof. If A, B, and C are all finite, then this follows from Proposition 2.1.5 and Lemma 2.4.4.

Now suppose that A, B, and C are all profinite. Let I, J, and K be directed sets indexing the finite quotients $A \twoheadrightarrow A_i$, $B \twoheadrightarrow B_j$, and $C \twoheadrightarrow C_k$, respectively. We define morphisms of directed sets $t : J \to I$ and $s : J \to K$ by

$$A_{t(j)} = \mathrm{im}(A \to B_j), \quad C_{s(j)} = B_j/A_{t(j)}.$$

Because the subgroup and quotient topologies on A and C agree with the profinite topologies, the images of t and s are cofinal in I and K, respectively. We therefore have:

$$\mathsf{H}^*(L^{\mathsf{S}}/L, A) = \varprojlim_{j\in J} \mathsf{H}^*(L^{\mathsf{S}}/L, A_{t(j)}), \quad \mathsf{H}^*(L^{\mathsf{S}}/L, C) = \varprojlim_{j\in J} \mathsf{H}^*(L^{\mathsf{S}}/L, C_{s(j)}).$$

For each j, we have a long exact sequence associated to the short exact sequence of finite Galois modules

$$0 \to A_{t(j)} \to B_j \to C_{s(j)} \to 0;$$

by Lemma 2.4.4, each term in the long exact sequence is finite. Since countable inverse limits of finite abelian groups are exact, taking limits completes the proof. The ind-finite case is completely analogous. □

2.5. **Selmer structures and patched Selmer groups.**

Definition 2.5.1. Let A be a countably profinite or ind-finite $\mathbb{Z}_p[G_L]$-module. A **generalized Selmer structure** $(\mathcal{F}, \mathsf{S})$ for A consists of:

- a finite set $\mathsf{S} \subset \mathsf{M}_L$ containing all Archimedean places, all places over p, and all ramified places for A;
- for each $\mathsf{v} \in \mathsf{M}_L$, a closed $\mathbb{Z}_p$-submodule (the **local condition**)

  $$\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{v}}, A) \subset \mathsf{H}^1(L_{\mathsf{v}}, A)$$

  such that

  $$\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{v}}, A) = \mathsf{H}^1_{\mathrm{unr}}(L_{\mathsf{v}}, A) := \ker\left(\mathsf{H}^1(L_{\mathsf{v}}, A) \to \mathsf{H}^1(L_{\mathsf{v}}^{nr}, A)\right)$$

  for all $\mathsf{v} \notin \mathsf{S}$.

If A is an R-module for some ring R and G_L acts on A by R-module automorphisms, a Selmer structure for A **over** R is a Selmer structure such that every local condition is an R-submodule.

2.5.2. If $B \subset A$ is any closed Galois-stable submodule, then a Selmer structure $(\mathcal{F}, \mathsf{S})$ for A induces Selmer structures on B and A/B defined in the usual way:

$$\begin{aligned} \mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{v}}, B) &= \ker\left(\mathsf{H}^1(L_{\mathsf{v}}, B) \to \frac{\mathsf{H}^1(L_{\mathsf{v}}, A)}{\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{v}}, A)}\right), \\ \mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{v}}, A/B) &= \mathrm{im}\left(\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{v}}, A) \to \mathsf{H}^1(L_{\mathsf{v}}, A/B)\right). \end{aligned}$$

Note that these Selmer structures are well-defined because, if A is unramified at v, then the unramified local condition for A induces the unramified local condition for B and A/B.

2.5.3. To a generalized Selmer structure we associate the **patched Selmer group**, defined by the exact sequence:

$$0 \to \mathrm{Sel}_{\mathcal{F}}(A) \to \mathsf{H}^1(L^{\mathsf{S}}/L, A) \to \prod_{\mathsf{s}\in\mathsf{S}} \frac{\mathsf{H}^1(L_{\mathsf{s}}, A)}{\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A)}, \tag{10}$$

or equivalently (by Proposition 2.4.5):

$$0 \to \mathrm{Sel}_{\mathcal{F}}(A) \to \mathsf{H}^1(L, A) \to \prod_{\mathsf{s}\in\mathsf{S}} \frac{\mathsf{H}^1(L_{\mathsf{s}}, A)}{\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A)} \times \prod_{\mathsf{s}\notin\mathsf{S}} \mathsf{H}^1(L_{\mathsf{s}}^{nr}, A). \tag{11}$$

(Note that the Selmer group attached to a Selmer structure does not depend on the choice of set S but only on the local conditions; we will therefore sometimes omit S from the notation when there is no risk of confusion.)

2.5.4. If $B \subset A$ is Galois-stable, and $B, A/B$ are equipped with the induced Selmer structures, then by definition there are natural maps:

$$\mathrm{Sel}_{\mathcal{F}}(B) \to \mathrm{Sel}_{\mathcal{F}}(A) \to \mathrm{Sel}_{\mathcal{F}}(A/B).$$

Proposition 2.5.5. *Let $(\mathcal{F}, \mathsf{S})$ be a generalized Selmer structure for A. If A is countably profinite and each continuous finite quotient $A \twoheadrightarrow A'$ is equipped with the Selmer structure induced by $\mathcal{F}$, then:*

$$\varprojlim \mathrm{Sel}_{\mathcal{F}}(A') \simeq \mathrm{Sel}_{\mathcal{F}}(A).$$

If instead A is ind-finite and each finite submodule $A' \subset A$ is given its induced Selmer structure, then:

$$\varinjlim \mathrm{Sel}_{\mathcal{F}}(A') \simeq \mathrm{Sel}_{\mathcal{F}}(A).$$

Proof. We show the countably profinite case; the ind-finite case is similar. By definition, $\mathrm{Sel}_{\mathcal{F}}(A)$ is the kernel of

$$\varprojlim \mathsf{H}^1(L^{\mathsf{S}}/L, A') \to \prod_{\mathsf{s}\in\mathsf{S}} \frac{\mathsf{H}^1(L_{\mathsf{s}}, A)}{\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A)},$$

whereas

$$\begin{aligned} \varprojlim \mathrm{Sel}_{\mathcal{F}}(A') &= \varprojlim \ker \left(\mathsf{H}^1(L^{\mathsf{S}}/L, A') \to \prod_{\mathsf{s}\in\mathsf{S}} \frac{\mathsf{H}^1(L_{\mathsf{s}}, A')}{\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A')} \right) \\ &= \ker \left(\mathsf{H}^1(L^{\mathsf{S}}/L, A) \to \varprojlim \frac{\mathsf{H}^1(L_{\mathsf{s}}, A')}{\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A')} \right). \end{aligned}$$

Since $\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A)$ is a closed subgroup of $\mathsf{H}^1(L_{\mathsf{s}}, A)$, it is isomorphic to the inverse limit:

$$\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A) = \varprojlim \operatorname{im}\left(\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A) \to \mathsf{H}^1(L_{\mathsf{s}}, A')\right) = \varprojlim \mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A').$$

This implies the result. $\square$

2.5.6. Given two Selmer structures $(\mathcal{F}, \mathsf{S})$ and $(\mathcal{G}, \mathsf{T})$ for A, we may define Selmer structures $(\mathcal{F}+\mathcal{G}, \mathsf{S}\cup\mathsf{T})$ and $(\mathcal{F}\cap\mathcal{G}, \mathsf{S}\cup\mathsf{T})$ by the local conditions:

$$\begin{aligned} \mathsf{H}^1_{\mathcal{F}+\mathcal{G}}(L_{\mathsf{v}}, A) &= \mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{v}}, A) + \mathsf{H}^1_{\mathcal{G}}(L_{\mathsf{v}}, A), \\ \mathsf{H}^1_{\mathcal{F}\cap\mathcal{G}}(L_{\mathsf{v}}, A) &= \mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{v}}, A) \cap \mathsf{H}^1_{\mathcal{G}}(L_{\mathsf{v}}, A). \end{aligned}$$

2.6. **Dual Selmer groups.**

2.6.1. Fix an ultraprime $\mathsf{v} \in \mathsf{M}_L$. If A is a countably profinite $\mathbb{Z}_p$-Galois module and A^* denotes the Cartier dual, then the cup product induces pairings:

$$\langle \cdot, \cdot \rangle_{\mathsf{v}} : \mathsf{H}^i(L_{\mathsf{v}}, A) \times \mathsf{H}^{2-i}(L_{\mathsf{v}}, A^*) \to \mathbb{Q}_p/\mathbb{Z}_p, \quad i = 0, 1, 2. \tag{12}$$

Proposition 2.6.2. *The pairing (12) is perfect if* v *is non-Archimedean. Moreover, the induced pairing*

$$\mathsf{H}^1(L^{\mathsf{S}}/L, A) \times \mathsf{H}^1(L^{\mathsf{S}}/L, A^*) \to \prod_{\mathsf{s} \in \mathsf{S}} \mathsf{H}^1(L_{\mathsf{s}}, A) \times \mathsf{H}^1(L_{\mathsf{s}}, A^*) \xrightarrow{\sum \langle \cdot, \cdot \rangle_{\mathsf{s}}} \mathbb{Q}_p/\mathbb{Z}_p$$

is identically zero.

Proof. For the perfectness of (12), the usual proof of Poitou-Tate duality applies equally well to G_{v}; alternatively, one may take limits using Proposition 2.3.2. The second claim is clear when A is finite by functoriality of the ultraproduct, and the general case follows by taking limits. □

2.6.3. Suppose that A is either countably profinite or countably ind-finite, i.e. the Pontryagin dual of a countably profinite Galois module. If $(\mathcal{F}, \mathsf{S})$ is any Selmer structure for A, then we define the dual Selmer structure $(\mathcal{F}^*, \mathsf{S})$ for A^* by:

$$\mathsf{H}^1_{\mathcal{F}^*}(L_{\mathsf{s}}, A^*) = \mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A)^{\perp}.$$

Here $\perp$ denotes the orthogonal complement under either the pairing of (2.6.1), or the usual modified Tate pairing of [15, Theorem 2.17] at Archimedian places. We observe that the dual Selmer structure to $(\mathcal{F}^*, \mathsf{S})$ is again $(\mathcal{F}, \mathsf{S})$. When A is finite, the dual Selmer groups are related by the Greenberg-Wiles formula:

Proposition 2.6.4. *Let* $(\mathcal{F}, \mathsf{S})$ *be a Selmer structure for a finite* $\mathbb{Z}_p[G_L]$*-module* A*. We have:*

$$\frac{\# \operatorname{Sel}_{\mathcal{F}}(A)}{\# \operatorname{Sel}_{\mathcal{F}^*}(A^*)} = \frac{\#\mathsf{H}^0(L^{\mathsf{S}}/L, A)}{\#\mathsf{H}^0(L^{\mathsf{S}}/L, A^*)} \prod_{\mathsf{s} \in \mathsf{S}} \frac{\#\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A)}{\#\mathsf{H}^0(L_{\mathsf{s}}, A)}.$$

Proof. This follows from [15, Theorem 2.19] by the exactness of ultraproducts and Proposition 2.1.4(ii). □

2.7. Selmer groups over discrete valuation rings.

2.7.1. Let R be a discrete valuation ring with uniformizer π which is a finite, flat extension of $\mathbb{Z}_p$, and suppose that $A = T$ is a free R-module of finite rank, with G_L action through R-module automorphisms. In particular, T is countably profinite. Suppose $\mathsf{S} \subset \mathsf{M}_L$ is a finite set containing all Archimedian places and all places over p, such that T is unramified outside S. If $T^\dagger = \operatorname{Hom}_{\mathbb{Z}_p}(T, \mathbb{Z}_p(1))$ is the dual, then the cup product induces a local Tate pairing

$$\langle \cdot, \cdot \rangle_{\mathsf{v}} : \mathsf{H}^1(L_{\mathsf{v}}, T) \times \mathsf{H}^1(L_{\mathsf{v}}, T^\dagger) \to \mathbb{Z}_p. \tag{13}$$

Proposition 2.7.2. *The kernels on the left and right of (13) are the* $\mathbb{Z}_p$*-torsion submodules; moreover, the induced pairing*

$$\mathsf{H}^1(L^{\mathsf{S}}/L, T) \times \mathsf{H}^1(L^{\mathsf{S}}/L, T^\dagger) \to \prod_{\mathsf{v} \in \mathsf{S}} \mathsf{H}^1(L_{\mathsf{v}}, T) \times \mathsf{H}^1(L_{\mathsf{v}}, T^\dagger) \xrightarrow{\sum \langle \cdot, \cdot \rangle_{\mathsf{v}}} \mathbb{Z}_p$$

is identically zero.

Proof. This follows from Proposition 2.6.2. □

Given a Selmer structure $(\mathcal{F}, \mathsf{S})$ for T over R, taking the orthogonal complement of each local condition under (13) yields a Selmer structure $(\mathcal{F}^\dagger, \mathsf{S})$ for $T^\dagger$.

Definition 2.7.3. A Selmer structure $(\mathcal{F}, \mathsf{S})$ for T is said to be **saturated** if the quotients

$$\frac{\mathsf{H}^1(L_{\mathsf{v}}, T)}{\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{v}}, T)}$$

are torsion-free for all v. (The condition is automatic for $\mathsf{v} \notin \mathsf{S}$.)

Proposition 2.7.4. *Let* $(\mathcal{F}, \mathsf{S})$ *be a saturated Selmer structure. Then, for all* j *and all* $\mathsf{v} \in \mathsf{M}_L$*,*

$$\mathsf{H}^1_{\mathcal{F}^*}(L_{\mathsf{v}}, T^*[\pi^j]) = \mathsf{H}^1_{\mathcal{F}^\dagger}(L_{\mathsf{v}}, T^\dagger/\pi^j)$$

under the natural identification $T^*[\pi^j] \simeq T^\dagger/\pi^j$*, and in particular*

$$\operatorname{Sel}_{\mathcal{F}^*}(T^*[\pi^j]) = \operatorname{Sel}_{\mathcal{F}^\dagger}(T^\dagger/\pi^j).$$

Hence, saturation ensures that all the induced local conditions on subquotients of $T^\dagger$ are Cartesian in the sense of [28, 38].

Proof. Although this fact is presumably standard, we give a proof for lack of a reference. For ease of notation, we abbreviate $\mathsf{H}^i(T^\dagger) = \mathsf{H}^i(L_\mathsf{v}, T^\dagger)$, etc. We have an identification $T^\dagger \otimes_{\mathbb{Z}_p} (\mathbb{Q}_p/\mathbb{Z}_p) \simeq T^*$ and an embedding $T^\dagger/\pi^j \hookrightarrow T^*$; let $\mathsf{H}^1_{\mathcal{F}^*}(T^\dagger/\pi^j)$ be the induced local condition from this embedding. Consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{H}^0(T^*)_{/\,\mathrm{div}} \arrow[r] \arrow[d, "\alpha"] & \mathsf{H}^1_{\mathcal{F}^\dagger}(T^\dagger) \arrow[r] \arrow[d, "\beta"] & \mathrm{Hom}(\mathsf{H}^1(T), \mathbb{Z}_p) \arrow[r] \arrow[d, "\gamma"] & \mathrm{Hom}(\mathsf{H}^1_{\mathcal{F}}(T), \mathbb{Z}_p) \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & \mathsf{H}^0(T^*)/\pi^j \arrow[r] & \mathsf{H}^1_{\mathcal{F}^*}(T^\dagger/\pi^j) \arrow[r] & \mathrm{Hom}(\mathsf{H}^1(T)/\pi^j, \mathbb{Q}_p/\mathbb{Z}_p) \arrow[r] & \mathrm{Hom}(\mathsf{H}^1_{\mathcal{F}}(T)/\pi^j, \mathbb{Q}_p/\mathbb{Z}_p) \arrow[r] & 0.
\end{tikzcd}$$

Here, the first horizontal map on each row is the Kummer map, and the subscript $/\,\mathrm{div}$ refers to the quotient by the maximal divisible submodule. By the hypothesis on $\mathsf{H}^1_{\mathcal{F}}(T)$, the maps $\operatorname{coker}\gamma \to \operatorname{coker}\delta$ and $\ker\gamma \to \ker\delta$ are injective and surjective, respectively. Also, α is clearly surjective. Breaking the diagram into two and applying the snake lemma, it follows that β is surjective. □

Proposition 2.7.5. *Let $(\mathcal{F}, \mathsf{S})$ be a Selmer structure for T over R. Then:*

$$\begin{aligned}\mathrm{rk}_R\,\mathrm{Sel}_{\mathcal{F}}(T) - \mathrm{rk}_R\,\mathrm{Sel}_{\mathcal{F}^\dagger}(T^\dagger) = \mathrm{rk}_R\, H^0(L, T) - \mathrm{rk}_R\, H^0(L, T^\dagger) + \\ \sum_{\mathsf{s}\in\mathsf{S}} \left(\mathrm{rk}_R\,\mathsf{H}^1_{\mathcal{F}}(L_\mathsf{s}, T) - \mathrm{rk}_R\,\mathsf{H}^0(L_\mathsf{s}, T)\right).\end{aligned}$$

Proof. We first reduce to the case that $(\mathcal{F}, \mathsf{S})$ is saturated. Indeed, if we define

$$\mathsf{H}^1_{\widetilde{\mathcal{F}}}(L_\mathsf{v}, T) := \left(\mathsf{H}^1(L_\mathsf{v}, T) \to \frac{\mathsf{H}^1(L_\mathsf{v}, T)}{\mathsf{H}^1_{\mathcal{F}}(L_\mathsf{v}, T)} \otimes \mathbb{Q}_p\right)$$

for all v, then $(\widetilde{\mathcal{F}}, \mathsf{S})$ is a saturated Selmer structure for T (the local conditions outside S do not change). Now, there is an exact sequence

$$0 \to \mathrm{Sel}_{\mathcal{F}}(T) \to \mathrm{Sel}_{\widetilde{\mathcal{F}}}(T) \to \prod_{\mathsf{v}\in\mathsf{S}} \frac{\mathsf{H}^1_{\widetilde{\mathcal{F}}}(L_\mathsf{v}, T)}{\mathsf{H}^1_{\mathcal{F}}(L_\mathsf{v}, T)}.$$

Since the final term is a finitely generated torsion R-module, we have

$$\mathrm{rk}_R\,\mathrm{Sel}_{\mathcal{F}}(T) = \mathrm{rk}_R\,\mathrm{Sel}_{\widetilde{\mathcal{F}}}(T).$$

So, replacing $\mathcal{F}$ with $\widetilde{\mathcal{F}}$ if necessary, we may assume $\mathcal{F}$ is saturated. By Propositions 2.6.4 and 2.7.4, we have for each j:

$$\begin{aligned}\lg \mathrm{Sel}_{\mathcal{F}}(T/\pi^j) - \lg \mathrm{Sel}_{\mathcal{F}^\dagger}(T^\dagger/\pi^j) = \lg \mathsf{H}^0(L, T/\pi^j) - \lg \mathsf{H}^0(L, T^\dagger/\pi^j) \\ + \sum_{\mathsf{v}\in\mathsf{S}} \left(\lg \mathsf{H}^1_{\mathcal{F}}(L_\mathsf{v}, T/\pi^j) - \lg \mathsf{H}^0(L_\mathsf{v}, T/\pi^j)\right).\end{aligned}$$

Since $\mathrm{Sel}_{\mathcal{F}}(T)$ is a finitely generated R-module, it follows from [38, Lemma 3.7.1] that

$$\lg \mathrm{Sel}_{\mathcal{F}}(T/\pi^j) = (\mathrm{rk}_R\,\mathrm{Sel}_{\mathcal{F}}(T)) \cdot \lg R/\pi^j + O(1)$$

as j varies, and likewise for $\mathrm{Sel}_{\mathcal{F}^\dagger}(T^\dagger)$ and each term on the right-hand side; the proposition follows. □

3. Bipartite Euler systems

3.1. **Admissible primes.**

3.1.1. Let f be a cuspidal newform of weight two, trivial character, and level N, and let $\wp \subset \mathcal{O}_f$ be a prime ideal of the ring of integers of its field of coefficients. We assume the rational prime p lying under $\wp$ is odd, and write $\mathcal{O}$ for the completion of $\mathcal{O}_f$ at $\wp$. The $\wp$-adic Galois representation V_f associated to f is equipped with a non-degenerate, symplectic, Galois-equivariant pairing:

$$V_f \times V_f \to \mathbb{Q}_p(1). \tag{14}$$

Fix a Galois-stable $\mathcal{O}$-lattice $T_f \subset V_f$, and let $\overline{T}_f$ be the residual representation $T_f/\wp$; we also write W_f for $T_f \otimes \mathbb{Q}_p/\mathbb{Z}_p$. Also let $K/\mathbb{Q}$ be an imaginary quadratic field. We assume throughout this section that $\overline{T}_f$ is absolutely irreducible as an $\mathcal{O}[G_K]$-module. Since the dual lattice to T_f is also $\mathcal{O}[G_K]$-stable, after rescaling we may assume that (15) restricts to a $\mathbb{Z}_p(1)$-valued pairing

$$T_f \times T_f \to \mathbb{Z}_p(1), \tag{15}$$

identifying T_f with $\mathrm{Hom}_{\mathbb{Z}_p}(T_f, \mathbb{Z}_p(1))$. We will sometimes use the condition:

(sclr) The image of the G_K action on $\overline{T}_f$ contains a nonzero scalar.

Definition 3.1.2. A nonconstant ultraprime $\mathsf{q} \in \mathsf{M}_{\mathbb{Q}}$ is said to be **admissible** with sign $\epsilon_{\mathsf{q}} = \pm 1$ for f if $\mathrm{Frob}_{\mathsf{q}}$ has nonzero image in $\mathrm{Gal}(K/\mathbb{Q})$, $\chi(\mathrm{Frob}_{\mathsf{q}}) \not\equiv 1 \pmod p$, and there is a rank-one direct summand $\mathrm{Fil}^+_{\mathsf{q},\epsilon_{\mathsf{q}}} T_f \subset T_f$ on which $\mathrm{Frob}_{\mathsf{q}}$ acts as $\chi(\mathrm{Frob}_{\mathsf{q}})\epsilon_{\mathsf{q}}$. (Equivalently, $\chi(\mathrm{Frob}_{\mathsf{q}}) \not\equiv 1 \pmod p$ and T_f admits a basis of eigenvectors for $\mathrm{Frob}_{\mathsf{q}}$ with eigenvalues ϵ_{q} and $\chi(\mathrm{Frob}_{\mathsf{q}})\epsilon_{\mathsf{q}}$.)

For example, if $\mathrm{Frob}_{\mathsf{q}} \in G_{\mathbb{Q}}$ is a complex conjugation, then q is admissible with either choice of ϵ_{q}. We abusively write q for the unique ultraprime in M_K lying over $\mathsf{q} \in \mathsf{M}_{\mathbb{Q}}$, whose Frobenius is $\mathrm{Frob}_{\mathsf{q}}^2$.

Definition 3.1.3. If q is admissible with sign ϵ_{q} for f, then we define the **ordinary** local condition (with sign ϵ_{q}) as:

$$\mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(K_{\mathsf{q}}, T_f) = \mathrm{im}\left(\mathsf{H}^1(K_{\mathsf{q}}, \mathrm{Fil}^+_{\mathsf{q},\epsilon_{\mathsf{q}}} T_f) \to \mathsf{H}^1(K_{\mathsf{q}}, T_f)\right).$$

The subscript ϵ_{q} will often be omitted (from this and future notation) when there is no risk of confusion.

Note that the ordinary local condition is well-defined because $\mathrm{Fil}^+_{\mathsf{q},\epsilon_{\mathsf{q}}}$ is uniquely determined by the pair $(\mathsf{q}, \epsilon_{\mathsf{q}})$.

Example 3.1.4. Suppose $\mathrm{Frob}_{\mathsf{q}} \in G_{\mathbb{Q}}$ is a complex conjugation and let e_1, e_2 be a basis of T_f such that $\mathrm{Frob}_{\mathsf{q}} e_1 = -e_1$ and $\mathrm{Frob}_{\mathsf{q}} e_2 = e_2$. Then as noted above, q is admissible with either choice of ϵ_{q}; if $\epsilon_{\mathsf{q}} = +1$, then $\mathrm{Fil}^+_{\mathsf{q},\epsilon_{\mathsf{q}}} T_f = \langle e_1 \rangle$, and if $\epsilon_{\mathsf{q}} = -1$, then $\mathrm{Fil}^+_{\mathsf{q},\epsilon_{\mathsf{q}}} T_f = \langle e_2 \rangle$. The local cohomology group $\mathsf{H}^1(K_{\mathsf{q}}, T_f)$ is free of rank 4, with a canonical decomposition into rank 2 subspaces:

$$\mathsf{H}^1(K_{\mathsf{q}}, T_f) = \mathsf{H}^1(K_{\mathsf{q}}, \langle e_1 \rangle) \oplus \mathsf{H}^1(K_{\mathsf{q}}, \langle e_2 \rangle).$$

The former is the ordinary local condition if $\epsilon_{\mathsf{q}} = +1$, and the latter if $\epsilon_{\mathsf{q}} = -1$. The unramified subspaces $\mathsf{H}^1_{\mathrm{unr}}(K_{\mathsf{q}}, \langle e_i \rangle)$ are free $\mathcal{O}$-modules of rank one.

Proposition 3.1.5. *Let q be admissible with sign ϵ_{q}. Then $\mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(K_{\mathsf{q}}, T_f)$ is its own exact annihilator under the local Tate pairing*

$$\mathsf{H}^1(K_{\mathsf{q}}, T_f) \times \mathsf{H}^1(K_{\mathsf{q}}, T_f) \to \mathcal{O}$$

induced by (13) and (15).

Proof. The Frobenius $\mathrm{Frob}_{\mathsf{q}} \in G_{\mathbb{Q}}$ acts on T_f with eigenvalues $\chi(\mathrm{Frob}_{\mathsf{q}})\epsilon_{\mathsf{q}}$ and ϵ_{q}. Let $e_1, e_2 \in T_f$ be generators of the corresponding eigenspaces, so $\mathrm{Fil}^+_{\mathsf{q},\epsilon_{\mathsf{q}}} T_f = \langle e_1 \rangle$. Then

$$\mathsf{H}^1(K_{\mathsf{q}}, T_f) = \mathsf{H}^1(K_{\mathsf{q}}, \langle e_1 \rangle) \oplus \mathsf{H}^1(K_{\mathsf{q}}, \langle e_2 \rangle), \tag{16}$$

and $\mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(K_{\mathsf{q}}, T_f) = \mathsf{H}^1(K_{\mathsf{q}}, \langle e_1 \rangle)$. Since the pairing (15) is symplectic, each of the direct summands in (16) is isotropic for the pairing on $\mathsf{H}^1(K_{\mathsf{q}}, T_f)$. Now note that:

$$\mathsf{H}^1(K_{\mathsf{q}}, T_f)_{\mathrm{tors}} \subset \mathsf{H}^1(K_{\mathsf{q}}, \langle e_1 \rangle). \tag{17}$$

Indeed, in the exact sequence

$$0 \to \mathsf{H}^1_{\mathrm{unr}}(K_{\mathsf{q}}, \langle e_2 \rangle) \to \mathsf{H}^1(K_{\mathsf{q}}, \langle e_2 \rangle) \to \mathsf{H}^1(I_{\mathsf{q}}, \langle e_2 \rangle),$$

the last term is automatically torsion-free, and the first is as well since $\mathrm{Frob}_{\mathsf{q}}^2$ acts trivially on e_2. We claim (17) implies the proposition. Suppose $y \in \mathsf{H}^1(K_{\mathsf{q}}, T_f)$ pairs trivially with $\mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(K_{\mathsf{q}}, T_f)$, and write

$$y = y_1 + y_2$$

in the decomposition (16). Since y_1 pairs trivially with $\mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(K_{\mathsf{q}}, T_f)$, y_2 does as well. But y_2 also pairs trivially with $\mathsf{H}^1(K_{\mathsf{q}}, \langle e_2\rangle)$, so y_2 lies in the kernel of the local Tate pairing, hence is a torsion class, hence trivial by (17). □

3.1.6. For any finite set $\mathsf{S} \subset \mathsf{M}_K$ such that T_f is unramified outside S, and any admissible $\mathsf{q} \notin \mathsf{S}$ with sign ϵ_{q}, define a localization map

$$\begin{aligned}\mathrm{loc}_{\mathsf{q},\epsilon_{\mathsf{q}}} : \mathsf{H}^1(K^{\mathsf{S}}/K, T_f) \to \mathsf{H}^1_{\mathrm{unr}}(K_{\mathsf{q}}, T_f) \to \\ \frac{\mathsf{H}^1_{\mathrm{unr}}(K_{\mathsf{q}}, T_f)}{\mathsf{H}^1_{\mathrm{unr}}(K_{\mathsf{q}}, T_f) \cap \mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(K_{\mathsf{q}}, T_f)} = \mathsf{H}^1_{\mathrm{unr}}(K_{\mathsf{q}}, T_f/\operatorname{Fil}^+_{\mathsf{q},\epsilon_{\mathsf{q}}} T_f) \approx \mathcal{O}.\end{aligned} \tag{18}$$

Define as well a residue map

$$\begin{aligned}\partial_{\mathsf{q},\epsilon_{\mathsf{q}}} : \mathsf{H}^1(K, T_f) \to \mathsf{H}^1(K_{\mathsf{q}}, T_f) \to \mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(K_{\mathsf{q}}, T_f) \to \\ \frac{\mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(K_{\mathsf{q}}, T_f)}{\mathsf{H}^1_{\mathrm{unr}}(K_{\mathsf{q}}, T_f) \cap \mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(K_{\mathsf{q}}, T_f)} = \mathsf{H}^1(I_{\mathsf{q}}, \operatorname{Fil}^+_{\mathsf{q},\epsilon_{\mathsf{q}}} T_f)^{\mathrm{Frob}_{\mathsf{q}}^2 = 1} \approx \mathcal{O},\end{aligned} \tag{19}$$

where the second map is given by the projection $T_f \twoheadrightarrow (\mathrm{Frob}_{\mathsf{q}} - \epsilon_{\mathsf{q}})T_f \simeq \operatorname{Fil}^+_{\mathsf{q},\epsilon_{\mathsf{q}}} T_f$. The maps $\mathrm{loc}_{\mathsf{q},\epsilon_q}$ and $\partial_{\mathsf{q},\epsilon_{\mathsf{q}}}$ may be extended in the obvious way to the patched cohomology for W_f and all $T_f/\wp^j$.

3.2. **Euler systems for anticyclotomic twists.**

3.2.1. Let R be a complete flat Noetherian local $\mathcal{O}$-algebra with finite residue field, equipped with an anticyclotomic character $\varphi : G_K \to R^\times$ which is trivial modulo the maximal ideal of R. We write T_φ for the anticyclotomic twist $T_f \otimes_{\mathcal{O}} R(\varphi)$, which is a countably profinite Galois module. If q is admissible with sign ϵ_{q}, then $\varphi(\mathrm{Frob}_{\mathsf{q}}^2) = 1$, so

$$\mathsf{H}^1(K_{\mathsf{q}}, T_\varphi) = \mathsf{H}^1(K_{\mathsf{q}}, T_f) \otimes_{\mathcal{O}} R.$$

We extend the ordinary local condition of the previous subsection by linearity to define the local condition $\mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(K_{\mathsf{q}}, T_\varphi)$, and likewise the maps $\mathrm{loc}_{\mathsf{q},\epsilon_{\mathsf{q}}}, \partial_{\mathsf{q},\epsilon_{\mathsf{q}}}$.

3.2.2. Suppose given a finite set $\mathsf{S} \subset \mathsf{M}_K$ and a generalized Selmer structure $(\mathcal{F}, \mathsf{S})$ for T_φ. Let $\mathsf{N} = \mathsf{N}_{\mathsf{S}}$ be the set of pairs $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\}$ where $\mathsf{Q} \subset \mathsf{M}_K - \mathsf{S}$ is a finite set of ultraprimes and $\epsilon_{\mathsf{Q}} : \mathsf{Q} \to \{\pm 1\}$ is a function such that q is admissible with sign $\epsilon_{\mathsf{Q}}(\mathsf{q})$ for all $\mathsf{q} \in \mathsf{Q}$. (We will drop the subscript S when it is clear from context, or when S contains only constant ultraprimes.) Given a pair $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$, define a generalized Selmer structure $(\mathcal{F}(\mathsf{Q}, \epsilon_{\mathsf{Q}}), \mathsf{S} \cup \mathsf{Q})$ for T_φ by the local conditions:

$$\mathsf{H}^1_{\mathcal{F}(\mathsf{Q},\epsilon_{\mathsf{Q}})}(K_{\mathsf{v}}, T_\varphi) = \begin{cases} \mathsf{H}^1_{\mathcal{F}}(K_{\mathsf{v}}, T_\varphi), & \mathsf{v} \notin \mathsf{Q} \\ \mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{Q}}(\mathsf{q})}(K_{\mathsf{q}}, T_\varphi), & \mathsf{v} = \mathsf{q} \in \mathsf{Q}. \end{cases} \tag{20}$$

For $\delta \in \mathbb{Z}/2\mathbb{Z}$, let $\mathsf{N}^\delta \subset \mathsf{N}$ be the collection of pairs $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$ such that $|\mathsf{Q}| \equiv \delta \pmod 2$. Also, given two pairs $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^\delta$ and $\{\mathsf{Q}', \epsilon_{\mathsf{Q}'}\} \in \mathsf{N}^{\delta'}$ such that $\mathsf{Q} \cap \mathsf{Q}' = \emptyset$, write

$$\{\mathsf{Q}\mathsf{Q}', \epsilon_{\mathsf{Q}\mathsf{Q}'}\} \in \mathsf{N}^{\delta+\delta'}$$

for the pair formed in the obvious way from $\mathsf{Q} \cup \mathsf{Q}'$ and the sign functions $\epsilon_{\mathsf{Q}}, \epsilon_{\mathsf{Q}'}$. The pair $\{\emptyset, \emptyset\} \in \mathsf{N}$ will be abbreviated as 1.

Definition 3.2.3. A **bipartite system** (κ, λ) for $(T_\varphi, \mathcal{F}, \mathsf{S})$ of parity $\delta \in \mathbb{Z}/2\mathbb{Z}$ consists of the following data:

(1) for each pair $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^\delta$, a cyclic submodule

$$(\kappa(\mathsf{Q}, \epsilon_{\mathsf{Q}})) \subset \mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi);$$

(2) for each pair $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\delta+1}$, a principal ideal

$$(\lambda(\mathsf{Q}, \epsilon_{\mathsf{Q}})) \subset R.$$

A **bipartite Euler system** is a bipartite system satisfying the "reciprocity laws":

(1) For each $\{\mathsf{Qq}, \epsilon_{\mathsf{Qq}}\} \in \mathsf{N}^{\delta+1}$,
$$\mathrm{loc}_{\mathsf{q}}((\kappa(\mathsf{Q}))) = (\lambda(\mathsf{Qq})) \subset R.$$

(2) For each $\{\mathsf{Qq}, \epsilon_{\mathsf{Qq}}\} \in \mathsf{N}^{\delta}$,
$$\partial_{\mathsf{q}}((\kappa(\mathsf{Qq}))) = (\lambda(\mathsf{Q})) \subset R.$$

We say (κ, λ) is **nontrivial** if there exists some $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$ such that either $\lambda(\mathsf{Q}, \epsilon_{\mathsf{Q}}) \neq 0$ or $\kappa(\mathsf{Q}, \epsilon_{\mathsf{Q}}) \neq 0$ depending on the parity of $|\mathsf{Q}| + \delta$.

3.3. Euler systems over discrete valuation rings.

3.3.1. Suppose that R is a discrete valuation ring with uniformizer π, and let $W_\varphi = T_\varphi \otimes \mathbb{Q}_p/\mathbb{Z}_p$. Exactly as in [27], there is a perfect pairing $T_\varphi \times T_\varphi \to R(1)$, G_K-equivariant up to a twist, which induces local pairings:
$$\mathsf{H}^1(K_{\mathsf{v}}, T_\varphi) \times \mathsf{H}^1(K_{\bar{\mathsf{v}}}, W_\varphi) \to R \otimes \mathbb{Q}_p/\mathbb{Z}_p,$$
$$\mathsf{H}^1(K_{\mathsf{v}}, T_\varphi/\pi^j) \times \mathsf{H}^1(K_{\bar{\mathsf{v}}}, W_\varphi[\pi^j]) \to R/\pi^j,$$
$$\mathsf{H}^1(K_{\mathsf{v}}, T_\varphi) \times \mathsf{H}^1(K_{\bar{\mathsf{v}}}, T_\varphi) \to R.$$

Here $\bar{\mathsf{v}} \in \mathsf{M}_K$ is the complex conjugate of v; the first two pairings are perfect, and the third is perfect modulo torsion. A Selmer structure $(\mathcal{F}, \mathsf{S})$ for T_φ induces a Selmer structure for W_φ, denoted the same way, by taking orthogonal complement local conditions.

