# Appendix

*Contains proofs for "Moment Inequalities and Their Application" by Pakes, Porter, Ho, and Ishii*

## Notation

Let $l$ denote a vector of ones. Below we suppress the random variable argument in all notation for the moment functions.

Let $\underline{\Omega}_F = \Omega_F(\underline{\theta}_F)$, $\underline{D}_F = D_F(\underline{\theta}_F)$, and $\underline{\Sigma}_F = \mathrm{Var}_F(m(\underline{\theta}_F))$ with estimators $Om\hat{e}ga_{J,F}$, $\hat{D}_{J,F}$, and $\hat{\Sigma}_{J,F}$ (and the dependence of the estimators on $F$ will usually be suppressed). We will assume the diagonal elements of $\hat{D}_{J,F}$ are positive.

## Assumptions

**Assumption A1** *(a) $\Theta$ is compact; and for all $F \in \mathcal{F}$ (b) for some $\epsilon > 0$, $\Theta^{\epsilon}_{0,F} \subset int(\Theta)$, where $\Theta^{\epsilon}_{0,F} = \{\theta \in \Theta : \inf_{\theta' \in \Theta_{0,F}} \|\theta - \theta'\| \leq \epsilon\}$; (c) $\Theta_{0,F}$ is closed; (d) $\underline{\theta}_F$ is a singleton.*

**Assumption A2** *For any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\inf_{F \in \mathcal{F}} \inf_{\theta \in (\Theta^{\epsilon}_{0,F})^c} \|\Big(\mathcal{P}_F m(\theta)\Big)_-\| > \delta.$$

**Assumption A3** *Define, for each $F \in \mathcal{F}$, $\mathcal{T}_F = \left\{\frac{\theta - \underline{\theta}_F}{\|\theta - \underline{\theta}_F\|} : \theta \in \Theta_{0,F}, \theta \neq \underline{\theta}_F\right\}$. Let $\bar{\delta} = \inf\{\tau_1 : \tau \in \mathcal{T}_F, F \in \mathcal{F}\}$. Assume $\bar{\delta} > 0$.*

**Assumption A4** *For some $\underline{\delta} > 0$, there exists $\eta_\Gamma, \varepsilon_\Gamma > 0$, and for each $F$ there is $\lambda_F$ with $\|\lambda_F\| = 1$ such that* **(a)** $\inf_F \min_{j:\underline{D}^{-1/2}_{j,F}\mathcal{P}_F m_j(\underline{\theta}_F) < \eta_\Gamma} \underline{D}^{-1/2}_{j,F}\underline{\Gamma}_{j,F}\lambda_F > \varepsilon_\Gamma$*;* **(b)** $\sup_F \sup_{\tau:\tau_1 \leq \underline{\delta}, \|\tau\|=1} \min_{j:\mathcal{P}_F m_j(\underline{\theta}_F)=0} \underline{D}^{-1/2}_{j,F}\underline{\Gamma}_{j,F}\tau < -\varepsilon_\Gamma$.

**Assumption A5** *For any $\delta > 0$,*

$$\sup_F \mathrm{Pr}_F\left(\sup_{\theta \in \Theta} \|\mathbb{P}_{J,F} m(\theta) - \mathcal{P}_F m(\theta)\| \geq \delta\right) \longrightarrow 0.$$

**Assumption A6** *(a) For some $\underline{d} > 0$ and $\delta > 0$, $\underline{d} \leq \inf_F \min_j Var_F(m_j(\underline{\theta}_F))$ and $\sup_F \max_j \mathcal{P}_F\|m(\underline{\theta}_F)\|^{2+\delta} < \infty$;*
*(b) for any $\delta > 0$,*

$$\sup_F \mathrm{Pr}_F\left(\left\|\hat{D}^{-1/2}_{J,F}\underline{D}^{1/2}_F - I\right\| \geq \delta\right) \longrightarrow 0.$$

**Assumption A7** *There exists $\nu > 0$ such that for all $F$, $\mathcal{P}_F m(\theta)$ is continuously differentiable in the neighborhood $\mathcal{N}_\nu^{\underline{\theta}_F}$ and there exists $C < \infty$ such that $\|\frac{\partial}{\partial\theta}\mathcal{P}_F m(\theta)\| \leq C$ for $\theta \in \mathcal{N}_\nu^{\underline{\theta}_F}$. For any $\delta \downarrow 0$,*

$$\sup_F \sup_{\theta:\|\theta-\underline{\theta}_F\|<\delta} \left\| \frac{\partial}{\partial\theta}\mathcal{P}_F m(\theta) - \frac{\partial}{\partial\theta}\mathcal{P}_F m(\underline{\theta}_F) \right\| = o(\delta)$$

*and there exists $\eta_d > 0$ such that*

$$\sup_F \sup_{\theta':\|\theta'-\underline{\theta}_F\|<\eta_d} \sup_{\theta:\|\theta-\theta'\|<\delta} \left\| \left[ \mathcal{P}_F m(\theta) - \mathcal{P}_F m(\theta') - \frac{\partial}{\partial\theta}\mathcal{P}_F m(\theta')(\theta-\theta') \right] \right\| = o(\delta).$$

**Assumption A8** *For any $\delta > 0$ and all sequences $\eta \downarrow 0$,*

$$\sup_F \Pr_F \left( \sup_{\theta:\|\theta-\underline{\theta}_F\|\leq\eta} \left\| \sqrt{J}\left[\mathbb{P}_{J,F} m(\theta) - \mathcal{P}_F m(\theta)) - (\mathbb{P}_{J,F} m(\underline{\theta}_F) - \mathcal{P} m(\underline{\theta}_F))\right] \right\| \geq \delta \right) \longrightarrow 0$$

## Proofs

PROOF OF THEOREM 2:

Let

$$\alpha^* = \liminf_{J\longrightarrow\infty} \inf_{F\in\mathcal{F}} \Pr\left(\sqrt{J}(\underline{\hat{\theta}}_{1,J,F} - \underline{\theta}_{1,F}) \leq q^*_{J,F}\right) \tag{1}$$

Then, there's a sequence $F_J$ such that

$$\Pr\left(\sqrt{J}(\underline{\hat{\theta}}_{1,J,F_J} - \underline{\theta}_{1,F_J}) \leq q^*_{J,F_J}\right) \longrightarrow \alpha^*$$

We will work along this sequence and its subsequences to show the desired result, see Andrews and Guggenberger (2009).

By continuity, for each $J$, $\mathcal{P}_{F_J} m_l(\underline{\theta}_{F_J}) = 0$ for some dimension $l$. Hence, for some dimension $l$, $\liminf_J \sqrt{J}\underline{D}_{l,F_J}^{-1/2}\mathcal{P}_{F_J} m_l(\underline{\theta}_{F_J}) = 0$. Such a moment is "classified" as binding. Consider the subsequence $F_{J'}$ such that $\mathcal{P}_{F_{J'}} m_l(\underline{\theta}_{F_{J'}}) = 0$. Now along this subsequence, search for any other moment $m_{l'}$ such that $\liminf_{J'} \sqrt{J'}\underline{D}_{l',F_{J'}}^{-1/2}\mathcal{P}_{F_{J'}} m_{l'}(\underline{\theta}_{F_{J'}}) = 0$. Pick a further subsequence such that $\mathcal{P}_{F_{J''}} m_{l'}(\underline{\theta}_{F_{J''}}) = 0$. Continue in this fashion until $\liminf_{J''} \sqrt{J''}\underline{D}l, F_{J''}{}^{-1/2}\mathcal{P}_{F_{J''}} m_l(\underline{\theta}_{F_{J''}}) > 0$ for all remaining moments $m_l$. The moments that have been "chosen" are called the binding moments. Let $m^a$ denote the vector containing the binding dimensions, and let $k_a$ denote the length of the vector.

Let $F_J$ denote the last subsequence that emerges from the above procedure. Along this subsequence, we now search among the remaining moments (not included in $m^a$) for a moment $m_l$ such that $\liminf_J \sqrt{J}\underline{D}_{l,F_J}^{-1/2}\mathcal{P}_{F_J} m_l(\underline{\theta}_{F_J})$ is finite. For such a moment, let $\gamma =$

$\liminf_J \sqrt{J}\underline{D}_{l,F_J}^{-1/2}\mathcal{P}_{F_J}m_l(\underline{\theta}_{F_J})$, and take a subsequence of $F_J$ such that $\sqrt{J'}\underline{D}_{l,F_{J'}}^{-1/2}\mathcal{P}_{F_{J'}}m_l(\underline{\theta}_{F_{J'}}) \longrightarrow \gamma$. Notice that $\gamma > 0$. Continue working along the subsequences to find such moments, just as in the procedure for the binding moments. The moments chosen are collected together with the binding moments to form the "nearly binding" moments and are denoted $m^0$ (with size $k_0$). Let $F_J$ denote the last subsequence taken in this procedure. If there are any remaining moments $m_l$ not contained in $m^0$, then $\sqrt{J}\underline{D}_{l,F_J}^{-1/2}\mathcal{P}_{F_J}m_l(\underline{\theta}_{F_J}) \longrightarrow \infty$. Such moments are called non-binding and are collected in the vector $m^1$ (with size $k_1$).

For notational convenience, assume that the moments are ordered with the binding moments first and the non-binding moments last. Also, an $a$ superscript will be used to denote the sub-vector or -sub-matrix corresponding to the rows from $m^a$ (similarly for superscripts 0 and 1).

Lastly, using boundedness implied by Assumptions A6 and A7, take a further subsequence such that $\underline{\Gamma}_{F_J}$ and $\underline{\Sigma}_{F_J}$ converge. Let $\underline{\Gamma}$ and $\underline{\Sigma}$ denote the limits.

The propositions below will be stated with respect to the final subsequence that results from this process. For notational convenience, we will denote it by $F_J$. Proposition 6 yields the desired result. □

Proof of Theorem 1:

Propositions 4 and 5 give the limiting distribution for $\sqrt{J}(\hat{\underline{\theta}}_J - \underline{\theta}_{F_J})$ along a sequence $F_J$. For Theorem 1, we only need the limiting distribution for a fixed $F$, which is a special case. Notice that for a fixed $F$, if $\mathcal{P}_F m_j(\underline{\theta}_F) > 0$, then $\sqrt{J}\mathcal{P}_F m_j(\underline{\theta}_F) \longrightarrow \infty$. So, in this case $m_j$ is in the non-binding moments $m^1$. As a result, $m^0$ consists only of moments $m_j$ such that $\mathcal{P}_F m_j(\underline{\theta}_F) = 0$. Thus, for a fixed $F$, $\mu^0 = 0$, which gives the form of the result stated, where $\underline{D}^0 = D_F^0(\underline{\theta}_F)$, $\underline{\Gamma}^0 = \Gamma_F^0(\underline{\theta}_F)$, and $\underline{\Omega}^0 = \Omega_F^0(\underline{\theta}_F)$. □

**Proposition 1** *Under Assumptions A1-A8, for any* $\epsilon > 0$, $\Pr_{F_J}(\|\hat{\underline{\theta}}_J - \underline{\theta}_{F_J}\| \geq \epsilon) \longrightarrow 0$.

Proof:

Without loss of generality, assume $\epsilon$ satisfies Assumption A1(b). First, we will show that the set estimator cannot be too much larger than the identified set, i.e. $\hat{\underline{\theta}}_J \in \Theta_{0,F_J}^{\delta/2}$ for some appropriately chosen $\delta > 0$ and $J$ large enough. Then we will show that the set estimator cannot be too small, implying $\hat{\underline{\theta}}_{1,J} \leq \underline{\theta}_{1,F_J} + \delta/2$. Together these two findings will yield the desired result.

In particular, given $\epsilon$ above and $\bar{\delta}$ defined in Assumption A3, let $\delta = \epsilon \cdot \min\{\bar{\delta}, \epsilon, 1\}/2$. Assumption A3 implies that if $\theta' \in \Theta_{0,F}$ and ${\theta_1}' - \underline{\theta}_{1,F} \leq \delta$, then $\|\theta' - \underline{\theta}_F\| \leq \delta/\bar{\delta} \leq \epsilon/2$. Then, $\Theta_{0,F}^{\delta/2} \cap \{\theta' \in \Theta : {\theta'}_1 < \underline{\theta}_{1,F} + \delta/2\} \subset \{\theta : \|\theta - \underline{\theta}_F\| < \epsilon\}$, so it remains to show that these last two events occur w.p.a. 1.

By Assumption A2, there exists $\delta' > 0$ such that $\inf_{\theta \in (\Theta_{0,F_J}^{\delta/2})^c} \frac{1}{\sqrt{2d}}\|(\mathcal{P}_{F_J}m(\theta))_-\| > \delta'$. By Assumption A5, $\sup_{\theta' \in \Theta} \sqrt{\frac{2}{d}}\|\mathbb{P}_J m(\theta') - \mathcal{P}_{F_J}m(\theta')\| < \delta'/2$ w.p.a. 1, and by Assumption A6,

$\frac{1}{\sqrt{2\bar{d}}} I < \hat{D}_J^{-1/2} < \sqrt{\frac{2}{\underline{d}}}\, I$ w.p.a. 1. Suppose these conditions hold, then

$$\begin{aligned}
&\inf_{\theta\in(\Theta_{0,F_J}^{\delta/2})^c} \|(\hat{D}_J^{-1/2}\mathbb{P}_J m(\theta))_-\| \\
&\geq \inf_{\theta\in(\Theta_{0,F_J}^{\delta/2})^c} \frac{1}{\sqrt{2\bar{d}}}\|(\mathcal{P}_{F_J} m(\theta))_-\| - \sup_{\theta'\in\Theta}\sqrt{\frac{2}{\underline{d}}}\,\|\mathbb{P}_J m(\theta') - \mathcal{P}_{F_J} m(\theta')\| \\
&> \delta' - \delta'/2 = \delta'/2.
\end{aligned}$$

And, for any $\theta' \in \Theta_{0,F_J}$, $\|(\hat{D}_J^{-1/2}\mathbb{P}_J m(\theta'))_-\| \leq \sqrt{\frac{2}{\underline{d}}}\,\|(\mathcal{P}_{F_J} m(\theta'))_-\| + \sqrt{\frac{2}{\underline{d}}}\,\|\mathbb{P}_J m(\theta') - \mathcal{P}_{F_J} m(\theta')\|$ $\leq \delta'/2$. Hence, $\hat{\theta} \notin (\Theta_{0,F_J}^{\delta/2})^c$, i.e. $\hat{\theta} \in \Theta_{0,F_J}^{\delta/2}$, and so $\underline{\hat{\theta}}_J \in \hat{\Theta}_J \subset \Theta_{0,F_J}^{\delta/2}$ w.p.a. 1.

Let $\zeta_J^1 = \min_j\{[\sqrt{J}(\underline{D}_{F_J}^1)^{-1/2}\mathcal{P}_{F_J} m^1(Z,\underline{\theta}_{F_J})]_j\}$ where $j$ indexes the elements of the vector $m^1$. Recall that $\zeta_J^1 \longrightarrow \infty$. Let $\zeta_J = \min\{\sqrt{J}, \zeta_J^1\}$, and define $\theta_J' = \underline{\theta}_{F_J} + \lambda_{F_J}\frac{\sqrt{\zeta_J}}{\sqrt{J}}$ for $\lambda_{F_J}$ as defined in Assumption A4. Note that by its definition, $\|\theta_J' - \underline{\theta}_{F_J}\| \longrightarrow 0$. We will show that $\underline{D}_{F_J}^{-1/2}\,\mathbb{P}_J m(\theta_J') > 0$ w.p.a. 1. By Assumption A6, it will follow that $\hat{D}_J^{-1/2}\,\mathbb{P}_J m(\theta_J') > 0$ w.p.a. 1.

By Assumptions A6 and A8 and Lyapunov's CLT, there exists $C_1 < \infty$ such that $\left\|\sqrt{J}\underline{D}_{F_J}^{-1/2}\left[\mathbb{P}_J m(\theta_J') - \mathcal{P}_{F_J} m(\theta_J')\right]\right\| \leq \left\|\sqrt{J}\underline{D}_{F_J}^{-1/2}\left[\mathbb{P}_J m(\underline{\theta}_{F_J}) - \mathcal{P}_{F_J} m(\underline{\theta}_{F_J})\right]\right\|$ $+\left\|\sqrt{J}\underline{D}_{F_J}^{-1/2}\left[\mathbb{P}_J m(\theta_J') - \mathcal{P}_{F_J} m(\theta_J') - \left(\mathbb{P}_J m(\underline{\theta}_{F_J}) - \mathcal{P}_{F_J} m(\underline{\theta}_{F_J})\right)\right]\right\| \leq C_1$ w.p.a. 1. By Assumptions A6 and A7,

$$\left\|\sqrt{J}\underline{D}_{F_J}^{-1/2}\left[\mathcal{P}_{F_J} m(\theta_J') - \mathcal{P}_{F_J} m(\underline{\theta}_{F_J}) - \underline{\Gamma}_{F_J}(\theta_J' - \underline{\theta}_{F_J})\right]\right\| = \sqrt{J}o(\|\theta_J' - \underline{\theta}_{F_J}\|) = o(\sqrt{\zeta_J}). \quad (2)$$