Definition 3.3.2. We say $(\mathcal{F}, \mathsf{S})$ is **self-dual** if, for all $\mathsf{v} \in \mathsf{M}_K$, $\mathsf{H}^1_{\mathcal{F}}(K_{\mathsf{v}}, T_\varphi)$ and $\mathsf{H}^1_{\mathcal{F}}(K_{\bar{\mathsf{v}}}, T_\varphi)$ are exact annihilators under the local pairing.

Proposition 3.3.3. *Suppose that $(\mathcal{F}, \mathsf{S})$ is a self-dual Selmer structure for T_φ. Then, for each $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$, $(\mathcal{F}(\mathsf{Q}), \mathsf{S})$ is self-dual. Moreover:*

(1) There is a non-canonical isomorphism of R-modules:
$$\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi) \approx (R \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{r_{\mathsf{Q}}} \oplus M_{\mathsf{Q}} \oplus M_{\mathsf{Q}}$$
for some finite-length R-module M_{Q} and an integer r_{Q}.

(2) $r_{\mathsf{Q}} = \mathrm{rk}_R \,\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi)$.

Proof. The self-duality claim is clear since $H^1_{\mathrm{ord}}(K_{\mathsf{q}}, T_f)$ is self-dual. For (i), the proof of [27, Theorem 1.4.2] applies without change to show that $\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi)[\pi^s]/\pi \,\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi)[\pi^{s+1}]$ is an even-dimensional vector space for all s. The claim follows from this since $\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi)[\pi]$ is finitely generated. For (ii), note that self-duality implies saturation. So by Proposition 2.7.4, we have
$$\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi/\pi^j) = \mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi)[\pi^j]$$
for all $j \geq 0$. Moreover, since $H^0(K, T_\varphi/\pi) = 0$, the natural map induces an isomorphism
$$\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi[\pi^j]) = \mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi)[\pi^j].$$
Because
$$\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi) = \varprojlim_j \mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi/\pi^j),$$
(ii) follows.

□

Proposition 3.3.4. *For any $\{\mathsf{Qq}, \epsilon_{\mathsf{Qq}}\} \in \mathsf{N}$, one of the following holds:*

(1) $\mathrm{loc}_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi)) = 0$, $\partial_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(T_\varphi)) \neq 0$, $r_{\mathsf{Qq}} = r_{\mathsf{Q}} + 1$, and
$$\mathrm{lg}_R M_{\mathsf{Qq}} = \mathrm{lg}_R M_{\mathsf{Q}} - \mathrm{lg}_R \mathrm{loc}_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi)).$$
Moreover,
$$\mathrm{lg}\, \mathrm{loc}_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi)) = \mathrm{lg}\, \mathrm{coker}\, \partial_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(T_\varphi)).$$

(2) $\mathrm{loc}_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi)) \neq 0$, $\partial_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(T_\varphi)) = 0$, $r_{\mathsf{Qq}} = r_{\mathsf{Q}} - 1$, *and*

$$\lg_R M_{\mathsf{Qq}} = \lg_R M_{\mathsf{Q}} + \lg_R \partial_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(W_\varphi)).$$

Moreover,

$$\lg \partial_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(W_\varphi)) = \lg \operatorname{coker} \mathrm{loc}_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi)).$$

Proof. Consider the Selmer structures $\mathcal{F}^{\mathsf{q}}(\mathsf{Q}) = \mathcal{F}(\mathsf{Q}) + \mathcal{F}(\mathsf{Qq})$ and $\mathcal{F}_{\mathsf{q}}(\mathsf{Q}) = \mathcal{F}(\mathsf{Q}) \cap \mathcal{F}(\mathsf{Qq})$. By Proposition 2.7.5,

$$\mathrm{rk}_R \mathrm{Sel}_{\mathcal{F}^{\mathsf{q}}(\mathsf{Q})}(T_\varphi) = \mathrm{rk}_R \mathrm{Sel}_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(T_\varphi) + 1.$$

Moreover, because $\mathcal{F}(\mathsf{Q})$ is self-dual, Proposition 2.7.2 implies that the image of

$$\frac{\mathrm{Sel}_{\mathcal{F}^{\mathsf{q}}(\mathsf{Q})}(T_\varphi)}{\mathrm{Sel}_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(T_\varphi)} \hookrightarrow \frac{\mathsf{H}^1_{\mathcal{F}^{\mathsf{q}}(\mathsf{Q})}(K_{\mathsf{q}}, T_\varphi)}{\mathsf{H}^1_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(K_{\mathsf{q}}, T_\varphi)} = \frac{\mathsf{H}^1_{\mathrm{unr}}(K_{\mathsf{q}}, T_\varphi)}{\mathsf{H}^1_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(K_{\mathsf{q}}, T_\varphi)} \oplus \frac{\mathsf{H}^1_{\mathrm{ord}}(K_{\mathsf{q}}, T_\varphi)}{\mathsf{H}^1_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(K_{\mathsf{q}}, T_\varphi)} \approx R^2$$

is self-annihilating under the induced local pairing, hence is contained either in the ordinary or unramified part. (The induced pairing is non-degenerate, and therefore cannot admit three distinct isotropic lines.)

For the relation between M_{Q} and M_{Qq}, we suppose we are in case (a), because the two arguments are identical. Using the perfect pairing between W_φ and T_φ, we see by Proposition 2.6.2 that

$$\mathrm{loc}_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi)) \oplus \partial_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi))$$

is the exact annihilator of $\partial_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(T_\varphi))$ under the perfect induced local pairing

$$\frac{\mathsf{H}^1_{\mathcal{F}^{\mathsf{q}}(\mathsf{Q})}(K_{\mathsf{q}}, T_\varphi)}{\mathsf{H}^1_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(K_{\mathsf{q}}, T_\varphi)} \times \frac{\mathsf{H}^1_{\mathcal{F}^{\mathsf{q}}(\mathsf{Q})}(K_{\mathsf{q}}, W_\varphi)}{\mathsf{H}^1_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(K_{\mathsf{q}}, W_\varphi)} \to R \otimes \mathbb{Q}_p/\mathbb{Z}_p.$$

This implies that $\partial_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(W_\varphi))$ is divisible and

$$\lg \mathrm{loc}_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi)) = \lg \operatorname{coker} \partial_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(T_\varphi)).$$

Now, for any short exact sequence of R-modules

$$0 \to A \to B \to C \to 0,$$

there is an induced exact sequence

$$0 \to \left(\frac{A \cap B_{\mathrm{div}}}{A_{\mathrm{div}}}\right) \to A_{/\mathrm{div}} \to B_{/\mathrm{div}} \to C_{/\mathrm{div}} \to 0, \tag{$*$}$$

where the subscript div denotes the maximal π-divisible submodule and $M_{/\mathrm{div}} = M/M_{\mathrm{div}}$ for any R-module M.

Consider the short exact sequences:

$$0 \to \mathrm{Sel}_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(W_\varphi) \to \mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi) \to \mathrm{loc}_{\mathsf{q}}\left(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi)\right) \to 0 \tag{21}$$

$$0 \to \mathrm{Sel}_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(W_\varphi) \to \mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(W_\varphi) \to \partial_{\mathsf{q}}\left(\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(W_\varphi)\right) \to 0. \tag{22}$$

By $(*)$ and the discussion above, we obtain the exact sequences of finite-length R-modules:

$$0 \to \mathrm{Sel}_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(W_\varphi)_{/\mathrm{div}} \to \mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi)_{/\mathrm{div}} \to \mathrm{loc}_{\mathsf{q}}\left(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi)\right) \to 0 \tag{23}$$

$$0 \to \operatorname{coker} \partial_{\mathsf{q}}\left(\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(T_\varphi)\right) \to \mathrm{Sel}_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(W_\varphi)_{/\mathrm{div}} \to \mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(W_\varphi)_{/\mathrm{div}} \to 0. \tag{24}$$

From this, we deduce

$$\lg_R \mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi)_{/\mathrm{div}} = \lg_R \mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(W_\varphi)_{/\mathrm{div}} + 2 \lg_R \mathrm{loc}_{\mathsf{q}}\left(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi)\right),$$

which gives the result. □

The following result will allow us to control the alternative in Proposition 3.3.4.

Theorem 3.3.5. *Let* $c \in \mathsf{H}^1(K^{\mathsf{T}}/K, T_\varphi)$ *be any nonzero element, where* $\mathsf{T} \supset \mathsf{S}$ *is a finite set. Then there are infinitely many admissible ultraprimes* $\mathsf{q} \notin \mathsf{T}$, *with associated signs* ϵ_{q}, *such that* $\mathrm{loc}_{\mathsf{q}}\, c \neq 0$.

The proof is via a series of lemmas.

Lemma 3.3.6. *There is an integer j such that, for all $n \geq 0$,*

$$\pi^j H^1(K(T_\varphi)/K, T_\varphi/\pi^n) = 0.$$

If (sclr) holds, then we may take $j = 0$.

Proof. Let $G = \mathrm{Gal}(K(T_\varphi)/K)$, and let $Z \subset G$ be its center; since T_f is absolutely irreducible over K, Z acts on T_φ by scalars. We claim:

$$Z \neq \{1\}\,. \tag{25}$$

Assuming (25), the lemma follows from the inflation-restriction exact sequence

$$H^1(G/Z, H^0(Z, T_\varphi/\pi^n)) \hookrightarrow H^1(G, T_\varphi/\pi^n) \to H^1(Z, T_\varphi/\pi^n).$$

Let us now prove (25). Let $G' = \mathrm{Gal}(K(T_f)/K)$, and let L/K be the Galois subfield of $K(T_f)$ cut out by the center $Z' = Z(G') \subset G'$. By a result of Momose [51], Z' is nontrivial. Let E/K be the Galois extension determined by the kernel of φ; then it suffices to show that EL/L and $K(T_f)/L$ are linearly disjoint. Both EL and $K(T_f)$ are Galois over $\mathbb{Q}$, so $G_\mathbb{Q}$ acts on $\mathrm{Gal}(EL/L)$ and $\mathrm{Gal}(K(T_f)/L)$ by conjugation. If $\tau \in G_\mathbb{Q}$ is a complex conjugation, then τ acts trivially on $\mathrm{Gal}(K(T_f)/L)$ but nontrivially on $\mathrm{Gal}(EL/L)$, so the two groups have no nontrivial common quotient compatible with the $G_\mathbb{Q}$-action; hence $EL \cap K(T_f) = L$. □

Lemma 3.3.7. *Suppose given a cocycle*

$$c \in H^1(K, T_\varphi/\pi^n)$$

such that $\pi^j c \neq 0$, where j is as in Lemma 3.3.6. Then, for any integer $N \geq n$, there exists a sign $\epsilon = \pm 1$ and infinitely many rational primes q such that:

1. *q is inert in K and unramified in the splitting field $\mathbb{Q}(T_f, c)$.*
2. *$\mathrm{Frob}_q \in \mathrm{Gal}(\mathbb{Q}(T_f)/\mathbb{Q})$ has distinct eigenvalues ± 1 on $T_f \otimes R/\pi^N$ (where R has trivial Galois action).*
3. *For any cocycle representative, $c(\mathrm{Frob}_q^2)$ has nonzero component in the ϵ eigenspace for Frob_q.*

Proof. Abbreviate $L = K(T_\varphi/\pi^N)$, and let $\phi \in \mathrm{Hom}_{G_K}(G_L, T_\varphi/\pi^n)$ be the image of c under restriction; by hypothesis $\phi \neq 0$. Without loss of generality, we may suppose that the image of ϕ is contained in $T_\varphi/\pi^n[\pi] \simeq T_\varphi/\pi$, which, since φ is residually trivial, is an extension of scalars $\overline{T}_f \otimes_{\mathcal{O}/\wp} k$. Now,

$$\mathrm{Hom}_{G_K}(G_L, \overline{T}_f \otimes k)$$

has a natural action of $\mathrm{Gal}(K/\mathbb{Q})$, and we may assume without loss of generality that ϕ lies in the ϵ eigenspace for some $\epsilon \in \{\pm 1\}$. Fix a complex conjugation $\tau \in G_\mathbb{Q}$. Since $\overline{T}_f$ is absolutely irreducible over G_K, there exists $g \in G_L$ such that $\phi(g)$ has nonzero component in the ϵ eigenspace of τ. Then

$$\phi(\tau g \tau g) = \epsilon\tau\phi(g) + \phi(g)$$

has nonzero component in the ϵ eigenspace as well. Any q with Frobenius τg in $L(\phi)$ satisfies the desired conditions. □

Remark 3.3.8. If $p \geq 5$ and the image of the Galois action on T_f is sufficiently large, then we can instead use primes q such that $p \nmid q^2 - 1$, as is more common in the literature.

Proof of Theorem 3.3.5. Since $H^0(K, T_\varphi/\pi) = 0$, Lemma 2.4.6 implies that

$$\mathsf{H}^1(K^\mathsf{T}/K, T_\varphi)[\pi] = 0.$$

Thus there exists some n such that the image $\overline{c}$ of c in $\mathsf{H}^1(K^\mathsf{T}/K, T_\varphi/\pi^n)$ satisfies $\pi^j\overline{c} \neq 0$, for some j as in Lemma 3.3.6. By definition, $\overline{c}$ is represented by a sequence of classes $c_m \in H^1(K^{T_m}/K, T_\varphi/\pi^n)$ such that $\pi^j c_m \neq 0$ for $\mathfrak{F}$-many m, where $\{T_m\}_{m\in\mathbb{N}}$ represents T. For each m, apply Lemma 3.3.7 (with $N = m$) to obtain a prime $q_m \notin T_m$ and a sign ϵ_m. If $\mathsf{q} \in \mathsf{M}_\mathbb{Q}$ is the equivalence class of the sequence $\{q_m\}_{m\in\mathbb{N}}$, and $\epsilon \in \mathcal{U}(\{\pm 1\}_{m\in\mathbb{N}}) \simeq \{\pm 1\}$ is the equivalence class of the sequence $\{\epsilon_m\}_{m\in\mathbb{N}}$, then the pair $\{\mathsf{q}, \epsilon\}$ has the desired properties. Since there are infinitely many choices for each q_m, there are also infinitely many choices for q. □

Corollary 3.3.9. *For any $\{\mathsf{Q}, \epsilon_\mathsf{Q}\} \in \mathsf{N}$, there exists some $\{\mathsf{Q}\mathsf{Q}', \epsilon_{\mathsf{Q}\mathsf{Q}'}\} \in \mathsf{N}$ such that $r_{\mathsf{Q}\mathsf{Q}'} = 0$.*

Proof. This is an obvious induction argument using Theorem 3.3.5 and Proposition 3.3.4. □

Combining Proposition 3.3.4 and Theorem 3.3.5 allows us to prove the main result of this subsection.

Theorem 3.3.10. *Suppose that $(\mathcal{F}, \mathsf{S})$ is self-dual and that (κ, λ) is a nontrivial bipartite Euler system with sign δ for $(T_\varphi, \mathcal{F}, \mathsf{S})$. Then there exists an integer C (possibly negative) such that:*

(1) For all $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^\delta$, r_{Q} is odd, $r_{\mathsf{Q}} = 1$ if and only if $\kappa(\mathsf{Q}) \neq 0$, and in that case

$$\lg_R M_{\mathsf{Q}} = \lg_R \left(\frac{\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi)}{(\kappa(\mathsf{Q}))} \right) + C.$$

(2) For all $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\delta+1}$, r_{Q} is even, $r_{\mathsf{Q}} = 0$ if and only if $\lambda(\mathsf{Q}) \neq 0$, and in that case

$$\lg_R M_{\mathsf{Q}} = \mathrm{ord}_\pi \lambda(\mathsf{Q}) + C.$$

In particular,

$$\delta = \mathrm{rk}_R \mathrm{Sel}_{\mathcal{F}}(T_\varphi) + 1 \pmod 2.$$

Proof. The proof will be in several steps.

Step 1. *If $\lambda(\mathsf{Q}) \neq 0$ for some $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\delta+1}$, then $r_{\mathsf{Q}} = 0$.*

Proof. If $0 \neq c \in \mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi)$, then by Theorem 3.3.5, there exists an admissible ultraprime q with sign ϵ_{q} such that $\mathrm{loc}_{\mathsf{q}}\, c \neq 0$. By Proposition 3.3.4, $\partial_{\mathsf{q}}(\kappa(\mathsf{Qq})) = 0$, which contradicts the reciprocity laws. ☐

Step 2. *If $\kappa(\mathsf{Q}) \neq 0$ for some $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^\delta$, then $r_{\mathsf{Q}} = 1$.*

Proof. Choose an admissible ultraprime q with sign ϵ_{q} such that $\mathrm{loc}_{\mathsf{q}}\, \kappa(\mathsf{Q}) \neq 0$. Then by the reciprocity laws, $\lambda(\mathsf{Qq}) \neq 0$, so by Step 1 $r_{\mathsf{Qq}} = 0$. Proposition 3.3.4 implies $r_{\mathsf{Q}} = 1$. ☐

Step 3. *For all $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$, $r_{\mathsf{Q}} \equiv \delta + |\mathsf{Q}| + 1 \pmod 2$.*

Proof. If $\{\mathsf{QQ}', \epsilon_{\mathsf{QQ}'}\} \in \mathsf{N}$, then by Proposition 3.3.4

$$r_{\mathsf{Q}} - r_{\mathsf{QQ}'} \equiv |\mathsf{Q}'| \pmod 2.$$

So Steps 1 and 2 imply Step 3. ☐

Step 4. *Suppose $r_{\mathsf{Q}} = 0$ for some $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\}$. Then, for all admissible ultraprimes $\mathsf{q} \notin \mathsf{Q} \cup \mathsf{S}$ with sign ϵ_{q}, $r_{\mathsf{Qq}} = 1$ and*

$$\lg_R M_{\mathsf{Qq}} + \mathrm{ord}_\pi \lambda(\mathsf{Q}) = \lg_R M_{\mathsf{Q}} + \lg_R \left(\frac{\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(T_\varphi)}{(\kappa(\mathsf{Qq}))} \right).$$

Proof. By Step 3, $\lambda(\mathsf{Q})$ and $\kappa(\mathsf{Qq})$ are well-defined. Then Step 4 follows from Proposition 3.3.4, since

$$\lg_R \left(\frac{\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(T_\varphi)}{(\kappa(\mathsf{Qq}))} \right) + \lg_R \mathrm{coker}\, \partial_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(T_\varphi) = \mathrm{ord}_\pi \lambda(\mathsf{Q}).$$

☐

The exact same reasoning implies:

Step 5. *Suppose that $r_{\mathsf{Q}} = 1$ and $\mathsf{q} \notin \mathsf{Q} \cup \mathsf{S}$ is an admissible ultraprime with sign ϵ_{q} such that $r_{\mathsf{Qq}} = 0$. Then*

$$\lg_R M_{\mathsf{Qq}} + \lg_R \left(\frac{\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi)}{(\kappa(\mathsf{Q}))} \right) = \lg_R M_{\mathsf{Q}} + \mathrm{ord}_\pi \lambda(\mathsf{Qq}).$$

Now consider the graph $\mathcal{X}$ whose vertices are the elements of N, and where the edges are between vertices of the form $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\}$ and $\{\mathsf{Qq}, \epsilon_{\mathsf{Qq}}\}$, for some admissible ultraprime q with sign ϵ_{q} (cf. [28]). We say $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\}$ is a **core vertex** if $r_{\mathsf{Q}} \leq 1$. The **core subgraph** $\mathcal{X}_0$ of $\mathcal{X}$ is the full subgraph on core vertices.

Step 6. *Assume $\mathcal{X}_0$ is path-connected. Then the theorem holds.*

Proof. For every $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathcal{X}_0$, set

$$C_{\mathsf{Q}} = \begin{cases} \lg_R M_{\mathsf{Q}} - \lg_R \left(\frac{\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi)}{(\kappa(\mathsf{Q}))} \right), & \text{if } \{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^\delta, \\ \lg_R M_{\mathsf{Q}} - \mathrm{ord}_\pi \lambda(\mathsf{Q}), & \text{if } \{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\delta+1}. \end{cases}$$

By Steps 4 and 5, C_{Q} is constant along paths contained in $\mathcal{X}_0$. Moreover, if (κ, λ) is nontrivial, $\mathcal{X}_0$ is nonempty by Steps 1 and 2. Under the additional assumption that $\mathcal{X}_0$ is path-connected, the common value of C_{Q} for $\mathsf{Q} \in \mathcal{X}_0$ is the global constant C of the theorem. ☐

In the rest of the proof, we will establish the path-connectedness of $\mathcal{X}_0$.

Step 7. *If $v = \{\mathsf{Q}, \epsilon_{\mathsf{Q}}\}$ and $v' = \{\mathsf{QQ}', \epsilon_{\mathsf{QQ}'}\}$ are core vertices, then they are connected by a path in $\mathcal{X}_0$.*

Proof. We proceed by induction on $|\mathsf{Q}'|$, where the base case is trivial. If $r_{\mathsf{QQ}'/\mathsf{q}} \leq 1$ for any $\mathsf{q} \in \mathsf{Q}'$, then we may apply the inductive hypothesis, so assume otherwise. By Proposition 3.3.4, $r_{\mathsf{QQ}'} = 1$ and $\partial_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{QQ}')}(T_\varphi)) = 0$ for all $\mathsf{q} \in \mathsf{Q}'$. Hence

$$\mathrm{Sel}_{\mathcal{F}(\mathsf{QQ}')}(T_\varphi) \subset \mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi).$$

Then, by Theorem 3.3.5 and Proposition 3.3.4, there exists an admissible ultraprime $\mathsf{q} \notin \mathsf{Q} \cup \mathsf{Q}' \cup \mathsf{S}$ with sign ϵ_{q} such that $r_{\mathsf{Qq}} = r_{\mathsf{QQ}'\mathsf{q}} = 0$. If $\mathsf{q}' \in \mathsf{Q}'$ is any factor, then $\left\{\mathsf{QQ}'\mathsf{q}/\mathsf{q}', \epsilon_{\mathsf{QQ}'\mathsf{q}/\mathsf{q}'}\right\} \in \mathsf{N}$ is a core vertex, which is connected to v' in $\mathcal{X}_0$. By the inductive hypothesis, $\left\{\mathsf{QQ}'\mathsf{q}/\mathsf{q}', \epsilon_{\mathsf{QQ}'\mathsf{q}/\mathsf{q}'}\right\}$ is also connected to the core vertex $\{\mathsf{Qq}, \epsilon_{\mathsf{Qq}}\}$, hence to v, by a path in $\mathcal{X}_0$. This completes the inductive step. ☐

Step 8. *If $v = \{\mathsf{Q}, \epsilon_{\mathsf{Q}}\}$ is a core vertex and $\mathsf{T} \subset \mathsf{M}_{\mathbb{Q}}$ is any finite set, then there exists a core vertex $v' = \{\mathsf{Q}', \epsilon_{\mathsf{Q}'}\}$ such that v and v' are connected by a path in $\mathcal{X}_0$ and $\mathsf{Q}' \cap \mathsf{T} = \emptyset$.*

Proof. By iterating, it suffices to assume that $\mathsf{Q} \cap \mathsf{T}$ consists of exactly one ultraprime $\mathsf{q} \in \mathsf{Q}$. If $r_{\mathsf{Q}/\mathsf{q}} \leq 1$, then the conclusion is obvious, so suppose otherwise. As in the proof of Step 7, choose an admissible ultraprime $\mathsf{q}' \notin \mathsf{Q} \cup \mathsf{S} \cup \mathsf{T}$ with associated sign $\epsilon_{\mathsf{q}'}$ such that $r_{\mathsf{Qq}'} = 0$, which implies $r_{\mathsf{Qq}'/\mathsf{q}} = 1$. The core vertex $v' = \left\{\mathsf{Qq}'/\mathsf{q}, \epsilon_{\mathsf{Qq}'/\mathsf{q}}\right\}$ has the desired properties. ☐

Finally, we have:

Step 9. *The core subgraph $\mathcal{X}_0$ is path-connected.*

Proof. Let $\{\mathsf{Q}_1, \epsilon_{\mathsf{Q}_1}\}$ and $\{\mathsf{Q}_2, \epsilon_{\mathsf{Q}_2}\}$ be two core vertices. Without loss of generality, by Step 8, we may assume $\mathsf{Q}_1 \cap \mathsf{Q}_2 = \emptyset$. (This step is necessary because the sign functions ϵ_{Q_1} and ϵ_{Q_2} need not agree on $\mathsf{Q}_1 \cap \mathsf{Q}_2$.) Consider $\{\mathsf{Q}_1\mathsf{Q}_2, \epsilon_{\mathsf{Q}_1\mathsf{Q}_2}\} \in \mathsf{N}$. This may not be a core vertex, but, by Corollary 3.3.9, there exists $\{\mathsf{Q}_3, \epsilon_{\mathsf{Q}_3}\} \in \mathsf{N}$ such that $\{\mathsf{Q}_1\mathsf{Q}_2\mathsf{Q}_3, \epsilon_{\mathsf{Q}_1\mathsf{Q}_2\mathsf{Q}_3}\}$ is a core vertex. We may then conclude by Step 6. ☐

☐

Proposition 3.3.11. *Under the hypotheses of Theorem 3.3.10, there exists a constant $C' \geq 0$ depending on $|\mathsf{S}|$, T_f, and the ramification index of $R/\mathcal{O}$, but not on φ, such that $C \geq -C'$. If (sclr) holds, then we may take $C = 0$.*

Proof. By Theorem 3.3.10, it suffices to show that there exists a constant with the desired dependencies and a pair $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^\delta$ such that $\lg_R M_{\mathsf{Q}} \leq C'$. We first note that the constant j in Lemma 3.3.7 depends only on T_f and the ramification index of $R/\mathcal{O}$, and can be taken to be 0 under (sclr).

Moreover, if k is the residue field of R, then $d = \dim_k \mathsf{H}^1(K^{\mathsf{S}}/K, W_\varphi[\pi])$ is also bounded with the desired uniformity. We now construct a sequence $\{\mathsf{Q}_i, \epsilon_{\mathsf{Q}_i}\}$ recursively (starting from $\mathsf{Q}_1 = 1$) by the following rules:

- If $r_{\mathsf{Q}_i} > 0$, then choose any $\mathsf{q}_{i+1} \notin \mathsf{Q}_i$ with sign $\epsilon_{\mathsf{q}_{i+1}}$ such that
$$\lg \operatorname{coker}(\mathrm{loc}_{\mathsf{q}} \mathrm{Sel}_{\mathcal{F}(\mathsf{Q}_i)}(T_\varphi)) \leq j.$$
- If $r_{\mathsf{Q}_i} = 0$ and the exponent of $\mathrm{Sel}_{\mathcal{F}(\mathsf{Q}_i)}(W_\varphi) \neq 0$ is $n_i > i \cdot j$, then choose any $\mathsf{q}_{i+1} \notin \mathsf{Q}_i$ with sign $\epsilon_{\mathsf{q}_{i+1}}$ such that the exponent of $\mathrm{loc}_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q}_i)}(W_\varphi))$ is at least $n_i - j$.

These choices are possible by Lemma 3.3.7. In either of the above two cases, set $\left\{\mathsf{Q}_{i+1}, \epsilon_{\mathsf{Q}_{i+1}}\right\} = \left\{\mathsf{Q}_i\mathsf{q}_{i+1}, \epsilon_{\mathsf{Q}_i\mathsf{q}_{i+1}}\right\}$; if neither holds, then end the construction. For each i, let r'_{Q_i} be the minimal number of generators of the R-module $\pi^{i \cdot j} M_{\mathsf{Q}_i}$. In the first case of the construction, $r'_{\mathsf{Q}_{i+1}} \leq r'_{\mathsf{Q}_i}$; in the second case, $r'_{\mathsf{Q}_{i+1}} < r'_{\mathsf{Q}_i}$ (by Proposition 3.3.4 respectively). After $r_1 \leq d$ steps, we alternate between the two cases of the construction, taking at most $2r'_1 \leq 2d$ more steps. Hence for some $i \leq 3d$, $r'_{\mathsf{Q}_i} = 0$ and $r_{\mathsf{Q}_i} = 0$, and the construction halts. For this i,

$$\lg_R M_{\mathsf{Q}_i} \leq ij \dim_k \mathrm{Sel}_{\mathcal{F}(\mathsf{Q}_i)}(W_\varphi)[\pi] \leq 3dj(d + 3d),$$

the last inequality by the reasoning of [28, Corollary 2.2.10]. (Less precisely, we could deduce the bound $3dj(d + 6d)$ directly from Proposition 3.3.4.)

Since d and j have bounds of the desired sort, the claim follows. ☐

3.4. **Euler systems over Λ.** Let Λ be the anticyclotomic Iwasawa algebra $\mathcal{O}[\![\mathrm{Gal}(K_\infty/K)]\!]$ with canonical character

$$\Psi : G_K \to \Lambda^\times.$$

If γ is a topological generator of $\mathrm{Gal}(K_\infty/K)$, then as a ring $\Lambda = \mathcal{O}[\![T]\!]$ where $T = \Psi(\gamma) - 1$. For each height-one prime $\mathfrak{P} \subset \Lambda$, let $S_{\mathfrak{P}}$ be the integral closure of $\Lambda/\mathfrak{P}$ in its field of fractions, so that Ψ induces a character $G_K \to \Lambda^\times \to S_{\mathfrak{P}}^\times$. We write $T_{\mathfrak{P}}$ for the twist $T_f \otimes_{\mathcal{O}} S_{\mathfrak{P}}(\Psi)$, $W_{\mathfrak{P}}$ for $T_{\mathfrak{P}} \otimes \mathbb{Q}_p/\mathbb{Z}_p$, and $\mathbf{T} = \mathbf{T}_f$ for the interpolated twist $T_f \otimes_{\mathcal{O}} \Lambda(\Psi)$. Also let $\mathbf{W} = \mathbf{W}_f = \mathbf{T}_f^*$ be the Cartier dual with Λ action twisted by the canonical involution ι, so that for each $\mathfrak{P}$ there is a natural map

$$W_{\mathfrak{P}} \to \mathbf{W}$$

of $\Lambda[G_K]$-modules (see, e.g., [27]). As in (3.3.1), a Selmer structure $\mathcal{F}_\Lambda$ for $\mathbf{T}$ induces a dual Selmer structure $\mathcal{F}_\Lambda^*$ for $\mathbf{W}$. The following definition is motivated by [38, Lemma 5.3.13] and its applications in [27, 28].

Definition 3.4.1. An **interpolated self-dual** Selmer structure

$$(\mathsf{S}, \mathcal{F}_\Lambda, \mathcal{F}_{\mathfrak{P}}, \Sigma_\Lambda)$$

for $\mathbf{T}$ consists of the following data:

- A finite set $\mathsf{S} \subset \mathsf{M}_K$.
- For each height-one prime $\mathfrak{P} \subset \Lambda$, a self-dual Selmer structure $(\mathcal{F}_{\mathfrak{P}}, \mathsf{S})$ for $T_{\mathfrak{P}}$.
- A finite set Σ_Λ of height-one primes $\mathfrak{P} \subset \Lambda$, such that $\wp\Lambda \in \Sigma_\Lambda$.
- A Selmer structure $(\mathcal{F}_\Lambda, \mathsf{S})$ for $\mathbf{T}$ such that, for all $\mathsf{v} \in \mathsf{M}_K$ and all $\mathfrak{P} \subset \Lambda$ with $\wp\Lambda \neq \mathfrak{P}$, the natural maps induce well-defined homomorphisms:

$$\begin{aligned} \mathsf{H}^1_{\mathcal{F}_\Lambda}(K_\mathsf{v}, \mathbf{T}/\mathfrak{P}) &\to \mathsf{H}^1_{\mathcal{F}_{\mathfrak{P}}}(K_\mathsf{v}, T_{\mathfrak{P}}), \\ \mathsf{H}^1_{\mathcal{F}^*_{\mathfrak{P}}}(K_\mathsf{v}, W_{\mathfrak{P}}) &\to \mathsf{H}^1_{\mathcal{F}^*_\Lambda}(K_\mathsf{v}, \mathbf{W}[\mathfrak{P}]). \end{aligned} \tag{26}$$

  Moreover, for all $\mathfrak{P} \notin \Sigma_\Lambda$, the maps (26) have finite kernel and cokernel with order bounded by a constant depending only on $[S_{\mathfrak{P}} : \Lambda/\mathfrak{P}]$ as $\mathfrak{P}$ varies.

Proposition 3.4.2. *Suppose $(\mathsf{S}, \mathcal{F}_{\mathfrak{P}}, \mathcal{F}_\Lambda, \Sigma_\Lambda)$ is an interpolated self-dual Selmer structure for $\mathbf{T}$. Then for all $\mathfrak{P} \subset \Lambda$ with $\wp\Lambda \neq \mathfrak{P}$, the natural map induces well-defined homomorphisms:*

$$\mathrm{Sel}_{\mathcal{F}_\Lambda}(\mathbf{T})/\mathfrak{P} \to \mathrm{Sel}_{\mathcal{F}_{\mathfrak{P}}}(T_{\mathfrak{P}})$$
$$\mathrm{Sel}_{\mathcal{F}_{\mathfrak{P}}}(W_{\mathfrak{P}}) \to \mathrm{Sel}_{\mathcal{F}_\Lambda}(\mathbf{W})[\mathfrak{P}].$$

Moreover, after possibly expanding the finite set Σ_Λ, the following holds: for all $\mathfrak{P} \notin \Sigma_\Lambda$, these maps have finite kernel and cokernel with a bound depending on $\mathcal{F}$ and on $[S_{\mathfrak{P}} : \Lambda/\mathfrak{P}]$, but not on $\mathfrak{P}$ itself.

Proof. See [38, Proposition 5.3.14]. □

3.4.3. Recall that, for any finitely generated Λ-module M, there exists a unique Λ-module N of the form $\Lambda^r \oplus \bigoplus \Lambda/\mathfrak{P}_i^{e_i}$ such that M admits a map to N with finite kernel and cokernel, where $\mathfrak{P}_i$ are height-one primes; we denote this relationship by $M \sim N$. The characteristic ideal $\mathrm{char}_\Lambda(M)$ is zero if $r \geq 1$, and equal to $\prod \mathfrak{P}_i^{e_i}$ otherwise. The following easy lemma is implicit in [38, p. 66].

Lemma 3.4.4. *Let $\mathfrak{P} \subset \Lambda$ be a height-one prime. Then there exists an integer d and a sequence of height-one primes $\mathfrak{P}_m$ such that, for all finitely generated torsion Λ-modules M,*

$$\mathrm{lg}_{\mathcal{O}}(M/\mathfrak{P}_m) = md\,\mathrm{ord}_{\mathfrak{P}}\,\mathrm{char}_\Lambda(M) + O(1)$$

as m varies (holding M fixed). Moreover $[S_{\mathfrak{P}_m} : \Lambda/\mathfrak{P}_m]$ is constant for large enough m, and if $\mathfrak{P} \neq \wp\Lambda$, then the rings $\Lambda/\mathfrak{P}_m$ are abstractly isomorphic.

Proof. If $\mathfrak{P} \neq \wp\Lambda$ is generated by a distinguished polynomial $f \in \Lambda$, and π is a uniformizer for $\mathcal{O}$, then we may take $\mathfrak{P}_m = f + \pi^m$ (for sufficiently large m) and $d = [S_{\mathfrak{P}} : \mathcal{O}]$. If $\mathfrak{P} = \wp\Lambda$, then we may take $\mathfrak{P}_m = T^m + \pi$ and $d = 1$. □

Proposition 3.4.5. *Suppose that $(\mathsf{S}, \mathcal{F}_{\mathfrak{P}}, \mathcal{F}_\Lambda, \Sigma_\Lambda)$ is an interpolated self-dual Selmer structure for $\mathbf{T}$. Then for all $\{\mathsf{Q}, \epsilon_\mathsf{Q}\} \in \mathsf{N}$:*

(1) $(\mathsf{S}\cup\mathsf{Q}, \mathcal{F}_{\mathfrak{P}}(\mathsf{Q}), \mathcal{F}_\Lambda(\mathsf{Q}), \Sigma_\Lambda)$ *is an interpolated self-dual Selmer structure for* $\mathbf{T}$ *and*

$$\mathrm{Sel}_{\mathcal{F}^*_\Lambda(\mathsf{Q})}(\mathbf{W})^\vee \sim \Lambda^{r_\mathsf{Q}} \oplus M_\mathsf{Q} \oplus M_\mathsf{Q}$$

for some torsion Λ*-module* M_Q *and an integer* r_Q*.*

(2) $r_\mathsf{Q} = \mathrm{rk}_\Lambda \mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{T})$.