By the bounds on $\underline{D}_{F_J}$ and $\underline{\Gamma}_{F_J}$ implied by Assumptions A6 and A7, there exists a bound $C_\Gamma$ such that $\|(\underline{D}_{F_J}^1)^{-1/2}\underline{\Gamma}_{F_J}^1\| \leq C_\Gamma$, so $(\underline{D}_{F_J}^1)^{-1/2}\underline{\Gamma}_{F_J}^1(\theta_J' - \underline{\theta}_{F_J}) = \sqrt{\zeta_J}(\underline{D}_{F_J}^1)^{-1/2}\underline{\Gamma}_{F_J}^1\lambda_{F_J}$ $\geq -\sqrt{\zeta_J}C_\Gamma l$. For large enough $J$, the $o(\sqrt{\zeta_J})$ term in (2) is bounded in absolute value by $\sqrt{\zeta_J}$ and $\zeta_J > (C_1+1) + \sqrt{\zeta_J} + C_\Gamma\sqrt{\zeta_J}$, so that

$$\begin{aligned}
\sqrt{J}(\underline{D}_{F_J}^1)^{-1/2}\mathbb{P}_J m^1(\theta_J') &\geq \sqrt{J}(\underline{D}_{F_J}^1)^{-1/2}\mathcal{P}_{F_J} m^1(\underline{\theta}_{F_J}) - \left\|\sqrt{J}(\underline{D}_{F_J}^1)^{-1/2}\left[\mathbb{P}_J m^1(\theta_J') - \mathcal{P}_{F_J} m^1(\theta_J')\right]\right\| l \\
&\quad - \left\|\sqrt{J}\underline{D}_{F_J}^{-1/2}\left[\mathcal{P}_{F_J} m(\theta_J') - \mathcal{P}_{F_J} m(\underline{\theta}_{F_J}) - \underline{\Gamma}_{F_J}(\theta_J' - \underline{\theta}_{F_J})\right]\right\| l \\
&\quad + (\underline{D}_{F_J}^1)^{-1/2}\underline{\Gamma}_{F_J}^1(\theta_J' - \underline{\theta}_{F_J}) \\
&\geq [\zeta_J - (C_1+1) - \sqrt{\zeta_J} - C_\Gamma\sqrt{\zeta_J}]l \\
&> 0
\end{aligned}$$

w.p.a. 1 and $\sqrt{J}(\hat{D}_J^1)^{-1/2}\mathbb{P}_J m^1(\theta_J') > 0$ w.p.a. 1.

For some $C_2 < \infty$, $\|\sqrt{J}(\underline{D}_{F_J}^0)^{-1/2}\mathcal{P}_{F_J} m^0(\underline{\theta}_{F_J})\| < C_2$. For $J$ large, the $o(\sqrt{\zeta_J})$ term in (2) is bounded in absolute value by $\frac{\varepsilon_\Gamma}{2}\sqrt{\zeta_J}$, $C_2/\sqrt{J} < \eta_\Gamma$, and $\frac{\varepsilon_\Gamma}{2}\sqrt{\zeta_J} > C_1 + C_2 + 1$ for $\varepsilon_\Gamma$

and $\eta_\Gamma$ defined in Assumption A4. Then w.p.a. 1,

$$\begin{aligned}
&\sqrt{J}(\underline{D}^0_{F_J})^{-1/2}\mathbb{P}_J m^0(\theta'_J)\\
&\geq \sqrt{J}(\underline{D}^0_{F_J})^{-1/2}\underline{\Gamma}^0_{F_J}(\theta'_J-\underline{\theta}_{F_J}) - \|\sqrt{J}(\underline{D}^0_{F_J})^{-1/2}[\mathbb{P}_J m^0(\theta'_J)-\mathcal{P}_{F_J}m^0(\theta'_J)]\| l\\
&\quad - \left\|\sqrt{J}(\underline{D}^0_{F_J})^{-1/2}\left[\mathcal{P}_{F_J}m^0(\theta'_J)-\mathcal{P}_{F_J}m^0(\underline{\theta}_{F_J})-\underline{\Gamma}^0_{F_J}(\theta'_J-\underline{\theta}_{F_J})\right]\right\| l\\
&\quad -\|\sqrt{J}(\underline{D}^0_{F_J})^{-1/2}\mathcal{P}_{F_J}m^0(\underline{\theta}_{F_J})\| l\\
&> \sqrt{\zeta_J}(\underline{D}^0_{F_J})^{-1/2}\underline{\Gamma}^0_{F_J}\lambda_{F_J} - (C_1+1)l - \frac{\varepsilon_\Gamma}{2}\sqrt{\zeta_J}l - C_2 l\\
&> 0
\end{aligned}$$

and $\sqrt{J}(\hat{D}^0_J)^{-1/2}\mathbb{P}_J m^0(\theta'_J) > 0$ w.p.a. 1.

Now, $\sqrt{J}\hat{D}_J^{-1/2}\mathbb{P}_J m(\theta'_J) > 0$ implies that $\hat{\underline{\theta}}_{1,J} \leq \theta'_{1,J} \leq \underline{\theta}_{1,F_J} + \sqrt{\zeta_J}/\sqrt{J} \leq \underline{\theta}_{1,F_J} + \delta/2$ w.p.a. 1. The result follows as argued above. □

**Proposition 2** *Under Assumptions A1-A8, for any $\epsilon > 0$, there exists $C < \infty$ such that* $\Pr_{F_J}(\|\sqrt{J}(\hat{\underline{\theta}}_J - \underline{\theta}_{F_J})\| \leq C) \geq 1-\epsilon$.

Proof:

First, we will show that for some $c_1 > 0$, $\hat{D}_J^{-1/2}\,\mathbb{P}_J m(\underline{\theta}_{F_J} + c_1\lambda_{F_J}/\sqrt{J}) \geq 0$ with probability approaching one (w.p.a. 1). Then, $\hat{\underline{\theta}}_{1,J} \leq \underline{\theta}_{1,F_J} + c_1\lambda_{1,F_J}/\sqrt{J}$ w.p.a 1. Second, we show $\Pr_{F_J}(\hat{\underline{\theta}} \notin B_{J,c_2}) \longrightarrow 1$ where $B_{J,c} = \left\{\theta : \frac{\theta_1 - \underline{\theta}_{1,F_J}}{\|\theta - \underline{\theta}_{F_J}\|} \leq \underline{\delta}, \|\theta - \underline{\theta}_{F_J}\| \geq c/\sqrt{J}\right\}$ and $\underline{\delta}$ as defined in Assumption A4.

Suppose, for some $c_2 > 0$, $\hat{\underline{\theta}}_{1,J} \leq \underline{\theta}_{1,F_J} + c_1\lambda_{1,F_J}/\sqrt{J}$ and $\hat{\underline{\theta}} \notin B_{J,c_2}$. If $\frac{\hat{\underline{\theta}}_{1,J} - \underline{\theta}_{1,F_J}}{\|\hat{\underline{\theta}} - \underline{\theta}_{F_J}\|} \leq \underline{\delta}$, then $\|\hat{\underline{\theta}} - \underline{\theta}_{F_J}\| \leq c_2/\sqrt{J}$. If $\frac{\hat{\underline{\theta}}_{1,J} - \underline{\theta}_{1,F_J}}{\|\hat{\underline{\theta}} - \underline{\theta}_{F_J}\|} > \underline{\delta}$, then $\|\hat{\underline{\theta}} - \underline{\theta}_{F_J}\| < (\hat{\underline{\theta}}_{1,J} - \underline{\theta}_{1,F_J})/\underline{\delta} \leq c_1/\underline{\delta}\sqrt{J}$. In either case, $\|\hat{\underline{\theta}} - \underline{\theta}_{F_J}\| \leq \max\{c_1/\underline{\delta}, c_2\}/\sqrt{J}$. So the above two conclusions will be sufficient to prove the result.

Let $\theta'_{F_J} = \underline{\theta}_{F_J} + c_1\lambda_{F_J}/\sqrt{J}$ and $\sqrt{J}(\underline{D}^0_{F_J})^{-1/2}\mathbb{P}_J m^0(\theta'_{F_J}) = (\underline{D}^0_{F_J})^{-1/2}c_1\underline{\Gamma}^0_{F_J}\lambda_{F_J} + \eta^0_J$. Note that $\|\eta^0_J\|$ is bounded w.p.a. 1 by Assumptions A6, A7, A8, and Lyapounov's central limit theorem. Hence, by Assumption A4, by choosing $c_1$ sufficiently large, the sum of the last two terms is nonnegative w.p.a. 1. So, using Assumption A6, we have $(\hat{D}^0_J)^{-1/2}\,\mathbb{P}_J m^0(\underline{\theta}_{F_J} + c_1\lambda_{F_J}/\sqrt{J}) \geq 0$ w.p.a. 1.

Let $\sqrt{J}(\underline{D}^1_{F_J})^{-1/2}\mathbb{P}_J m^1(\theta'_{F_J}) = \sqrt{J}(\underline{D}^1_{F_J})^{-1/2}\mathcal{P}_{F_J}m^1(\underline{\theta}_{F_J}) + c_1(\underline{D}^1_{F_J})^{-1/2}\underline{\Gamma}^1_{F_J}\lambda_{F_J} + \eta^1_J$. By the bounds in Assumptions A6 and A4, there exists $C_\Gamma$ such that $(\underline{D}^1_{F_J})^{-1/2}\underline{\Gamma}^1_{F_J}\lambda_{F_J} \geq -C_\Gamma l$, so $\sqrt{J}(\underline{D}^1_{F_J})^{-1/2}\mathbb{P}_J m^1(\theta'_{F_J}) \geq \sqrt{J}(\underline{D}^1_{F_J})^{-1/2}\mathcal{P}_{F_J}m^1(\underline{\theta}_{F_J}) - c_1C_\Gamma l - \|\eta^1_J\|$. As with the $m^0$ argument above, $\|\eta^1_J\|$ can be bounded w.p.a. 1. Every element of $\sqrt{J}(\underline{D}^1_{F_J})^{-1/2}\mathcal{P}_{F_J}m^1(\underline{\theta}_{F_J})$ is approaching infinity, so with $J$ large enough the whole expression is nonnegative with probability arbitrarily close to one. Again, using Assumption A6 and putting the $m^0$ and $m^1$ results together, we have the desired first result that $\hat{D}_J^{-1/2}\mathbb{P}_J m(\underline{\theta}_{F_J} + c_1\lambda_{F_J}/\sqrt{J}) \geq 0$ w.p.a. 1.