Proof. At places $\mathfrak{q}\in\mathsf{Q}$,

$$\mathsf{H}^1_{\mathcal{F}_\Lambda(\mathsf{Q})}(K_\mathfrak{q}, \mathbf{T}) = \mathsf{H}^1_{\mathrm{ord}}(K_\mathfrak{q}, T_f)\otimes\Lambda$$

and

$$\mathsf{H}^1_{\mathcal{F}_\mathfrak{P}(\mathsf{Q})}(K_\mathfrak{q}, T_\mathfrak{P}) = \mathsf{H}^1_{\mathrm{ord}}(K_\mathfrak{q}, T_f)\otimes S_\mathfrak{P},$$

so we clearly have local maps with kernel and cokernels bounded as desired (and similarly for $\mathbf{W}_f$ and $W_\mathfrak{P}$); so indeed $(\mathsf{S}\cup\mathsf{Q}, \mathcal{F}_{\mathfrak{P}}(\mathsf{Q}), \mathcal{F}_\Lambda(\mathsf{Q}), \Sigma_\Lambda)$ is an interpolated self-dual Selmer structure. The rest of the claims are deduced from Proposition 3.4.2 and Proposition 3.3.3 exactly as in [27, Theorem 2.2.10]. □

Theorem 3.4.6. *Suppose that* $(\mathsf{S}, \mathcal{F}_\mathfrak{P}, \mathcal{F}_\Lambda, \Sigma_\Lambda)$ *is an interpolated self-dual Selmer structure for* $\mathbf{T}$ *and* $\{\boldsymbol{\kappa}, \boldsymbol{\lambda}\}$ *is a nontrivial bipartite Euler system with parity* δ *for the triple* $(\mathbf{T}, \mathcal{F}_\Lambda, \mathsf{S})$*. Then there exists a nonzero fractional ideal* $I\subset\Lambda\otimes\mathbb{Q}_p$ *such that:*

(1) For all $\{\mathsf{Q}, \epsilon_\mathsf{Q}\}\in\mathsf{N}^\delta$*,* r_Q *is odd,* $r_\mathsf{Q}=1$ *if and only if* $\boldsymbol{\kappa}(\mathsf{Q})\neq 0$*, and in that case*

$$\mathrm{char}_\Lambda(M_\mathsf{Q})\cdot I = \mathrm{char}_\Lambda\left(\frac{\mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{T})}{(\boldsymbol{\kappa}(\mathsf{Q}))}\right).$$

(2) For all $\{\mathsf{Q}, \epsilon_\mathsf{Q}\}\in\mathsf{N}^{\delta+1}$*,* r_Q *is even,* $r_\mathsf{Q}=0$ *if and only if* $\boldsymbol{\lambda}(\mathsf{Q})\neq 0$*, and in that case*

$$\mathrm{char}_\Lambda(M_\mathsf{Q})\cdot I = (\boldsymbol{\lambda}(\mathsf{Q})).$$

In particular,

$$\delta = \mathrm{rk}_R\,\mathrm{Sel}_\mathcal{F}(\mathbf{T}) + 1 \pmod 2.$$

If (sclr) holds, then $I\subset\Lambda$*.*

Proof. Let $\mathfrak{P}\subset\Lambda$ be any height-one prime; via the natural maps $\mathrm{Sel}_{\mathcal{F}_\Lambda}(\mathbf{T})\to\mathrm{Sel}_{\mathcal{F}_\mathfrak{P}}(T_\mathfrak{P})$ and $\Lambda\to S_\mathfrak{P}$, the Euler system $(\boldsymbol{\kappa}, \boldsymbol{\lambda})$ defines an Euler system $(\kappa_\mathfrak{P}, \lambda_\mathfrak{P})$ of parity δ for the triple $(T_\mathfrak{P}, \mathcal{F}_\mathfrak{P}, \mathsf{S})$. In particular, Theorem 3.3.10 applies.

Since $\overline{T}_f$ is absolutely irreducible, $\mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{T})$ is torsion-free (by the long exact sequence in Lemma 2.4.6). Hence, if $\boldsymbol{\kappa}(\mathsf{Q})\neq 0$, then by Proposition 3.4.2 $\kappa_\mathfrak{P}(\mathsf{Q})\neq 0$ for all but finitely many $\mathfrak{P}$. Similarly, if $\boldsymbol{\lambda}(\mathsf{Q})\neq 0$, then clearly $\lambda_\mathfrak{P}(\mathsf{Q})\neq 0$ for all but finitely many $\mathfrak{P}$. Because

$$\mathrm{rk}_\Lambda\,\mathrm{Sel}_{\mathcal{F}_\Lambda}(\mathbf{T}) \le \mathrm{rk}_{S_\mathfrak{P}}\,\mathrm{Sel}_{\mathcal{F}_\mathfrak{P}}(T_\mathfrak{P})$$

with equality for all but finitely many $\mathfrak{P}$, the claims about r_Q follow from Theorem 3.3.10.

For any $\mathfrak{P}$ and $\{\mathsf{Q}, \epsilon_\mathsf{Q}\}\in\mathsf{N}^{\delta+1}$ such that $\boldsymbol{\lambda}(\mathsf{Q})\neq 0$, by Proposition 3.4.5 and Lemma 3.4.4 we have

$$\begin{aligned} e_\mathfrak{P}(\mathsf{Q}) &\coloneqq \mathrm{ord}_\mathfrak{P}(\boldsymbol{\lambda}(\mathsf{Q})) - \mathrm{ord}_\mathfrak{P}\,\mathrm{char}_\Lambda(M_\mathsf{Q}) \\ &= \lim_{m\to\infty}\frac{\mathrm{lg}_\mathcal{O}(S_{\mathfrak{P}_m}/\lambda_{\mathfrak{P}_m}(\mathsf{Q})) - \mathrm{lg}_\mathcal{O} M_{\mathsf{Q},\mathfrak{P}_m}}{md}. \end{aligned}$$

Applying Theorem 3.3.10, this quantity does not depend on $\{\mathsf{Q}, \epsilon_\mathsf{Q}\}$ (as long as $\boldsymbol{\lambda}(\mathsf{Q})\neq 0$); it is also clearly zero for almost all $\mathfrak{P}$, so that $\prod_\mathfrak{P}\mathfrak{P}^{e_\mathfrak{P}}$ defines a fractional ideal I of Λ satisfying (ii). The same calculation shows that I satisfies (i) as well, and the integrality properties follow from Proposition 3.3.11. □

4. Geometry of modular Jacobians

4.1. **Multiplicity one.**

4.1.1. Let N_1 and N_2 be coprime positive integers, with N_2 squarefree. Consider the Hecke algebra $\mathbb{T} = \mathbb{T}_{N_1,N_2}$ generated over $\mathbb{Z}$ by operators T_ℓ for all primes $\ell \nmid N = N_1N_2$ and U_ℓ for all $\ell|N$, acting on the modular forms of weight two and level $\Gamma_0(N)$ which are new at all factors $\ell|N_2$. If I is the kernel of the projection $\mathbb{T}_{N_1N_2,1} \to \mathbb{T}$, then we set

$$J_{\min}^{N_1,N_2} \coloneqq J_0(N)/IJ_0(N), \tag{27}$$

an abelian variety with a (faithful) action of $\mathbb{T}$. If N_1, N_2 are clear from context, we will omit the superscript.

For any abelian variety A with an action of $\mathbb{T}$, and any maximal ideal $\mathfrak{m} \subset \mathbb{T}$, the $\mathfrak{m}$-adic Tate module is defined to be the localization

$$T_\mathfrak{m} A \coloneqq T_p A \otimes_\mathbb{T} \mathbb{T}_\mathfrak{m}, \tag{28}$$

where p is the residue characteristic of $\mathfrak{m}$. (Note that this is dual to the notation of [25].) For any $\mathfrak{m}$ which is non-Eisenstein with odd residue characteristic $p \nmid N$, it follows from [62] that $T_\mathfrak{m} J_{\min}$ is free of rank two over $\mathbb{T}_\mathfrak{m}$; by [25, Corollary 4.7], the natural map then induces an isomorphism

$$\mathbb{T}_\mathfrak{m} \xrightarrow{\sim} \mathrm{End}_\mathbb{T}(J_{\min})_\mathfrak{m}. \tag{29}$$

4.1.2. Now suppose that A is an abelian variety with faithful $\mathbb{T}$-action, admitting a $\mathbb{T}$-equivariant isogeny to $J_{\min}$. For any $\ell|N_2$, let $\mathcal{A}_{/\mathbb{Z}_\ell}$ be the Néron model of A. The neutral connected component $\mathcal{A}^0_{\mathbb{F}_\ell}$ of the special fiber of $\mathcal{A}$ is a torus, and we write $\mathcal{X}_\ell(A) = \mathrm{Hom}(\mathcal{A}^0_{\mathbb{F}_\ell}, \mathbb{G}_m)$ for its character group. The association $A \mapsto \mathcal{X}_\ell(A)$ is contravariantly functorial.

Proposition 4.1.3 ((Helm))**.** *Let $\mathfrak{m} \subset \mathbb{T}$ be non-Eisenstein of residue characteristic $p \nmid 2N$. Then the natural maps induce $\mathbb{T}_\mathfrak{m}$-module isomorphisms:*

$$T_\mathfrak{m} J_{\min} \otimes \mathrm{Hom}(J_{\min}, A)_\mathfrak{m} \xrightarrow{\sim} T_\mathfrak{m} A,$$
$$\mathcal{X}_\ell(J_{\min}^\vee) \otimes \mathrm{Hom}(J_{\min}, A)_\mathfrak{m} \xrightarrow{\sim} \mathcal{X}_\ell(A^\vee),$$
$$\mathrm{Hom}(A, J_{\min})_\mathfrak{m} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{T}_\mathfrak{m}}\left(\mathrm{Hom}(J_{\min}, A)_\mathfrak{m}, \mathrm{End}(J_{\min})_\mathfrak{m}\right).$$

Here, all Hom-sets are understood to be $\mathbb{T}$-equivariant morphisms, and tensor products are taken modulo $\mathbb{Z}$-torsion.

Proof. This follows by duality from [25, Corollary 4.1, Theorem 4.11, Proposition 4.14]. □

We record the following elementary lemma for later use.

Lemma 4.1.4. *Let $\mathcal{X} = \mathcal{X}_\ell(J_{\min}^\vee)_\mathfrak{m}$ for some $\ell|N_2$ and $\mathfrak{m} \subset \mathbb{T}$, where $\mathfrak{m}$ is non-Eisenstein of odd residue characteristic p. If the associated residual representation $\overline{\rho}_\mathfrak{m}$ is ramified at ℓ, then $\mathcal{X}$ is free of rank one over $\mathbb{T}_\mathfrak{m}$. In general, there exist $\mathbb{T}_\mathfrak{m}$-module maps*

$$\phi_i : \mathcal{X} \to \mathbb{T}_\mathfrak{m}, \ \ \psi_i : \mathbb{T}_\mathfrak{m} \to \mathcal{X}, \ \ i = 1, 2$$

such that

$$\phi_i \circ \psi_i = \psi_i \circ \phi_i = t_i \in \mathbb{T}_\mathfrak{m} \subset \mathrm{End}(\mathcal{X})$$

and

$$t_1 + t_2 = \ell - 1 \in \mathbb{T}_\mathfrak{m}.$$

Proof. If $\ell - 1$ is a p-adic unit, or if $\overline{\rho}_\mathfrak{m}$ is ramified, then this follows from [25, Lemma 6.5]. In general, we have

$$\mathcal{X} = \mathrm{Hom}((\mathcal{J}_{\min}^\vee)^0_{\mathbb{F}_\ell}[\mathfrak{m}^\infty], \mu_{p^\infty}) \tag{30}$$

so that $\mathcal{X}$ may be identified with a $\mathbb{T}_\mathfrak{m}[G_{\mathbb{Q}_\ell}]$-module quotient

$$\pi : T_\mathfrak{m} J_{\min} \to \mathcal{X};$$

the Galois action on $\mathcal{X}$ is unramified and Frobenius acts as U_ℓ, which is a constant ± 1 because the residue characteristic of $\mathfrak{m}$ is $p > 2$.

Because $T_\mathfrak{m} J_{\min}$ is free of rank two over $\mathbb{T}_\mathfrak{m}$, it may be equipped with a basis $\{e_1, e_2\}$, and moreover an alternating $\mathbb{T}_\mathfrak{m}$-module pairing

$$\langle \cdot, \cdot \rangle : T_\mathfrak{m} J_{\min} \times T_\mathfrak{m} J_{\min} \to \mathbb{T}_\mathfrak{m} \tag{31}$$

such that

$$y = \langle e_1, y\rangle e_2 - \langle e_2, y\rangle e_1 \tag{32}$$

for all $y \in T_{\mathfrak{m}} J_{\min}$. Define maps

$$\begin{aligned} \phi_i : T_{\mathfrak{m}} J_{\min} &\to \mathbb{T}_{\mathfrak{m}}, \quad i = 1, 2 \\ \phi_1 : y &\mapsto \langle y, (F - U_\ell)e_2\rangle \\ \phi_2 : y &\mapsto \langle y, (F - U_\ell)e_1\rangle, \end{aligned}$$

where $F \in G_{\mathbb{Q}_\ell}$ is any lift of Frobenius. We first claim that the maps ϕ_i factor through π. Since $\mathbb{T}_{\mathfrak{m}}$ is p-torsion-free, it suffices to check this after inverting p. On $T_{\mathfrak{m}} J_{\min} \otimes \mathbb{Q}_p$, F acts with distinct eigenvalues U_ℓ and ℓU_ℓ, and $\pi \otimes \mathbb{Q}_p : T_{\mathfrak{m}} J_{\min} \otimes \mathbb{Q}_p \to \mathcal{X} \otimes \mathbb{Q}_p$ coincides with the projection onto the U_ℓ-eigenspace. Since $\langle \cdot, \cdot \rangle$ is alternating and $\mathbb{T}_{\mathfrak{m}}$-linear, it follows that each ϕ_i does indeed descend to a $\mathbb{T}_{\mathfrak{m}}$-module map $\mathcal{X} \to \mathbb{T}_{\mathfrak{m}}$. Now define maps

$$\begin{aligned} \psi_i : \mathbb{T}_{\mathfrak{m}} &\to \mathcal{X}, \quad i = 1, 2 \\ \psi_1 : 1 &\mapsto U_\ell \pi(e_1) \\ \psi_2 : 1 &\mapsto -U_\ell \pi(e_2). \end{aligned}$$

We claim that ψ_i and ϕ_i satisfy the conclusion of the lemma. One readily calculates:

$$\begin{aligned} \phi_1 \circ \psi_1(1) &= U_\ell \langle e_1, (F - U_\ell)e_2\rangle \\ \psi_1 \circ \phi_1(e_1) &= U_\ell \langle e_1, (F - U_\ell)e_2\rangle \pi(e_1) \\ \psi_1 \circ \phi_1(e_2) &= U_\ell \langle e_2, (F - U_\ell)e_2\rangle \pi(e_1) \\ &= U_\ell \langle e_1, (F - U_\ell)e_2\rangle \pi(e_2) - U_\ell (F - U_\ell)\pi(e_2) \\ &= U_\ell \langle e_1, (F - U_\ell)e_2\rangle \pi(e_2), \end{aligned}$$

where in the last two steps we have used (32) and the fact that $F = U_\ell$ on $\mathcal{X}$. Similarly,

$$\phi_2 \circ \psi_2 = \psi_2 \circ \phi_2 = -U_\ell \langle e_2, (F - U_\ell)e_1\rangle,$$

and

$$U_\ell \langle e_1, (F - U_\ell)e_2\rangle - U_\ell \langle e_2, (F - U_\ell)e_1\rangle = \mathrm{tr}_{T_{\mathfrak{m}} J_{\min}} U_\ell (F - U_\ell) = \ell - 1.$$

□

4.2. **Shimura curves.**

4.2.1. If $\nu(N_2)$ is even, then there exists a Shimura curve X_{N_1,N_2}, with $\Gamma_0(N_1)$ level structure, associated to the indefinite quaternion algebra $B = B_{N_2}$ over $\mathbb{Q}$ of discriminant N_2. Let

$$J^{N_1,N_2} := J(X_{N_1,N_2}),$$

an abelian variety with a natural action of $\mathbb{T}$ by correspondences (induced by Picard functoriality). When N_1 and N_2 are understood, we abbreviate $J = J^{N_1,N_2}$. By the Jacquet-Langlands correspondence and Faltings' Theorem [18], there is a noncanonical Hecke-equivariant isogeny $J \to J_{\min}$. Consider the following technical hypothesis on the residual representation $\overline{\rho}_{\mathfrak{m}} : G_{\mathbb{Q}} \to GL_2(\mathbb{T}/\mathfrak{m})$ associated to $\mathfrak{m}$:

(∗) If $p = 3$ and $\overline{\rho}_{\mathfrak{m}}$ is induced from a character of $G_{\mathbb{Q}(\sqrt{-3})}$, $\exists\, \ell || N_2$ such that either $\ell \equiv -1 \pmod 3$ or $\overline{\rho}_{\mathfrak{m}}$ is ramified at ℓ.

Theorem 4.2.2 ((Helm))**.** *Let $\mathfrak{m} \subset \mathbb{T}$ be a non-Eisenstein maximal ideal of residue characteristic $p \nmid 2N$ satisfying (∗). Then there is an isomorphism of $\mathbb{T}_{\mathfrak{m}}$-modules:*

$$\mathrm{Hom}(J_{\min}, J)_{\mathfrak{m}} \simeq \otimes_{\ell | N_2} \mathcal{X}_\ell (J_{\min}^\vee)_{\mathfrak{m}},$$

modulo $\mathbb{Z}$-torsion on the right-hand side. Here, the tensor products are over $\mathbb{T}_{\mathfrak{m}}$.

Proof. This is essentially [25, Theorem 8.7]; to complete the case $p = 3$, by [25, Remark 8.12] one only needs a level-raising input that is provided by [16]. □

4.3. **Shimura sets.**

4.3.1. Let $B = B_{N_2}$ be the quaternion algebra over $\mathbb{Q}$ ramified at N_2 (and possibly ∞). Recall [40, 52] that an oriented Eichler order (R, ϕ) of level N_1 in B is an Eichler order R of level N_1 equipped with a local orientation ϕ_ℓ for each $\ell | N$. If $\ell | N_1$, then $R \otimes \mathbb{Z}_\ell$ is of the form $R_1 \cap R_2$ for a uniquely determined unordered pair (R_1, R_2) of maximal orders of $B \otimes \mathbb{Q}_\ell$; ϕ_ℓ is the data of a choice of ordering of (R_1, R_2). If $\ell | N_2$, then ϕ_ℓ is the data of an isomorphism of the residue field of R with a fixed field $\mathbb{F}_{\ell^2}$ of cardinality ℓ^2.

4.3.2. If $\nu(N_2)$ is odd, then we define the Shimura set X_{N_1,N_2} to be the set of isomorphism classes of oriented Eichler orders of level N_1 in B. Because $B^\times(\mathbb{A}_f)$ has a natural transitive action on the set of oriented Eichler orders of level N_1, choosing an oriented Eichler order (R, ϕ) as a base point identifies X_{N_1,N_2} with the finite double coset space

$$B^\times(\mathbb{Q}) \backslash B^\times(\mathbb{A}_\mathbb{Q}) / \widehat{R}^\times. \tag{33}$$

When N_1 and N_2 are clear from context, the subscripts on X_{N_1,N_2} may be omitted.

4.3.3. The $\mathbb{Z}$-module $\mathbb{Z}[X]^0$ of formal degree-zero divisors in X has two natural actions of $\mathbb{T} = \mathbb{T}_{N_1,N_2}$ by correspondences: an "Albanese" action induced by viewing an element of $\mathbb{Z}[X]^0$ as a formal sum of points in a double coset space, and a "Picard" action induced by identifying $\mathbb{Z}[X] = \mathrm{Hom}_{\mathrm{Set}}(X, \mathbb{Z})$. We will consider $\mathbb{Z}[X]^0$ as a $\mathbb{T}$-module through the latter action. The analogue of Theorem 4.2.2 is:

Theorem 4.3.4. *Let $\mathfrak{m} \subset \mathbb{T}$ be a non-Eisenstein maximal ideal of residue characteristic $p \nmid 2N$ satisfying $(*)$. Then there is an isomorphism of $\mathbb{T}_\mathfrak{m}$-modules:*

$$\mathbb{Z}[X]^0_\mathfrak{m} \simeq \otimes_{\ell | N_2} \mathcal{X}_\ell(J^\vee_{\min})_\mathfrak{m},$$

modulo $\mathbb{Z}$-torsion on the right-hand side. Here, the tensor products are taken over $\mathbb{T}_\mathfrak{m}$.

Proof. Choose any prime $q | N_2$, so that $\nu(N_2/q)$ is even. Let $\mathbb{T}' = \mathbb{T}_{N_1 q, N_2/q}$, and write $\mathfrak{m}$ as well for the maximal ideal of $\mathbb{T}'$ induced by the map $\mathbb{T}' \to \mathbb{T}$.

Applying Theorem 4.2.2 to the pair $N_1 q, N_2/q$, we obtain an isomorphism of $\mathbb{T}'_\mathfrak{m}$-modules (modulo $\mathbb{Z}$-torsion)

$$\mathrm{Hom}(J^{N_1 q, N_2/q}_{\min}, J^{N_1 q, N_2/q})_\mathfrak{m} \simeq \otimes_{\ell | N_2/q} \mathcal{X}_\ell(J^{N_1 q, N_2/q, \vee}_{\min}). \tag{34}$$

By [25, Corollary 5.3, Lemma 8.2], this implies an isomorphism of $\mathbb{T}_\mathfrak{m}$-modules

$$\mathrm{Hom}(J^{N_1, N_2}_{\min}, J^{N_1 q, N_2/q}_{q\text{-new}})_\mathfrak{m} \simeq \otimes_{\ell | N_2/q} \mathcal{X}_\ell(J^{N_1, N_2, \vee}_{\min}), \tag{35}$$

where $J^{N_1 q, N_2/q}_{q\text{-new}}$ is the q-new quotient of $J^{N_1 q, N_2/q}$. Then, by Proposition 4.1.3, we have

$$\mathcal{X}_q(J^{N_1 q, N_2/q, \vee}_{q\text{-new}})_\mathfrak{m} \simeq \mathcal{X}_q(J^{N_1, N_2, \vee}_{\min})_\mathfrak{m} \otimes_{\ell | N_2/q} \mathcal{X}_\ell(J^{N_1, N_2, \vee}_{\min})_\mathfrak{m}. \tag{36}$$

By [2, Proposition 5.3], $\mathcal{X}_q(J^{qN_1, N_2/q, \vee})$ is identified with $\mathbb{Z}[X_{N_1,N_2}]^0$. It remains to show that the inclusion $J^{N_1 q, N_2/q, \vee}_{q\text{-new}} \hookrightarrow J^{N_1 q, N_2/q, \vee}$ induces an isomorphism on character groups at q. Indeed, since $J^{N_1 q, N_2/q, \vee}_{q\text{-new}}$ has purely toric reduction at q, there is a surjection of character groups

$$\mathcal{X}_q(J^{N_1 q, N_2/q, \vee}) \twoheadrightarrow \mathcal{X}_q(J^{N_1 q, N_2/q, \vee}_{q\text{-new}}). \tag{37}$$

After tensoring both sides with $\mathbb{Q}$, (37) is an isomorphism because the q-old isogeny factors of $J^{N_1 q, N_2/q, \vee}$ have good reduction at q. Since the source of (37) is a free $\mathbb{Z}$-module, the surjection is an isomorphism. $\square$

4.4. **Special fibers of Shimura curves.**

4.4.1. Assume $\nu(N_2)$ is **even**. We choose an oriented maximal order $\mathcal{O}_B \subset B = B_{N_2}$, and a positive involution $*$ on $\mathcal{O}_B$. Then the Shimura curve X_{N_1,N_2} has a moduli interpretation as a space of triples (A, ι, C), where A is an abelian surface, ι is an embedding $\mathcal{O}_B \hookrightarrow \mathrm{End}(A)$, and C is a $\Gamma_0(N_1)$ structure in the sense of [52]. Details can be found, e.g., in [5]. In this subsection, we will recall (following [52]) the geometry of the special fiber of the canonical model of X_{N_1,N_2} over $\mathbb{Z}_q$ in two cases: $q \nmid N_1 N_2$, and $q | N_2$.

4.4.2. Consider the following hypothesis on the residual representation $\overline{\rho}_{\mathfrak{m}}$ associated to a non-Eisenstein maximal ideal $\mathfrak{m} \subset \mathbb{T}_{N_1,N_2}$ of residue characteristic p (for any Hecke algebra $\mathbb{T}_{N_1,N_2}$):

$$\text{(TW)} \qquad \text{if } p = 3, \text{ then } \overline{\rho}_{\mathfrak{m}} \text{ is absolutely irreducible over } \mathbb{Q}(\sqrt{-3}).$$

Note that this is strictly stronger than condition $(*)$ above.

Proposition 4.4.3. *Suppose $\nu(N_2)$ is even, and fix a prime $q \nmid N_1N_2$. Then:*

(1) The supersingular locus $X_{N_1,N_2}(\overline{\mathbb{F}_{q^2}})^{ss} = X_{N_1,N_2}(\mathbb{F}_{q^2})^{ss}$ is canonically identified with X_{N_1,N_2}. This identification is compatible with the action of $\mathbb{T}_{N_1q,N_2}$, where U_q acts on $X_{N_1,N_2}(\mathbb{F}_{q^2})^{ss}$ by Frob_q.

(2) If $\mathfrak{m} \subset \mathbb{T}_{N_1q,N_2}$ is a non-Eisenstein maximal ideal of residue characteristic $p \nmid 2Nq$ satisfying (TW), then the induced map

$$\mathbb{Z}[X_{N_1,N_2q}]^0_{\mathfrak{m}} \to J^{N_1,N_2}(\mathbb{F}_{q^2})_{\mathfrak{m}}$$

is surjective.

Proof. The first part is proven in [52], but we recall the construction for use below in Proposition 4.6.12. If (A,ι,C) is a point of $X_{N_1,N_2}(\overline{\mathbb{F}_{q^2}})^{ss}$, then $\mathrm{End}^0(A,\iota)$ is isomorphic to B_{N_2q}, and $R := \mathrm{End}(A,\iota,C) \subset \mathrm{End}^0(A,\iota)$ is an Eichler order of level N_1. Moreover, R has a natural orientation at all primes dividing $\ell | Nq$, which we now recall. For $\ell | N_1$, the local orientation is determined by the inclusion $R \subset \mathrm{End}(A,\iota)$, where the latter is a maximal order.

For $\ell | N_2$, if $\mathfrak{m}_\ell \subset \mathcal{O}_B$ is the unique maximal ideal of residue characteristic ℓ, then $A[\mathfrak{m}_\ell]$ is a vector space of dimension one over $\mathcal{O}_B/\mathfrak{m}_\ell \simeq \mathbb{F}_{\ell^2}$, where the isomorphism is chosen according to the orientation of $\mathcal{O}_B$. The action of R on $A[\mathfrak{m}_\ell]$ therefore defines a homomorphism $R \to \mathbb{F}_{\ell^2}$, which we take to be the orientation of R at ℓ.

Finally, the Lie algebra of A is a $\overline{\mathbb{F}_{q^2}}$-vector space of dimension 2, on which R acts by scalars valued in $\mathbb{F}_{q^2}$, cf. [52, p. 24]. This defines a map $R \to \mathbb{F}_{q^2}$, which we take to be the local orientation q. Thus for every $(A,\iota,C) \in X_{N_1,N_2}(\overline{\mathbb{F}_{q^2}})^{ss}$, we have described an oriented Eichler order of level N_1 in B_{N_2q}, well-defined up to the choice of isomorphism $\mathrm{End}^0(A,\iota) \simeq B_{N_2q}$, i.e. up to $B^\times_{N_2q}(\mathbb{Q})$-conjugacy. This describes a map

$$X_{N_1,N_2}(\overline{\mathbb{F}_{q^2}})^{ss} \to X_{N_1,N_2q},$$

and [52, Theorem 3.4] shows that this map is an isomorphism.

The Hecke compatibility for operators coprime to q is clear from the construction. We can also see that replacing (A,ι,C) by its Frobenius twist has the effect of switching the orientation of $\mathrm{End}(A,\iota,C)$ at q, which is precisely the action of U_q on X_{N_1,N_2q}. In particular, Frob_q^2 acts trivially on $X_{N_1,N_2}(\overline{\mathbb{F}_{q^2}})^{ss}$, so all the supersingular points are in fact defined over $\mathbb{F}_{q^2}$.

Part (ii) is a well-known application of Ihara's Lemma which can be deduced from the argument in [2, Proposition 9.2]: we add auxiliary level of the form $\Gamma_1(\ell)$, where $\ell \nmid N$ is a prime such that $\ell - 1, T_\ell - \ell - 1 \notin \mathfrak{m}$. That such a prime exists follows from condition (TW) by [16, Lemma 3]. $\square$

4.4.4. Now suppose instead that $q | N_2$. The Shimura curve X_{N_1,N_2} has a canonical, semistable integral model over $\mathbb{Z}_q$. We denote by $X^{\pm}_{N_1,N_2/q}$ the set $X_{N_1,N_2/q} \times \{\pm\}$.

Proposition 4.4.5. *The set of irreducible geometric components of the special fiber at q of X_{N_1,N_2} is canonically identified with $X^{\pm}_{N_1,N_2/q}$. Each component is defined over $\mathbb{F}_{q^2}$, and the Frobenius action switches the sign without changing the value in $X_{N_1,N_2/q}$.*

This identification is equivariant for the action of $\mathbb{T}_{N_1q,N_2/q}$, where U_q acts on $X^{\pm}_{N_1,N_2/q}$ as the correspondence

$$\begin{pmatrix} T_q & q \\ -1 & 0 \end{pmatrix}.$$

Proof. This follows from [52, Theorem 5.4], but once again we recall the construction for use in Proposition 4.6.12 below. First, [52, Theorem 5.3] identifies the set of irreducible components with the set of so-called pure triples (A,ι,C), where A is a superspecial abelian surface over $\mathbb{F}_{q^2}$ with an embedding $\iota : \mathcal{O}_B \hookrightarrow \mathrm{End}(A)$ and C is a $\Gamma_0(N_1)$-level structure. The purity condition means that $\mathcal{O}_B$ acts on the 2-dimensional $\mathbb{F}_{q^2}$-vector space $\mathrm{Lie}\, A$ via scalars, i.e. through a homomorphism $\mathcal{O}_B \to \mathbb{F}_{q^2}$. Since $\mathcal{O}_B$ is given with an orientation at q, we say a pure triple is of type $+$ if this homomorphism agrees with the orientation, and of type $-$ otherwise.

Now for any pure triple (A, ι, C), $\mathrm{End}^0(A, \iota)$ is isomorphic to $B_{N_2/q}$, and $\mathrm{End}(A, \iota, C)$ is canonically an oriented Eichler order of level N_1 in $\mathrm{End}^0(A, \iota)$. (The orientation is defined as in the proof of Proposition 4.4.3.) Thus we have a well-defined map from the set of irreducible components to $X^{\pm}_{N_1,N_2/q}$, sending a pure triple (A, ι, C) of type $\delta \in \{\pm\}$ to $(\mathrm{End}(A, \iota, C), \delta)$. This map is an isomorphism by [52, Theorem 4.13], and the Frobenius action is described in the following remark of loc. cit. The Hecke equivariance it is clear away from q, and at q it is given by [2, Proposition 5.8(ii)]. (The U_q operator there is the adjoint of ours since they are describing the Picard action.) $\square$

4.4.6. We continue to assume $q|N_2$ and $\nu(N_2)$ is even. The Néron model $\mathcal{J}_q$ of the Jacobian J^{N_1,N_2} has purely toric reduction, and we write $\mathcal{X}$ and Φ for the character group and the group of connected components, respectively, of its special fiber. Recall the rigid-analytic uniformization of J^{N_1,N_2}, which gives rise to an exact sequence:

$$0 \to \mathcal{X} \to \mathcal{X}^\dagger \otimes \overline{\mathbb{Q}}_q \to J^{N_1,N_2}(\overline{\mathbb{Q}}_q) \to 0. \tag{38}$$

Here, $\mathcal{X}^\dagger = \mathrm{Hom}(\mathcal{X}, \mathbb{Z})$, and the maps are Hecke-equivariant if $\mathcal{X}$ is given Hecke action through Albanese functoriality, and the actions on $\mathcal{X}^\dagger$ and $J_q(\overline{\mathbb{Q}}_q)$ are induced by Picard functoriality. Importantly for our later applications, (38) is compatible with the Galois action of $G_{\mathbb{Q}_q}$ [4], where the action on $\mathcal{X}$ is unramified and Frobenius acts through U_q [53, Proposition 3.8]. The rigid analytic uniformization is related to the monodromy pairing $j : \mathcal{X} \to \mathcal{X}^\dagger$ of Grothendieck [24] by the commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{X} \arrow[r] \arrow[d, equal] & \mathcal{X}^\dagger \otimes \mathbb{Q}_{q^2} \arrow[r] \arrow[d, "\mathrm{ord}"] & J^{N_1,N_2}(\mathbb{Q}_{q^2}) \arrow[r] \arrow[d, "\mathrm{Sp}_q"] & 0 \\ 0 \arrow[r] & \mathcal{X} \arrow[r, "j"] & \mathcal{X}^\dagger \arrow[r] & \Phi \arrow[r] & 0. \end{tikzcd} \tag{39}$$

Since $U_q^2 = 1$ on Φ, Proposition 4.4.5 induces a canonical map of $\mathbb{T}_{N_1q,N_2/q}$-modules

$$\frac{\mathbb{Z}[X^{\pm}_{N_1,N_2/q}]^0}{(U_q^2 - 1)} \to \Phi.$$

Proposition 4.4.7. *Suppose* $\mathfrak{m} \subset \mathbb{T}_{N_1q,N_2}$ *is a non-Eisenstein maximal ideal. Then the induced map*

$$\frac{\mathbb{Z}[X^{\pm}_{N_1,N_2/q}]^0_{\mathfrak{m}}}{(U_q^2 - 1)} \to \Phi_{\mathfrak{m}}$$

is an isomorphism.

Proof. This is [2, Proposition 5.13]. $\square$

4.5. Geometric level raising.

4.5.1. Let f be a cuspidal eigenform of weight two and trivial character, new of level N, and let $\wp \subset \mathcal{O}_f$ be a prime ideal of the ring of integers of its coefficient field, with residue characteristic $p \nmid 2N$. We write $\mathcal{O}$ for the completion of $\mathcal{O}_f$ at $\wp$, and π for a uniformizer of $\mathcal{O}$. We assume the residual representation $\overline{T}_f$ associated to f and $\wp$ is absolutely irreducible. We also fix a factorization $N = N_1N_2$, where N_1 and N_2 are coprime, and N_2 is squarefree. If Q and Q' are coprime squarefree positive integers, then we abbreviate $\mathbb{T}^Q_{Q'} = \mathbb{T}_{N_1Q,N_2Q'}$, omitting any superscript or subscript which is equal to 1.

Definition 4.5.2. We say a prime $q \nmid N$ is **weakly j-admissible** with sign $\epsilon_q = \pm 1$ if $a_q \equiv \epsilon_q(q + 1) \pmod{\pi^j}$ and $q \not\equiv 1 \pmod p$. In this case, $T_j := T_f/\pi^j$ has a unique subspace $\mathrm{Fil}^+_{q,\epsilon_q} T_j$, free of rank one over $\mathcal{O}_j := \mathcal{O}/\pi^j$, on which Frob_q acts as $q\epsilon_q$. We will omit the subscript ϵ_q when there is no risk of confusion. We say (q, ϵ_q) is **weakly admissible** if it is j-admissible for some $j \geq 1$. A **weakly admissible pair** $\{Q, \epsilon_Q\}$ is an ordered pair of a squarefree number Q and a function $\epsilon_Q : \{q|Q\} \to \{\pm 1\}$ such that q is weakly admissible with sign $\epsilon_Q(q)$ for all $q|Q$. If $\{Q, \epsilon_Q\}$ is a weakly admissible pair, then for all $q|Q$, there is a unique root $u_q \in \mathcal{O}$ of the polynomial $y^2 - ya_q + q$ such that $u_q \equiv \epsilon_Q(q) \pmod{\wp}$. We view $\mathcal{O}$ as a $\mathbb{T}^Q$-algebra by letting U_q act through u_q, and letting the other Hecke operators act through their eigenvalues on f; let $\mathfrak{m}^{\epsilon_Q}_Q$ be the associated maximal ideal (we will usually drop the superscript). Finally, a weakly admissible pair $\{Q, \epsilon_Q\}$ is called j**-level-raising** if

$$\mathrm{lg}_{\mathcal{O}}\left(\mathbb{T}_Q \otimes_{\mathbb{T}^Q} \mathcal{O}\right) \geq j.$$

Remark 4.5.3. If $\{Q, \epsilon_Q\}$ is j-level-raising, then each $q|Q$ is j-admissible. Indeed, in $\mathbb{T}_Q$ we have $U_q^2 = 1$, but U_q acts on $\mathcal{O}$ by the unique root of $y^2 - ya_q + q$ congruent to $\epsilon_Q(q)$; hence $a_q = \epsilon_q(q+1)$ in $\mathbb{T}_Q \otimes_{\mathbb{T}^Q} \mathcal{O}$.