Next, we show that for some positive constant $c_2$, $\hat{\underline{\theta}}_J \notin B_{J,c_2}$ w.p.a. 1.

By consistency, there exists $K_J \downarrow 0$ such that $\|\hat{\underline{\theta}}_J - \underline{\theta}_{F_J}\| < K_J$ w.p.a. 1. Let $A_{J,c} = \left\{\theta : \|\theta - \underline{\theta}_{F_J}\| < K_J, \frac{\theta_1 - \underline{\theta}_{1,F_J}}{\|\theta - \underline{\theta}_{F_J}\|} \leq \underline{\delta}, \|\theta - \underline{\theta}_{F_J}\| \geq c/\sqrt{J}\right\}$. So, it will suffice to show $\hat{\underline{\theta}}_J \notin A_{J,c_2}$ w.p.a. 1. We'll prove this by showing that w.p.a. 1 for some $\eta_1 > 0$, $\|(\sqrt{J}\hat{D}_J^{-1/2}\mathbb{P}_J m(\underline{\theta}_{F_J}))_-\| \leq \eta_1$ while $\|(\sqrt{J}\hat{D}_J^{-1/2}\mathbb{P}_J m(\theta))_-\| \geq 2\eta_1$ for all $\theta \in A_{J,c_2}$.

Note that

$$\sqrt{J}(\underline{D}^1_{F_J})^{-1/2}\mathbb{P}_J m^1(\underline{\theta}_{F_J}) \geq \sqrt{J}(\underline{D}^1_{F_J})^{-1/2}\mathcal{P}_{F_J} m^1(\underline{\theta}_{F_J}) - \|\sqrt{J}(\underline{D}^1_{F_J})^{-1/2}[\mathbb{P}_J m^1(\underline{\theta}_{F_J}) - \mathcal{P}_{F_J} m^1(\underline{\theta}_{F_J})]\|.$$

The last term is bounded w.p.a. 1, and since the first term is tending to infinity, $\sqrt{J}(\underline{D}^1_{F_J})^{-1/2}\mathbb{P}_J m^1(\underline{\theta}_{F_J}) > 0$ w.p.a. 1. And, by Assumption A6, $\sqrt{J}(\hat{D}^1_J)^{-1/2}\mathbb{P}_J m^1(\underline{\theta}_{F_J}) > 0$ w.p.a. 1. So, the following holds w.p.a. 1,

$$\begin{aligned}
&\|(\sqrt{J}\hat{D}_J^{-1/2}\mathbb{P}_J m(\underline{\theta}_{F_J}))_-\| = \|(\sqrt{J}(\hat{D}^0_J)^{-1/2}\mathbb{P}_J m^0(\underline{\theta}_{F_J}))_-\| \\
&\leq \|\sqrt{J}(\hat{D}^0_J)^{-1/2}\mathbb{P}_J m^0(\underline{\theta}_{F_J})\| \leq \sqrt{2}\|\sqrt{J}(\underline{D}^0_{F_J})^{-1/2}\mathbb{P}_J m^0(\underline{\theta}_{F_J})\| \\
&\leq \sqrt{2}\|\sqrt{J}(\underline{D}^0_{F_J})^{-1/2}(\mathbb{P}_J m^0(\underline{\theta}_{F_J}) - \mathcal{P}_{F_J} m^0(\underline{\theta}_{F_J}))\| + \sqrt{2}\|\sqrt{J}(\underline{D}^0_{F_J})^{-1/2}\mathcal{P}_{F_J} m^0(\underline{\theta}_{F_J})\|
\end{aligned}$$

The last two terms are bounded w.p.a. 1, and we can set $\eta_1$ equal to this bound.

Next, we show that $\sup_{\theta \in A_{J,c_2}} \min_{j \in \{1,\ldots,k_a\}} \sqrt{J}\hat{D}_{j,J}^{-1/2}\mathbb{P}_J m^a_j(\theta) \leq -2\eta_1$ with probability approaching one. Noting that $\mathcal{P}_{F_J} m^a(\underline{\theta}_{F_J}) = 0$,

$$\begin{aligned}
&\sup_{\theta \in A_{J,c_2}} \min_{j \in \{1,\ldots,k_a\}} \sqrt{J}\underline{D}_{j,F_J}^{-1/2}\mathbb{P}_J m^a_j(\theta)) \\
&\leq \sup_{\theta \in A_{J,c_2}} \min_{j \in \{1,\ldots,k_a\}} \sqrt{J}\underline{D}_{j,F_J}^{-1/2}\left[\mathcal{P}_{F_J} m^a_j(\theta) - \mathcal{P}_{F_J} m^a_j(\underline{\theta}_{F_J})\right] \qquad (3) \\
&\quad + \underline{d}^{-1/2} \sup_{\theta \in A_{J,c_2}} \|\sqrt{J}[\mathbb{P}_J m_a(\theta)) - \mathcal{P}_{F_J} m_a(\theta) - (\mathbb{P}_J m_a(\underline{\theta}_{F_J}) - \mathcal{P}_{F_J} m_a(\underline{\theta}_{F_J}))]\| \\
&\quad + \|\sqrt{J}(\underline{D}^a_{F_J})^{-1/2}[\mathbb{P}_J m_a(\underline{\theta}_{F_J}) - \mathcal{P}_{F_J} m_a(\underline{\theta}_{F_J})]\|
\end{aligned}$$

By Assumptions A6 and A8, the last two terms are bounded w.p.a. 1.

It will, then, suffice to show that for large enough $J$,

$$\sup_{\theta \in A_{J,c_2}} \min_{j \in \{1,\ldots,k_a\}} \sqrt{J}\underline{D}_{j,F_J}^{-1/2}\left[\mathcal{P}_{F_J} m^a_j(\theta) - \mathcal{P}_{F_J} m^a_j(\underline{\theta}_{F_J})\right]$$

is less than an arbitrary negative number through choice of $c_2$. Given appropriate $c_2$, we will then have $\sup_{\theta \in A_{J,c_2}} \min_{j \in \{1,\ldots,k_a\}} \sqrt{J}\hat{D}_{j,J}^{-1/2}\mathbb{P}_J m^a_j(\theta)) \leq -2\eta_1$ w.p.a. 1.

For any $\theta \neq \underline{\theta}_F$, there exists $w_{\theta,F} > 0$ and $\lambda_{\theta,F}$ with $\|\lambda_{\theta,F}\| = 1$ such that $\theta = \underline{\theta}_F + w_{\theta,F}\lambda_{\theta,F}$. For any $\lambda$ with $\|\lambda\| = 1$, let $j_{\lambda,F}$ be a solution to $\min_{j\in\{1,\ldots,k_a\}} \underline{D}_{j,F}^{-1/2}\underline{\Gamma}_{j,F}\lambda$. Then,

$$\begin{aligned}
&\limsup_{J\longrightarrow\infty} \sup_{\theta\in A_{J,c_2}} \min_{j\in\{1,\ldots,k_a\}} \sqrt{J}\underline{D}_{j,F_J}^{-1/2}\left[\mathcal{P}_{F_J} m_j^a(\theta) - \mathcal{P}_{F_J} m_j^a(\underline{\theta}_{F_J})\right] \\
&\leq \limsup_{J\longrightarrow\infty} \sup_{\theta\in A_{J,c_2}} \sqrt{J}\underline{D}_{j_{\lambda_\theta},F_J}^{-1/2}\left[\mathcal{P}_{F_J} m_{j_{\lambda_\theta}}^a(\theta) - \mathcal{P}_{F_J} m_{j_{\lambda_\theta}}^a(\underline{\theta}_{F_J})\right] = \overline{\nu}
\end{aligned}$$

Let $J'$ be a subsequence along with $\theta_{J'}$ such that $j_{\lambda_{\theta_{J'}}}$ is the same value for all $J'$, ie $j^* = j_{\lambda_{\theta_{J'}}}$ for all $J'$, and $\sqrt{J'}\underline{D}_{j^*,F_{J''}}^{-1/2}\left[\mathcal{P}_{F_{J'}} m_{j^*}^a(\theta_{J'}) - \mathcal{P}_{F_{J'}} m_{j^*}^a(\underline{\theta}_{F_{J'}})\right] \longrightarrow \overline{\nu}$.

For the sequence $\theta_{J'}$ defined above, there exist corresponding $w_{\theta_{J'},F_{J'}}$ and $\lambda_{\theta_{J'},F_{J'}}$ (for notational simplicity, we suppress the dependence on $F_{J'}$). Define $\bar{\theta}_{J'}$ as

$$\bar{\theta}_{J'} = \underline{\theta}_{F_{J'}} + \frac{c_2}{\sqrt{J'}}\lambda_{\theta_{J'}}$$

and $\bar{w}_{\theta_{J'}} = \frac{c_2}{\sqrt{J'}}$. Now note, for $J'$ large enough,

$$\begin{aligned}
&\sqrt{J'}\underline{D}_{j^*,F_{J'}}^{-1/2}\left[\mathcal{P}_{F_{J'}} m_{j^*}^a(\theta_{J'}) - \mathcal{P}_{F_{J'}} m_{j^*}^a(\underline{\theta}_{F_{J'}})\right] - \sqrt{J'}\underline{D}_{j^*,F_{J'}}^{-1/2}\left[\mathcal{P}_{F_{J'}} m_{j^*}^a(\bar{\theta}_{J'}) - \mathcal{P}_{F_{J'}} m_{j^*}^a(\underline{\theta}_{F_{J'}})\right] \\
&= \sqrt{J'}\underline{D}_{j^*,F_{J'}}^{-1/2}\frac{\partial}{\partial\theta}\mathcal{P}_{F_{J'}} m_{j^*}^a(\underline{\theta}_{F_{J'}})\lambda_{\theta_{J'}}(w_{\theta_{J'}} - \bar{w}_{\bar{\theta}_{J'}}) \\
&\quad + \sqrt{J'}\underline{D}_{j^*,F_{J'}}^{-1/2}\left[\frac{\partial}{\partial\theta}\mathcal{P}_{F_{J'}} m_{j^*}^a(\tilde{\theta}_{J'}) - \frac{\partial}{\partial\theta}\mathcal{P}_{F_{J'}} m_{j^*}^a(\underline{\theta}_{F_{J'}})\right]\lambda_{\theta_{J'}}(w_{\theta_{J'}} - \bar{w}_{\bar{\theta}_{J'}}) \\
&\leq 0
\end{aligned}$$

where $\tilde{\theta}_{J'}$ is a mean value and $\theta_{J'} - \bar{\theta}_{J'} = \lambda_{\theta_{J'}}(w_{\theta_{J'}} - \bar{w}_{\bar{\theta}_{J'}})$. By Assumptions A6 and A7, $\underline{D}_{j^*,F_{J'}}^{-1/2}\left[\frac{\partial}{\partial\theta}\mathcal{P}_{F_{J'}} m_{j^*}^a(\tilde{\theta}_{J'}) - \frac{\partial}{\partial\theta}\mathcal{P}_{F_{J'}} m_{j^*}^a(\underline{\theta}_{F_{J'}})\right]\lambda_{\theta_{J'}} \longrightarrow 0$. By Assumption A4, $\underline{D}_{j^*,F_{J'}}^{-1/2}\frac{\partial}{\partial\theta}\mathcal{P}_{F_{J'}} m_{j^*}^a(\underline{\theta}_{F_{J'}})\lambda_{\theta_{J'}} < -\varepsilon_\Gamma$, leading to the last inequality. So,

$$\begin{aligned}
&\lim_{J'\longrightarrow\infty} \sqrt{J'}\underline{D}_{j^*,F_{J'}}^{-1/2}\left[\mathcal{P}_{F_{J'}} m_{j^*}^a(\theta_{J'}) - \mathcal{P}_{F_{J'}} m_{j^*}^a(\underline{\theta}_{F_{J'}})\right] \\
&\leq \limsup_{J'\longrightarrow\infty} \sqrt{J'}\underline{D}_{j^*,F_{J'}}^{-1/2}\left[\mathcal{P}_{F_{J'}} m_{j^*}^a(\bar{\theta}_{J'}) - \mathcal{P}_{F_{J'}} m_{j^*}^a(\underline{\theta}_{F_{J'}})\right] \\
&= \lim_{J''\longrightarrow\infty} \sqrt{J''}\underline{D}_{j^*,F_{J''}}^{-1/2}\left[\mathcal{P}_{F_{J''}} m_{j^*}^a(\bar{\theta}_{J''}) - \mathcal{P}_{F_{J''}} m_{j^*}^a(\underline{\theta}_{F_{J''}})\right] \\
&\quad \text{(for some subsequence } J''\text{)} \\
&= c_2 \lim_{J''\longrightarrow\infty} \underline{D}_{j^*,F_{J''}}^{-1/2}\underline{\Gamma}_{j^*,F_{J''}}\lambda_{\bar{\theta}_{J''}} \\
&\leq -c_2\varepsilon_\Gamma
\end{aligned}$$

By choosing $c_2$ large enough, we can conclude that for large enough $J$,

$$\sup_{\theta\in A_{J,c_2}} \min_{j\in\{1,\ldots,k_0\}} \sqrt{J}\underline{D}_{j,F_J}^{-1/2}\left[\mathcal{P}_{F_J} m_j^0(\theta) - \mathcal{P}_{F_J} m_j^0(\underline{\theta}_{F_J})\right]$$

is bounded above by any given negative number. □

Define $\theta_1^* = \min\{\theta_1 : D_{0,F_J}^{-1/2}\underline{\Gamma}_{0,F_J}(\theta - \underline{\theta}_{F_J}) + D_{0,F_J}^{-1/2}\mathbb{P}_J m_0(\underline{\theta}_{F_J}) \geq 0\}$ and take $\theta^*$ to be any vector in the infimum set with first element $\theta_1^*$.

**Proposition 3** *Under Assumptions A1-A8, for any $\epsilon > 0$, there exists $C < \infty$ such that $\Pr_{F_J}(\|\sqrt{J}(\theta_J^* - \underline{\theta}_{F_J})\| \leq C) \geq 1 - \epsilon$.*

Proof:

We can choose $J$ large enough that $(\underline{D}_{F_J}^0)^{-1/2}\mathcal{P}_{F_J}m^0(\underline{\theta}_{F_J}) < \eta_\Gamma$, so that by Assumption A4, for $c$ large enough, $\sqrt{J}(\underline{D}_{F_J}^0)^{-1/2}\underline{\Gamma}_{F_J}^0(c\lambda_{F_J}/\sqrt{J}) + \sqrt{J}(\underline{D}_{F_J}^0)^{-1/2}\mathbb{P}_J m^0(\underline{\theta}_{F_J}) \geq 0$. Hence, $\theta_J^*$ is well-defined.

Given $\rho > 0$. Choose $c$ such that $\Pr_{F_J}(c(\underline{D}_{F_J}^0)^{-1/2}\underline{\Gamma}_{F_J}^0\lambda_{F_J} - \|\sqrt{J}(\underline{D}_{F_J}^0)^{-1/2}\mathbb{P}_J m^0(\underline{\theta}_{F_J})\|l > 0) > 1 - \rho$ for $J \geq \bar{J}$ for some $\bar{J}$.

Now, $c(\underline{D}_{F_J}^0)^{-1/2}\underline{\Gamma}_{F_J}^0\lambda_{F_J} + \sqrt{J}(\underline{D}_{F_J}^0)^{-1/2}\mathbb{P}_J m^0(\underline{\theta}_{F_J}) \geq c(\underline{D}_{F_J}^0)^{-1/2}\underline{\Gamma}_{F_J}^0\lambda_{F_J} - \|\sqrt{J}(\underline{D}_{F_J}^0)^{-1/2}\mathbb{P}_J m^0(\underline{\theta}_{F_J})\|l \geq 0$ implies that $\underline{\theta}_{F_J} + c\lambda_{F_J}/\sqrt{J}$ is in the infimum set over which $\theta_J^*$ is chosen, so $\sqrt{J}(\theta_{1,J}^* - \underline{\theta}_{1,F_J}) \leq c\lambda_{1,F_J} \leq c$.

Also, $c(\underline{D}_{F_J}^0)^{-1/2}\underline{\Gamma}_{F_J}^0\lambda_{F_J} - \sqrt{J}(\underline{D}_{F_J}^0)^{-1/2}\mathbb{P}_J m^0(\underline{\theta}_{F_J}) \geq c(\underline{D}_{F_J}^0)^{-1/2}\underline{\Gamma}_{F_J}^0\lambda_{F_J} - \|\sqrt{J}(\underline{D}_{F_J}^0)^{-1/2}\mathbb{P}_J m^0(\underline{\theta}_{F_J})\|l \geq 0$ implies $(\underline{D}_{F_J}^0)^{-1/2}\underline{\Gamma}_{F_J}^0[\sqrt{J}(\theta_J^* - \underline{\theta}_{F_J}) + c\lambda_{F_J}] \geq 0$ and $(\underline{D}_{F_J}^a)^{-1/2}\underline{\Gamma}_{F_J}^a[\sqrt{J}(\theta_J^* - \underline{\theta}_{F_J}) + c\lambda_{F_J}] \geq 0$. By Assumption A4, this last event would imply $\|\sqrt{J}(\theta_J^* - \underline{\theta}_{F_J}) + c\lambda_{F_J}\|\underline{\delta} \leq \sqrt{J}(\theta_{1,J}^* - \underline{\theta}_{1,F_J}) + c\lambda_{1,F_J}$. By the triangle inequality, $\|\sqrt{J}(\theta_J^* - \underline{\theta}_{F_J})\| \leq \|\sqrt{J}(\theta_J^* - \underline{\theta}_{F_J}) + c\lambda_{F_J}\| + \|c\lambda_{F_J}\|$.