4.5.4. In light of the structural similarity of Theorems 4.2.2 and 4.3.4, let

$$M_Q = \begin{cases} \mathrm{Hom}(J_{\min}^{N_1, N_2 Q}, J^{N_1, N_2 Q}), & \nu(N_2 Q) \text{ even}, \\ \mathbb{Z}[X_{N_1, N_2 Q}]^0, & \nu(N_2 Q) \text{ odd}. \end{cases} \tag{40}$$

It is well-known that M_Q is a faithful $\mathbb{T}_Q$-module, and indeed $M_Q \otimes \mathbb{Q}$ is free of rank one over $\mathbb{T}_Q \otimes \mathbb{Q}$.

Lemma 4.5.5. *Suppose $\{Q, \epsilon_Q\}$ is a weakly admissible pair, and let*

$$C = \sum_{\substack{\ell | N_2 \\ \overline{T}_f \text{ unram at } \ell}} \mathrm{ord}_\pi(\ell - 1).$$

Then there exists an $\mathcal{O}$-module map

$$M_Q \otimes_{\mathbb{T}^Q} \mathcal{O} \to \mathbb{T}_Q \otimes_{\mathbb{T}^Q} \mathcal{O}$$

with kernel and cokernel annihilated by π^C; in particular, $\pi^C (M_Q \otimes_{\mathbb{T}^Q} \mathcal{O})$ is principal of length at least $\mathrm{lg}(\mathbb{T}_Q \otimes_{\mathbb{T}^Q} \mathcal{O}) - 2C$.

Proof. We may assume that $\mathfrak{m}_Q \subset \mathbb{T}^Q$ descends to $\mathbb{T}_Q$. Now, by Theorems 4.2.2 and 4.3.4, we have

$$M_{Q, \mathfrak{m}_Q} \simeq \otimes_{\ell | N_2 Q} \mathcal{X}_\ell \left(J_{\min}^{N_1, N_2 Q, \vee}\right)_{\mathfrak{m}_Q},$$

modulo $\mathbb{Z}$-torsion on the right. Lemma 4.1.4 implies that there exists a collection of $\mathbb{T}_Q$-module maps

$$\phi_i : M_{Q, \mathfrak{m}_Q} \to \mathbb{T}_{Q, \mathfrak{m}_Q}, \;\; \psi_i : \mathbb{T}_{Q, \mathfrak{m}_Q} \to M_{Q_n, \mathfrak{m}_{Q_n}}, \;\; i = 1, \ldots, r$$

such that

$$\phi_i \circ \psi_i = \psi_i \circ \phi_i = t_i \in \mathbb{T}_{Q, \mathfrak{m}_Q} \subset \mathrm{End}(M_{Q_n, \mathfrak{m}_{Q_n}})$$

and

$$t_1 + \ldots + t_r = \prod_{\substack{\ell | N_2 \\ \overline{T}_f \text{ unram at } \ell}} (\ell - 1) \in \mathbb{T}_{Q, \mathfrak{m}_Q}.$$

Since $\mathcal{O}$ is principal, we may choose some i such that the image of t_i in $\mathbb{T}_Q \otimes_{\mathbb{T}^Q} \mathcal{O}$ divides π^C. Then ϕ_i and ψ_i induce $\mathcal{O}$-module maps

$$M_Q \otimes_{\mathbb{T}^Q} \mathcal{O} \to \mathbb{T}_Q \otimes_{\mathbb{T}^Q} \mathcal{O}, \;\; \mathbb{T}_Q \otimes_{\mathbb{T}^Q} \mathcal{O} \to M_Q \otimes_{\mathbb{T}^Q} \mathcal{O}$$

whose composition in either direction is multiplication by a divisor of π^C, which implies the result. □

The following corollary is not needed for geometric level raising, but will be used later in the construction of bipartite Euler systems.

Corollary 4.5.6. *Let $\{Q, \epsilon_Q\}$ be a weakly admissible pair that is $(j + 2C)$-level raising. Then there exists a map of $\mathbb{T}^Q$-Galois modules*

$$T_{\mathfrak{m}_Q} J^{N_1, N_2 Q} \to T_j$$

the factors through multiplication by π^C and is surjective after $\mathcal{O}$-linearization.

Proof. By Lemma 4.5.5 and Proposition 4.1.3, there is a unique (up to scalars) map of $\mathbb{T}^Q[G_{\mathbb{Q}}]$-modules $T_{\mathfrak{m}_Q} J^Q \to T_{\mathfrak{m}_Q} J_{\min}^{N^+, N^- Q} \otimes_{\mathbb{T}^Q} \mathcal{O}$ that is surjective after $\mathcal{O}$-linearization. Since $T_{\mathfrak{m}_Q} J_{\min}^{N^+, N^- Q}$ is free of rank 2 over $\mathbb{T}_{Q, \mathfrak{m}_Q}$ and satisfies the Eichler-Shimura relation

$$\mathrm{Frob}_\ell^2 - T_\ell \,\mathrm{Frob}_\ell + \ell = 0 \text{ on } T_{\mathfrak{m}_Q} J_{\min}^{N^+, N^- Q}, \; \forall \ell \nmid Np,$$

it follows that $T_{\mathfrak{m}_Q} J_{\min}^{N^+, N^- Q} \otimes_{\mathbb{T}^Q} \mathcal{O}$ is isomorphic to T_j. (Here we are using the absolute irreduciblity of $\overline{T}_f$.) □

Theorem 4.5.7. *Assume $\overline{T}_f$ satisfies (TW).*

(1) If $\{Qq, \epsilon_{Qq}\}$ is a weakly admissible pair, then

$$\lg(\mathbb{T}_{Qq} \otimes_{\mathbb{T}^{Qq}} \mathcal{O}) \geq \lg\left(\frac{\mathbb{T}_Q \otimes_{\mathbb{T}^Q} \mathcal{O}}{(a_q - \epsilon_q(q+1))}\right) - C,$$

where C is the number of Lemma 4.5.5.

(2) If $\{Q, \epsilon_Q\}$ is a weakly admissible pair such that q is j-admissible with sign $\epsilon_Q(q)$ for all $q|Q$, then $\{Q, \epsilon_Q\}$ is $(j - \nu(Q) \cdot C)$-level-raising.

Proof. (ii) follows from (i) by induction, so we prove (i). There are two cases, depending on the parity of $\nu(N_2Q)$.

Case 1. *$\nu(N_2Q)$ is even.*

Let us abbreviate $J^Q = J^{N_1, N_2Q}$ and $J^Q_{\min} = J^{N_1,N_2Q}_{\min}$. Consider the composite

$$M_{Qq,\mathfrak{m}_{Qq}} \to J^Q(\mathbb{F}_{q^2})_{\mathfrak{m}_{Qq}} \to H^1\left(\mathbb{F}_q^2, T_{\mathfrak{m}_Q} J^Q\right)_{\mathfrak{m}_{Qq}} \simeq M_Q \otimes \frac{T_{\mathfrak{m}_Q} J^Q_{\min}}{(U_q - \epsilon_q)}$$

induced from Proposition 4.4.3, the Kummer map, and Proposition 4.1.3. These are surjective maps of $\mathbb{T}^q_Q$-modules, where U_q acts on the three latter modules through Frob_q. Since $T_{\mathfrak{m}_Q} J^Q_{\min}$ is free of rank two over $\mathbb{T}_{Q,\mathfrak{m}_Q}$ and Frob_q acts with the characteristic polynomial $\mathrm{Frob}_q^2 - T_q \mathrm{Frob}_q + q$ (whose roots are distinct modulo $\mathfrak{m}_Q$), we may fix an identification

$$\frac{T_{\mathfrak{m}_Q} J^Q_{\min}}{(U_q - \epsilon_q)} \simeq \frac{\mathbb{T}_{Q,\mathfrak{m}_Q}}{(T_q - \epsilon_q(q+1))},$$

considered as a $\mathbb{T}^q_{Q,\mathfrak{m}_{Qq}}$-module again through U_q acting by ϵ_q. Tensoring with $\mathcal{O}$, we obtain a surjective map

$$M_{Qq} \to M_Q \otimes_{\mathbb{T}^{Qq}} \frac{\mathcal{O}}{(a_q - \epsilon_q(q+1))},$$

hence (by Lemma 4.5.5) a map of $\mathbb{T}^{Qq}$-modules $M_{Qq} \to \frac{\mathbb{T}_Q \otimes \mathcal{O}}{(a_q - \epsilon_q(q+1))}$ with cokernel annihilated by π^C. Since the action of $\mathbb{T}^{Qq}$ on M_{Qq} factors through $\mathbb{T}_{Qq}$, we obtain, by taking eigenvalues, a surjection $\mathbb{T}_{Qq} \to \mathcal{O}/\pi^j$ for some

$$j \geq \lg\left(\frac{\mathbb{T}_Q \otimes_{\mathbb{T}^Q} \mathcal{O}}{(a_q - \epsilon_q(q+1))}\right) - C.$$

Case 2. *$\nu(N_2Q)$ is odd.*

By Proposition 4.4.7, the action of $\mathbb{T}^q_{Q,\mathfrak{m}_{Qq}}$ on

$$M_{Q,\mathfrak{m}_Q} \otimes_{\mathbb{T}_Q} \left(\mathbb{T}_Q^2 / \operatorname{im}\begin{pmatrix} T_q - \epsilon_q & q \\ -1 & -\epsilon_q \end{pmatrix}\right),$$

with U_q acting by ϵ_q, factors through $\mathbb{T}_{Qq,\mathfrak{m}_{Qq}}$. Hence the action of $\mathbb{T}^q_Q$ on

$$A = M_Q \otimes_{\mathbb{T}^Q} \frac{\mathcal{O}}{(a_q - \epsilon_q(q+1))}$$

likewise factors through $\mathbb{T}_{Qq}$ (again with U_q acting by ϵ_q). The conclusion of Lemma 4.5.5 implies that A has a $\mathbb{T}^q_Q$-module map to

$$\frac{\mathbb{T}_Q \otimes_{\mathbb{T}^Q} \mathcal{O}}{(a_q - \epsilon_q(q+1))}$$

with cokernel annihilated by π^C, from which the result follows. □

Remark 4.5.8. If $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$ in the notation of (3.2.2), then for $\mathfrak{F}$-many n there is a corresponding weakly admissible pair $\{Q_n, \epsilon_{Q_n}\}$, where Q_n is a sequence representing Q. To be precise, if $\mathsf{Q} = \{\mathsf{q}_1, \ldots, \mathsf{q}_r\}$, we choose sequences q_i^n representing each q_i; for $\mathfrak{F}$-many n, the product $Q_n = q_1^n \cdots q_r^n$, equipped with sign function $\epsilon_{Q_n}(q_i^n) = \epsilon_{\mathsf{Q}}(\mathsf{q}_i)$, forms a weakly admissible pair $\{Q_n, \epsilon_{Q_n}\}$. It follows from the definition of N and from the theorem that, for any $j \geq 0$, there exist $\mathfrak{F}$-many n which are j-level-raising. We say that a sequence of weakly admissible pairs $\{Q_n, \epsilon_{Q_n}\}$ (defined for $\mathfrak{F}$-many n) represents the pair $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\}$ if it is obtained from this construction for some choice of representatives q_i^n.

4.6. **CM points.**

4.6.1. Let us now fix an imaginary quadratic field $K \subset \overline{\mathbb{Q}}$, and a positive integer $N = N^+N^-$ such that every prime factor of N^+ is split in K, and N^- is a squarefree product primes inert in K. Let $B = B_{N^-}$ be the quaternion algebra over $\mathbb{Q}$ ramified exactly of the factors of N^-, and possibly ∞. For each $\ell | N$, we have a fixed embedding $K \hookrightarrow \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_\ell}$. If $\ell | N^+$, this determines a distinguished prime $\mathfrak{l}$ of K above ℓ. We write $\mathfrak{l}^c$ for its conjugate. If $\ell | N^-$, this determines a distinguished isomorphism $\mathcal{O}_K/\ell \simeq \mathbb{F}_{\ell^2}$. Finally, for each positive integer m, let $\mathcal{O}_{m,K} \subset \mathcal{O}_K$ be the order of conductor m.

4.6.2. Assume $\nu(N^-)$ is **even**. We choose an oriented maximal order $\mathcal{O}_B \subset B$ to give a moduli interpretation to X_{N^+,N^-} as in (4.4.1).

Definition 4.6.3. Let m be a positive integer coprime to N, and write $K[m]$ for the ring class field of conductor m. The set of (positively oriented) CM points of conductor m,

$$\mathrm{CM}_{N^+,N^-}(m) \subset X_{N^+,N^-}(K[m]),$$

is the set of triples (A, ι, C) over $K[m]$ admitting an isomorphism $\mathcal{O}_{m,K} \xrightarrow{\sim} \mathrm{End}(A, \iota, C)$ such that:

(1) The action of $\mathcal{O}_{m,K}$ on the $K[m]$-vector space $\mathrm{Lie}\, A$ agrees with the fixed embedding $\mathcal{O}_{m,K} \subset K \hookrightarrow \overline{\mathbb{Q}} \hookrightarrow K[m]$.
(2) For all $\ell | N^+$, $\mathcal{O}_{m,K} \otimes \mathbb{Z}_\ell = \mathcal{O}_{K,\mathfrak{l}} \times \mathcal{O}_{K,\mathfrak{l}^c}$ acts on the ℓ-primary component $C_\ell \subset C$ through the projection to $\mathcal{O}_{K,\mathfrak{l}}$.
(3) For all $\ell | N^-$, let $\mathfrak{m}_\ell \subset \mathcal{O}_B$ be the unique maximal ideal of residue characteristic ℓ. Then $A[\mathfrak{m}_\ell](\overline{K[m]})$ is a rank-one vector space over $\mathcal{O}_B/\mathfrak{m}_\ell \simeq \mathbb{F}_{\ell^2}$, where the isomorphism comes from the orientation of $\mathcal{O}_B$ at ℓ. We require that the action of $\mathcal{O}_{m,K}/\ell$ on this vector space correspond to our distinguished isomorphism $\mathcal{O}_{m,K}/\ell \simeq \mathbb{F}_{\ell^2}$.

4.6.4. If A is an abelian variety, any element $\gamma \in (\mathrm{End}(A) \otimes \mathbb{A}_f)^\times$ defines an abelian variety A_γ with a map $f : A_\gamma \to A$ in the isogeny category of abelian varieties such that $f_*(T_\ell A_\gamma) = \gamma_\ell T_\ell A$ for all ℓ. In this way, we obtain a canonical action of

$$\mathrm{Pic}\, \mathcal{O}_{m,K} = K^\times \backslash \mathbb{A}_{f,K}^\times / \widehat{\mathcal{O}}_{f,K}^\times$$

on $\mathrm{CM}_{N^+,N^-}(m)$. We denote by

$$\mathrm{rec} : \mathrm{Gal}(K[m]/K) \to K^\times \backslash \mathbb{A}_{f,K}^\times / \widehat{\mathcal{O}}_{f,K}^\times$$

the reciprocity map of class field theory, normalized so that uniformizers correspond to geometric Frobenius elements.

Proposition 4.6.5. *(1) Via the reciprocity map, the action of* $\mathrm{Gal}(K[m]/K)$ *on* $\mathrm{CM}_{N^+,N^-}(m)$ *agrees with the action of* $K^\times \backslash \mathbb{A}_{f,K}^\times / \widehat{\mathcal{O}}_{m,K}^\times$ *described above.*
(2) $\mathrm{CM}_{N^+,N^-}(m)$ *is a torsor under the action of* $\mathrm{Gal}(K[m]/K)$.

Proof. Part (i) follows from Shimura's reciprocity law. For (ii), see the discussion in [67, p. 55]; it is an elementary exercise using the complex uniformization of X_{N^+,N^-} to see that our definition of the positively oriented CM points of conductor m agrees with the adelic description given in loc. cit. (Recall that any $\mathbb{C}$-valued point of X_{N^+,N^-} admitting extra endomorphisms by $\mathcal{O}_{m,K}$ is automatically defined over $K[m]$.) □

4.6.6. Now assume that $\nu(N^-)$ is **odd**. In this case, we fix an embedding $K \hookrightarrow B$.

Definition 4.6.7. Suppose m is coprime to N. Then $\mathrm{CM}_{N^+,N^-}(m)$ is defined as the set of isomorphism classes of oriented Eichler orders (R, ϕ) of B of level N^+ such that:

(1) $R \cap K = \mathcal{O}_{m,K}$.
(2) For primes $\ell | N^+$, let $R_1 \supset R \otimes \mathbb{Z}_\ell$ be the maximal order determined by ϕ_ℓ. If $\ell^k || N$, then we require that the natural map $\mathcal{O}_{m,K} \to (R \otimes \mathbb{Z}_\ell)/\ell^k R_1 \simeq \mathbb{Z}/\ell^k$ is given by the projection $\mathcal{O}_{m,K}/\ell^k \twoheadrightarrow \mathcal{O}_{m,K}/\mathfrak{l}^k$.
(3) For all $\ell | N^-$, let $\mathfrak{m}_\ell \subset R$ be the unique ideal of residue characteristic ℓ. Then we require that the isomorphism $\mathcal{O}_{m,K}/\ell = R/\mathfrak{m}_\ell \simeq \mathbb{F}_{\ell^2}$ determined by ϕ_ℓ agrees with the fixed isomorphism $\mathcal{O}_K/\ell$ chosen above.

Here the equivalence relation is conjugation by $K^\times$. (Properties (i)-(iii) are not stable under conjugation by $B^\times(\mathbb{Q})$.)

Notice that $\mathrm{CM}_{N^+,N^-}(m)$ comes equipped with a natural projection map

$$\mathrm{CM}_{N^+,N^-}(m) \to X_{N^+,N^-}. \tag{41}$$

4.6.8. We define an action of $\mathrm{Gal}(K[m]/K)$ on $\mathrm{CM}_{N^+,N^-}(m)$ as follows. Let

$$\mathrm{rec} : \mathrm{Gal}(K[m]/K) \xrightarrow{\sim} K^\times \backslash \mathbb{A}_{f,K}^\times / \widehat{\mathcal{O}}_{m,K}^\times$$

be the reciprocity map of class field theory, normalized so that uniformizers correspond to geometric Frobenius elements. Then for $\sigma \in \mathrm{Gal}(K[m]/K)$, $\sigma \cdot (R,\phi)$ is $\mathrm{rec}(\sigma) \cdot (R,\phi)$: that is, the Eichler order $\mathrm{rec}(\sigma)\widehat{R}\,\mathrm{rec}(\sigma)^{-1} \cap B(\mathbb{Q})$ with the induced orientation.

Proposition 4.6.9. *Suppose m is coprime to N. Then $\mathrm{CM}_{N^+,N^-}(m)$ is a torsor for $\mathrm{Gal}(K[m]/K)$.*

Proof. It is clear that $\mathrm{Gal}(K[m]/K)$ acts with trivial stabilizers on $\mathrm{CM}_{N^+,N^-}(m)$, so we will show transitivity. Let (R,ϕ) and (R',ϕ') be two elements of $\mathrm{CM}_{N^+,N^-}(m)$. Since the orientations ϕ and ϕ' are determined by properties (ii) and (iii) of the definition of $\mathrm{CM}_{N^+,N^-}(m)$, it suffices to show that there exists $k \in \mathbb{A}_{f,K}^\times$ such that

$$k\widehat{R}'k^{-1} = \widehat{R}.$$

We do this by working locally at all primes ℓ. To ease notation, abbreviate

$$R_\ell = R \otimes \mathbb{Z}_\ell,\ R'_\ell = R' \otimes \mathbb{Z}_\ell,\ B_\ell = B \otimes \mathbb{Q}_\ell,\ \mathcal{O}_{m,K,\ell} = \mathcal{O}_{m,K} \otimes \mathbb{Z}_\ell,\ K_\ell = K \otimes \mathbb{Q}_\ell.$$

Suppose first that $\ell \nmid N$. Because $R \cap K = R' \cap K = \mathcal{O}_{m,K}$, [13, Lemma 6.2] implies there exists $k_\ell \in K_\ell^\times$ such that $k_\ell R'_\ell k_\ell^{-1} = R_\ell$. For $\ell | N^-$, the maximal order in B_ℓ is unique, so $R'_\ell = R_\ell$. For $\ell | N^+$, let $j = \mathrm{ord}_\ell(N^+)$, and fix an isomorphism $B_\ell \cong M_2(\mathbb{Q}_\ell)$ that identifies $\mathcal{O}_{K,\ell}$ with

$$\begin{pmatrix} \mathbb{Z}_\ell & 0 \\ 0 & \mathbb{Z}_\ell \end{pmatrix} \subset M_2(\mathbb{Q}_\ell).$$

Eichler orders of level ℓ^j in B_ℓ are all of the form $\mathrm{End}(L_1) \cap \mathrm{End}(L_2)$, where $L_1 \subset L_2$ are lattices in $\mathbb{Q}_\ell^2$ with $L_2/L_1 \simeq \mathbb{Z}/\ell^j\mathbb{Z}$. If $\mathrm{End}(L)$ contains $\mathcal{O}_{m,K,\ell}$, then L is of the form $\mathbb{Z}_\ell \oplus \ell^n\mathbb{Z}_\ell$ (up to homothety). The possible Eichler orders of level ℓ^j containing $\mathcal{O}_{m,K,\ell}$ are therefore

$$\begin{pmatrix} \mathbb{Z}_\ell & \ell^{-n}\mathbb{Z}_\ell \\ \ell^{n+j}\mathbb{Z}_\ell & \mathbb{Z}_\ell \end{pmatrix},\ n \in \mathbb{Z}.$$

These are evidently all conjugate by diagonal matrices, so we may choose $k_\ell \in K_\ell^\times$ such that $k_\ell R'_\ell k_\ell^{-1} = R_\ell$. Setting $k = \prod_{\ell \nmid N^-} k_\ell$, we have

$$k\widehat{R}'k^{-1} \cap B(\mathbb{Q}) = \widehat{R}.$$

□

Remark 4.6.10. We will soon be varying N^- (keeping K and N^+ fixed). The choices made in the definition of the CM points – i.e. the maximal order $\mathcal{O}_B$ if $\nu(N^-)$ is even, and the embedding $K \hookrightarrow B$ if $\nu(N^-)$ is odd – will be considered to be fixed, once and for all, for each possible N^-.

4.6.11. In the remainder of this section, we recall the geometric ingredients for the explicit reciprocity laws originally studied in [2].

Proposition 4.6.12. *Suppose $\nu(N^-)$ is even. Let m be coprime to N, and let $q \nmid Nm$ be a prime inert in K, with $\mathfrak{q}$ a prime of $K[m]$ above q. Then there is an isomorphism $t_{N^+,N^-,q} : \mathrm{CM}_{N^+,N^-}(m) \xrightarrow{\sim} \mathrm{CM}_{N^+,N^-q}(m)$ of $\mathrm{Gal}(K[m]/K)$-torsors fitting into a canonical commutative diagram:*

$$\begin{tikzcd}
\mathrm{CM}_{N^+,N^-}(m) \arrow[rrr, hook] \arrow[d, "t_{N^+,N^-,q}"] & & & X_{N^+,N^-}(K[m]) \arrow[d, "\mathrm{Red}_\mathfrak{q}"] \\
\mathrm{CM}_{N^+,N^-q}(m) \arrow[r] & X_{N^+,N^-q} \arrow[r, "\sim"] & X_{N^+,N^-q}(\mathbb{F}_{q^2})^{ss} \arrow[r, hook] & X_{N^+,N^-q}(\mathbb{F}_{q^2}).
\end{tikzcd}$$

Proof. Choose any $(A,\iota,C)\in \mathrm{CM}_{N^+,N^-}(m)$, and let (A_0,ι_0,C_0) denote its reduction modulo $\mathfrak{q}$. Since q is inert in K, A_0 is supersingular. Moreover we have a distinguished action $\mathcal{O}_{m,K}\hookrightarrow \mathrm{End}(A_0,\iota_0,C_0)$ coming from the reduction of the complex multiplication. Choose an isomorphism $\mathrm{End}^0(A_0,\iota_0,C_0)\simeq B_{N^-q}$ identifying the corresponding embedding $K\hookrightarrow \mathrm{End}^0(A_0,\iota_0,C_0)$ with our fixed inclusion $K\hookrightarrow B_{N^-q}$. The choice of this isomorphism is unique up to $K^\times$-conjugacy. Therefore $\mathrm{End}(A_0,\iota_0,C_0)$ yields a well-defined point of $\mathrm{CM}_{N^+,N^-q}(m)$ – note that conditions (ii) and (iii) of Definition 4.6.7 are satisfied by conditions (ii) and (iii) of Definition 4.6.3. This defines the map $t_{N^+,N^-,q}$, and it is Galois-equivariant by Shimura's reciprocity law for the Galois action on CM points, cf. [66, §5.3-5.4]. Since the $\mathrm{Gal}(K[m]/K)$-action is simply transitive, it is automatically an isomorphism. The commutativity of the diagram follows from the construction in the proof of Proposition 4.4.3. □

Proposition 4.6.13. *Suppose $\nu(N^-)$ is odd. Let m be coprime to N, and let $q\nmid Nm$ be a prime inert in K, with $\mathfrak{q}$ a prime of $K[m]$ above q. Then every point of $\mathrm{CM}_{N^+,N^-q}(m)$ lies in $X_{N^+,N^-q}(K[m]/K)^{sm}$, the subset of points which reduce modulo $\mathfrak{q}$ to smooth points of the special fiber. Moreover, there is an isomorphism $s_{N^+,N^-,q}:\mathrm{CM}_{N^+,N^-q}(m)\xrightarrow{\sim}\mathrm{CM}_{N^+,N^-}$ of $\mathrm{Gal}(K[m]/K)$-torsors fitting into a canonical commutative diagram:*

$$\begin{array}{ccc}
\mathrm{CM}_{N^+,N^-q}(m) & \longrightarrow & X_{N^+,N^-q}(K[m])^{sm} \\
\big\downarrow{\scriptstyle s_{N^+,N^-,q}} & & \big\downarrow{\scriptstyle \mathrm{Sp}_{\mathfrak{q}}} \\
\mathrm{CM}_{N^+,N^-}(m) & \xrightarrow{(41)\times\{+\}} & X^{\pm}_{N^+,N^-}.
\end{array}$$

Proof. That each point of $\mathrm{CM}_{N^+,N^-q}(m)$ has smooth reduction modulo $\mathfrak{q}$ follows from [2, p. 55]. For the rest, let (A,ι,C) be a point of $\mathrm{CM}_{N^+,N^-q}(m)$, and (A_0,ι_0,C_0) its reduction modulo $\mathfrak{q}$. Since (A_0,ι_0,C_0) is a nonsingular $\mathcal{O}_{K[m]}/\mathfrak{q}$-valued point of the special fiber, by [52, Proposition 4.4, Theorem 5.3], there is a unique $\mathcal{O}_{B_{N^-q}}$-stable subgroup scheme $H\subset A_0$ which is isomorphic to α_p; since H is unique, it is automatically $\mathcal{O}_{m,K}$-stable as well. Let $\bar\iota_0$ and $\overline{C}_0$ denote the induced $\mathcal{O}_{B_{N^-q}}$-action and $\Gamma_0(N^+)$-structure on A_0/H. Then $(A_0/H,\bar\iota_0,\overline{C}_0)$ is a pure triple over $\mathcal{O}_{K[m]}/\mathfrak{q}\simeq\mathbb{F}_{q^2}$ in the notation of the proof of Proposition 4.4.5, and the irreducible component of the special fiber of X_{N^+,N^-q} containing (A_0,ι_0,C_0) is parameterized by the q-Frobenius twist $(A_0,\bar\iota_0,\overline{C}_0)^{(q)}$. As in the proof of Proposition 4.6.12, the induced $\mathcal{O}_{m,K}$-action on $(A_0/H)^{(q)}$ allows us to view $\mathrm{End}(A_0/H,\bar\iota_0,\overline{C}_0)$ as a point of $\mathrm{CM}_{N^+,N^-}(m)$, and the resulting map $\mathrm{CM}_{N^+,N^-q}(m)\to\mathrm{CM}_{N^+,N^-}(m)$ is then an isomorphism of $\mathrm{Gal}(K[m]/K)$-torsors (by Proposition 4.6.5).

To finish the proof, we must show that $(A_0/H,\bar\iota_0,\overline{C}_0)^{(q)}$ is pure of type $+$, or equivalently that $(A_0/H,\bar\iota_0,\overline{C}_0)$ is pure of type $-$. By [52, Proposition 4.7], it suffices to show that H is of type $+$ in the following sense: if $\mathcal{M}$ is the Dieudonné module of A_0, then H corresponds to a submodule $(F,V)\mathcal{M}\subset\mathcal{N}\subset\mathcal{M}$; $\mathcal{O}_{B_{N^-q}}$ on the one-dimensional $\overline{\mathbb{F}}_{q^2}$-vector space $\mathcal{M}/\mathcal{N}$ by the map $\mathcal{O}_{B_{N^-q}}\to\mathbb{F}_{q^2}\subset\overline{\mathbb{F}}_{q^2}$ determined by the fixed orientation. Indeed, let $\mathcal{A}$ denote the Néron model of A over $\mathcal{O}_{K[m],\mathfrak{q}}$, with special fiber A_0. By Raynaud's Theorem [50] and the orientation condition of Definition 4.6.3(iii), the actions of $\mathcal{O}_{B_{N^-q}}$ and $\mathcal{O}_{m,K}$ on the group scheme $\mathcal{A}[\mathfrak{m}_q]$ coincide under the fixed composite isomorphism $\mathcal{O}_{B_{N^-q}}/\mathfrak{m}_q\simeq\mathbb{F}_{q^2}\simeq\mathcal{O}_{m,K}/q\mathcal{O}_{m,K}$. So it suffices to show that the action of $\mathcal{O}_{m,K}$ on $\mathcal{M}/\mathcal{N}$ is by the fixed isomorphism $\mathbb{F}_{q^2}\simeq\mathcal{O}_{m,K}/q\mathcal{O}_{m,K}$. But $\mathcal{M}/F\mathcal{M}$ is dual to $\mathrm{Lie}\,A_0=\mathrm{Lie}\,\mathcal{A}\otimes\mathcal{O}_{K[m]}/\mathfrak{q}$, and $\mathcal{O}_{m,K}$ acts on $\mathrm{Lie}\,\mathcal{A}$ by the canonical embedding $\mathcal{O}_{m,K}\hookrightarrow\mathcal{O}_{K[m],\mathfrak{q}}$ (using the orientation condition of Definition 4.6.3(i)). Hence $\mathcal{O}_{m,K}$ acts on $\mathcal{M}/F\mathcal{M}$ via the reduction map to $\mathcal{O}_{m,K}/\mathfrak{q}\simeq\mathbb{F}_{q^2}$. Since $\mathcal{M}/F\mathcal{M}$ surjects onto $\mathcal{M}/\mathcal{N}$, this completes the proof. □

5. Construction of bipartite Euler systems

5.1. **The CM class construction.**

5.1.1. Fix a quadratic imaginary field $K\subset\overline{\mathbb{Q}}$, and let f be as in (4.5.1), such that $\overline{T}_f$ satisfies (TW). We assume that N admits a factorization $N=N^+N^-$, where all $\ell|N^+$ are split in K, and N^- is a squarefree product of primes inert in K. We continue the notation of §4.5 (using $N_1=N^+$ and $N_2=N^-$). Fix an integer m which is coprime to N, and let $G_m=\mathrm{Gal}(K[m]/K)$. If $q\nmid m$ is a prime inert in K, we fix a prime $\mathfrak{q}$ of $K[m]$ above q; for instance, this can be done by choosing an embedding $K[m]\hookrightarrow\overline{\mathbb{Q}}$. If q is weakly

j-admissible with sign ϵ_q, we define the ordinary subspace:

$$H^1_{\mathrm{ord},\epsilon_q}(K_q, T_j) = \mathrm{im}\left(H^1(K_q, \mathrm{Fil}^+_{q,\epsilon_q} T_j) \to H^1(K_q, T_j)\right). \tag{42}$$

Using the map obtained from Shapiro's Lemma (e.g. [60, §3.1.2])

$$\mathrm{Res}_{\mathfrak{q}} : H^1(K[m], T_j) \to \mathrm{Hom}_{\mathrm{Set}}(G_m, H^1(K_q, T_j)), \tag{43}$$

we also have maps:

$$\partial_{q,\epsilon_q} : H^1(K[m], T_j) \to \mathrm{Hom}_{\mathrm{Set}}(G_m, H^1(I_q, \mathrm{Fil}^+_q T_j)) \approx \mathcal{O}_j[G_m],$$
$$\mathrm{loc}_{q,\epsilon_q} : H^1(K[m]^\Sigma/K[m], T_j) \to \mathrm{Hom}_{\mathrm{Set}}(G_m, T_j/\mathrm{Fil}^+_q T_j), \approx \mathcal{O}_j[G_m], \quad q \notin \Sigma,$$

defined as in (18,19).

For notational convenience, we temporarily denote by N_j the collection of weakly admissible pairs $\{Q, \epsilon_Q\}$ which are $(j + 2C)$-level-raising, and such that all primes $q|Q$ are inert in K.

Construction 5.1.2. *If $\Sigma \subset M_{\mathbb{Q}}$ is the set of places dividing $Np\infty$, then for all $\{Q, \epsilon_Q\} \in N_j$, there exist maps (well-defined to a unit scalar):*

$$\kappa_j(\cdot, Q, \epsilon_Q) : \mathrm{CM}_{N^+,N^-Q}(m) \to H^1(K[m]^{\Sigma\cup Q}/K[m], T_j), \qquad \nu(N^-Q) \text{ even},$$
$$\lambda_j(\cdot, Q, \epsilon_Q) : \mathrm{CM}_{N^+,N^-Q}(m) \to \mathrm{Hom}_{Set}(G_m, \mathcal{O}_j) = \mathcal{O}_j[G_m], \qquad \nu(N^-Q) \text{ odd},$$

compatible under the natural reduction maps for $j' \le j$, $\mathrm{Gal}(K[m]/K)$-equivariant, and such that the following properties hold.

(1) If $\{Q, \epsilon_Q\} \in N_j$ where $\nu(N^-Q)$ is even, then for all $q|Q$ and all $y \in \mathrm{CM}_{N^+,N^-Q}(m)$,

$$\mathrm{Res}_{\mathfrak{q}}(\kappa_j(y, Q, \epsilon_Q)) \in H^1_{\mathrm{ord},\epsilon_Q(q)}(K_q, T_j).$$

(2) If $\{Qq, \epsilon_{Qq}\}, \{Q, \epsilon_Q\} \in N_j$ where $\epsilon_Q = \epsilon_{Qq}|_Q$ and $\nu(N^-Qq)$ is even, then there is an isomorphism $i : H^1(I_q, \mathrm{Fil}^+_{q,\epsilon_Q(q)} T_j) \simeq \mathcal{O}_j$ such that, for all $y \in \mathrm{CM}_{N^+,N^-Qq}(m)$,

$$i\left(\partial_{q,\epsilon_{Qq}(q)}\kappa_j(y, Qq, \epsilon_{Qq})\right) = \lambda_j(s_{N^+,N^-Q,q}(y), Q, \epsilon_Q).$$

Here $s_{N^+,N^-Q,q} : \mathrm{CM}_{N^+,N^-Qq}(m) \xrightarrow{\sim} \mathrm{CM}_{N^+,N^-Q}(m)$ is the map of Proposition 4.6.13.

(3) If $\{Qq, \epsilon_{Qq}\}, \{Q, \epsilon_Q\} \in N_j$ where $\epsilon_Q = \epsilon_{Qq}|_Q$ and $\nu(N^-Qq)$ is odd, then there is an isomorphism $i : H^1_{\mathrm{unr}}(K_q, T_j/\mathrm{Fil}^+_{q,\epsilon_Q(q)} T_j) \simeq \mathcal{O}_j$ such that, for all $y \in \mathrm{CM}_{N^+,N^-Q}(m)$,

$$i\left(\mathrm{loc}_{q,\epsilon_{Qq}(q)}(\kappa_j(y, Q, \epsilon_Q))\right) = \lambda_j(t_{N^+,N^-Q,q}(y), Q, \epsilon_Q).$$

Here $t_{N^+,N^-Q,q} : \mathrm{CM}_{N^+,N^-Q}(m) \xrightarrow{\sim} \mathrm{CM}_{N^+,N^-Qq}(m)$ is the map of Proposition 4.6.12.