For $J \geq \bar{J}$,

$$\begin{aligned}
&\Pr\left(\|\sqrt{J}(\theta_J^* - \underline{\theta}_{F_J})\| \leq \frac{2c\lambda_{1,F_J}}{\underline{\delta}} + \|c\lambda_{F_J}\|\right) \geq \Pr\left(\|\sqrt{J}(\theta_J^* - \underline{\theta}_{F_J}) + c\lambda_{F_J}\| \leq \frac{2c\lambda_{1,F_J}}{\underline{\delta}}\right) \\
&\geq \Pr(\{\underline{\delta}\|\sqrt{J}(\theta_J^* - \underline{\theta}_{F_J}) + c\lambda_{F_J}\| \leq \sqrt{J}(\theta_{1,J}^* - \underline{\theta}_{1,F_J}) + c\lambda_{1,F_J}\} \cap \{\sqrt{J}(\theta_{1,J}^* - \underline{\theta}_{1,F_J}) \leq c\lambda_{1,F_J}\}) \\
&\geq \Pr(c\underline{\Gamma}_{F_J}^0\lambda_{F_J} - \|\sqrt{J}\mathbb{P}_J m^0(\underline{\theta}_{F_J})\|l \geq 0) \\
&\geq 1 - \rho
\end{aligned}$$

So set $C = (2c/\underline{\delta}) + c \geq (2c\lambda_{1,F_J}/\underline{\delta}) + \|c\lambda_{F_J}\|$ to show $\Pr(\|\sqrt{J}(\theta_J^* - \underline{\theta}_{F_J})\| \leq C) \geq 1 - \rho$ for $J \geq \bar{J}$ and yield $\|\sqrt{J}(\theta_J^* - \underline{\theta}_{F_J})\| = O_p(1)$. □

**Proposition 4** *Let $\hat{\tau}_1 = \min\{\tau_1 : (\underline{D}^0)^{-1/2}\underline{\Gamma}^0\tau + \mathcal{Z} + \mu^0 \geq 0\}$, where $\mathcal{Z} \sim N(0, \underline{\Omega}^0)$ and $\mu^0 = \lim_{J\longrightarrow\infty}\sqrt{J}(\underline{D}_{F_J}^0)^{-1/2}\mathcal{P}_{F_J}m^0(\underline{\theta}_{F_J})$. Under Assumptions A1–A8, $\sqrt{J}(\theta_{1,J}^* - \underline{\theta}_{1,F_J}) \xrightarrow{d} \hat{\tau}_1$.*

Proof:

For $J$ large enough, $(\underline{D}_{F_J}^0)^{-1/2}\mathcal{P}_{F_J}m^0(\underline{\theta}_{F_J}) < \eta_\Gamma$ and $(\underline{D}_{F_J}^0)^{-1/2}\underline{\Gamma}_{F_J}^0\lambda_{F_J} > \varepsilon_\Gamma$ by Assumption A4. Since $\|\lambda_{F_J}\| = 1$, there exists a convergent subsequence $\lambda_{F_{J'}} \longrightarrow \underline{\lambda}$ with $\|\underline{\lambda}\| = 1$, $\underline{\lambda}_1 > 0$, and $(\underline{D}^0)^{-1/2}\underline{\Gamma}^0\underline{\lambda} > \varepsilon_\Gamma/2$. By continuity, $\sup_{\tau_1 \leq \underline{\delta}, \|\tau\|=1}\min_{j\in\{1,\ldots,k_a\}}\underline{D}_j^{-1/2}\underline{\Gamma}_j\tau <$

$-\varepsilon_\Gamma/2$. And, for $\|D^{-1/2}\Gamma - (\underline{D}^0)^{-1/2}\underline{\Gamma}^0\|$ small enough, $\sup_{\tau_1 \leq \underline{\delta}, \|\tau\|=1} \min_{j\in\{1,\dots,k_a\}} D_j^{-1/2}\Gamma_j\tau < 0$. Continuity of the optimal value of the linear program is then shown in Lemma A1. The result then follows by the definition of $\theta_1^*$, Lyapunov's CLT and Assumption A6, and the continuous mapping theorem. Note that the distribution of the solution to the linear program is the same if the equations are scaled by $(\underline{D}^0_{F_J})^{-1/2}$. □

**Proposition 5** *Under Assumptions A1-A8, for any $\epsilon > 0$, $\Pr_{F_J}(\|\sqrt{J}(\hat{\underline{\theta}}_{1,J} - \theta^*_{1,J})\| \geq \epsilon) \longrightarrow 0$.*

Proof:

Let $L_J(\theta) = (\underline{D}^0_{F_J})^{-1/2}\underline{\Gamma}^0_{F_J}\sqrt{J}(\theta - \underline{\theta}_{F_J}) + \sqrt{J}(\underline{D}^0_{F_J})^{-1/2}\mathbb{P}_J m^0(\underline{\theta}_{F_J})$. First, show there exists $h_J \downarrow 0$ such that $L_J(\hat{\underline{\theta}}_J + h_J\lambda_{F_J}/\sqrt{J}) \geq \sqrt{J}(\underline{D}^0_{F_J})^{-1/2}\mathbb{P}_J m^0(\hat{\underline{\theta}}_J)$ w.p.a. 1 and it will follow that $\theta^*_{1,J} \leq \hat{\underline{\theta}}_{1,J} + h_J\lambda_{1,F_J}/\sqrt{J}$ w.p.a. 1. Second, show there exists $r_J \downarrow 0$ such that $\sqrt{J}(\underline{D}^0_{F_J})^{-1/2}\mathbb{P}_J m^0(\theta^*_J + r_J\lambda_{F_J}/\sqrt{J}) \geq L_J(\theta^*_J)$ w.p.a. 1. Also, $\sqrt{J}(\hat{D}^1_J)^{-1/2}\mathbb{P}_J m^1(\theta^*_J + r_J\lambda_{F_J}/\sqrt{J}) > 0$ w.p.a. 1 by the $\sqrt{J}$-consistency of $\theta^*_J$. It will follow that $\hat{\underline{\theta}}_{1,J} \leq \theta^*_{1,J} + r_J\lambda_{1,F_J}/\sqrt{J}$ w.p.a. 1. Then, $-h_J\lambda_{1,F_J}/\sqrt{J} \leq \hat{\underline{\theta}}_{1,J} - \theta^*_{1,J} \leq r_J\lambda_{1,F_J}/\sqrt{J}$ w.p.a. 1, and the conclusion will follow.

Define

$$\begin{aligned}\varepsilon_J &= -\sqrt{J}(\underline{D}^0_{F_J})^{-1/2}[\mathcal{P}_{F_J}m^0(\hat{\underline{\theta}}_J) - \mathcal{P}_{F_J}m^0(\underline{\theta}_{F_J}) - \underline{\Gamma}^0_{F_J}(\hat{\underline{\theta}}_J - \underline{\theta}_{F_J})]\\ &\quad -\sqrt{J}(\underline{D}^0_{F_J})^{-1/2}[\mathbb{P}_J m^0(\hat{\underline{\theta}}_J) - \mathcal{P}_{F_J}m^0(\hat{\underline{\theta}}_J) - (\mathbb{P}_J m^0(\underline{\theta}_{F_J}) - \mathcal{P}m^0(\underline{\theta}_{F_J}))],\end{aligned}$$

so $L_J(\hat{\underline{\theta}}_J + \lambda_{F_J}h_J/\sqrt{J}) - \sqrt{J}(\underline{D}^0_{F_J})^{-1/2}\mathbb{P}_J m^0(\hat{\underline{\theta}}_J) = (\underline{D}^0_{F_J})^{-1/2}\underline{\Gamma}^0_{F_J}\lambda_{F_J}h_J + \varepsilon_J$.

By $\sqrt{J}$-consistency of $\hat{\underline{\theta}}_J$ and Assumptions A6, A7, and A8, there exists a sequence $\gamma_J \downarrow 0$ such that $\|\varepsilon_J\| \leq \gamma_J$ w.p.a. 1. Set $h_J = \gamma_J/\varepsilon_\Gamma$, then $L_J(\hat{\underline{\theta}}_J + h_J\lambda_{F_J}/\sqrt{J}) - \sqrt{J}(\underline{D}^0_{F_J})^{-1/2}\mathbb{P}_J m^0(\hat{\underline{\theta}}_J) \geq 0$. Using Assumption A6 and the result from the proof of consistency that $(\hat{D}^0_J)^{-1/2}\mathbb{P}_J m^0(\hat{\underline{\theta}}_J) \geq 0$ w.p.a. 1, we conclude that $L_J(\hat{\underline{\theta}}_J + h_J\lambda_{F_J}/\sqrt{J}) \geq 0$ w.p.a. 1. Hence, w.p.a. 1, $\theta^*_{1,J} = \inf_{\theta: L_J(\theta)\geq 0}\theta_1 \leq \hat{\underline{\theta}}_{1,J} + h_J\lambda_{1,F_J}/\sqrt{J}$.