Proof. The specifications ϵ_Q will be dropped to ease notation. We fix throughout a prime $\ell_0 \nmid Nmp$ such that $a_{\ell_0}(f) - \ell_0 - 1$ is a unit in $\mathcal{O}$.

Suppose first that $\nu(N^-Q)$ is odd. By Lemma 4.5.5, there is a unique map (up to scalars) $M_Q \to \mathcal{O}_j$ of $\mathbb{T}^Q$-modules that factors through multiplication by π^C and is surjective after $\mathcal{O}$-linearization. For $y \in \mathrm{CM}_{N^+,N^-Q}(m)$ and $g \in G_m$, we define $\lambda_j(y)(g)$ to be the image of gy by the composite map

$$\mathrm{CM}_{N^+,N^-Q}(m) \to X_{N^+,N^-Q} \xrightarrow{T_{\ell_0} - \ell_0 - 1} M_Q \to \mathcal{O}_j.$$

(The notation M_Q was defined in (4.5.4).)

Now suppose that $\nu(N^-Q)$ is even. For each $y \in \mathrm{CM}_{N^+,N^-Q}(m)$, $(T_{\ell_0} - \ell_0 - 1)y$ is a degree zero divisor on X_{N^+,N^-Q}, and its image in the Jacobian J^Q is defined over $K[m]$. Let

$$d(y, Q) \in H^1(K[m]^{\Sigma\cup Q}/K[m], T_{\mathfrak{m}_Q} J^Q)$$

be the Kummer image. We define $\kappa_j(y, Q)$ to be the image of $d(y, Q)$ under the map

$$H^1(K[m]^{\Sigma\cup Q}/K[m], T_{\mathfrak{m}_Q} J^Q) \to H^1(K[m]^{\Sigma\cup Q}/K[m], T_j)$$

induced by Corollary 4.5.6.

We now establish properties (i)-(iii).

(1) This follows from the rigid analytic uniformization (38) by the argument of [23, p. 15]. Indeed, the argument there shows that the image of the Kummer map

$$J^Q(\mathbb{Q}_{q^2}) \to H^1(\mathbb{Q}_{q^2}, T_{\mathfrak{m}_Q} J^Q)$$

agrees with the image of the map

$$H^1(\mathbb{Q}_{q^2}, \mathcal{X}(J^Q)(1)) \to H^1(\mathbb{Q}_{q^2}, T_{\mathfrak{m}_Q} J^Q),$$

where $\mathcal{X}(J^Q)(1) \to (T_{\mathfrak{m}_Q} J^Q)[\mathrm{Frob}_q - U_q q]$ is a canonical isomorphism. Since $p \nmid q-1$, the surjection

$$T_{\mathfrak{m}_Q} J^Q \twoheadrightarrow T_j$$

induces

$$\mathcal{X}(J^Q)(1) \twoheadrightarrow \mathrm{Fil}_q^+ T_j,$$

and the claim follows.

(2) We claim that the composite

$$J^{Qq}(\mathbb{Q}_{q^2}) \to H^1(\mathbb{Q}_{q^2}, T_{\mathfrak{m}_{Qq}} J^{Qq}) \to H^1(I_q, T_j)$$

factors through

$$\mathrm{Sp}_q : J^{Qq}(\mathbb{Q}_{q^2}) \to \Phi_{\mathfrak{m}_Q}.$$

Indeed, the target of the composite map has Frobenius eigenvalue U_q, and, because $p \nmid q-1$, a diagram chase using (38) shows that the pro-p part of the kernel of Sp_q has Frobenius eigenvalue $-U_q$ (if it is nontrivial at all).

By Proposition 4.6.13, $\partial_q(\kappa_j(y, Qq))(g)$ is therefore the image of gy under the composite the natural map

$$\mathrm{CM}_{N^+,N^-Q}(m) \to X_{N^+,N^-Q} \xrightarrow{T_{\ell_0} - \ell_0 - 1} M_Q$$

with *some* surjective map of Hecke modules $M_Q \to \mathcal{O}_j$, which factors through multiplication by π^C by the choice of map $T_{\mathfrak{m}_{Qq}} J^{Qq} \to T_j$. We may conclude by Lemma 4.5.5.

(3) By Proposition 4.6.12, $\mathrm{loc}_q\, \kappa_j(y, Q)(g)$ is the image of $t_{N^+,N^-Q,q}(y)$ under the composite map

$$\begin{gathered}\mathrm{CM}_{N^+,N^-Qq}(m) \to X_{N^+,N^-Qq} \xrightarrow{T_{\ell_0} - \ell_0 - 1} M_{Qq} \twoheadrightarrow J^Q(\mathbb{F}_{q^2})^{ss} \twoheadrightarrow \\ H^1_{\mathrm{unr}}(\mathbb{F}_{q^2}, T_{\mathfrak{m}_Q} J^Q) \twoheadrightarrow H^1_{\mathrm{unr}}(\mathbb{F}_{q^2}, T_j) \twoheadrightarrow H^1_{\mathrm{unr}}(\mathbb{F}_{q^2}, T_j / \mathrm{Fil}_q^+ T_j).\end{gathered} \tag{44}$$

(The third arrow is surjective by Proposition 4.4.3(ii).) The final map in this composition projects onto the ϵ_q eigenspace of Frob_q; hence, applying Proposition 4.4.3(i) for the U_q action, the composite

$$M_{Qq} \twoheadrightarrow H^1_{\mathrm{unr}}(\mathbb{F}_{q^2}, T_j / \mathrm{Fil}_q^+ T_j) \approx \mathcal{O}_j \tag{45}$$

is equivariant for the full Hecke algebra $\mathbb{T}^q_{Q,\mathfrak{m}_{Qq}}$. Hence by Lemma 4.5.5 it coincides (up to a unit scalar) with the map used to construct $\lambda_j(y, Qq)$, which gives (iii).

□

5.2. p-adic interpolation.

5.2.1. Suppose for this subsection that:

(spl) $$p \text{ splits in } K$$

and

(ord) $$a_p \notin \wp.$$

We denote by $K_m \subset K[p^m]$ the mth layer of the anticyclotomic $\mathbb{Z}_p$-extension.

Proposition 5.2.2. *Suppose Q is a squarefree product of primes inert in K. Then there exists a sequence $y(m) \in \mathrm{CM}_{N^+,N^-Q}(p^m)$ such that*

$$T_p y(m) = \mathrm{tr}_{K[p^{m+1}]/K[p^m]}\, y(m+1) + y(m-1), \ \ \forall m \geq 1,$$

as formal divisors on X_{N^+,N^-Q}.

Proof. This is a standard calculation, but we give a sketch of the proof for lack of a precise reference. The set

$$\mathrm{CM}_{N^+,N^-Q}(p^\infty) \coloneqq \cup_{m\geq 0}\,\mathrm{CM}_{N^+,N^-Q}(p^m)$$

has a natural action of the Hecke operator T_p, compatible with its action on X_{N^+,N^-Q} and commuting with the action of $\mathrm{Gal}(K[p^\infty]/K)$. If y is a CM point of conductor p^m with $m\geq 1$, then (since p is split in K) $T_p y$ contains a CM point of conductor p^{m-1}, and another of conductor p^{m+1}. Since $T_p y$ is fixed by $\mathrm{Gal}(K[p^{m+1}]/K[p^m])$, the proposition follows formally. □

5.2.3. Suppose given any $\{\mathsf{Q},\epsilon_{\mathsf{Q}}\}\in\mathsf{N}$, and let $\{Q_n,\epsilon_{Q_n}\}$ be a representative sequence of weakly admissible pairs as in Remark 4.5.8, with each Q_n a squarefree product of primes inert in K. For each n, let $y(m)_n\in\mathrm{CM}_{N^+,N^-Q_n}(p^m)$ be a sequence of CM points which are compatible in the sense of Proposition 5.2.2.

Since $T_p\notin\mathfrak{m}$, Hensel's Lemma implies that the Hecke algebras $\mathbb{T}_{Q_n,\mathfrak{m}_{Q_n}}$ contain a (unique) element $u\notin\mathfrak{m}_{Q_n}$ such that $u^2-ua_p+p=0$. Let $\alpha_p\in\mathcal{O}^\times$ be the image of u.

5.2.4. We now suppose that $|\mathsf{Q}|+\nu(N^-)$ is even. Adopting the notation of Construction 5.1.2, it follows from the compatibility relation of the $y(m)_n$ that the classes

$$d(m,Q_n)\coloneqq\mathrm{Cores}_{K[p^m]/K_m}\left(u^{-m+1}d(y(m)_n,Q_n)-u^{-m}\,\mathrm{Res}_{K[p^m]/K[p^{m-1}]}\,d(y(m-1)_n,Q_n)\right)$$

are compatible under the corestriction maps

$$H^1(K_m,T_{\mathfrak{m}_{Q_n}}J^{Q_n})\to H^1(K_{m-1},T_{\mathfrak{m}_{Q_n}}J^{Q_n}).$$

Let $\kappa_j(m,Q_n)$ be the image of $d(m,Q_n)$ under the map of Corollary 4.5.6; this is well-defined for $\mathfrak{F}$-many n depending on j, and the classes $\kappa_j(m,Q_n)$ are compatible under corestriction. We let

$$\boldsymbol{\kappa}(\mathsf{Q})\in\varprojlim_{m,j}\mathsf{H}^1(K_m^{\Sigma\cup\mathsf{Q}}/K_m,T_j)\simeq\mathsf{H}^1(K,T_f\otimes\Lambda(\Psi))$$

be the class represented by the family $\kappa_j(m,Q_n)$. (The isomorphism follows from Shapiro's Lemma.)

5.2.5. Similarly, if $|\mathsf{Q}|+\nu(N^-)$ is odd, the elements

$$\lambda_j(m,Q_n)\coloneqq\alpha_p^{-m+1}\lambda_j(y(m)_n,Q_n)-\alpha_p^{-m}\lambda_j(y(m-1)_n,Q_n)\in\mathcal{O}_j[\mathrm{Gal}(K[p^m]/K)]$$

are compatible under the natural projection maps

$$\mathcal{O}_j[\mathrm{Gal}(K[p^m]/K)]\to\mathcal{O}_j[\mathrm{Gal}(K[p^{m-1}]/K)].$$

Taking the trace to $\mathrm{Gal}(K_m/K)$, we then obtain an element

$$\boldsymbol{\lambda}(\mathsf{Q})\in\varprojlim_{m,j}\mathcal{U}\left(\{\mathcal{O}_j[\mathrm{Gal}(K_m/K)]\}_{n\in\mathbb{N}}\right)\simeq\mathcal{O}[\![\mathrm{Gal}(K_\infty/K)]\!]\simeq\Lambda.$$

5.2.6. Let $\mathsf{S}\subset\mathsf{M}_K$ be the set of constant ultraprimes $\underline{v}$ such that $v|Np\infty$. We define a Selmer structure $(\mathcal{F}_\Lambda,\mathsf{S})$ for $\mathbf{T}_f\coloneqq T_f\otimes\Lambda$ as follows:

$$\mathsf{H}^1_{\mathcal{F}_\Lambda}(K_{\mathsf{v}},\mathbf{T}_f)=\begin{cases}\mathrm{im}\left(H^1(K_v,\mathrm{Fil}_v^+T\otimes\Lambda)\to H^1(K_v,\mathbf{T}_f)\right), & \mathsf{v}=\underline{v},v|p,\\ H^1(K_v,\mathbf{T}_f), & \mathsf{v}=\underline{v},v|N\infty,\\ \ker\left(\left(\mathsf{H}^1(K_{\mathsf{v}},\mathbf{T}_f)\to\mathsf{H}^1(I_{\mathsf{v}},\mathbf{T}_f)\otimes\mathbb{Q}_p\right)\right), & \text{otherwise.}\end{cases}\tag{46}$$

Here, if $v|p$, $\mathrm{Fil}_v^+T\subset T$ is the unique free, rank-one direct summand on which I_v acts by the cyclotomic character. This Selmer structures is well-defined because, if I_{v} acts trivially on $\mathbf{T}_f$, then $\mathsf{H}^1(I_{\mathsf{v}},\mathbf{T}_f)$ is torsion-free; hence for $\mathsf{v}\notin\mathsf{S}$, $\mathsf{H}^1_{\mathcal{F}_\Lambda}(K_{\mathsf{v}},\mathbf{T}_f)=\mathsf{H}^1_{\mathrm{unr}}(K_{\mathsf{v}},\mathbf{T}_f)$.

Remark 5.2.7. (1) The local conditions at $\mathsf{v}=\underline{v}$ for $v|N\infty$ are a reformulation of those in [28, §3.1]. Indeed, if $v|N^+$, then $H^1(K_v,\mathbf{T}_f)=H^1_{\mathrm{unr}}(K_v,\mathbf{T}_f)$ by [55, Corollary B.3.4]. If $v|N^-$, then $H^1(K_v,\mathbf{T}_f)=H^1(K_v,T_f)\otimes_{\mathcal{O}}\Lambda$, and a direct calculation shows that the whole local cohomology group $H^1(K_v,T_f)$ is ordinary in the sense of [28].

(2) Some authors take the unramified local condition at $v|N^-$ when defining a Selmer structure. As explained in [47], the effect of this alternative definition is to increase the μ-invariant of $\mathrm{Sel}_{\mathcal{F}_\Lambda^*}(K,\mathbf{W}_f)^\vee$ (since restricting the local condition for $\mathbf{T}_f$ corresponds to relaxing the local condition for $\mathbf{W}_f$).

By [28, Proposition 3.3.1, Lemma 3.3.4], $\mathcal{F}_\Lambda$ fits into an interpolated self-dual Selmer structure $(\mathsf{S},\mathcal{F}_\Lambda,\mathcal{F}_{\mathfrak{P}},\Sigma_\Lambda)$ for $\mathbf{T}_f$.

Proposition 5.2.8. *The pair $(\boldsymbol{\kappa}, \boldsymbol{\lambda})$ is a nontrivial bipartite Euler system for the triple $(\mathbf{T}_f, \mathcal{F}_\Lambda, \mathsf{S})$.*

Proof. We first show that $\boldsymbol{\kappa}(\mathsf{Q})$ lies in $\mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{T}_f)$ for all $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\nu(N^-)}$. The only local conditions to verify are those at $v|p$; the local conditions for $\mathsf{q} \in \mathsf{Q}$ follow from Construction 5.1.2(i), and the rest are trivial. If Q is represented by the sequence Q_n, let $\mathrm{Fil}_v^+ T_{\mathfrak{m}_{Q_n}} J^{Q_n}$ be the maximal $\mathbb{T}_{\mathfrak{m}_{Q_n}}$ submodule on which I_v acts by the cyclotomic character (adopting the notation of Construction 5.1.2 and if necessary restricting our attention to $\mathfrak{F}$-many n). As in [12, Proposition 4.7], it suffices to show that, for all m and n and a fixed extension of v to K_∞, the image $d_{n,m}$ of the class $d(m, Q_n)$ under the composite

$$H^1(K_m, T_{\mathfrak{m}_{Q_n}} J^{Q_n}) \to H^1(K_{m,v}, T_{\mathfrak{m}_{Q_n}} J^{Q_n} / \mathrm{Fil}^+ T_{\mathfrak{m}_{Q_n}} J^{Q_n})$$

is trivial. Since $d(m, Q_n)$ is a $\mathbb{T}_{\mathfrak{m}}$-linear combination of Kummer images over K_m, by [3, Example 3.11] and [42, Proposition 12.5.8] $d_{n,m}$ lies in the kernel of

$$H^1(K_{m,v}, T_{\mathfrak{m}_{Q_n}} J^{Q_n} / \mathrm{Fil}^+ T_{\mathfrak{m}_{Q_n}} J^{Q_n}) \to H^1(K_{m,v}, \mathbb{Q}_p \otimes T_{\mathfrak{m}_{Q_n}} J^{Q_n} / \mathrm{Fil}^+ T_{\mathfrak{m}_{Q_n}} J^{Q_n}).$$

Since the classes $d_{n,m}$ are corestriction-compatible as m varies, the argument of [29, Proposition 2.4.5] shows that indeed $d_{n,m} = 0$ for all n, m.

The explicit reciprocity laws are a consequence of Construction 5.1.2(ii,iii), and the nonvanishing of either $\boldsymbol{\kappa}(1)$ or $\boldsymbol{\lambda}(1)$ (according to the parity of $\nu(N^-)$) is due to the work of Cornut [13] and Vatsal [64]. ☐

5.3. **Kolyvagin classes.**

5.3.1. Before defining the Kolyvagin classes in patched cohomology, we begin by recalling a calculation explained in [22].

Let m be a squarefree product of primes ℓ inert in K. We have

$$\mathrm{Gal}(K[m]/K[1]) \simeq \prod_{\ell | m} \mathrm{Gal}(K[\ell]/K[1]);$$

each $\mathrm{Gal}(K[\ell]/K[1])$ is cyclic of order $\ell + 1$. For a place λ of $K[m]$ over some $\ell|m$, let $\mathrm{Frob}_\lambda \in G_{\mathbb{Q}}$ be a lift of absolute Frobenius, and let $\sigma_\lambda \in I_\lambda \subset G_K$ be a generator of $\mathrm{Gal}(K[\ell]/K[1])$. Recall the Kolyvagin derivative operators [22]:

$$D_\ell = \sum_{i=1}^{\ell} i\sigma_\lambda^i \in \mathbb{Z}[\mathrm{Gal}(K[\ell]/K[1])], \quad D_m = \prod_{\ell | m} D_\ell.$$

Finally, let Q be a squarefree product of primes inert in K that is coprime to Nmp, and choose a CM point $y(m, Q) \in \mathrm{CM}_{N^+, N^- Q}(m)$. We define

$$P(m, Q) = D_m y(m, Q),$$

viewed as a formal divisor on $X_{N^+, N^- Q}$.

Proposition 5.3.2. *For any $\ell | m$, there exists a CM point $y(m/\ell, Q) \in \mathrm{CM}_{N^+, N^- Q}(m/\ell)$ such that:*

(1) $(\sigma_\lambda - 1) P(m, Q) = (\ell + 1) D_{m/\ell} y(m, Q) - T_\ell P(m/\ell, Q)$.

(2) If $\nu(N^- Q)$ is even, then

$$D_{m/\ell} y(m, Q) \equiv \mathrm{Frob}_\lambda P(m/\ell, Q) \pmod{\lambda}.$$

Proof. This is [22, p. 240] in the modular curve case; the same reasoning applies to Shimura curves by [43, Proposition 4.13]. The argument for (i) formally applies to Shimura sets as well, along the lines of Proposition 5.2.2. ☐

Fix ℓ_0 as in the proof of Construction 5.1.2, and let

$$P'(m, Q) = (T_{\ell_0} - \ell_0 - 1) P(m, Q),$$

a formal degree zero divisor on $X_{N^+, N^- Q}$.

Proposition 5.3.3. *Suppose $\mathfrak{m}_Q \subset \mathbb{T}_Q = \mathbb{T}_{N^+, N^- Q}$ is a maximal ideal whose associated residual representation has no $G_{K[m]}$-fixed points, and let $I_m \subset \mathbb{T}_Q$ be the ideal generated by $\ell + 1$ and T_ℓ for all $\ell | m$. Then if $\nu(N^- Q)$ is even:*

(1) Restriction induces an isomorphism

$$\mathrm{Res}_m : H^1(K[1], T_{\mathfrak{m}_Q} J^Q / I_m) \xrightarrow{\sim} H^1(K[m], T_{\mathfrak{m}_Q} J^Q / I_m)^{\mathrm{Gal}(K[m]/K[1])}.$$

(2) The Kummer image $d(m,Q)$ of $P'(m,Q)$ in $H^1(K[m], T_{\mathfrak{m}_Q}J^Q/I_m)$ lies in the image of Res_m.
(3) If $c(m,Q) = \mathrm{Cores}_{K[1]/K}\,\mathrm{Res}_m^{-1}\,d(m,Q)$, then for all $\ell|m$ and any choices of representatives,

$$c(m,Q)(\sigma_\lambda) = \mathrm{Frob}_\lambda^{-1}\,d(m/\ell,Q)(\mathrm{Frob}_\lambda^2) \pmod{I_m}.$$

(4) The class $c(m,Q)$ is unramified at any place $v \nmid NpmQ\infty$.

Proof. (i) follows from the inflation restriction exact sequence as in [22], and (ii) is immediate from Proposition 5.3.2. Also (iv) is clear from the construction. For (iii), it suffices to check the corresponding statement for $c'(m,Q) = \mathrm{Res}_m^{-1}\,d(m,Q)$. The proof is a modification of the argument in [39]. Fix division points $\frac{P'(m,Q)}{\ell+1}$ and $\frac{P'(m/\ell,Q)}{\ell+1}$ in $J^Q(\overline{K})$, and a lift $\widetilde{\sigma}_\lambda$ of σ_λ to $\mathrm{Gal}(\overline{K}/K[1])$. One may verify that $c'(m,Q)(\sigma_\lambda)$ is the unique element $A \in T_{\mathfrak{m}_Q}J^Q/I_m$ such that, for all $g \in G_{K[m/\ell]}$,

$$(g-1)A \equiv (g-1)(\widetilde{\sigma}_\lambda - 1)\frac{P'(m,Q)}{\ell+1} \in T_{\mathfrak{m}_Q}J^Q/I_m.$$

By Proposition 5.3.2(i), A is also the image of the (unique) point $T \in J^Q[\ell+1]$ such that

$$T \equiv D_{m/\ell}\,\mathrm{CM}_{N^+,N^-Q}(P'(m,Q)) - T_\ell\frac{P'(m/\ell,Q)}{\ell+1} \pmod{\lambda}.$$

But by Proposition 5.3.2(ii), this is equivalent to

$$T \equiv \mathrm{Frob}_\lambda^{-1}\,P(m/\ell,Q) - T_\ell\frac{P'(m/\ell,Q)}{\ell+1} \pmod{\lambda}. \tag{47}$$

By the Eichler-Shimura relation, and the fact that ℓ splits completely in $K[m/\ell]$, the image of T in $T_{\mathfrak{m}_Q}J^Q/I_m$ is precisely

$$\mathrm{Frob}_\lambda^{-1}\,d(m/\ell,Q)(\mathrm{Frob}_\lambda^2).$$

□

Definition 5.3.4. For a squarefree product m of primes inert in K, let $I_m(f) \subset \mathcal{O}$ be the ideal generated by $a_\ell(f)$ and $\ell+1$ for all $\ell|m$. Suppose given $\{Q,\epsilon_Q\} \in \mathcal{N}_j$ (notation as before Construction 5.1.2), with all $q|Q$ inert in K and with $j \geq v_\wp(I_m(f))$. If $\nu(N^-Q)$ is even, then the Kolyvagin class

$$\bar{c}(m,Q) \in H^1(K^{\Sigma\cup Q\cup m}/K, T_f/I_m(f)) \tag{48}$$

is defined to be the image of $c(m,Q)$. If $\nu(N^-Q)$ is odd, then Construction 5.1.2, extended linearly to formal sums of CM points, defines an element

$$\lambda_j(P(m,Q),Q) \in \mathcal{O}_j[\mathrm{Gal}(K[m]/K)].$$

Its reduction modulo I_m is constant on cosets of $\mathrm{Gal}(K[m]/K[1])$ by Proposition 5.3.2(i) and therefore descends to

$$\lambda'(m,Q) \in (\mathcal{O}/I_m)[\mathrm{Gal}(K[1]/K)]. \tag{49}$$

The Kolyvagin element is then defined as:

$$\lambda(m,Q) = \mathrm{tr}_{K[1]/K}\,\lambda'(m,Q) \in \mathcal{O}/I_m. \tag{50}$$

Remark 5.3.5. When $Q=1$ and $\nu(N^-)$ is even, this agrees with Kolyvagin's construction [36].

Recall that ϵ_f is the global root number of f. For applications to the parity conjecture, we will require the following:

Proposition 5.3.6. *If $\nu(N^-)$ is even, then $\bar{c}(m,1)$ lies in the $\epsilon_f\cdot(-1)^{\nu(m)+1}$-eigenspace for the action of τ. If $\nu(N^-)$ is odd and $\lambda(m,1)\neq 0$, then $\epsilon_f = (-1)^{\nu(m)}$.*

Proof. Since f is a newform of level N, the maps $\varphi : T_{\mathfrak{m}}J^{N^+,N^-} \to T_f/\pi^j$ or $\varphi : \mathbb{Z}[X_{N^+,N^-}]^0 \to \mathcal{O}/\pi^j$ used in Construction 5.1.2 satisfy

$$\varphi(U_Nx) = \epsilon_f\cdot(-1)^{\nu(N^-)+1}\varphi(x),$$

where $U_N = \prod_{\ell|N}U_\ell$ is the Atkin-Lehner involution; the minus signs appear because the local root number of f at $\ell|N^-$ is the *negative* of the U_ℓ eigenvalue of f when viewed as automorphic representation of $B_{N^-}^\times$. (We note that this Atkin-Lehner equivariance may not be true of the corresponding maps at level N^+N^-Q, which are not necessarily reductions of genuine modular parameterizations.)

Now suppose $\nu(N^-)$ is even. If $\tau \in \mathrm{Gal}(K[m]/K)$ is a lift of complex conjugation, then, for all $y(m) \in \mathrm{CM}_{N^+,N^-}(m)$, we claim that $U_N\tau y(m)$ lies in $\mathrm{CM}_{N^+,N^-}(m)$ as well. Indeed, this is clear from Definition 4.6.3: applying $U_N\tau$ reverses all the orientation conditions (i)-(iii), but then we can replace the action of $\mathcal{O}_{m,K}$ with its complex conjugate so the conditions are again satisfied. Since $\overline{c}(m,1)$ is independent of the choice of $y(m)$ by Proposition 5.3.3, the calculation in [22, Proposition 5.4] applies to show that $\tau\overline{c}(m,1) = -\epsilon_f \cdot (-1)^{\nu(m)}\overline{c}(m,1)$, as desired.

The case when $\nu(N^-)$ is odd is similar: if $y(m) \in \mathrm{CM}_{N^+,N^-}(m)$ is represented by a pair (R,ϕ) satisfying Definition 4.6.7 for the fixed embedding $K \hookrightarrow B$, then $U_N y(m)$ is represented by the pair (R,ϕ^{op}), with all orientations reversed. But (R,ϕ^{op}) satisfies Definition 4.6.7 for the embedding $K \xrightarrow{\tau} K \hookrightarrow B$, which is conjugate to $K \hookrightarrow B$ by an element of $B^\times(\mathbb{Q})$, so $U_N y(m)$ lies in the image of $\mathrm{CM}_{N^+,N^-}(m) \to X_{N^+,N^-}$. Applying formally the calculations in [22, Proposition 5.4], it follows that $\lambda(m,1) = \epsilon_f \cdot (-1)^{\nu(m)}\lambda(m,1)$, which gives the claim. ☐

Definition 5.3.7. An ultraprime l is called Kolyvagin-admissible if

$$\mathrm{Frob}_{\mathsf{l}} \in \mathrm{Gal}(K(T_f)/\mathbb{Q})$$

is a complex conjugation. A Kolyvagin-admissible set is a finite set of Kolyvagin-admissible ultraprimes, and the collection of all Kolyvagin-admissible sets is denoted K.

5.3.8. If l is Kolyvagin-admissible, then the local cohomology

$$\mathsf{H}^1(K_{\mathsf{l}}, T_f)$$

is free of rank four over $\mathcal{O}$, and carries a natural action of the complex conjugation $\tau \in \mathrm{Gal}(K/\mathbb{Q})$. It has a canonical splitting of the finite-singular exact sequence:

$$\mathsf{H}^1(K_{\mathsf{l}}, T_f) = \mathsf{H}^1_{\mathrm{unr}}(K_{\mathsf{l}}, T_f) \oplus \mathsf{H}^1_{\mathrm{tr}}(K_{\mathsf{l}}, T_f),$$

defined as follows. If the sequence ℓ_n represents l, then for any j and for $\mathfrak{F}$-many n, Frob_{ℓ_n} acts as complex conjugation on T_f/π^j, and

$$H^1_{\mathrm{tr}}(K_{\ell_n}, T_f/\pi^j) = \ker\left(H^1(K_{\ell_n}, T_f/\pi^j) \to H^1(K[\ell_n]_{\lambda_n}, T_f/\pi^j)\right)$$

is isomorphic to $H^1(I_{\ell_n}, T_f/\pi^j)^{\mathrm{Frob}^2_{\ell_n}=1}$, where λ_n is a prime of $K[\ell_n]$ over ℓ_n. Then

$$H^1_{\mathrm{tr}}(K_{\mathsf{l}}, T_f) = \varprojlim \mathcal{U}\left(\left\{H^1_{\mathrm{tr}}(K_{\ell_n}, T_f/\pi^j)\right\}_{n\in\mathbb{N}}\right) \subset \mathsf{H}^1(K_{\mathsf{l}}, T_f)$$

is our transverse subspace. We denote by $\mathrm{loc}^\pm_{\mathsf{l}}$ and $\partial^\pm_{\mathsf{l}}$ the composites $\mathsf{H}^1(K,T_f) \to \mathsf{H}^1_{\mathrm{unr}}(K_{\mathsf{l}}, T_f)^\pm$ and $\mathsf{H}^1(K,T_f) \to \mathsf{H}^1_{\mathrm{tr}}(K_{\mathsf{l}}, T_f)^\pm$, respectively, where $\pm$ is the Frobenius eigenvalue. The codomain of each is free of rank one over $\mathcal{O}$.

Let $\mathsf{S} \subset \mathsf{M}_K$ be the set of constant ultraprimes $\underline{v}$ such that $v|Np\infty$. We will consider the Kolyvagin-transverse Selmer structure $(\mathcal{F}(\mathsf{m}), \mathsf{S}\cup\mathsf{m})$ on T_f, for any $\mathsf{m} \in \mathsf{K}$:

$$\mathsf{H}^1_{\mathcal{F}(\mathsf{m})}(K_{\mathsf{v}}, T_f) = \begin{cases} \ker\left(H^1(K_v,T_f) \to \frac{H^1(K_v,V_f)}{H^1_f(K_v,V_f)}\right), & \mathsf{v} = \underline{v}, \\ \mathsf{H}^1_{\mathrm{tr}}(K_{\mathsf{l}}, T_f), & \mathsf{v} = \mathsf{l} \in \mathsf{m}, \\ \mathsf{H}^1_{\mathrm{unr}}(K_{\mathsf{v}}, T_f), & \text{otherwise.} \end{cases} \tag{51}$$

Here $H^1_f(K_v, V_f)$ is the Bloch-Kato local condition on $V_f = T_f \otimes \mathbb{Q}_p$. Note that $(\mathcal{F}(\mathsf{m}), \mathsf{S}\cup\mathsf{m})$ is a self-dual Selmer structure by the self-duality of $H^1_f(K_v,V_f)$ – the transverse local conditions at m are self-dual by [38, Proposition 1.3.2]. If $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}_{\mathsf{m}}$, then we denote by $(\mathcal{F}(\mathsf{m},\mathsf{Q}), \mathsf{S}\cup\mathsf{m}\cup\mathsf{Q})$ the modified Selmer structure of (3.2.2).

5.3.9. Let $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\nu(N^-)}_{\mathsf{m}}$, and fix representatives Q_n and m_n, which we may assume to be disjoint. Our patched Kolyvagin class is the element

$$\kappa(\mathsf{m},\mathsf{Q}) \in \mathsf{H}^1(K^{\mathsf{S}\cup\mathsf{m}\cup\mathsf{Q}}/K, T_f)$$

whose image in T_j is represented by the sequence of images of the classes $\overline{c}(m_n, Q_n)$, well-defined for $\mathfrak{F}$-many n.

If $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\nu(N^-)+1}$, then we similarly set

$$\lambda(\mathsf{m}, \mathsf{Q}) \in \mathcal{O} \simeq \varprojlim \mathcal{U}\left(\{\mathcal{O}/\pi^j\}\right)$$

to be the element whose image in $\mathcal{O}/\pi^j$ is represented by the sequence $\lambda(m_n, Q_n)$.

Proposition 5.3.10. *For any* $\mathsf{m} \in \mathsf{K}$ *and* $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}_{\mathsf{m}}^{\nu(N^-)}$,

$$(\kappa(\mathsf{m}, \mathsf{Q})) \subset \mathrm{Sel}_{\mathcal{F}(\mathsf{m},\mathsf{Q})}(T_f).$$

Moreover:

(1) *For all* $\mathsf{l} \in \mathsf{m}$,
$$(\mathrm{loc}_{\mathsf{l}}^{\pm}(\kappa(\mathsf{m}/\mathsf{l}, \mathsf{Q}))) = (\partial_{\mathsf{l}}^{\mp}(\kappa(\mathsf{m}, \mathsf{Q})))$$
as submodules of $\mathcal{O}$.

(2) *For all* $\mathsf{q} \in \mathsf{Q}$,
$$(\partial_{\mathsf{q}}(\kappa(\mathsf{m}, \mathsf{Q}))) = (\lambda(\mathsf{m}, \mathsf{Q}/\mathsf{q}))$$
as submodules of $\mathcal{O}$.

(3) *For all* $\mathsf{q} \notin \mathsf{Q}$, *admissible with sign* ϵ_{q},
$$(\mathrm{loc}_{\mathsf{q}}(\kappa(\mathsf{m}, \mathsf{Q}))) = (\lambda(\mathsf{m}, \mathsf{Q}\mathsf{q}))$$
as submodules of $\mathcal{O}$.

In particular, for any fixed m, $(\kappa(\mathsf{m}, \cdot), \lambda(\mathsf{m}, \cdot))$ *forms a bipartite Euler system with sign* $\nu(N^-)$ *for the triple* $(T_f, \mathcal{F}(\mathsf{m}), \mathsf{S} \cup \mathsf{m})$.

Proof. We verify the local conditions for each $\mathsf{v} \in \mathsf{S} \cup \mathsf{m} \cup \mathsf{Q}$. If $\mathsf{v} = \underline{v}$ for a prime $v | N\infty$, then the local condition is all of $H^1(K_v, T_f)$, so there is nothing to show. (Indeed, $H^1(K_v, T_f)$ is torsion for any $v|N\infty$.) If $v|p$, then by [19] it suffices to show, for all j, that the image c_j of $\mathrm{Res}_v\, \kappa(\mathsf{m}, \mathsf{Q})$ in $H^1(K_v, T_f/\pi^j)$ lies in the image of the canonical map

$$\mathrm{Ext}^1_{\text{f.f.g.s.}}(\underline{\mathcal{O}/\pi^j}, T_f/\pi^j) \to \mathrm{Ext}^1_{G_{K_v}}(\mathcal{O}/\pi^j, T_f/\pi^j) = H^1(K_v, T_f/\pi^j).$$

Let E be the extension of $\mathcal{O}/\pi^j$ by T_f/π^j corresponding to c_j, viewed as a group scheme over $\mathrm{Spec}\, K_v$. Now, for $\mathfrak{F}$-many n, the restriction of c_j to $K[m_n]_v$ is the image of a Kummer class in $H^1(K[m_n]_v, J^{Q_n}[p^M])$ by a map of Galois representations $J^{Q_n}[p^M] \to T_f/\pi^j$, which extends to a map of finite flat group schemes by Raynaud's theorem [50]. (Here we have extended v to a place of $K[m_n]$. Let $\mathcal{O}_{K_v}$ and $\mathcal{O}_{K[m_n]_v}$ denote the corresponding completions of the rings of integers of K and $K[m_n]$, respectively.) As a consequence, for $\mathfrak{F}$-many n E extends to a finite flat group scheme $\mathcal{E}$ over $\mathrm{Spec}\, \mathcal{O}_{K[m_n]_v}$. Then by Raynaud's theorem once again [50], the action of G_{K_v} on E gives a descent datum for $\mathcal{E}$ over $\mathrm{Spec}\, \mathcal{O}_{K_v}$. Hence E extends to a finite flat group scheme over $\mathrm{Spec}\, \mathcal{O}_{K_v}$, as desired.

If $\mathsf{v} = \mathsf{l} | \mathsf{m}$, then, adopting as well the notation of (5.3.1), the class $c(P(m_n), Q_n)$ is zero when restricted to $K[m_n]_{\lambda_n}$ because $D_{\ell_n} = \ell_n(\ell_n + 1)$ on $\mathbb{F}_{\lambda_n}$; hence $\mathrm{Res}_{\mathsf{v}}\, \kappa(\mathsf{m}, \mathsf{Q}) \in \mathsf{H}^1_{\mathrm{tr}}(K_{\mathsf{l}}, T_f)$. The local conditions at $\mathsf{q} \in \mathsf{Q}$ are satisfied because every factor of Q_n splits completely in $K[m_n]$; for the same reason, (ii, iii) follow from Construction 5.1.2(ii, iii). Moreover (i) is clear from Proposition 5.3.3(iii). $\square$

Remark 5.3.11. The Euler system $(\kappa(1, \cdot), \lambda(1, \cdot))$ may be viewed as a specialization of $(\boldsymbol{\kappa}, \boldsymbol{\lambda})$. Indeed, by the usual Heegner point norm relations [14, Proposition 3.10], if p splits in K, $\mathbb{1}(\boldsymbol{\lambda}(\mathsf{Q})) = (\alpha_p - 1)^2(\lambda(1, \mathsf{Q}))$ and $\mathbb{1}(\boldsymbol{\kappa}(\mathsf{Q})) = (\alpha_p - 1)^2(\kappa(1, \mathsf{Q}))$ when $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\nu(N^-)+1}$ and $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\nu(N^-)}$, respectively. (Here $\mathbb{1} : \Lambda \to \mathcal{O}$ is specialization at the trivial character.)