By the $\sqrt{J}$-consistency of $\theta^*_J$, there exists $C_0$ such that

$$\sup_{\omega\in[0,1]} \sqrt{J}\left\|\theta^*_J + \frac{\omega\lambda_{F_J}}{\sqrt{J}} - \underline{\theta}_{F_J}\right\| \leq C_0$$

w.p.a. 1. By Assumptions A6 and A7, there is $\beta_J \downarrow 0$ and $C_\Gamma$ such that

$$\sup_{\|\theta - \underline{\theta}_{F_J}\| \leq C_0/\sqrt{J}} \underline{D}^{-1/2}_{F_J}\|\mathcal{P}_{F_J}m(\theta) - \mathcal{P}_{F_J}m(\underline{\theta}_{F_J}) - \underline{\Gamma}_{F_J}(\theta - \underline{\theta}_{F_J})\| \leq \frac{\beta_J}{\sqrt{J}}$$

and $(\underline{D}^1_{F_J})^{-1/2}\underline{\Gamma}^1_{F_J}\tau \geq -C_\Gamma l$ when $\|\tau\| \leq C_0$. By Assumptions A6 and A8, there is $\delta_J \downarrow 0$ such that

$$\sup_{\|\theta - \underline{\theta}_{F_J}\| \leq C_0/\sqrt{J}} \sqrt{J}\underline{D}^{-1/2}_{F_J}\|\mathbb{P}_J m(\theta) - \mathcal{P}_{F_J}m(\theta) - [\mathbb{P}_J m(\underline{\theta}_{F_J}) - \mathcal{P}_{F_J}m(\underline{\theta}_{F_J})]\| < \delta_J$$

w.p.a. 1. By Lyapunov's Central Limit Theorem, there is $C_1$ such that $\|(\underline{D}^1_{F_J})^{-1/2}\sqrt{J}[\mathbb{P}_J m^1(\underline{\theta}_{F_J}) - \mathcal{P}_{F_J} m^1(\underline{\theta}_{F_J})]\| \leq C_1$ w.p.a. 1.

Let $r_J = (\delta_J + \beta_J)/\varepsilon_\Gamma$. If $\sup_{\|\theta - \underline{\theta}_{F_J}\| \leq C_0/\sqrt{J}} \sqrt{J}\underline{D}^{-1/2}_{F_J}\|\mathbb{P}_J m(\theta) - \mathcal{P}_{F_J} m(\theta) - [\mathbb{P}_J m(\underline{\theta}_{F_J}) - \mathcal{P}_{F_J} m(\underline{\theta}_{F_J})]\| < \delta_J$ and $\|\theta^*_J + r_J\lambda_{F_J}/\sqrt{J} - \underline{\theta}_{F_J}\| \leq C_0/\sqrt{J}$, then $\sqrt{J}(\underline{D}^0_{F_J})^{-1/2}\mathbb{P}_J m^0(\theta^*_J + r_J\lambda_{F_J}/\sqrt{J}) - L_J(\theta^*_J) \geq (\underline{D}^0_{F_J})^{-1/2}\underline{\Gamma}^0_{F_J}\lambda_{F_J} r_J - (\beta_J + \delta_J)l \geq 0$. And if $\|(\underline{D}^1_{F_J})^{-1/2}\sqrt{J}[\mathbb{P}_J m^1(\underline{\theta}_{F_J}) - \mathcal{P}_{F_J} m^1(\underline{\theta}_{F_J})]\| \leq C_1$, then $\sqrt{J}(\underline{D}^1_{F_J})^{-1/2}\mathbb{P}_J m^1(\theta^*_J + r_J\lambda_{F_J}/\sqrt{J}) \geq \sqrt{J}(\underline{D}^1_{F_J})^{-1/2}\mathcal{P}_{F_J} m^1(\underline{\theta}_{F_J}) - (\beta_J + \delta_J + C_\Gamma + C_1)l$. For $J$ large enough $\sqrt{J}(\underline{D}^1_{F_J})^{-1/2}\mathcal{P}_{F_J} m^1(\underline{\theta}_{F_J}) - (\beta_J + \delta_J + C_\Gamma + C_1)l \geq 0$. Thus, by the above and Assumption A6, $\sqrt{J}\hat{D}^{-1/2}_J\mathbb{P}_J m(\theta^*_J + r_J\lambda_{F_J}/\sqrt{J}) \geq 0$ w.p.a. 1, and so $\hat{\underline{\theta}}_{1,J} \leq \theta^*_{1,J} + r_J\lambda_{1,F_J}/\sqrt{J}$ w.p.a. 1. □

**Proposition 6** *Suppose Assumptions A1-A8 hold and $\alpha \in (0,1)$. Also, for any $\delta > 0$, $\sup_F \Pr_F\left(\|(\hat{\Gamma}_{J,F}, \hat{\Sigma}_{J,F}) - (\underline{\Gamma}_{F_J}, \underline{\Sigma}_{F_J})\| \geq \delta\right) \longrightarrow 0$. Then,*

$$\liminf_{J\longrightarrow\infty} \Pr_{F_J}\left(\sqrt{J}(\hat{\underline{\theta}}_{1,J} - \underline{\theta}_{1,F_J}) \leq \underline{q}^*_{\alpha,J}\right) \geq \alpha. \tag{4}$$

Proof:

Let $Z^{0*} \sim N(0, \underline{\Omega}^0)$ and $\hat{Z}^* \sim N(0, \hat{\Omega}_J)$, and define the infeasible simulation estimators:
$\bar{\tau}_1 = \min\{\tau_1 : 0 \leq (\underline{D}_0)^{-1/2}\underline{\Gamma}^0\tau + Z^{0*}\}$
$\tilde{\tau}_1 = \min\{\tau_1 : 0 \leq (\hat{D}^0_J)^{-1/2}\hat{\Gamma}^0_J\tau + \hat{Z}^{0*} + (r_J(\hat{D}^0_J)^{-1/2}\mathbb{P}_J m^0(\hat{\underline{\theta}}_J))_+\}$.
Recall $\underline{\tau}^*_1 = \min\{\tau_1 : 0 \leq \hat{D}^{-1/2}_{s,J}\hat{\Gamma}_{s,J}\tau + \hat{Z}^*_s + (r_J\hat{D}^{-1/2}_{s,J}\mathbb{P}_J m_s(\hat{\underline{\theta}}_J))_+\}$. Let $q_\alpha$ denote the $\alpha^{th}$ quantile of the limit distribution $\hat{\tau}_1$. Let $\underline{q}^*_\alpha$, $\bar{q}_\alpha$, and $\tilde{q}_\alpha$ denote the $\alpha^{th}$ quantile of the $\underline{\tau}^*_1$, $\bar{\tau}_1$, and $\tilde{\tau}_1$ distributions. If the inequalities satisfying $\tilde{\tau}_1$ have no solution, a similar elimination of moments as used in the definition of $\underline{\tau}^*_1$ can be used.

Since $\mu^0 \geq 0$, $\bar{q}_{\alpha-\varepsilon} \geq q_{\alpha-\varepsilon}$. By Lemma A2, $\Pr_{F_J}(\sqrt{J}(\hat{\underline{\theta}}_{1,J} - \underline{\theta}_{1,F_J}) \leq q_{\alpha-\varepsilon}) \longrightarrow \alpha - \varepsilon$. So there exists $J_a$ such that for $J \geq J_a$, $\Pr_{F_J}(\sqrt{J}(\hat{\underline{\theta}}_{1,J} - \underline{\theta}_{1,F_J}) \leq q_{\alpha-\varepsilon}) \geq \alpha - 2\varepsilon$. Next, let $\eta = \bar{q}_\alpha - \bar{q}_{\alpha-\varepsilon} > 0$. Then, by Lemma A3, there exists $\delta > 0$ such that $|q^{\Gamma,\mu,\Sigma}_\alpha - q^{\underline{\Gamma}^0,0,\underline{\Sigma}^0}_\alpha| \leq \eta$ for $(\Gamma, \mu, \Sigma) \in \mathcal{N}^{\underline{\Gamma}^0,0,\underline{\Sigma}^0}_\delta$. For $J$ large enough, $\|(\underline{\Gamma}^0_{F_J}, \underline{\Sigma}^0_{F_J}) - (\underline{\Gamma}^0, \underline{\Sigma}^0)\| \leq \delta/3$ and $\|(\hat{\Gamma}^0_J, \hat{\Sigma}^0_J) - (\underline{\Gamma}^0_{F_J}, \underline{\Sigma}^0_{F_J})\| \leq \delta/3$ w.p.a. 1.

By Assumptions A6, A7, and A8 along with the $\sqrt{J}$-consistency of $\hat{\underline{\theta}}_J$, $\|\sqrt{J}\underline{D}^{-1/2}_{j,F_J}\mathbb{P}_J m_j(\hat{\underline{\theta}}_J) - \sqrt{J}\underline{D}^{-1/2}_{j,F_J}\mathcal{P}_{F_J} m_j(\underline{\theta}_{F_J})\|$ can be bounded with probability approaching one. Since $\sqrt{J}(\underline{D}^0_{F_J})^{-1/2}\mathcal{P}_{F_J} m^0(\underline{\theta}_{F_J})$ is bounded, $\|r_J(\hat{D}^0_J)^{-1/2}\mathbb{P}_J m^0(\hat{\underline{\theta}}_J)\| \leq \delta/3$ w.p.a. 1. It follows that $\bar{q}_{\alpha-\varepsilon} \leq \tilde{q}_{\alpha,J}$ w.p.a. 1.