6. Deformation theory

Theorem 4.5.7 allows us to produce weak eigenforms (i.e. ring maps) $\mathbb{T}_{N^+, N^-Q} \to \mathcal{O}/\pi^j$ for arbitrarily large j, simply by requiring sufficiently deep congruence conditions on all $q|Q$. However, in general these maps do not lift to characteristic zero. To prove the main results, we also need to be able to π-adically approximate f by genuine level-raised newforms. In this section, we provide this input via the relative deformation theory of Fakhruddin-Khare-Patrikis [17].

6.1. **Review of relative deformation theory.**

6.1.1. We recall the setting of (3.1.1). Let f be a cuspidal newform of weight 2, trivial character, and level N, and let $\wp \subset \mathcal{O}_f$ be a prime ideal of the ring of integers of its field of coefficients, with uniformizer π and residue characteristic $p > 2$. (For this section, we allow $p|N$.) As usual, write $\mathcal{O}$ for the completion of $\mathcal{O}_f$ at $\wp$, and T_f for a $\mathcal{O}$-stable lattice in the $\wp$-adic Galois representation associated to f, with a perfect $\mathbb{Z}_p$-valued pairing. We assume throughout that the residual representation $\overline{T}_f \coloneqq T_f/\pi$ is absolutely irreducible. We will also consider the hypotheses:

(non-CM) f does not have complex multiplication.

(TW) If $p = 3$, then $\overline{T}_f$ is absolutely irreducible when restricted to $G_{\mathbb{Q}(\sqrt{-3})}$.

Consider the adjoint representation

$$L = \mathrm{ad}^0\, T_f$$

and its $\mathbb{Z}_p$-dual, $L^\dagger \simeq L(1)$, and let $\overline{L}$ and $\overline{L}^* \simeq L^\dagger/\pi$ be the associated residual representations.

6.1.2. We now recall the construction in [17, Proposition 4.7] of certain local conditions for the Galois cohomology of L. For all primes $\ell \neq p$, let $\widetilde{R}_\ell$ denote the framed universal deformation ring of $\overline{T}_f|_{G_{\mathbb{Q}_\ell}}$, of fixed determinant χ. (As usual χ is the p-adic cyclotomic character.) For $\ell = p$, let $\widetilde{R}_p$ denote the framed potentially semistable deformation ring [33] of $\overline{T}_f|_{G_{\mathbb{Q}_p}}$, with fixed Hodge-Tate weights 0 and 1, fixed determinant χ, and fixed Galois type agreeing with that of T_f. For any ℓ, the generic fiber $\widetilde{R}_\ell[1/\pi]$ is of pure dimension $3 + \delta_{\ell=p}$. A choice of framing for T_f defines a formally smooth point y_ℓ of $\operatorname{Spec} \widetilde{R}_\ell[1/\pi]$ by [1, Theorem D, Proposition 1.2.2]; let $\operatorname{Spec} R_\ell \subset \operatorname{Spec} \widetilde{R}_\ell$ be the Zariski closure of the irreducible component of $\operatorname{Spec} \widetilde{R}_\ell[1/\pi]$ containing y_ℓ. The following is proved in [17, Proposition 4.7].

Proposition 6.1.3. *There exists a nonempty open set $Y_\ell \subset \operatorname{Spec} R_\ell(\mathcal{O})$ containing y_ℓ, and a collection of submodules $Z_r \subset Z^1(G_{\mathbb{Q}_\ell}, L/\pi^r)$ which are free of rank $3 + \delta_{\ell=p}$ over $\mathcal{O}/\pi^r$ for all $r \geq 0$, satisfying the following properties.*

1. *Let Y_n^ℓ be the image of Y_ℓ in $\operatorname{Spec} R(\mathcal{O}/\pi^n)$ and denote by $\pi_{n,r}^{Y_\ell} : Y_{n+r}^\ell \to Y_n^\ell$ the reduction maps for $n, r \geq 0$. Then given $r_0 > 0$, there exists $n_0 > 0$ such that, for all $n \geq n_0$ and all $0 \leq r \leq r_0$, the fibers of $\pi_{n,r}^{Y_\ell}$ are nonempty principal homogeneous spaces for Z_r.*
2. *The natural $\mathcal{O}$-module maps $\mathcal{O}/\pi^r \to \mathcal{O}/\pi^{r-1}$ and $\mathcal{O}/\pi^{r-1} \to \mathcal{O}/\pi^r$ induce surjections $Z_r \twoheadrightarrow Z_{r-1}$ and inclusions $Z_{r-1} \hookrightarrow Z_r$.*

We define the local condition

$$H^1_{\mathcal{S}}(\mathbb{Q}_\ell, L/\pi^r) \subset H^1(\mathbb{Q}_\ell, L/\pi^r)$$

to be the image of Z_r for all ℓ; if $\ell \nmid Np$, these are just the unramified local conditions. The proposition implies that, taking inverse limits, we obtain a saturated local condition

$$H^1_{\mathcal{S}}(\mathbb{Q}_\ell, L) \subset H^1(\mathbb{Q}_\ell, L).$$

(In [17], these local conditions are denoted $L_{\rho,r}$.)

6.1.4. Now suppose q is a j-admissible prime with sign ϵ_q, and let $\underline{\epsilon}_q$ denote the unramified character of $G_{\mathbb{Q}_q}$ sending Frob_q to ϵ_q. Let $\operatorname{Spec} R^{\mathrm{ord}}_{q,\epsilon_q} \subset \operatorname{Spec} \widetilde{R}_q$ be the Zariski closure of the unique irreducible component of $\operatorname{Spec} \widetilde{R}_q[1/\pi]$ containing a Steinberg representation.

Recall from Definition 4.5.2 the uniquely determined subspace $\mathrm{Fil}^+_{q,\epsilon_q} T_j \subset T_j$ free of rank one over $\mathcal{O}/\pi^j$, on which the Galois group acts by $\chi\underline{\epsilon}_q$. We define

$$\mathrm{Fil}^+_{q,\epsilon_q} L/\pi^j = \mathrm{Hom}\left(T_j/\mathrm{Fil}^+_{q,\epsilon_q} T_j, \mathrm{Fil}^+_{q,\epsilon_q} T_j\right) \subset L/\pi^j,$$

a free $\mathcal{O}/\pi^j$-submodule of rank one, and

$$H^1_{\mathrm{ord},\epsilon_q}(\mathbb{Q}_q, L/\pi^j) = \mathrm{im}\left(H^1(\mathbb{Q}_q, \mathrm{Fil}^+_{q,\epsilon_q} L/\pi^j) \to H^1(\mathbb{Q}_q, L/\pi^j)\right).$$

As always the subscripts ϵ_q will usually be omitted when they are clear from context.

Proposition 6.1.5. *Suppose q is j-admissible with sign ϵ_q and $\rho_n \in \operatorname{Spec} R_q^{\mathrm{ord}}(\mathcal{O}/\pi^n)$ is a lift of $T_j|_{G_{\mathbb{Q}_q}}$ (with any framing). Then for all $r \le j$, the fiber of the reduction map*

$$\operatorname{Spec} R_q^{\mathrm{ord}}(\mathcal{O}/\pi^{n+r}) \to \operatorname{Spec} R^{\mathrm{ord}}(\mathcal{O}/\pi^n)$$

over ρ_n is a principal homogeneous space under

$$Z^1_{\mathrm{ord}}(\mathbb{Q}_q, L/\pi^r) := \ker\left(Z^1(\mathbb{Q}_q, L/\pi^r) \to \frac{H^1(\mathbb{Q}_q, L/\pi^r)}{H^1_{\mathrm{ord}}(\mathbb{Q}_q, L/\pi^r)}\right),$$

which is free of rank 3 over $\mathcal{O}/\pi^r$.

Proof. Without loss of generality, we choose a basis for T_j such that Frob_q acts via the diagonal matrix $\begin{pmatrix} q\epsilon_q & 0 \\ 0 & \epsilon_q \end{pmatrix}$.

By the explicit calculations in [58, Lemma 5.4, Proposition 5.6], R_q^{ord} is a power series ring $\mathcal{O}[\![X, Y, B]\!]$ with universal deformation

$$\rho_q^{\mathrm{ord}}(\sigma) = \begin{pmatrix} 1 & X \\ Y & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & X \\ Y & 1 \end{pmatrix}$$
$$\rho_q^{\mathrm{ord}}(\phi) = \begin{pmatrix} 1 & X \\ Y & 1 \end{pmatrix}^{-1} \begin{pmatrix} q\epsilon_q & 0 \\ 0 & \epsilon_q \end{pmatrix} \begin{pmatrix} 1 & X \\ Y & 1 \end{pmatrix},$$

where σ is a generator of tame inertia and ϕ is a lift of arithmetic Frobenius. (We note this calculation crucially uses $q \not\equiv 1 \pmod p$.) In particular, $\operatorname{Spec} R_q^{\mathrm{ord}}$ is formally smooth, and by the discussion in [17, §4.1, Lemma 4.5], the fiber is a principal homogeneous space under a submodule Z_r of $Z^1(\mathbb{Q}_q, L/\pi^r)$ which is free of rank three over $\mathcal{O}/\pi^r$ and contains all coboundaries. It is also clear that the cocycles of the form $\phi \mapsto 0, \sigma \mapsto \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$ are contained in Z_r, and these generate $H^1_{\mathrm{ord}}(\mathbb{Q}_q, L/\pi^r)$. By counting dimensions, we find $Z_r = Z^1_{\mathrm{ord}}(\mathbb{Q}_q, L/\pi^r)$. $\square$

6.1.6. Let N_j denote the set of weakly admissible pairs $\{Q, \epsilon_Q\}$ such that each $q|Q$ is weakly j-admissible with sign $\epsilon_Q(q)$. If $\{Q, \epsilon_Q\} \in N_j$ is a weakly admissible pair, we will consider the (non-patched) Selmer groups

$$\mathrm{Sel}_Q(\mathbb{Q}, L/\pi^j) := \ker\left(H^1(\mathbb{Q}^{S\cup Q}, L/\pi^j) \to \prod_{v|Np\infty} \frac{H^1(\mathbb{Q}_v, L/\pi^j)}{H^1_{\mathcal{S}}(\mathbb{Q}_v, L/\pi^j)} \times \prod_{q|Q} \frac{H^1(\mathbb{Q}_q, L/\pi^j)}{H^1_{\mathrm{ord},\epsilon_Q(q)}(\mathbb{Q}_q, L/\pi^j)}\right),$$

where S is the set of places dividing $NpQ_n\infty$. We also have the dual Selmer group $\mathrm{Sel}_Q(\mathbb{Q}, L^*[\pi^j])$ defined using orthogonal complement local conditions.

Finally, define, for any finite set of places Σ containing all $v|Np\infty$:

$$\text{Ш}^1_\Sigma(\overline{L}^*) = \ker\left(H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, \overline{L}^*) \to \prod_{v\in\Sigma} H^1(\mathbb{Q}_v, \overline{L}^*)\right). \tag{52}$$

Proposition 6.1.7. *There exists a finite set of places Σ, containing all $v|Np\infty$, such that*

$$\text{Ш}^1_\Sigma = 0.$$

Proof. We claim it suffices to show that

$$H^1(\mathbb{Q}(\overline{L}^*)/\mathbb{Q}, \overline{L}^*) = 0. \tag{53}$$

Indeed, suppose Σ is a finite set of places containing all $v|Np\infty$, and

$$c \in \text{Ш}^1_\Sigma$$

is nonzero. Then by (53), the restriction map

$$H^1(G_{\mathbb{Q}}, \overline{L}^*) \to H^1(G_{\mathbb{Q}(\overline{L}^*)}, \overline{L}^*)$$

is injective, so c restricts to a nonzero homomorphism $c' : G_{\mathbb{Q}(\overline{L}^*)} \to \overline{L}^*$. Let $\ell \notin \Sigma$ be a prime which is totally split in $\mathbb{Q}(\overline{L}^*)$ but not in the extension cut out by c' (which is possible by the Chebotarev Density Theorem).

Then $\mathrm{loc}_\ell\, c \neq 0$, hence the inclusion $Ш^1_{\Sigma\cup\{\ell\}} \subset Ш^1_\Sigma$ is strict. Since $Ш^1_\Sigma$ is always finite dimensional, iterating this process produces a set Σ such that $Ш^1_\Sigma = 0$.

We now show (53). If $\mu_p \not\subset \mathbb{Q}(\overline{L})$, then the center of $\mathrm{Gal}(\mathbb{Q}(\overline{L}^*)/\mathbb{Q})$ contains elements that act by nontrivial scalars on $\overline{L}^*$, and (53) follows from inflation-restriction. So suppose that $\mu_p \subset \mathbb{Q}(\overline{L})$; then the projective image $\overline{G} = \mathrm{Gal}(\mathbb{Q}(\overline{L})/\mathbb{Q})$ of the (irreducible) Galois action on $\overline{T}_f$ has a cyclic quotient of order $p-1$, and a classical result of Dickson implies that $p = 3$ and $\overline{G}$ is either a dihedral group, or S_4. In the former case, the order of $\mathrm{Gal}(\mathbb{Q}(\overline{L}^*)/\mathbb{Q})$ is prime to p, so (53) still holds. We are left to consider the case $\overline{G} = S_4$ and $p = 3$. Let $G = \mathrm{Gal}(\mathbb{Q}(\overline{T}_f)/\mathbb{Q})$ be the image of the Galois action; since we have assumed that $\det : G \to \mathbb{F}_3^\times$ factors through $\overline{G}$, a complex conjugation c in G projects to a transposition in $\overline{G}$. Let $\overline{H} \subset \overline{G}$ be a copy of S_3 containing the image of c, and H the normalizer of its preimage in G, which is contained in a unique Borel subgroup B. Let N be the unipotent radical of $B \cap G$. To prove (53), it suffices to check that

$$H^1(H, \overline{L}^*) = H^1(N, \overline{L}^*)^{H/N} = 0.$$

This holds because $\mathrm{im}(N-1)$ is isomorphic to the subgroup $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subset \overline{L}^*$, while $c \in H$ acts on N by -1 and on $\overline{L}^*/\,\mathrm{im}(N-1)$ by 1. $\square$

Theorem 6.1.8. *Let f be as above, satisfying (TW). Suppose given a weakly admissible pair $\{Q, \epsilon_Q\} \in N_j$ and an integer $k \le j$ satisfying the following conditions:*

(1) The maps

$$\mathrm{Sel}_Q(L/\pi^k) \to \mathrm{Sel}_Q(\overline{L}), \;\; \mathrm{Sel}_Q(L^*[\pi^k]) \to \mathrm{Sel}_Q(\overline{L}^*)$$

are identically zero.

(2) For each $\ell | Np$, let $n_0(\ell, k)$ be the number guaranteed by Proposition 6.1.3 for $r_0 = k$, and let $N_0(k) = \max_{\ell|Np}\{n_0(\ell,k)\}$. Then $j - k + 1 \ge N_0(k)$.

Then there is a newform g of weight 2, level NQ, and trivial character, with a prime $\wp_g$ of the ring of integers of its coefficient field $\mathcal{O}_g$, such that:

- *The completion $\mathcal{O}_{g,\wp_g}$ is a subring of $\mathcal{O}$.*
- *There is a congruence of Galois representations (in some basis)*

$$\rho_f \equiv \rho_{g,\wp_g} \pmod{\pi^{j-k+1}}.$$

- *The inertial types of $\rho_{g,\wp_g}|_{G_{\mathbb{Q}_\ell}}$ and $\rho_f|_{G_{\mathbb{Q}_\ell}}$ agree for all $\ell | N$ with $\ell \neq p$.*
- *$\rho_{g,\wp_g}|_{G_{\mathbb{Q}_p}}$ has the same Galois type as ρ_f and is potentially crystalline if and only if ρ_f is.*
- *For all $q|Q$, $\rho_{g,\wp_g}|_{G_{\mathbb{Q}_q}}$ is a Steinberg representation twisted by the unramified character $\mathrm{Frob}_q \mapsto \epsilon_Q(q)$.*

Proof. We will construct a Galois representation

$$\tau : G_{\mathbb{Q}} \to GL_2(\mathcal{O})$$

satisfying the following properties:

- $\tau \equiv \rho_f \pmod{\pi^{j-k+1}}$ (for some choice of basis of T_f).
- $\det \tau = \chi$.
- For all $\ell \nmid Q$, $\tau|_{G_{\mathbb{Q}_\ell}}$ defines a point of $\mathrm{Spec}\, R_\ell$.
- For all $q|Q$, $\tau|_{G_{\mathbb{Q}_q}}$ defines a point of $\mathrm{Spec}\, R^{\mathrm{ord}}_{q,\epsilon_Q(q)}$.

Let us first show that the existence of the representations τ is sufficient for the theorem. Since τ is odd and potentially semistable with distinct Hodge-Tate weights 0 and 1, we may apply the modularity lifting theorem of [45, Theorem 1.0.4] (see [63] for the case $p = 3$, using (TW)) to conclude that τ arises from a modular form g, which is automatically of weight two and trivial character. Now by modularity and [1, Theorem D, Proposition 1.2.2], $\tau|_{G_{\mathbb{Q}_\ell}}$ defines not only a point of $\mathrm{Spec}\, R_\ell$ but a smooth point of $\mathrm{Spec}\, \widetilde{R}_\ell$ for all ℓ. Since the potentially crystalline locus of $\mathrm{Spec}\, \widetilde{R}_p$ is a union of irreducible components (cf. [34]), τ is potentially crystalline if and only if ρ_f is. By construction, $\tau|_{G_{\mathbb{Q}_p}}$ has the same Galois type as $\rho_f|_{G_{\mathbb{Q}_p}}$. Hence g and f have the same conductor at p [56]. For $\ell \neq p$, the inertial type is constant on components of $\mathrm{Spec}\, \widetilde{R}_\ell[1/\pi]$, except possibly at the nonsmooth points (cf. [58]); it follows that τ has the same inertial type as ρ_f at all $\ell \nmid Q$, and is Steinberg for all $q|Q$. Since for all ℓ, the ℓ-part of the conductor of g is the conductor of the Weil-Deligne representation associated to $\tau|_{G_{\mathbb{Q}_\ell}}$ [8], we see that g has level NQ.

We now construct τ, as the inverse limit of representations

$$\tau_m : G_{\mathbb{Q}} \to GL_2(\mathcal{O}/\pi^m),$$

compatible under reduction maps, with the following key property: for all m, $\tau_m|_{G_{\mathbb{Q}_\ell}}$ lies in the subset Y_m^ℓ of Proposition 6.1.3 if $\ell \nmid Q$, and $\tau_m|_{G_{\mathbb{Q}_q}}$ defines a point of $\operatorname{Spec} R_q^{\mathrm{ord}}(\mathcal{O}/\pi^m)$ if $q|Q$. The representations τ_m are constructed inductively, *but* when constructing τ_{m+1}, we will allow ourselves to modify the representations $\tau_{m-k+2}, \ldots, \tau_m$. (This is the "relative" aspect of the construction.) Before we begin the construction, let us fix once and for all a set Σ of places containing all $v|NpQ\infty$ such that $Ш^1_\Sigma = 0$ (possible by Proposition 6.1.7). Our base case is $\tau_j = \rho_f \pmod{\pi^j}$, with any fixed framing for ρ_f. We use the same global framing for the local constructions in (6.1.2). Suppose we have defined τ_m for some $m \geq j$. For each $\ell \in \Sigma$, we may fix a local lift $\rho_{m+1,\ell}$ of $\tau_m|_{G_{\mathbb{Q}_\ell}}$ with the following property: if $\ell \nmid Q$, then $\rho_{m+1,\ell}$ lies in Y_{m+1}^ℓ, and if $\ell = q|Q$, then $\rho_{m+1,q}$ lies in $\operatorname{Spec} R_q^{\mathrm{ord}}(\mathcal{O}/\pi^{m+1})$. This is possible by Propositions 6.1.3 and 6.1.5, and by the key property of τ_m. In particular, the obstruction to lifting ρ_m modulo π^{m+1} vanishes locally. Then since $Ш^1_\Sigma = 0$, there exists by [48, p. 551] a representation $\rho_{m+1} : G_{\mathbb{Q}} \to GL_2(\mathcal{O}/\pi^{m+1})$ which is unramified outside Σ. Comparing $\rho_{m+1,\ell}$ to ρ_{m+1} as lifts of $\tau_{m-k+1}|_{G_{\mathbb{Q}_\ell}}$, they differ by local cocycles

$$(c_v) \in \bigoplus_{v\in\Sigma} (\mathbb{Q}_v, L/\pi^k).$$

Moreover, since ρ_{m+1} lifts τ_m, Propositions 6.1.3 and 6.1.5 imply that (c_v) has trivial image in

$$\bigoplus_{v\in\Sigma} \frac{H^1(\mathbb{Q}_v, L/\pi^{k-1})}{H^1_{\mathcal{S}}(\mathbb{Q}_v, L/\pi^{k-1})}.$$

By the argument in [17, p. 3578], it follows from the hypothesis on the vanishing of the Selmer group that there is a global cocycle $c \in H^1(\mathbb{Q}, L/\pi^k)$ whose localizations at $v \in \Sigma$ agree with (c_v). Adjusting ρ_{m+1} by the cocycle c, we obtain a representation τ_{m+1} with the desired key property. (Note that we are using $m+1-k \geq j+1-k \geq N_0$ to apply Proposition 6.1.3.) We now *redefine* $\tau_{m-k+2}, \ldots, \tau_m$ to be the reductions of τ_{m+1}; since τ_{m+1} is a lift of τ_{m-k+1} by construction, the representations $\tau_1, \ldots, \tau_{m-k+1}$ do not need to be redefined. This completes the inductive step of the construction, hence the proof of the theorem. □

6.2. **Patching adjoint Selmer groups.**

6.2.1. Patched Selmer groups provide a convenient framework to produce weekly admissible pairs $\{Q, \epsilon_Q\}$ satisfying the conditions of Theorem 6.1.8.

Suppose that q is an admissible ultraprime with sign ϵ_{q}. Using the exact sequence of $\mathcal{O}[G_{\mathsf{q}}]$-modules in Definition 3.1.2,

$$0 \to \operatorname{Fil}_{\mathsf{q}}^+ T_f \to T_f \to T_f / \operatorname{Fil}_{\mathsf{q}}^+ T_f \to 0,$$

we define

$$\operatorname{Fil}_{\mathsf{q}}^+ L = \operatorname{Hom}(T_f / \operatorname{Fil}_{\mathsf{q}}^+ T_f, \operatorname{Fil}_{\mathsf{q}}^+ T_f) \subset L$$

and

$$\mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L) = \operatorname{im}\left(\mathsf{H}^1(\mathbb{Q}_{\mathsf{q}}, \operatorname{Fil}_{\mathsf{q}}^+ L) \to \mathsf{H}^1(\mathbb{Q}_{\mathsf{q}}, L)\right).$$

It is clear that, if q is represented by sequence q_n, then

$$\mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L) = \varprojlim_j \mathcal{U}\left(H^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(\mathbb{Q}_{q_n}, L/\pi^j)\right),$$

where $H^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(\mathbb{Q}_{q_n}, L/\pi^j)$ is well-defined for all n such that q_n is j-admissible with sign ϵ_{q}. We also define $\mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L^\dagger)$ as the orthogonal complement of $\mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L)$ under the local Tate pairing. Note that, since $\mathsf{H}^1(\mathbb{Q}_{\mathsf{q}}, L)$ is torsion-free, $\mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L^\dagger)$ and $\mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L)$ are exact annihilators. We will require the restriction maps

$$\begin{aligned} \mathrm{loc}_{\mathsf{q}} &: \mathsf{H}^1(\mathbb{Q}, L) \to \frac{\mathsf{H}^1(\mathbb{Q}_{\mathsf{q}}, L)}{\mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L) \cap \mathsf{H}^1_{\mathrm{unr}}(\mathbb{Q}_{\mathsf{q}}, L)}, \\ \mathrm{loc}_{\mathsf{q}}^\dagger &: \mathsf{H}^1(\mathbb{Q}, L^\dagger) \to \frac{\mathsf{H}^1(\mathbb{Q}_{\mathsf{q}}, L^\dagger)}{\mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L^\dagger) \cap \mathsf{H}^1_{\mathrm{unr}}(\mathbb{Q}_{\mathsf{q}}, L^\dagger)}. \end{aligned} \tag{54}$$

Analogously, if $q \in M_{\mathbb{Q}}$ is j-admissible with sign ϵ_q, then we may define the local conditions $H^1_{\mathrm{ord}}(\mathbb{Q}_q, L/\pi^j)$, $H^1_{\mathrm{ord}}(\mathbb{Q}_q, L^\dagger/\pi^j)$, and the localization maps loc_q, $\mathrm{loc}_q^\dagger$.

6.2.2. Let $\mathsf{S} \subset \mathsf{M}_{\mathbb{Q}}$ be the set of constant ultraprimes $\underline{v}$ for $v|Np\infty$. For any $\{\mathsf{PQR}, \epsilon_{\mathsf{PQR}}\} \in \mathsf{N}$, we define the Selmer structure $(\mathcal{S}^{\mathsf{R}}_{\mathsf{P}}(\mathsf{Q}), \mathsf{S} \cup \mathsf{PQR})$ for L:

$$\mathsf{H}^1_{\mathcal{S}^{\mathsf{P}}_{\mathsf{R}}(\mathsf{Q})}(\mathbb{Q}_{\mathsf{v}}, L) = \begin{cases} \mathsf{H}^1_{\mathcal{S}}(\mathbb{Q}_{\mathsf{v}}, L), & \mathsf{v} \notin \mathsf{PQR} \\ \mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L), & \mathsf{v} = \mathsf{q} \in \mathsf{Q}, \\ \mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L) + \mathsf{H}^1_{\mathrm{unr}}(\mathbb{Q}_{\mathsf{q}}, L), & \mathsf{v} = \mathsf{q} \in \mathsf{P}, \\ \mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L) \cap \mathsf{H}^1_{\mathrm{unr}}(\mathbb{Q}_{\mathsf{q}}, L), & \mathsf{v} = \mathsf{q} \in \mathsf{R}. \end{cases} \tag{55}$$

The corresponding dual Selmer structure for $L^\dagger$ will be written $\mathcal{S}^{\mathsf{R},\dagger}_{\mathsf{P}}(\mathsf{Q})$. When P, Q, or R is empty, it is omitted from the notation.

Proposition 6.2.3. *For all $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$,*

$$d_{\mathsf{Q}} := \mathrm{rk}_{\mathcal{O}} \mathrm{Sel}_{\mathcal{S}(\mathsf{Q})}(L) = \mathrm{rk}_{\mathcal{O}} \mathrm{Sel}_{\mathcal{S}^\dagger(\mathsf{Q})}(L^\dagger).$$

Proof. It follows from Proposition 6.1.3 that

$$\mathrm{rk}_{\mathcal{O}}\, H^1_{\mathcal{S}}(\mathbb{Q}_\ell, L) = \mathrm{rk}_{\mathcal{O}}\, H^0(\mathbb{Q}_\ell, L) + \delta_{\ell=p}.$$

Since

$$\mathrm{rk}_{\mathcal{O}}\, \mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L) = \mathrm{rk}_{\mathcal{O}}\, \mathsf{H}^0(\mathbb{Q}_{\mathsf{q}}, L) = 1$$

for all $\mathsf{q} \in \mathsf{Q}$, the claim results from Proposition 2.7.5. □

In the language of patching, we can reformulate Theorem 6.1.8 as follows.

Theorem 6.2.4. *Assume f satisfies (TW), and suppose given a pair $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$ such that $d_{\mathsf{Q}} = 0$. Fix a sequence $\{Q_n, \epsilon_{Q_n}\}$ of weakly admissible pairs representing $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\}$ and an integer $j \geq 0$. Then there is a sequence (defined for $\mathfrak{F}$-many n) of newforms g_n of weight 2, level NQ_n, and trivial character, with a prime $\wp_{g_n}$ of the ring of integers of its coefficient field $\mathcal{O}_{g_n}$, such that:*

- *The completion $\mathcal{O}_{g_n,\wp_{g_n}}$ is a subring of $\mathcal{O}$.*
- *The inertial types of $\rho_{g_n,\wp_{g_n}}|_{G_{\mathbb{Q}_\ell}}$ and $\rho_f|_{G_{\mathbb{Q}_\ell}}$ agree for all $\ell|N$ with $\ell \neq p$.*
- *$\rho_{g_n,\wp_{g_n}}|_{G_{\mathbb{Q}_p}}$ has the same Galois type as ρ_f and is potentially crystalline if and only if ρ_f is.*
- *For all $q_n|Q_n$, $\rho_{g_n,\wp_{g_n}}|_{G_{\mathbb{Q}_{q_n}}}$ is a Steinberg representation twisted by the unramified character $\mathrm{Frob}_{q_n} \mapsto \epsilon_{Q_n}(q_n)$.*
- *For any fixed j, there is a congruence of Galois representations (in some basis)*

  $$\rho_f \equiv \rho_{g_n,\wp_{g_n}} \pmod{\pi^j}$$

  for $\mathfrak{F}$-many n.

In particular, the maps

$$\mathbb{T}_{Q_n} = \mathbb{T}_{N^+,N^-Q_n} \to \mathcal{O}/\pi^j$$

of Remark 4.5.8 admit $\mathcal{O}$-valued lifts for $\mathfrak{F}$-many n.

Proof. Since $d_{\mathsf{Q}} = 0$, Proposition 2.5.5 implies that there exists some $k \geq 0$ such that the natural maps

$$\mathrm{Sel}_{\mathcal{S}(\mathsf{Q})}(L/\pi^k) \to \mathrm{Sel}_{\mathcal{S}(\mathsf{Q})}(\overline{L}),\ \ \mathrm{Sel}_{\mathcal{S}(\mathsf{Q})}(L^\dagger/\pi^k) \to \mathrm{Sel}_{\mathcal{S}(\mathsf{Q})}(\overline{L}^*) \tag{56}$$

are identically zero. Thus for $\mathfrak{F}$-many n, the maps

$$\mathrm{Sel}_{Q_n}(L/\pi^k) \to \mathrm{Sel}_{Q_n}(\overline{L})$$

are identically zero. Also, since Proposition 6.1.3 implies $\mathcal{S}$ is saturated, Proposition 2.7.4 shows that $\mathrm{Sel}_{\mathcal{S}(\mathsf{Q})}(L^\dagger/\pi^k) = \mathrm{Sel}_{\mathcal{S}(\mathsf{Q})}(L^*[\pi^k])$. Hence for $\mathfrak{F}$-many n, the maps

$$\mathrm{Sel}_{Q_n}(L^*[\pi^k]) \to \mathrm{Sel}_{Q_n}(\overline{L}^*)$$

are also identically zero. Since k is fixed independently of n, the theorem is now immediate from Theorem 6.1.8.

□

6.3. **Annihilating two Selmer groups.**

6.3.1. For our application of Theorem 6.2.4, we will want to choose $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$ such that $d_{\mathsf{Q}} = 0$, and $r_{\mathsf{Q}} = \mathrm{rk}_{\mathcal{O}} \operatorname{Sel}_{\mathcal{F}(\mathsf{Q})}(T_f) = 0$, where $\mathcal{F}$ is a self-dual Selmer structure for T_f. In this subsection, we show that such a choice is possible (Proposition 6.3.5 below). The proof is inspired by [9], and begins with a series of lemmas. For the entirety of this subsection, we assume (non-CM).

Lemma 6.3.2. *There exists an integer j that, for all $n \geq 0$,*

$$\pi^j H^1(K(T_f)/\mathbb{Q}, L/\pi^n) = \pi^j H^1(K(T_f)/\mathbb{Q}, L^\dagger/\pi^n) = 0.$$

Proof. Let $E = \mathbb{Q}(\mu_{p^\infty}) \subset K(T_f)$, and note that L and $L^\dagger$ are isomorphic G_E-modules. Since f is non-CM, $(L \otimes \mathbb{Q}_p)^{G_E} = 0$, and so $(L/\pi^n)^{G_E}$ is uniformly bounded in n.

The pro-p-Sylow subgroup of $\mathrm{Gal}(K(T_f)/E)$ is a compact p-adic Lie group with semisimple Lie algebra; hence, by [17, Lemma B.1], the cohomology $H^1(K(T_f)/E, L/\pi^n)$ is uniformly bounded in n.

Now, by inflation-restriction, we have exact sequences

$$\begin{aligned} &0 \to H^1(E/\mathbb{Q}, (L/\pi^n)^{G_E}) \to H^1(K(T_f)/\mathbb{Q}, L/\pi^n) \to H^1(K(T_f)/E, L/\pi^n), \\ &0 \to H^1(E/\mathbb{Q}, (L^\dagger/\pi^n)^{G_E}) \to H^1(K(T_f)/\mathbb{Q}, L^\dagger/\pi^n) \to H^1(K(T_f)/E, L^\dagger/\pi^n), \end{aligned} \tag{57}$$

where the outer terms are isomorphic and uniformly bounded in n; the lemma follows. ☐

For the next lemma, we abbreviate $L_m := L/\pi^m$, $L_m^\dagger := L^\dagger/\pi^m$, and $T_m := T_f/\pi^m$. Moreover, if $y \in M$ for any torsion $\mathcal{O}$-module M, let $\mathrm{ord}(y)$ be the smallest integer $t \geq 0$ such that $\pi^t y = 0$.

Lemma 6.3.3. *There is a constant C, depending only on T_f, with the following property. Given cocycles $\phi \in H^1(\mathbb{Q}, L_m)$, $\psi \in H^1(\mathbb{Q}, L_m^\dagger)$, and $c_1, c_2 \in H^1(K, T_m)^\delta$ for some $\delta = \pm 1$, there exist infinitely many primes $q \nmid Np$ such that all the cocycles are unramified at q and:*

- *The Frobenius of q in $\mathrm{Gal}(K(T_m)/\mathbb{Q})$ is a complex conjugation; in particular, q is m-admissible with sign δ.*
- $\mathrm{ord}\,\mathrm{loc}_q \phi \geq \mathrm{ord}\,\phi - C$.
- $\mathrm{ord}\,\mathrm{loc}_q^\dagger \psi \geq \mathrm{ord}\,\psi - C$, *or* $\mathrm{loc}_q^\dagger \psi = 0$, *as desired.*
- $\mathrm{ord}\,\mathrm{loc}_q c_i \geq \mathrm{ord}\, c_i - C$ *for $i = 1, 2$.*

Proof. Let us first fix a complex conjugation $c \in G_{\mathbb{Q}}$ and choose a basis for T_m in which c acts as $\begin{pmatrix} -\delta & 0 \\ 0 & \delta \end{pmatrix}$.

The restriction of the cocycles ϕ, ψ, c_i to $G_{K(T_m)}$ may be considered as a homomorphism

$$h : G_{K(T_m)} \to L_m \oplus L_m^\dagger \oplus (T_m)^2$$

compatible with the action of G_K; let H be the image of this homomorphism. Since there exists an element of $g_z \in G_K$ that acts by a scalar $z \neq \pm 1$ on T_f, we have:

$$\begin{aligned} H \supset &(g_z - z)(g_z - z^2)H + (g_z - z)(g_z - 1)H + (g_z - z^2)(g_z - 1)H \\ &\supset (z-1)(z^2-1)(z^2-z)\left(\pi_{L_m}(H) \oplus \pi_{L_m^\dagger}(H) \oplus \pi_{T_m^2}(H),\right) \end{aligned}$$

where $\pi_\bullet$ are the projection operators. Now, since L and $L^\dagger$ are absolutely irreducible, the natural maps $\mathbb{Q}_p[G_K] \to \mathrm{End}(L \otimes \mathbb{Q}_p)$ and $\mathbb{Q}_p[G_K] \to \mathrm{End}(L^\dagger \otimes \mathbb{Q}_p)$ are surjective. Combining these observations with Lemma 6.3.2, we see that, for some constant C depending only on T_f, there exists $\gamma \in G_{K(T_f)}$ satisfying:

- The $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ component of $\phi(\gamma)$ has order at least $\mathrm{ord}\,\phi - C$.
- The $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ component of $\psi(\gamma_\psi)$ has order at least $\mathrm{ord}\,\psi - C$, or is 0, as desired.
- The components of $c_i(\gamma)$ and $c_2(\gamma)$ in the δ eigenspace have order at least $\mathrm{ord}\, c_i - C$, where $i = 1, 2$.

For the final item, we are using the elementary fact that a group cannot be the union of two nontrivial subgroups, as well as the irreducibility of T_f.

Since $\phi(c^2) = c\phi(c) + \phi(c) = 0$, $\phi(c)$ lies in the -1 eigenspace for complex conjugation, whereas $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ has eigenvalue 1; hence the $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ component of $\phi(c\gamma)$ has order at least $\mathrm{ord}\,\phi - C$. Similarly, the $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ component of $\psi(c\gamma_\psi)$ has order at least $\mathrm{ord}\,\psi - C$, or is 0, as desired.

Any prime with Frobenius $c\gamma$ in $\ker h$ satisfies the conclusion of the lemma; cf. the proof of Lemma 3.3.7 for the assertions about c_i. $\square$

Corollary 6.3.4. *Suppose given a finite set of ultraprimes* T *and non-torsion cocycles:*

- $\phi \in \mathsf{H}^1(\mathbb{Q}^{\mathsf{T}}/\mathbb{Q}, L)$;
- $\psi \in \mathsf{H}^1(\mathbb{Q}^{\mathsf{T}}/\mathbb{Q}, L^\dagger)$;
- $c_1, c_2 \in \mathsf{H}^1(K^{\mathsf{T}}/K, T_f)^\delta$ *for* $\delta = \pm 1$.

Then there exist infinitely many admissible ultraprimes $\mathsf{q} \notin \mathsf{T}$ *with sign* δ *such that:*

- $\mathrm{loc}_{\mathsf{q}}\, \phi \neq 0$.
- *Either* $\mathrm{loc}_{\mathsf{q}}^\dagger\, \psi \neq 0$ *or* $\mathrm{loc}_{\mathsf{q}}^\dagger\, \psi = 0$, *as desired.*
- $\mathrm{loc}_{\mathsf{q}}\, c_i \neq 0$.

Proof. Choose a sequence T_n representing T and sequences $\phi_n, \psi_n, c_n^1, c_n^2$ representing the respective cocycles in $H^1(\mathbb{Q}^{T_n}/\mathbb{Q}, L/\pi^n)$, etc. For each n, apply Lemma 6.3.3 with $m = n$ and the appropriate desideratum for ψ_n; by definition, any resulting admissible ultraprime q, represented by a sequence q_n, satisfies the desired conclusion. $\square$

Proposition 6.3.5. *Assume f satisfies (non-CM), and suppose given a self-dual Selmer structure $(\mathcal{F}, \mathsf{T})$ for T_f. Then there exists $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}_{\mathsf{T}}$ such that*

$$r_{\mathsf{Q}} = d_{\mathsf{Q}} = 0.$$

(Recall that $r_{\mathsf{Q}} = \mathrm{rk}_{\mathcal{O}}\, \mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_f)$.)

Proof. Without loss of generality, by Corollary 3.3.9 we may assume that $r_1 = 0$; for if not, choose any $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}_{\mathsf{T}}$ with $r_{\mathsf{Q}} = 0$, and then relabel $\mathcal{F}(\mathsf{Q})$ as $\mathcal{F}$.

We will show that, if $d_1 > 0$, we may find $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\}$ such that $r_{\mathsf{Q}} = 0$ and $d_{\mathsf{Q}} < d_1$; this clearly suffices by induction. By Proposition 6.2.3, there exist non-torsion elements $\phi \in \mathrm{Sel}_{\mathcal{S}}(L)$, $\psi \in \mathrm{Sel}_{\mathcal{S}^\dagger}(L^\dagger)$. We choose any admissible $\mathsf{q} \notin \mathsf{T}$ with sign ϵ_{q} such that $\mathrm{loc}_{\mathsf{q}}\, \phi \neq 0$, $\mathrm{loc}_{\mathsf{q}}^\dagger\, \psi \neq 0$. Then by Proposition 2.7.5,

$$\mathrm{rk}_{\mathcal{O}}\, \mathrm{Sel}_{\mathcal{S}^{\dagger,\mathsf{q}}}(L^\dagger) + \mathrm{rk}_{\mathcal{O}}\, \mathrm{Sel}_{\mathcal{S}^{\mathsf{q}}}(L) = 2 + \mathrm{rk}_{\mathcal{O}}\, \mathrm{Sel}_{\mathcal{S}^\dagger_{\mathsf{q}}}(L^\dagger) + \mathrm{rk}_{\mathcal{O}}\, \mathrm{Sel}_{\mathcal{S}_{\mathsf{q}}}(L)$$

(in the notation of (55)). The images of the localization maps

$$\mathrm{loc}_{\mathsf{q}} : \frac{\mathrm{Sel}_{\mathcal{S}^{\mathsf{q}}}(L)}{\mathrm{Sel}_{\mathcal{S}_{\mathsf{q}}}(L)} \hookrightarrow \frac{\mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L)}{\mathsf{H}^1_{\mathrm{ord}\,\cap\,\mathrm{unr}}(\mathbb{Q}_{\mathsf{q}}, L)} \oplus \frac{\mathsf{H}^1_{\mathrm{unr}}(\mathbb{Q}_{\mathsf{q}}, L)}{\mathsf{H}^1_{\mathrm{ord}\,\cap\,\mathrm{unr}}(\mathbb{Q}_{\mathsf{q}}, L)}$$

and

$$\mathrm{loc}_{\mathsf{q}}^\dagger : \frac{\mathrm{Sel}_{\mathcal{S}^{\dagger,\mathsf{q}}}(L^\dagger)}{\mathrm{Sel}_{\mathcal{S}^\dagger_{\mathsf{q}}}(L^\dagger)} \hookrightarrow \frac{\mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L^\dagger)}{\mathsf{H}^1_{\mathrm{ord}\,\cap\,\mathrm{unr}}(\mathbb{Q}_{\mathsf{q}}, L^\dagger)} \oplus \frac{\mathsf{H}^1_{\mathrm{unr}}(\mathbb{Q}_{\mathsf{q}}, L^\dagger)}{\mathsf{H}^1_{\mathrm{ord}\,\cap\,\mathrm{unr}}(\mathbb{Q}_{\mathsf{q}}, L^\dagger)}$$

have total rank two and annihilate each other under the induced Tate pairing by Proposition 2.7.2. Hence the image in the ordinary part is zero for both maps, and $d_{\mathsf{q}} < d_1$. However, by adding q, we have made $r_{\mathsf{q}} = 1$. Let $c \in \mathrm{Sel}_{\mathcal{F}(\mathsf{q})}(T_f)$ be a generator; since $\partial_{\mathsf{q}} c \neq 0$ by Proposition 3.3.4, c has nonzero component in the ϵ_{q} eigenspace for τ.

Now consider the set P of admissible ultraprimes s with sign $\epsilon_{\mathsf{s}} = \epsilon_{\mathsf{q}}$ such that $\mathrm{loc}_{\mathsf{s}}\, c \neq 0$. If, for any $\mathsf{s} \in \mathsf{P}$, $d_{\mathsf{qs}} \leq d_{\mathsf{q}}$, then we may take $\mathsf{Q} = \mathsf{qs}$ and complete our induction step. For example, this will occur provided $d_{\mathsf{q}} > 0$, by the argument above; so without loss of generality, $d_{\mathsf{q}} = 0$ and $d_{\mathsf{qs}} = 1$ for all $\mathsf{s} \in \mathsf{P}$. By definition, we therefore have non-torsion elements $\phi(\mathsf{s}) \in \mathrm{Sel}_{\mathcal{S}(\mathsf{qs})}(L)$ and $\psi(\mathsf{s}) \in \mathrm{Sel}_{\mathcal{S}^\dagger(\mathsf{qs})}(L^\dagger)$ such that $\mathrm{loc}_{\mathsf{s}}\, \phi(s)$ and $\mathrm{loc}_{\mathsf{s}}\, \psi(s)$ do not lie in the unramified subspace of the ordinary cohomology.

Choose any $\mathsf{s}_1 \in \mathsf{P}$, and then choose $\mathsf{s}_2 \in \mathsf{P}$ such that $\mathrm{loc}_{\mathsf{s}_2}\, \phi(\mathsf{s}_1) \neq 0$ but $\mathrm{loc}_{\mathsf{s}_2}\, \psi(\mathsf{s}_1) = 0$. By another application of Proposition 3.3.4, $r_{\mathsf{qs}_1\mathsf{s}_2} = 1$, and a generator c' of $\mathrm{Sel}_{\mathcal{F}(\mathsf{qs}_1\mathsf{s}_2)}(T_f)$ again has nonzero component in the ϵ_{q} eigenspace. We now choose $\mathsf{s}_3 \in \mathsf{P}$ such that $\mathrm{loc}_{\mathsf{s}_3}\, c' \neq 0$, $\mathrm{loc}_{\mathsf{s}_3}\, \phi(\mathsf{s}_2) \neq 0$, and $\mathrm{loc}_{\mathsf{s}_3}\, \psi(\mathsf{s}_1) \neq 0$. Clearly $r_{\mathsf{qs}_1\mathsf{s}_2\mathsf{s}_3} = 0$. Note that $\mathrm{rk}_{\mathcal{O}}\, \mathrm{Sel}_{\mathcal{S}^{\mathsf{s}_1\mathsf{s}_2\mathsf{s}_3}(\mathsf{q})} = 3$; up to torsion, $\phi(\mathsf{s}_i)$ are generators. So to show that $d_{\mathsf{qs}_1\mathsf{s}_2\mathsf{s}_3} = d_{\mathsf{q}}$, it suffices to show that the images of $\phi(\mathsf{s}_i)$ form a rank-three subspace of

$$S := \bigoplus_{i=1}^{3} \frac{\mathsf{H}^1_{\mathrm{unr}\,+\,\mathrm{ord}}(\mathbb{Q}_{\mathsf{s}_i}, L)}{\mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{s}_i}, L)}$$

under the localization

$$\mathrm{loc} : \frac{\mathrm{Sel}_{\mathcal{S}^{\mathsf{s}_1\mathsf{s}_2\mathsf{s}_3}}(\mathsf{q})(L)}{\mathrm{Sel}_{\mathcal{S}_{\mathsf{s}_1\mathsf{s}_2\mathsf{s}_3}}(\mathsf{q})(L)} \hookrightarrow S.$$

By pairing $\phi(\mathsf{s}_i)$ and $\psi(\mathsf{s}_j)$ for $i \neq j$ and applying Proposition 2.7.2 once more, we see that $\mathrm{loc}_{\mathsf{s}_i} \phi(\mathsf{s}_j) \neq 0$ if and only if $\mathrm{loc}_{\mathsf{s}_j} \psi(\mathsf{s}_i) \neq 0$. Hence, the images of $\phi(\mathsf{s}_i)$ in S are of the form:

$$\begin{aligned} \mathrm{loc}(\phi(\mathsf{s}_1)) &= (0, *, \cdot) \\ \mathrm{loc}(\psi(\mathsf{s}_2)) &= (0, 0, *), \\ \mathrm{loc}(\psi(\mathsf{s}_3)) &= (*, 0, 0), \end{aligned}$$

where $*$ is nonzero and $\cdot$ may or may not be zero. This completes the inductive step since $d_{\mathsf{q}\mathsf{s}_1\mathsf{s}_2\mathsf{s}_3} = d_{\mathsf{q}} < d_1$. □

7. Proof of main results: split ordinary case

For this section, let f be a non-CM cuspidal eigenform of weight two and trivial character, with ring of integers $\mathcal{O}_f$ of its coefficient field. Suppose f is new of level N, and let $\wp \subset \mathcal{O}_f$ be an ordinary prime lying over $p \nmid 2N$. Denote by $\mathcal{O}$ the completion.

7.1. **A result of Skinner-Urban.** The following result is a corollary to the proof of the main conjecture [60].

Theorem 7.1.1 ((Skinner-Urban))**.** *Let K be an imaginary quadratic field of discriminant prime to Np in which p splits. Assume that $\wp$ is ordinary for f and that:*

- *the mod $\wp$ representation $\overline{T}_f$ is absolutely irreducible;*
- *$N = N_1N_2$, where every factor of N_1 is split in K and N_2 is the squarefree product of an odd number primes inert in K.*

If $\mathrm{Sel}_{\mathcal{F}^*_\Lambda}(\mathbf{W}_f)$ is Λ-cotorsion, then

$$\mathrm{char}_\Lambda \mathrm{Sel}_{\mathcal{F}^*_\Lambda}(\mathbf{W}_f)^\vee \subset (\boldsymbol{\lambda}(1))^2$$

as ideals of Λ, where $\boldsymbol{\lambda}(1) \in \Lambda$ is the element constructed in (5.2).

Here the Selmer structure $\mathcal{F}^*_\Lambda$ for $\mathbf{W}_f$ is the dual of the Selmer structure in (46).

Proof. We must explain some details and notations of [60], in which it is assumed that $\overline{T}_f$ is ramified at every $\ell | N_2$. As in [60], we let O_L be the ring of integers of a suitable finite extension of $\mathbb{Q}_p$ and consider f as a specialization of a suitable Hida family $\mathbf{f}$. This family is parametrized by $\mathbb{I}$, which is a normal domain and a finite integral extension of $O_L[\![W]\!]$. We write $\Gamma_K = \Gamma_K^+ \times \Gamma_K^-$ for the Galois group of the maximal $\mathbb{Z}_p$-extension of K and its decomposition into cyclotomic/anticyclotomic components. For a sufficiently large finite set of primes Σ containing all those dividing $N\infty$, there is [60, Theorem 12.3.1] a three-variable p-adic L-function $\mathcal{L}^\Sigma_{\mathbf{f},K} \in \mathbb{I}[\![\Gamma_K]\!]$. (Here the superscript Σ refers to removing Euler factors at primes in Σ, or relaxing local conditions for a Selmer group.) Letting γ^- be a topological generator of Γ_K^-, we may expand:

$$\mathcal{L}^\Sigma_{\mathbf{f},K} = a_0 + a_1(\gamma^- - 1) + a_2(\gamma^- - 1)^2 + \dots \tag{58}$$

where $a_i \in \mathbb{I}[\![\Gamma_K^+]\!]$. Let $Ch^\Sigma_{K_\infty}(\mathbf{f}) \subset \mathbb{I}[\![\Gamma_K]\!]$ be the characteristic ideal of the three-variable Selmer group as considered in [60]. Skinner and Urban deduce

$$Ch^\Sigma_{K_\infty}(\mathbf{f}) \subset (\mathcal{L}^\Sigma_{\mathbf{f},K}) \tag{59}$$

by proving (see [60, Theorem 6.5.4, Proposition 12.3.6, Proposition 13.4.1]):

(1) If $P \subset \mathbb{I}[\![\Gamma_K]\!]$ is a height one prime which is not of the form $P_+\mathbb{I}[\![\Gamma_K]\!]$ for some $P_+ \subset \mathbb{I}[\![\Gamma_K^+]\!]$, then

$$\mathrm{ord}_P\, Ch^\Sigma_{K_\infty}(\mathbf{f}) \geq \mathrm{ord}_P(\mathcal{L}^\Sigma_{\mathbf{f},K}).$$

(2) If $\overline{T}_f$ is ramified at every $\ell | N_2$, then $\mathrm{ord}_P(\mathcal{L}^\Sigma_{\mathbf{f},K}) = 0$ for all height one primes P of the form $P_+\mathbb{I}[\![\Gamma_K]\!]$ for some $P_+ \subset \mathbb{I}[\![\Gamma_K^+]\!]$.

Although (ii) does not apply, we claim that we may replace (59) by the weaker inclusion:

$$Ch^{\Sigma}_{K_\infty}(\mathbf{f}) \cdot (a_i) \supset (\mathcal{L}^{\Sigma}_{\mathbf{f},K}) \tag{60}$$

where a_i is any of the terms in (58). Indeed, because both sides of (60) are divisorial, it suffices to check that $\mathrm{ord}_P(\mathcal{L}^{\Sigma}_{\mathbf{f},K}) \leq \mathrm{ord}_P(a_i)$ for all P as in (ii). But this is clear: if $(\mathcal{L}^{\Sigma}_{\mathbf{f},K})$ is zero modulo P^k for such a prime P, then a_i is as well. By [64] a_i may be chosen so that its image under the specialization map $\mathbb{1} : \mathbb{I}[\![\Gamma^+_K]\!] \to O_L$ is nonzero. Fix such a choice $\widetilde{\alpha}$.

The divisibility (60) also (trivially) implies a divisibility for the Fitting ideal of the 3-variable Selmer group:

$$(\widetilde{\alpha}) Fitt^{\Sigma}_{K_\infty}(\mathbf{g}) \subset (\mathcal{L}^{\Sigma}_{\mathbf{g},K}). \tag{61}$$

Specializing (61) to the anticyclotomic variable, we obtain

$$\mathrm{char}_\Lambda \mathrm{Sel}^{\Sigma}(K_\infty, f) \subset L^{\Sigma}_p(K_\infty, f) \text{ in } \Lambda \otimes \mathbb{Q}_p,$$

where $L^{\Sigma}_p(K_\infty, f)$ is a certain Σ-primitive anticyclotomic L-function, and $\mathrm{Sel}^{\Sigma}(K_\infty, f)$ is the Σ-primitive Selmer group. Replacing [60, Proposition 3.3.19] by [47, Proposition A.2] (and using the hypothesis that the Selmer group is Λ-cotorsion), we may convert this to an imprimitive divisibility

$$\mathrm{char}_\Lambda \mathrm{Sel}_{\mathcal{F}^*_\Lambda}(\mathbf{W}_f)^\vee \subset (\boldsymbol{\lambda}(1))^2 \text{ in } \Lambda \otimes \mathbb{Q}_p. \tag{62}$$

NB: The anticyclotomic p-adic L-function appearing in [60, §12.3.5], which emerges naturally from the specialization of the three-variable p-adic L-function, is normalized using Hida's canonical period, whereas $\boldsymbol{\lambda}(1)^2$ is the L-function constructed in [2], normalized using Gross's period. However, these L-functions differ only by a power of p. Similarly, our $\mathrm{Sel}_{\mathcal{F}^*_\Lambda}(\mathbf{W}_f)$ differs from the $\mathrm{Sel}(K_\infty, f)$ used in [60] by the local conditions used at $\ell | N_2$. Since the local cohomology groups $H^1(K_\ell, \mathbf{W}_f)$ for $\ell | N_2$ have characteristic ideal a power of $\wp\Lambda$, this choice of local condition does not change the characteristic ideal in $\Lambda \otimes \mathbb{Q}_p$. See §8 and [47] for a detailed discussion.

To upgrade (62) to a divisibility in Λ, we simply note that the μ-invariant of $\boldsymbol{\lambda}(1)$ is 0 by [64]. ☐

7.2. **The Heegner point main conjecture.** In this subsection, we prove the following main theorem.

Theorem 7.2.1. *Let f be a non-CM cuspidal eigenform of weight two and trivial character, new of level N, with ring of integers $\mathcal{O}_f$ of its coefficient field. Let $\wp \subset \mathcal{O}_f$ be a prime of ordinary residue characteristic p, and let K be an imaginary quadratic field. Suppose:*

- *$N = N^+N^-$, where every factor of N^+ is split in K, and N^- is a squarefree product of primes inert in K.*
- *p does not divide $2D_K N$, and p splits in K.*
- *The modulo $\wp$ representation $\overline{T}_f$ associated to f is absolutely irreducible; if $p = 3$, assume that $\overline{T}_f$ is not induced from a character of $G_{\mathbb{Q}(\sqrt{-3})}$.*

Then, for all $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\nu(N^-)}$ such that $(\boldsymbol{\kappa}(\mathsf{Q}, \epsilon_{\mathsf{Q}})) \neq 0$, we have

$$\mathrm{rk}_\Lambda \mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{T}_f) = \mathrm{crk}_\Lambda \mathrm{Sel}_{\mathcal{F}^*_\Lambda(\mathsf{Q})}(\mathbf{W}_f) = 1$$

and

$$\mathrm{char}_\Lambda \left(\left(\mathrm{Sel}_{\mathcal{F}^*_\Lambda(\mathsf{Q})}(\mathbf{W}_f)^\vee \right)_{\mathrm{tors}} \right) = \mathrm{char}_\Lambda \left(\frac{\mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{T}_f)}{(\boldsymbol{\kappa}(\mathsf{Q}))} \right)^2 \text{ in } \Lambda \otimes \mathbb{Q}_p.$$

For all $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\nu(N^-)+1}$ such that $\boldsymbol{\lambda}(\mathsf{Q}) \neq 0$,

$$\mathrm{rk}_\Lambda \mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{T}_f) = \mathrm{crk}_\Lambda \mathrm{Sel}_{\mathcal{F}^*_\Lambda(\mathsf{Q})}(\mathbf{W}_f) = 0$$

and

$$\mathrm{char}_\Lambda \left(\mathrm{Sel}_{\mathcal{F}^*_\Lambda(\mathsf{Q})}(\mathbf{W}_f)^\vee \right) = (\boldsymbol{\lambda}(\mathsf{Q}))^2 \text{ in } \Lambda \otimes \mathbb{Q}_p.$$

If moreover the image of the $G_{\mathbb{Q}}$ action on $\overline{T}_f$ contains a nontrivial scalar, then the equalities hold in Λ.

Proof. Given f, apply Proposition 6.3.5 to the standard Selmer structure $(\mathcal{F}, \mathsf{S})$ on T_f (with local conditions the image of the Kummer map at all $\underline{v}$ such that $v | Np$). Let $\{Q_n, \epsilon_{Q_n}\}$ be a sequence of weakly admissible pairs representing the resulting pair $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$. Let g_n be the resulting sequence of newforms of level NQ_n obtained from Theorem 6.2.4; g_n may only be defined for $\mathfrak{F}$-many n.

Step 1. $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\nu(N^-)+1}$.

Proof. The prime (T), corresponding to the trivial character, does not lie in the exceptional set Σ for the interpolated self-dual Selmer structure associated to $\mathcal{F}_\Lambda$ (see the proofs of [38, Lemma 5.3.13] and [27, Lemma 2.2.7]). Hence $\mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{T}_f) = 0$, which by Theorem 3.4.6 and the nontriviality of $(\boldsymbol{\kappa}, \boldsymbol{\lambda})$ implies the claim. $\square$

Step 2. *For any fixed j,*

$$(\boldsymbol{\lambda}(\mathsf{Q})) \equiv (\boldsymbol{\lambda}_{g_n}(1)) \pmod{\pi^j, T^j}$$

for $\mathfrak{F}$-many n.

Proof. Recall notations of (4.5) and (5.2). By definition, the image of $\boldsymbol{\lambda}(\mathsf{Q})$ modulo (π^j, T^j) is a map $\mathrm{Gal}(K_j/K) \to \mathcal{O}$ obtained, for $\mathfrak{F}$-many n, by evaluating a surjective map $F_n : M_{Q_n} \otimes_{\mathbb{T}^{Q_n}} \mathcal{O}(f) \to \mathcal{O}(f)/\pi^j$ of $\mathbb{T}^{Q_n}$-modules at certain CM points, where $\mathcal{O}(f)$ is $\mathcal{O}$ with $\mathbb{T}^{Q_n}$-action by f. Recall that the map is chosen to factor through multiplicatation by $\mathcal{O}(f)/\pi^{j+C}$ for the constant C of Lemma 4.5.5, and that $\pi^C(M_{Q_n} \otimes_{\mathbb{T}^{Q_n}} \mathcal{O}(f))$ is principal. When g_n has a sufficiently deep congruence to f, $\mathcal{O}(g_n)/\pi^{j+C} = \mathcal{O}(f)/\pi^{j+C}$ as $\mathbb{T}^{Q_n}$-modules, and the composite $G_n : M_{Q_n} \to \mathcal{O}(g_n) \to \mathcal{O}(g_n)/\pi^j$ induces a unit multiple of F_n, where $M_{Q_n} \to \mathcal{O}(g_n)$ is the quaternionic modular form associated to g_n. But G_n is the very map whose evaluation at CM points is used to define $\boldsymbol{\lambda}_{g_n}(1)$, and the claim follows. $\square$

Step 3. *For any fixed j,*

$$\mathrm{Fitt}_\Lambda \mathrm{Sel}_{\mathcal{F}^*_\Lambda(\mathsf{Q})}(\mathbf{W}_f)^\vee \equiv \mathrm{Fitt}_\Lambda \mathrm{Sel}_{\mathcal{F}^*_{g_n,\Lambda}}(\mathbf{W}_{g_n})^\vee \pmod{\pi^j, T^j}$$

for $\mathfrak{F}$-many n.

Proof. Since fitting ideals are stable under base change and $\overline{T}_f$ has no G_K-fixed points, it suffices to show that

$$\mathrm{Fitt}_\Lambda \left(\mathrm{Sel}_{\mathcal{F}^*_\Lambda(\mathsf{Q})}(\mathbf{W}_f[\pi^j, T^j])\right) = \mathrm{Fitt}_\Lambda \left(\mathrm{Sel}_{\mathcal{F}^*_{g_n,\Lambda}}(\mathbf{W}_{g_n}[\pi^j, T^j])\right) \tag{63}$$

for $\mathfrak{F}$-many n. Note that, for $\mathfrak{F}$-many n, $\mathbf{W}_f[\pi^j, T^j] = \mathbf{W}_{g_n}[\pi^j, T^j]$ as finite Galois modules, and

$$\mathrm{Sel}_{\mathcal{F}^*_\Lambda(\mathsf{Q})}(\mathbf{W}_f[\pi^j, T^j])$$

is isomorphic to a submodule of $H^1(K^{\Sigma \cup Q_n}/K, \mathbf{W}_f[\pi^j, T^j])$ defined by certain local conditions. We will show that these local conditions coincide with the ones defining $\mathrm{Sel}_{\mathcal{F}_{g_n,\Lambda}}(\mathbf{W}_{g_n}[\pi^j, T^j])$. At a j-admissible prime $q_n | Q_n$, this is clear. At $v|N$, since $\mathsf{H}^1_{\mathcal{F}_\Lambda}(K_{\overline{v}}, \mathbf{T}_f) = H^1(K_{\overline{v}}, \mathbf{T}_f)$, we have $\mathsf{H}^1_{\mathcal{F}^*_\Lambda}(K_v, \mathbf{W}_f) = 0$, and similarly for g_n. So the induced local conditions are simply the kernels

$$\begin{gathered} \ker\left(H^1(K_v, \mathbf{W}_f[\pi^j, T^j]) \to H^1(K_v, \mathbf{W}_f)\right), \\ \ker\left(H^1(K_v, \mathbf{W}_{g_n}[\pi^j, T^j]) \to H^1(K_v, \mathbf{W}_{g_n})\right). \end{gathered} \tag{64}$$

If $v|N$ is a prime of multiplicative reduction for f, then $\mathbf{W}_f = \mathbf{W}_{g_n}$ as G_{K_v} modules for $\mathfrak{F}$-many n, so the local conditions clearly coincide. For other places of bad reduction, the inertia co-invariants T_{f,I_v} and T_{g_n,I_v} are finite and isomorphic for $\mathfrak{F}$-many n. If T_{f,I_v} is annihilated by π^{M-j} for some $M \geq 0$, then the local conditions are also given by the kernels of

$$\begin{gathered} \ker\left(H^1(K_v, \mathbf{W}_f[\pi^j, T^j]) \to H^1(K_v, \mathbf{W}_f[\pi^M])\right), \\ \ker\left(H^1(K_v, \mathbf{W}_{g_n}[\pi^j, T^j]) \to H^1(K_v, \mathbf{W}_{g_n}[\pi^M])\right), \end{gathered} \tag{65}$$

which agree for $\mathfrak{F}$-many n.

For primes $v|p$, it suffices to compare the kernels

$$\begin{gathered} \left(H^1(K_v, \mathrm{gr}\, \mathbf{W}_f[\pi^j, T^j]) \to H^1(K_v, \mathrm{gr}\, \mathbf{W}_f)\right), \\ \ker\left(H^1(K_v, \mathrm{gr}\, \mathbf{W}_{g_n}[\pi^j, T^j]) \to H^1(K_v, \mathrm{gr}\, \mathbf{W}_{g_n})\right). \end{gathered} \tag{66}$$

A similar argument as above applies provided that

$$H^0(K_v, \mathrm{gr}\, \mathbf{W}_f) = \prod_{w|v} H^0(K_{\infty,w}, \mathrm{gr}\, W_f)$$

is finite, which it is because a_p cannot be a root of unity. $\square$

Step 4. *Conclusion of the proof.*

Step 1 shows that N^-Q_n is the squarefree product of an odd number of primes inert in K for $\mathfrak{F}$-many n, and by Theorem 3.4.6 applied to the Euler system $(\boldsymbol{\kappa}_{g_n}, \boldsymbol{\lambda}_{g_n})$ for T_{g_n}, $\mathrm{Sel}_{\mathcal{F}^*_{g_n,\Lambda}}(\mathbf{W}_{g_n})$ is then Λ-cotorsion. By Theorem 7.1.1, for such n we have:

$$\mathrm{Fitt}_\Lambda \mathrm{Sel}_{\mathcal{F}^*_{g_n,\Lambda}}(\mathbf{W}_{g_n})^\vee \subset (\boldsymbol{\lambda}_{g_n}(1))^2 \subset \Lambda. \tag{67}$$

On the other hand, by Theorem 3.4.6, the theorem would follow from:

$$\mathrm{Fitt}_\Lambda \mathrm{Sel}_{\mathcal{F}^*_\Lambda(\mathsf{Q})}(\mathbf{W}_f)^\vee \subset (\boldsymbol{\lambda}(\mathsf{Q}))^2 \subset \Lambda. \tag{68}$$

For the passage between characteristic ideal and fitting ideals, recall that the characteristic ideal of any Λ-module is the smallest divisorial ideal containing the Fitting ideal; cf. [60, Corollary 3.2.9]. Steps 2 and 3 allow us to pass from (67) to (68). ☐

Corollary 7.2.2. *Under the hypotheses of Theorem 7.2.1, if additionally $\nu(N^-)$ is even, then the Heegner point main conjecture holds for f in $\Lambda \otimes \mathbb{Q}_p$; that is, there is a pseudo-isomorphism of Λ-modules:*

$$\mathrm{Sel}_{\mathcal{F}^*_\Lambda}(\mathbf{W}_f)^\vee \approx \Lambda \oplus M \oplus M$$

for some torsion Λ-module M, and

$$\mathrm{char}_\Lambda\left(\frac{\mathrm{Sel}_{\mathcal{F}_\Lambda}(K, \mathbf{T}_f)}{\Lambda\boldsymbol{\kappa}(1)}\right) = \mathrm{char}_\Lambda(M)$$

as ideals of $\Lambda \otimes \mathbb{Q}_p$. If additionally the image of the Galois action on $\overline{T}_f$ contains a nontrivial scalar, then the equality is true in Λ.

Corollary 7.2.3. *Under the hypotheses of Theorem 7.2.1, if additionally $\nu(N^-)$ is odd, then the anticyclotomic main conjecture holds for f in $\Lambda \otimes \mathbb{Q}_p$; that is, there is a pseudo-isomorphism of Λ-modules:*

$$\mathrm{Sel}_{\mathcal{F}^*_\Lambda}(\mathbf{W}_f)^\vee \approx M \oplus M$$

for some torsion Λ-module M, and

$$(\boldsymbol{\lambda}(1)) = \mathrm{char}_\Lambda(M)$$

as ideals of $\Lambda \otimes \mathbb{Q}_p$. If additionally the image of the Galois action on $\overline{T}_f$ contains a nontrivial scalar, then the equality is true in Λ.

Corollary 7.2.4. *Under the hypotheses of Theorem 7.2.1, the bipartite Euler system*

$$(\kappa(1, \cdot), \lambda(1, \cdot))$$

of (5.3.9) is nontrivial.

Proof. Let $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$ be such that $\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_f) = 0$, where again $\mathcal{F}$ is the standard Selmer structure for T_f (cf. Corollary 3.3.9). By the same reasoning as in Step 1 of the proof of Theorem 7.2.1, we have $\mathrm{char}_\Lambda(\mathrm{Sel}_{\mathcal{F}^*_\Lambda(\mathsf{Q})}(\mathbf{W}_f) \not\subset (T)$, so $(\boldsymbol{\lambda}(\mathsf{Q})) \not\subset (T)$; this implies $\lambda(1, \mathsf{Q}) \neq 0$ by Remark 5.3.11. ☐

Corollary 7.2.4 is generalized by Theorem 8.1.1 below.

7.3. **Kolyvagin's conjecture.** Let f, $\wp$, K, and N^+N^- be as in (3.1.1) and (4.5.1).

7.3.1. For any $\mathsf{m} \in \mathsf{K}$ and any $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}_{\mathsf{m}}$, define the m-transverse Selmer ranks

$$r^{\pm}_{\mathsf{m}}(\mathsf{Q}) = \mathrm{rk}_{\mathcal{O}} \mathrm{Sel}_{\mathcal{F}(\mathsf{m},\mathsf{Q})}(T_f)^{\pm}, \tag{69}$$

where $\pm$ refers to the τ eigenvalue; note that this is well-defined because the local conditions defining $\mathcal{F}(\mathsf{m})$ are all τ-stable. When $\mathsf{Q} = 1$, we simply write $r^{\pm}_{\mathsf{m}}$. When $\mathsf{m} = 1$, the $r^{\pm}_1$ are the classical Selmer ranks of f.

Proposition 7.3.2. *For all $\mathsf{ml} \in \mathsf{K}$ and $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}_{\mathsf{ml}}$, and for each $\delta \in \{\pm\}$, either:*

- $r^\delta_{\mathsf{ml}}(\mathsf{Q}) = r^\delta_{\mathsf{m}}(\mathsf{Q}) - 1$, $\mathrm{loc}^\delta_{\mathsf{l}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{m},\mathsf{Q})}(T_f))^\delta \neq 0$, *and* $\partial^\delta_{\mathsf{l}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{ml},\mathsf{Q})}(T_f))^\delta = 0$.
- $r^\delta_{\mathsf{ml}}(\mathsf{Q}) = r^\delta_{\mathsf{m}}(\mathsf{Q}) + 1$, $\mathrm{loc}^\delta_{\mathsf{l}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{m},\mathsf{Q})}(T_f))^\delta = 0$, *and* $\partial^\delta_{\mathsf{l}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{ml},\mathsf{Q})}(T_f))^\delta \neq 0$.

Proof. If $\mathcal{F}^{\mathsf{l}}(\mathsf{m},\mathsf{Q}) = \mathcal{F}(\mathsf{ml},\mathsf{Q}) + \mathcal{F}(\mathsf{m},\mathsf{Q})$ and $\mathcal{F}_{\mathsf{l}}(\mathsf{m},\mathsf{Q}) = \mathcal{F}(\mathsf{ml},\mathsf{Q}) \cap \mathcal{F}(\mathsf{m},\mathsf{Q})$, then we have a τ-equivariant exact sequence

$$0 \to \mathrm{Sel}_{\mathcal{F}_{\mathsf{l}}(\mathsf{m},\mathsf{Q})}(T_f) \to \mathrm{Sel}_{\mathcal{F}^{\mathsf{l}}(\mathsf{m},\mathsf{Q})}(T_f) \to \mathsf{H}^1(K_{\mathsf{l}}, T_f),$$

where the image of the final arrow has rank two and is self-annihilating under the local Tate pairing by Propositions 2.7.2 and 2.7.5. Since the Tate pairing of two classes with opposite τ eigenvalues is necessarily zero, the proposition follows as in the proof of Proposition 3.3.4. □

Lemma 7.3.3. *Suppose given elements $c^{\pm} \in \mathsf{H}^1(K, T_f)^{\pm}$, where $\pm$ is the τ eigenvalue. Then there exists a Kolyvagin-admissible ultraprime l such that*

$$c^{\pm} \neq 0 \implies \mathrm{loc}_{\mathsf{l}}^{\pm} c^{\pm} \neq 0.$$

If (sclr) holds for T_f, then the same is true for elements $c^{\pm} \in \mathsf{H}^1(K, T_f/\pi^j)$.

Proof. The proof of Theorem 3.3.5 applies almost verbatim, except that in the proof of Lemma 3.3.7 we will have two homomorphisms

$$\phi^{\pm} \in \mathrm{Hom}_{G_K}(G_L, \overline{T}_f)^{\pm},$$

and we must choose $g \in G_L$ so that $\phi^{\epsilon}(g)$ has nonzero component in the ϵ eigenspace of τ for both signs ϵ (unless ϕ^{ϵ} is itself 0); for each ϵ, this condition is satisfied outside a proper subgroup of G_L, so indeed there exists $g \in G_L$ such that both conditions are satisfied. With this modification, the rest of the proof applies unchanged. □

Lemma 7.3.4. *Suppose that the bipartite Euler system $(\kappa(1,\cdot), \lambda(1,\cdot))$ of (5.3.9) is nontrivial. Then, for all $\mathsf{m} \in \mathsf{K}$, $(\kappa(\mathsf{m},\cdot), \lambda(\mathsf{m},\cdot))$ is nontrivial.*

In particular, for all $\mathsf{m} \in \mathsf{K}$ and $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}_{\mathsf{m}}$:

$$\mathrm{rk}_{\mathcal{O}} \mathrm{Sel}_{\mathcal{F}(\mathsf{m},\mathsf{Q})}(T_f) \equiv \nu(N^-) + 1 \pmod 2;$$

and

$$\mathrm{rk}_{\mathcal{O}} \mathrm{Sel}_{\mathcal{F}(\mathsf{m},\mathsf{Q})}(T_f) \leq 1 \iff \begin{cases} \kappa(\mathsf{m},\mathsf{Q}) \neq 0, & \nu(N^-) + |\mathsf{Q}| \text{ even} \\ \lambda(\mathsf{m},\mathsf{Q}) \neq 0, & \nu(N^-) + |\mathsf{Q}| \text{ odd.} \end{cases}$$

Proof. Recall that, for fixed m, the pair $(\kappa(\mathsf{m},\cdot), \lambda(\mathsf{m},\cdot))$ forms a bipartite Euler system with sign $\nu(N^-)$ for the self-dual Selmer structure $(\mathcal{F}(\mathsf{m}), \mathsf{S} \cup \mathsf{m})$ on T_f. We will prove that, for any $\mathsf{ml} \in \mathsf{K}$, if $(\kappa(\mathsf{m},\cdot), \lambda(\mathsf{m},\cdot))$ is nontrivial then so is $(\kappa(\mathsf{ml},\cdot), \lambda(\mathsf{ml},\cdot))$.

Choose $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}_{\mathsf{m}}^{\nu(N^-)+1}$ such that $\mathrm{Sel}_{\mathcal{F}(\mathsf{m},\mathsf{Q})}(T_f) = 0$ and $\mathsf{l} \notin \mathsf{Q}$; this is possible by Corollary 3.3.9. By Proposition 7.3.2, we may choose a nonzero

$$d \in \mathrm{Sel}_{\mathcal{F}(\mathsf{ml},\mathsf{Q})}(T_f).$$

Applying Theorem 3.3.5 to d, let q be admissible with sign ϵ_{q} such that $\mathsf{q} \notin \mathsf{Qml}$ and $\mathrm{loc}_{\mathsf{q}} d \neq 0$. By Proposition 3.3.4 for the Selmer structures $\mathcal{F}(\mathsf{m},\mathsf{Qq})$ and $\mathcal{F}(\mathsf{m},\mathsf{Q})$,

$$\mathrm{rk}_{\mathcal{O}} \mathrm{Sel}_{\mathcal{F}(\mathsf{m},\mathsf{Qq})}(T_f) = 1. \tag{70}$$

Hence, by hypothesis, $\kappa(\mathsf{m},\mathsf{Qq})$ generates $\mathrm{Sel}_{\mathcal{F}(\mathsf{m},\mathsf{Qq})}(T_f)$ up to finite index, and in particular $\partial_{\mathsf{q}}\kappa(\mathsf{m},\mathsf{Qq}) \neq 0$. Now, taking the sum of local pairings and using Proposition 2.7.2,

$$0 = \sum_{\mathsf{v}} \langle d, \kappa(\mathsf{m},\mathsf{Qq})\rangle_{\mathsf{v}} = \langle d, \kappa(\mathsf{m},\mathsf{Qq})\rangle_{\mathsf{l}} + \langle d, \kappa(\mathsf{m},\mathsf{Qq})\rangle_{\mathsf{q}}. \tag{71}$$

Since the latter pairing is nonzero by construction, the former is as well, and so, by Proposition 5.3.10(i),

$$\mathrm{Res}_{\mathsf{l}}\, \kappa(\mathsf{m},\mathsf{Qq}) \neq 0 \implies \kappa(\mathsf{ml},\mathsf{Qq}) \neq 0.$$

□

7.3.5. For any $\mathsf{m} \in \mathsf{K}$, define the vanishing order of the Kolyvagin system at m:

$$\nu_{\mathsf{m}} = \begin{cases} \min\{|\mathsf{n}| \,:\, \mathsf{mn} \in \mathsf{K},\, \lambda(\mathsf{mn}, 1) \neq 0\}, & \nu(N^-) \text{ odd}, \\ \min\{|\mathsf{n}| \,:\, \mathsf{mn} \in \mathsf{K},\, \kappa(\mathsf{mn}, 1) \neq 0\}, & \nu(N^-) \text{ even}. \end{cases} \tag{72}$$

Theorem 7.3.6. *If $(\kappa(1,\cdot), \lambda(1,\cdot))$ is nontrivial, and in particular under the hypotheses of Theorem 8.1.1, we have for all $\mathsf{m} \in \mathsf{K}$:*

- *If $\nu(N^-)$ is odd, then $\nu_{\mathsf{m}} = \max\{r_{\mathsf{m}}^+, r_{\mathsf{m}}^-\}$ and $\epsilon_f = (-1)^{r_{\mathsf{m}}^\pm + |\mathsf{m}|}$.*
- *If $\nu(N^-)$ is even, then $\nu_{\mathsf{m}} = \max\{r_{\mathsf{m}}^+, r_{\mathsf{m}}^-\} - 1$ and $\epsilon_f \cdot (-1)^{|\mathsf{m}| + \nu_{\mathsf{m}} + 1}$ is the larger τ eigenspace.*

In particular, if $\operatorname{rk}_{\mathcal{O}} \operatorname{Sel}(K, T_f) = 1$, then $L'(f/K, 1) \neq 0$.

Proof. For ease of notation, let $\delta = 0$ if $\nu(N^-)$ is odd, and $\delta = 1$ if $\nu(N^-)$ is even. Suppose given $\mathsf{mn} \in \mathsf{K}$ such that $\lambda(\mathsf{mn}, 1)$ or $\kappa(\mathsf{mn}, 1)$ is nontrivial; then

$$\operatorname{rk}_{\mathcal{O}} \operatorname{Sel}_{\mathcal{F}(\mathsf{mn})}(T_f) = \delta$$

by Lemma 7.3.4. In particular, the kernel of the localization map

$$\operatorname{Sel}_{\mathcal{F}(\mathsf{m})}(T_f)^\pm \to \bigoplus_{\mathsf{l} \in \mathsf{n}} H^1_{\mathrm{unr}}(K_{\mathsf{l}}, T_f)^\pm$$

has rank at most δ. It follows that $\max\{r_{\mathsf{m}}^+, r_{\mathsf{m}}^-\} - \delta \leq \nu_{\mathsf{m}}$. We now show that equality holds by induction on $\max\{r_{\mathsf{m}}^+, r_{\mathsf{m}}^-\}$. If $\max\{r_{\mathsf{m}}^+, r_{\mathsf{m}}^-\} \leq \delta$, then Lemma 7.3.4 implies $\nu_{\mathsf{m}} = 0 = \max\{r_{\mathsf{m}}^+, r_{\mathsf{m}}^-\} - \delta$.

Now suppose that $\max\{r_{\mathsf{m}}^+, r_{\mathsf{m}}^-\} > \delta$, and let ϵ be the sign of the larger value of $r_{\mathsf{m}}^\pm$ (choose either if they agree). If $r_{\mathsf{m}}^{-\epsilon} > 0$, then by Lemma 7.3.3 and Proposition 7.3.2, there exists $\mathsf{l} \in \mathsf{K}$ not in m such that $r_{\mathsf{ml}}^\pm = r_{\mathsf{m}}^\pm - 1$. In this case, $\max\left\{r_{\mathsf{ml}}^+, r_{\mathsf{ml}}^-\right\} = \max\{r_{\mathsf{m}}^-, r_{\mathsf{m}}^-\} - 1$. Hence (by the inductive hypothesis)

$$\nu_{\mathsf{m}} \leq \nu_{\mathsf{ml}} + 1 = \max\left\{r_{\mathsf{ml}}^+, r_{\mathsf{ml}}^-\right\} - \delta + 1 = \max\left\{r_{\mathsf{m}}^+, r_{\mathsf{m}}^-\right\} - \delta.$$

Since we have already shown the opposite equality, this completes the inductive step under the assumption $r_{\mathsf{m}}^{-\epsilon} > 0$.

If on the other hand $r_{\mathsf{m}}^{-\epsilon} = 0$, then $r_{\mathsf{m}}^\epsilon \geq \delta + 2$, since $r_{\mathsf{m}}^+ + r_{\mathsf{m}}^- \equiv \delta \pmod 2$. Then by Lemma 7.3.3 and Proposition 7.3.2 again, we may choose $\mathsf{l} \in \mathsf{K}$ such that $r_{\mathsf{ml}}^\epsilon = r_{\mathsf{m}}^\epsilon - 1$, while necessarily $r_{\mathsf{ml}}^{-\epsilon} = 1 \leq \delta + 1 \leq r_{\mathsf{ml}}^\epsilon$. Hence $\max\left\{r_{\mathsf{ml}}^+, r_{\mathsf{ml}}^-\right\} = r_{\mathsf{m}}^\epsilon - 1$, and the same argument as above again completes the inductive step.

Finally, we consider the parity assertions of the theorem. If $\nu(N^-)$ is even, i.e. if $\delta = 1$, then the Selmer ranks $r_{\mathsf{m}}^\pm$ are always distinct. As we pass from m to ml in the inductive step above, the sign of the larger eigenspace is preserved, and $\nu_{\mathsf{m}} + |\mathsf{m}| = \nu_{\mathsf{ml}} + |\mathsf{ml}|$. It therefore suffices to show that $\epsilon \cdot (-1)^{|\mathsf{m}| + \nu_{\mathsf{m}} + 1}$ is the sign of the larger τ eigenspace when $\nu_{\mathsf{m}} = 0$, i.e. when $\kappa(\mathsf{m}, 1) \neq 0$. In this case it follows from Proposition 5.3.6.

When $\nu(N^-)$ is odd, then whenever $\lambda(\mathsf{mn}) \neq 0$, Proposition 5.3.6 implies that $\epsilon_f = (-1)^{|\mathsf{mn}|}$. Hence

$$\epsilon_f = (-1)^{\nu_{\mathsf{m}} + |\mathsf{m}|} = (-1)^{\max\left\{r_{\mathsf{m}}^+, r_{\mathsf{m}}^-\right\} + |\mathsf{m}|},$$

which proves the claim since $r_{\mathsf{m}}^\pm$ have the same parity when $\nu(N^-)$ is odd.

For the final statement, take $\mathsf{m} = 1$. If $r_1^+ + r_1^- = 1$, the theorem implies $\nu_1 = 0$, so $\kappa(1, 1) \neq 0$. Since $\kappa(1, 1)$ is the Kummer image of the classical Heegner point y_K, the result follows from the Gross-Zagier theorem of [66]. □

It remains to relate the vanishing of the patched Kolyvagin classes to the classical vanishing order

$$\nu_{\mathrm{classical}} := \begin{cases} \min\left\{\nu(m) \,:\, \overline{\lambda}(m, 1) \neq 0\right\}, & \nu(N^-) \text{ odd}, \\ \min\left\{\nu(m) \,:\, \overline{c}(m, 1) \neq 0\right\}, & \nu(N^-) \text{ even}. \end{cases} \tag{73}$$

Corollary 7.3.7. *If $(\kappa(1,\cdot), \lambda(1,\cdot))$ is nontrivial, and in particular under the hypotheses of Theorem 8.1.1, $\nu_{classical}$ is finite. If (sclr) holds for f, then $\nu_{classical} = \nu_1$, and in particular:*

- *If $\nu(N^-)$ is odd, then $\nu_{classical} = \max\left\{r_1^+, r_1^-\right\}$ and $r_1^\pm \equiv \frac{\epsilon_f - 1}{2} \pmod 2$.*
- *If $\nu(N^-)$ is even, then $\nu_{classical} = \max\left\{r_1^+, r_1^-\right\} - 1$ and $\epsilon_f \cdot (-1)^{1 + \nu_{classical}}$ is the larger τ eigenspace.*

Proof. The finiteness of the classical vanishing order is clear by construction: if a patched Kolyvagin class or element is nontrivial, then infinitely many of the classical Kolyvagin classes or elements defining it are nontrivial. This also shows $\nu_{\text{classical}} \leq \nu_1$. We will check that equality holds under the condition (sclr). Suppose first $\nu(N^-)$ is even. We abbreviate by $c_j(m,1) \in H^1(K,T_j)$ the image of $\bar{c}(m,1)$ when $v_\wp(I_m) \geq j$. Given some nonzero $c_j(m,1)$, one may show as in [39, p. 309] that there exist classes $c_j(m_n,1) \neq 0$ with $v_\wp(I_{m_n}) \to \infty$ and $\nu(m_n) = \nu(m)$. (In [39], additional hypotheses are put on the image of the Galois action, but the argument goes through by invoking Lemma 7.3.3.) In particular, the sequence m_n defines a nonzero $\kappa(\mathsf{m},1)$ witnessing $\nu_1 \leq \nu_{\text{classical}}$.

Now suppose that $\nu(N^-)$ is odd, and that $\lambda_j(m,1) \neq 0$ where $\nu(m) = \nu_{\text{classical}}$. We choose an auxiliary $\{\mathsf{q}, \epsilon_{\mathsf{q}}\} \in \mathsf{N}$ with the following properties:

- ϵ_{q} is the sign of the larger τ eigenspace in $\mathrm{Sel}_{\mathcal{F}}(T_f)$.
- The localization map $\mathrm{loc}_{\mathsf{q}}$ is trivial on $\mathrm{Sel}_{\mathcal{F}}(T_f)$.

To ensure the second condition, we may choose $\mathrm{Frob}_{\mathsf{q}} \in G_{\mathbb{Q}}$ to be a complex conjugation. Let $\{q_n, \epsilon_{q_n}\}$ represent $\{\mathsf{q}, \epsilon_{\mathsf{q}}\}$ as in Remark 4.5.8. Once again, the argument of [39, p. 309] implies that, for each n, there exists m_n with $c_j(m_n, q_n) \neq 0$ and $v_\wp(I_{m_n}) \to \infty$. We therefore obtain a nonzero patched class $\kappa(\mathsf{m},\mathsf{q})$ with $|\mathsf{m}| = \nu_{\text{classical}}$. Repeating the argument of Lemma 7.3.4, it follows that $\nu_1 \leq \nu_{\text{classical}} + 1$. For contradiction, we assume that

$$\nu_{\text{classical}} = \nu_1 - 1 = r^{\epsilon_{\mathsf{q}}} - 1.$$

This implies $\partial_{\mathsf{q}}\kappa(\mathsf{m},\mathsf{q}) = 0$, so by Lemma 7.3.4 and Proposition 3.3.4, we conclude

$$\mathrm{rk}_{\mathcal{O}}\,\mathrm{Sel}_{\mathcal{F}(\mathsf{m},1)}(T_f) = 2.$$

However, by Proposition 7.3.2 and the assumption $|\mathsf{m}| = \nu_1 - 1 = r^{\epsilon_{\mathsf{q}}} - 1$, $\mathrm{rk}_{\mathcal{O}}\,\mathrm{Sel}_{\mathcal{F}(\mathsf{m},1)}(T_f)^{\epsilon_{\mathsf{q}}}$ is odd, hence equal to one. Proposition 7.3.2 then implies

$$\mathrm{Sel}_{\mathcal{F}(\mathsf{m},1)}(T_f)^{\epsilon_{\mathsf{q}}} \subset \mathrm{Sel}_{\mathcal{F}}(T_f)^{\epsilon_{\mathsf{q}}},$$

so

$$\mathrm{loc}_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{m},1)}(T_f)) = 0$$

by the choice of q. However, this contradicts Proposition 3.3.4, so we must have $\nu_1 = \nu_{classical}$.

□

8. Kolyvagin's conjecture for inert or non-ordinary p

8.1. **The main result.** In this section, we shall prove the following main result:

Theorem 8.1.1. *Let f be a non-CM cuspidal eigenform of weight two and trivial character, new of level N, with ring of integers $\mathcal{O}_f$ of is coefficient field. Let $\wp \subset \mathcal{O}_f$ be a prime, and let K be an imaginary quadratic field. Assume:*

- *$N = N^+N^-$, where every factor of N^+ is split in K, and N^- is a squarefree product of an even number of primes inert in K.*
- *The residue characteristic p of $\wp$ does not divide $2D_KN$.*
- *The modulo $\wp$ representation $\overline{T}_f$ associated to f is absolutely irreducible; if $p = 3$, assume that $\overline{T}_f$ is not induced from a character of $G_{\mathbb{Q}(\sqrt{-3})}$.*
- *If p is inert in K, then there exists some prime $\ell_0 || N$.*
- *If a_p is not a $\wp$-adic unit, then either there exists $\ell_0 || N^+$ such that $\overline{T}_f|_{G_{\mathbb{Q}_{\ell_0}}}$ is ramified and $\overline{T}_f^{G_{\mathbb{Q}_{\ell_0}}} = 0$; or there exist primes $\ell_1, \ell_2 | N^-$ such that $\overline{T}_f|_{G_{\mathbb{Q}_{\ell_i}}}$ is ramified for $i = 1, 2$, $\overline{T}_f^{G_{\mathbb{Q}_{\ell_1}}} = 0$, and $\overline{T}_f^{G_{\mathbb{Q}_{\ell_2}}} \neq 0$.*

Then $(\kappa(1,\cdot), \lambda(1,\cdot))$ is nontrivial.

8.1.2. If p is split in K, then this is simply Corollary 7.2.4. If p is non-ordinary or inert in K, then the anticyclotomic main conjecture is currently not known in full generality; however, since all we are interested in specialization the trivial character, we will show that the result may instead be obtained, more circuitously, by combining main conjectures for quadratic twists of f. The proof applies equally well to the split ordinary case.

8.2. **Comparing periods.**

8.2.1. Let f be a modular form of weight two, level N, and trivial character, with ring of integers $\mathcal{O}_f$ of its coefficient field, and let $\wp \subset \mathcal{O}_f$ be an ordinary prime lying over $p \nmid 2N$, with associated completion $\mathcal{O}$ and uniformizer π; we assume that $\overline{T}_f$ is absolutely irreducible. There are two ways to normalize the anticyclotomic p-adic L-function, as explained in [64, 47]. For any factorization $N = N_1N_2$, where N_1 and N_2 are coprime, the congruence ideal $\eta_f(N_1, N_2) \subset \mathcal{O}_f$ is defined as

$$\pi_f(\mathrm{Ann}_{\mathbb{T}_{N_1,N_2}}(\ker \pi_f))\cdot, \tag{74}$$

where $\pi_f : \mathbb{T}_{N_1,N_2} \to \mathcal{O}_f$ is the projection giving the Hecke eigenvalues of f. Hida's canonical period [26] is defined (up to $\wp$-adic units) by:

$$\Omega_f^{can} = \frac{(f,f)}{\eta_f(N,1)}, \tag{75}$$

where (f,f) is the Peterson inner product.

On the other hand, if $N = N_1N_2$ where N_2 is squarefree with an odd number of prime factors, then the f-isotypic part of the Hecke module $\mathcal{O}_f[X_{N_1,N_2}]$ is locally free of rank one, say with φ_{f,N_2} a generator up to $\wp$-adic units of $\mathcal{O}_f$. Gross's period is defined (up to $\wp$-adic units) by:

$$\Omega_{f,N_2} = \frac{(f,f)}{\langle \varphi_{f,N_2}, \varphi_{f,N_2}\rangle}, \tag{76}$$

where $\langle \cdot, \cdot \rangle$ is the canonical intersection pairing on $\mathcal{O}_f[X_{N_1,N_2}]$. This period occurs naturally in anticyclotomic Iwasawa theory due to the well-known special value formula of Gross.

Proposition 8.2.2. *Let K be an imaginary quadratic field of discriminant prime to Np, and suppose that $N = N^+N^-$ where all factors of N^+ are split in K, and N^- is a squarefree product of an odd number primes inert in K. Then $L(f/K, 1) \in \overline{\mathbb{Q}} \cdot \Omega_{f,N^-}$ and the element $\lambda(1) \in \Lambda$ constructed in (5.3.9) satisfies:*

$$\frac{L(f/K,1)}{\Omega_{f,N^-}} = \lambda(1)^2 \tag{77}$$

up to $\wp$-adic units.

Proof. This is [68, Theorem 7.1]. □

8.2.3. For any $\ell | N$, let $c_f(\ell)$ be the maximal exponent e such that T_f/π^e is unramified at ℓ. The following theorem generalizes [54, 31] and [47, Theorem 6.8].

Proposition 8.2.4. *Suppose $N = N_1N_2$ where N_2 is squarefree with an odd number of prime factors and coprime to N_1. For any $\ell_0 || N$, we have:*

$$v_\wp \eta_f(N,1) - v_\wp \langle \varphi_{f,N_2}, \varphi_{f,N_2}\rangle \geq \sum_{\ell | N_2} c_f(\ell) - \sigma(N_2)c_f(\ell_0).$$

Proof. For a decomposition $N = N_1'N_2'$ with N_2' the squarefree product of an even number of primes that do not divide N_1', one defines $\delta(N_1', N_2') \subset \mathcal{O}$ to be the "degree" of an optimal modular parametrization $J_{N_1',N_2'} \to A_f$ as explained in [47, 31]. By [47, Proposition 6.6], we have:

$$v_\wp \delta(N,1) = c_f(\ell_0) + v_\wp \langle \varphi_{f,\ell_0}, \varphi_{f,\ell_0}\rangle. \tag{78}$$

On the other hand, [15, Lemma 4.17] implies that:

$$v_\wp \eta_f(N/\ell_0, \ell_0) \geq v_\wp \langle \varphi_{f,\ell_0}, \varphi_{f,\ell_0}\rangle. \tag{79}$$

Because $\mathbb{T}_{N/\ell_0,\ell_0}$ is a quotient of $\mathbb{T}_{N,1}$, we conclude that:

$$v_\wp \eta_f(N,1) \geq v_\wp \delta(N,1) - c_f(\ell_0). \tag{80}$$

We apply [47, Proposition 6.6] again, this time to the decomposition $N = N_1N_2$ and any $r | N_2$. This yields:

$$v_\wp \delta(N_1 r, N_2/r) = c_f(r) + v_\wp \langle \varphi_{f,N_2}, \varphi_{f,N_2}\rangle. \tag{81}$$

If N_2 is prime, this is sufficient to conclude. If not, we may choose $r \neq \ell_0$.

The results of Ribet-Takahashi and Khare [54, 31] imply that:

$$v_\wp \delta(N,1) \geq v_\wp \delta(N_1 r, N_2/r) + \sum_{\ell | N_2/r} c_f(\ell) - \nu(N_2/r) \cdot c_f(\ell_0). \tag{82}$$

(If $\ell_0|N_1$, we are using the fact that both r and ℓ_0 exactly divide $N_1 r$.) Combining (80), (81), and (82) completes the proof.

□

Remark 8.2.5. If ℓ_0 is residually ramified, the inequality is an equality. In [69, Theorem 6.4], more restrictive conditions are given under which this result holds.

8.2.6. We will also require the following related result:

Proposition 8.2.7. *Let f and $\wp$ be as above and suppose $\ell_0||N$. Then, in the notation of the proof of Theorem 8.2.4,*

$$v_\wp \eta_f(N,1) - v_\wp \langle \varphi_{f,\ell_0}, \varphi_{f,\ell_0}\rangle \le c_f(\ell_0).$$

Proof. Let $J = J_0(N)$ be the modular Jacobian and let $\mathbb{T}$ be the full Hecke algebra of level N. Write $\pi_f : \mathbb{T} \to \mathcal{O}_f$ for the projection associated to f, with associated maximal ideal $\mathfrak{m}$, and let $I = \ker \pi_f$. The claim will follow from (78) once we establish

$$v_\wp \eta_f(N,1) \le v_\wp \delta(N,1). \tag{83}$$

Indeed, if $J \to A_f$ is an optimal parametrization, then the dual map $A_f^\vee \to J^\vee$ is an inclusion. The composition

$$\phi : J \to A_f \to A_f^\vee \to J^\vee \xrightarrow{w_N} J$$

is a Hecke-equivariant endomorphism; by (29), its image in $\mathrm{End}(J)_\mathfrak{m}$ may be identified with some $y \in \mathbb{T}_\mathfrak{m}$. Because $\operatorname{im}\phi \subset J[I]$, we have $y \in \mathrm{Ann}(I)$. By the definition of $\delta(N,1)$,

$$(\pi(y)) = \delta(N,1) \subset \mathcal{O}.$$

This implies (83). □

8.3. Ordinary cyclotomic Iwasawa theory.

8.3.1. Let $\Lambda_{\mathbb{Q}_\infty} = \mathcal{O}[\![\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]\!]$ be the cyclotomic Iwasawa algebra. We denote by $\mathbb{1} : \Lambda_{\mathbb{Q}_\infty} \to \mathcal{O}$ and $\mathbb{1} : \Lambda \to \mathcal{O}$ the specializations at the trivial character. If $\wp$ is ordinary and Σ is a finite set of rational primes, we consider the Σ-ordinary cyclotomic Selmer group

$$\mathrm{Sel}^\Sigma(\mathbb{Q}_\infty, W_f) = \ker\left(H^1(\mathbb{Q}, \mathbf{W}_f) \to \prod_{v \notin \Sigma \cup \{p\}} H^1(I_v, \mathbf{W}_f) \times \frac{H^1(\mathbb{Q}_p, \mathbf{W}_f)}{H^1_{\mathrm{ord}}(\mathbb{Q}_p, \mathbf{W}_f)}\right),$$

and denote by $Ch^\Sigma_{\mathbb{Q}_\infty,f} \subset \Lambda_{\mathbb{Q}_\infty}$ the characteristic ideal of its Pontryagin dual. Kato [30] has proven one direction of the main conjecture in this setting:

Theorem 8.3.2 ((Kato))**.** *Let f be a modular form of weight two, level N, and trivial character, and $\wp \subset \mathcal{O}_f$ a prime of good ordinary reduction with odd residue characteristic. Then $\mathrm{Sel}(\mathbb{Q}_\infty, W_f)$ is $\Lambda_{\mathbb{Q}_\infty}$-cotorsion and*

$$L_p(\mathbb{Q}_\infty, f) \subset Ch_{\mathbb{Q}_\infty,f}$$

in $\Lambda_{\mathbb{Q}_\infty} \otimes \mathbb{Q}_p$. If the image of the Galois action on T_f contains $SL_2(\mathbb{Z}_p)$, then all of the inclusions hold in $\Lambda_{\mathbb{Q}_\infty}$.

For the opposite direction of the main conjecture, we deduce the following result from the work of Skinner-Urban [60].

Theorem 8.3.3 ((Skinner-Urban))**.** *Let K be an imaginary quadratic field of discriminant prime to Np in which p splits. Assume that $\wp$ is good ordinary for f and that:*

- *the mod $\wp$ representation $\overline{T}_f$ is absolutely irreducible;*
- *$N = N_1 N_2$, where every factor of N_1 is split in K and N_2 is the squarefree product of an odd number primes inert in K.*

Then there exists an element $\alpha \in \Lambda_{\mathbb{Q}_\infty}$ such that $\mathbb{1}(\alpha)$ divides

$$\frac{\Omega_{f,N_2}}{\Omega_f^{can}} \sim \frac{\eta_f(N,1)}{\langle \varphi_{f,N_2}, \varphi_{f,N_2}\rangle}$$

in $\mathcal{O}$ and

$$(\alpha) Ch_{\mathbb{Q}_\infty,f} Ch_{\mathbb{Q}_\infty,f\otimes\chi_K} \subset (L_p(\mathbb{Q}_\infty, f))(L_p(\mathbb{Q}_\infty, f \otimes \chi_K)).$$

Proof. Let Σ be a finite set of primes containing all factors of $N\infty$. Recall the divisibility established in the course of the proof of Theorem 7.1.1 for the Fitting ideal of the 3-variable Selmer group:

$$(\widetilde{\alpha})Fitt^{\Sigma}_{K_\infty}(\mathbf{g}) \subset (\mathcal{L}^{\Sigma}_{\mathbf{f},K}), \tag{84}$$

where $\widetilde{\alpha} \in \mathbb{I}[\Gamma^+_K]$ may be chosen such that $\widetilde{\alpha}$ specializes to a unit multiple of $\Omega_{f,N_2}/\Omega^{can}_f$ at the trivial character (by [64]). By Lemma 3.2.5, Corollary 3.2.9(i), and Corollary 3.2.20(iii) of [60], specializing to the cyclotomic variable yields a divisibility

$$(\alpha)Ch^{\Sigma}_{\mathbb{Q}_\infty,f}Ch^{\Sigma}_{\mathbb{Q}_\infty,f\otimes\chi_K} \subset (L^{\Sigma}_p(\mathbb{Q}_\infty,f))(L^{\Sigma}_p(\mathbb{Q}_\infty,f\otimes\chi_K)), \tag{85}$$

where α is the image of $\widetilde{\alpha}$. The desired divisibility for the imprimitive L-functions and Selmer groups follows by [60, Proposition 3.2.18]. $\square$

8.3.4. Denote by $\mu(f)$ the μ-invariant of $Ch_{\mathbb{Q}_\infty,f}$. To control the powers of $\wp$ in Theorem 8.3.2, we will use the following.

Lemma 8.3.5. *Let f and g be modular forms of weight two and trivial character such that $\overline{T}_f$ is absolutely irreducible. Suppose that f and g have a congruence modulo π^j, i.e. there is a common completion $\mathcal{O}$ of $\mathcal{O}_f$ and $\mathcal{O}_g$ and, in some basis, a congruence of $\mathcal{O}$-valued associated Galois representations*

$$T_f \equiv T_g \pmod{\pi^j}.$$

If f and g are $\wp$-ordinary and $\mu(f) < j$, then $\mu(g) = \mu(f)$.

Proof. By [21], $\mu(f)$ is also the μ-invariant of $Ch^{\Sigma}_{\mathbb{Q}_\infty,f}$ for any finite set of primes Σ, and likewise for g. If Σ contains all primes dividing the level of either f or g, then we have:

$$\mathrm{Sel}^{\Sigma}(\mathbb{Q}_\infty,W_f)[\pi^j] \simeq \mathrm{Sel}^{\Sigma}(\mathbb{Q}_\infty,W_g)[\pi^j] \tag{86}$$

as $\Lambda_{\mathbb{Q}_\infty}$-modules. Let M_f and M_g be the Pontryagin duals of $\mathrm{Sel}^{\Sigma}(\mathbb{Q}_\infty,W_f)$ and $\mathrm{Sel}^{\Sigma}(\mathbb{Q}_\infty,W_g)[\wp^j]$, respectively, and let $\mathfrak{P} = (\wp) \subset \Lambda_{\mathbb{Q}_\infty}$. Then we have an isomorphism

$$M_f \otimes \Lambda_{\mathbb{Q}_\infty}/\mathfrak{P}^j \simeq M_g \otimes \Lambda_{\mathbb{Q}_\infty}/\mathfrak{P}^j.$$

Since $\mu(f) = \lg M_{f,(\mathfrak{P})} < j$, where $(\mathfrak{P})$ denotes the localization,

$$M_{f,(\mathfrak{P})} \otimes \Lambda_{\mathbb{Q}_\infty}/\mathfrak{P}^j = M_{f,(\mathfrak{P})} \otimes \Lambda_{\mathbb{Q}_\infty}/\mathfrak{P}^{j-1}, \tag{87}$$

which implies the same for g. Therefore $M_{g,(\mathfrak{P})} \otimes \Lambda_{\mathbb{Q}_\infty}/\mathfrak{P}^j = M_{g,(\mathfrak{P})}$ and the result follows. $\square$

Proof of Theorem 8.1.1. If $\wp$ is ordinary, fix once and for all an auxiliary quadratic imaginary field $\mathcal{K}$, not contained in the fixed field $K(T_f)$, such that ℓ_0 is inert in $\mathcal{K}$ and every other factor of Np is split in $\mathcal{K}$. As in the proof of Theorem 7.2.1, begin by applying Proposition 6.3.5 and Theorem 6.2.4 to obtain some $\{\mathsf{Q},\epsilon_{\mathsf{Q}}\} \in \mathsf{N}$, represented by Q_n, and a resulting sequence of newforms g_n of NQ_n; we make sure to choose each $\mathsf{q} \in \mathsf{Q}$ such that $\mathrm{Frob}_{\mathsf{q}}$ has trivial image in $\mathrm{Gal}(\mathcal{K}/\mathbb{Q})$, which is clearly possible.

Claim. There exists a constant C, depending only on f, such that

$$v_\wp\left(\frac{L(g_n/K,1)}{\Omega^{can}_{g_n}}\right) \leq \lg_{\mathcal{O}} \mathrm{Sel}(K,W_{g_n}) + 2\sum_{\ell | NQ_n} c_{g_n}(\ell) + C \tag{88}$$

for $\mathfrak{F}$-many n.

Here $\mathrm{Sel}(K,W_{g_n})$ is defined using the dual local conditions to (51), with $\mathsf{m} = 1$.

Proof of claim. The non-ordinary case (in fact with $C = 0$) follows from the main result of [20], applied to g_n and $g_n \otimes \chi_K$; see Corollary 5.4 of loc. cit. Additional details on the passage between the equivariant Tamagawa number conjecture and the BSD formula may be found in [32].

Now consider the ordinary case. By Lemma 8.3.5 and Theorem 8.3.2,

$$\wp^{\mu(f\otimes\chi_{\mathcal{K}})} \cdot (L_p(\mathbb{Q}_\infty,g_n\otimes\chi_{\mathcal{K}})) \subset Ch_{\mathbb{Q}_\infty,g_n\otimes\chi_{\mathcal{K}}} \text{ in } \Lambda_{\mathbb{Q}_\infty} \tag{89}$$

for $\mathfrak{F}$-many n. By Theorem 8.3.3 for g_n, for $\mathfrak{F}$-many n we have

$$(\alpha) \cdot \wp^{\mu(f\otimes\chi_{\mathcal{K}})} \cdot Ch_{\mathbb{Q}_\infty,g_n} \cdot Ch_{\mathbb{Q}_\infty,g_n\otimes\chi_{\mathcal{K}}} \subset (L_p(\mathbb{Q}_\infty,g_n)) \cdot Ch_{\mathbb{Q}_\infty,g_n\otimes\chi_{\mathcal{K}}}, \tag{90}$$

where, by Proposition 8.2.7, $\mathbb{1}(\alpha)$ divides $\wp^{c_f(\ell_0)}$ in $\mathcal{O}$. Since $Ch_{\mathbb{Q}_\infty, g_n\otimes\chi_K} \neq 0$, and since characteristic ideals are divisorial, we conclude that

$$(\alpha)\cdot\wp^{\mu(f\otimes\chi_K)}\cdot Ch_{\mathbb{Q}_\infty,g_n}\subset(L_p(\mathbb{Q}_\infty,g_n)). \tag{91}$$

Applying the same argument to $g_n\otimes\chi_K$, we have:

$$\begin{aligned}(\alpha)^2\cdot\wp^{\mu(f\otimes\chi_K\otimes\chi_K)+\mu(f\otimes\chi_K)}\cdot Ch_{\mathbb{Q}_\infty,g_n}\cdot Ch_{\mathbb{Q}_\infty,g_n\otimes\chi_K}\\ \subset(L_p(\mathbb{Q}_\infty,g_n)\cdot(L_p(\mathbb{Q}_\infty,g_n\otimes\chi_K)).\end{aligned} \tag{92}$$

The result now follows from standard interpolation properties of both sides of (92), cf. e.g. [60, Theorem 3.6.11].

□

As in Step 3 of the proof of Theorem 7.2.1,

$$\#\,\mathrm{Sel}(K,W_{g_n}[\pi^j])=\#\,\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_f[\pi^j])$$

for any j and for $\mathfrak{F}$-many n (the local conditions at $v|N$ may be compared in the same way, and at $v|p$ we use [23, Lemma 7]). Since $\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_f)$ is finite, it follows that $\#\,\mathrm{Sel}(K,W_{g_n})=\#\,\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_f)$ for $\mathfrak{F}$-many n.

Now, by combining the claim above with Proposition 8.2.4, we have for $\mathfrak{F}$-many n:

$$v_\wp\left(\frac{L(g_n/K,1)}{\Omega_{g_n,N^-Q_n}}\right)\leq \lg_{\mathcal{O}}\mathrm{Sel}(K,W_{g_n})+2\sum_{\ell|N^+}c_{g_n}(\ell)+C' \tag{93}$$

for a constant C' that does not depend on n. In particular, for $\mathfrak{F}$-many n, $L(g_n/K,1)\neq 0$, which by parity considerations implies that $\nu(N^-)+|\mathsf{Q}|$ is odd. Exactly as in Step 2 of the proof of Theorem 7.2.1, we then conclude from (93) that $\lambda(1,\mathsf{Q})\neq 0$.

□

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