Since $\sqrt{J}\underline{D}^{-1/2}_{j,F_J}\mathbb{P}_J m_j(\underline{\theta}_{F_J}) \longrightarrow \infty$ for each $j \geq k_0+1$, we have $\max_{j\in\{1,\ldots,k_0\}} \sqrt{J}\hat{D}^{-1/2}_{j,J}\mathbb{P}_J m_j(\hat{\underline{\theta}}_J) < \min_{l\in\{k_0+1,\ldots,k\}} \sqrt{J}\hat{D}^{-1/2}_{l,J}\mathbb{P}_J m_l(\hat{\underline{\theta}}_J)$ w.p.a. 1. This means that, in the definition of $\underline{\tau}^*_1$, the procedure to eliminate inequalities until a solution to the system of inequalities is found will eliminate the moments in $m^1$ first with probability approaching one. If inequalities corresponding to $m^0$ in the definition of $\tilde{\tau}_1$ have a solution, and the $m^0$ inequalities are contained in the $m_s$ inequalities in the definition of $\underline{\tau}^*_1$, then $\underline{q}^*_{\alpha,J} \geq \tilde{q}_{\alpha,J}$ since the possibly additional inequalities determining $\underline{\tau}_1$ can only increase the minimizing solution. So, $\tilde{q}_{\alpha,J} \leq \underline{q}^*_{\alpha,J}$ w.p.a. 1.

So, for $J$ large enough, we can ensure that $\Pr_{F_J}(\bar{q}_{\alpha-\varepsilon} \leq \underline{q}^*_{\alpha,J}) \geq 1-\varepsilon$. It follows that for $J$ large $\Pr_{F_J}(\sqrt{J}(\hat{\underline{\theta}}_{1,J} - \underline{\theta}_{1,F_J}) \leq \underline{q}^*_{\alpha,J}) \geq \alpha - 2\varepsilon$. Since this holds for any $\varepsilon > 0$, the conclusion follows. □

## Lemmas

Define

$$\tau_1^{lp}(\Gamma, Z) = \min_{\tau: 0 \leq \Gamma\tau + Z} \tau_1,$$

and use the following notation for a neighborhood, $\mathcal{N}_\delta^{\bar{A}} = \{A : \|A - \bar{A}\| \leq \delta\}$.

For each Lemma below, we make the following assumptions:
(a) Given $\underline{\Gamma}_0$, assume there exists $\lambda$ with $\|\lambda\| = 1$ and $\lambda_1 > 0$ such that $\underline{\Gamma}_0\lambda > 0$;
(b) Assume that for each $\Gamma$ in some open neighborhood of $\underline{\Gamma}_0$, the unique solution to $\min_{\tau:\Gamma\tau\geq 0} \tau_1$ is zero.

**Lemma A1** *Under the assumptions above, there exists $\eta > 0$ such that $\tau_1^{lp}(\Gamma, Z)$ is continuous at each $\Gamma \in \mathcal{N}_\eta^{\Gamma_0}$ and $Z \in \mathbb{R}^m$.*

Proof: Find $\eta > 0$ such that given $\bar{\Gamma} \in \mathcal{N}_\eta^{\Gamma_0}$, any $\bar{Z}$, and $\varepsilon > 0$, we can find a $\delta > 0$ such that $|\tau_1^{lp}(\Gamma, Z) - \tau_1^{lp}(\bar{\Gamma}, \bar{Z})| < \varepsilon$ for all $(\Gamma, Z) \in \mathcal{N}_\delta^{\bar{\Gamma},\bar{Z}}$. There exists $\delta_a > 0$ such that $\mathcal{N}_{\delta_a}^{\Gamma_0}$ is contained in the open neighborhood in the assumptions and $\Gamma\lambda > 0$ for all $\Gamma \in \mathcal{N}_{\delta_a}^{\Gamma_0}$. We note that by Kall (1970) Theorem 4, for all $\Gamma \in \mathcal{N}_{\delta_a}^{\Gamma_0}$ and any $Z$, $\tau_1^{lp}(\Gamma, Z)$ is well defined and a solution to the corresponding linear program exists.

Next show (uniform) boundedness of the solutions of the linear programs on some neighborhood contained in $\mathcal{N}_{\delta_a}^{\Gamma_0,\bar{Z}}$, following the approach in the proof of Bereanu (1976) Lemma 2.1. In fact, it will be useful (for other results in the paper) to show a stronger result. Take any constant $C_z > 0$ we will show uniform boundedness for $(\Gamma, Z) \in \mathcal{N}_\delta^{\Gamma_0} \times \{Z : \|Z\| \leq C_z\}$, for some $\delta \leq \delta_a$ (take $C_z$ large enough that $\|\bar{Z}\| + \delta_a \leq C_z$). If solutions are not bounded on any such set, then there exists a sequence $\delta_n \longrightarrow 0$ and $(\Gamma_n, Z_n) \in \mathcal{N}_{\delta_n}^{\Gamma_0} \times \{Z : \|Z\| \leq C_z\}$ such that for some solution $\tau_n^*$ to the linear program $\min\{\tau_1 : \Gamma_n\tau + Z_n \geq 0\}$, $\|\tau_n^*\| \longrightarrow \infty$. By the Duality Theorem (of Linear Programming), for any solution $\tau_n^*$ to the primal linear program, there exists a corresponding solution $\beta_n^*$ to the dual linear program. The compactness of $\{Z : \|Z\| \leq C_z\}$ implies that there exists a convergent subsequence $\{n'\}$ such that $Z_{n'} \longrightarrow \tilde{Z} \in \{Z : \|Z\| \leq C_z\}$. Let $A_n = [\Gamma_n \quad -\Gamma_n]$, $A_0 = [\underline{\Gamma}_0 \quad -\underline{\Gamma}_0]$, $e_1{}' = (1, 0, \ldots, 0)$, $c' = (e_1{}', -e_1{}')$, and

$$B_n = \begin{bmatrix} A_n & 0 \\ 0 & -A_n{}' \\ -c' & Z_n{}' \end{bmatrix}, \quad B_0 = \begin{bmatrix} A_0 & 0 \\ 0 & -A_0{}' \\ -c' & \tilde{Z}' \end{bmatrix} \quad d_n = \begin{pmatrix} -Z_n \\ -c \\ 0 \end{pmatrix}.$$

Then there is $u_n^* = (\tau_n^+, \tau_n^-, \beta_n^*) \geq 0$ such that $B_n u_n^* \geq d_n$ and $\tau_n^* = \tau_n^+ - \tau_n^-$. Also, $B_n \frac{u_n^*}{\|u_n^*\|} \geq \frac{1}{\|u_n^*\|} d_n$ (and $\|u_n^*\| \longrightarrow \infty$). Since $u_{n'}^*/\|u_{n'}^*\|$ is a sequence on the compact unit ball, it has a convergent subsequence $\{n''\}$ such that $u_{n''}^*/\|u_{n''}^*\| \longrightarrow u^* = (\tau^{*+}, \tau^{*-}, \beta^*)$, $\|u^*\| = 1$, $u^* \geq 0$ and $B_0 u^* \geq 0$. The last conclusion implies $\underline{\Gamma}_0' \beta^* = 0$. Note that if $\beta^* \neq 0$ (recalling $\beta^* \geq 0$), then $\beta^{*\prime} \underline{\Gamma}_0 \lambda > 0$ and hence $\beta^{*\prime} \underline{\Gamma}_0 \neq 0$. So, $\beta^* = 0$. So we have $\underline{\Gamma}_0 \tau^* \geq 0$ and $\tau_1^* \leq 0$ (with $\tau^* \neq 0$), which contradicts an assumption of the lemma. Hence, for some $\delta_b \leq \delta_a$, there exists $C$ such that for any solution, $\bar{\tau}$, of the linear program $\min\{\tau_1 : \Gamma\tau + Z \geq 0\}$ with $(\Gamma, Z) \in \mathcal{N}_{\delta_b}^{\Gamma_0} \times \{Z : \|Z\| \leq C_z\}$, $\|\bar{\tau}\| \leq C$.

Let $\gamma = \min_{\Gamma \in \mathcal{N}_{\delta_b}^{\Gamma_0}} \min_j [\Gamma\lambda]_j > 0$. Take any $\bar{\Gamma} \in \mathcal{N}_{\delta_b/2}^{\Gamma_0}$. Choose $\rho > 0$ small enough that $\rho\lambda_1 < \varepsilon$. And take $\delta > 0$ such that $\delta \leq \delta_b/2$ and $\|(\bar{\Gamma} - \Gamma)\tau\| + \|\bar{Z} - Z\| \leq \rho\gamma$ for all $\|\tau\| \leq C$ and $(\Gamma, Z) \in \mathcal{N}_\delta^{\bar{\Gamma}, \bar{Z}}$. Take any $(\Gamma, Z) \in \mathcal{N}_\delta^{\bar{\Gamma}, \bar{Z}}$ and let $\bar{\tau}$ be any solution to the linear program $\min\{\tau_1 : \Gamma\tau + Z \geq 0\}$. Then,

$$\bar{\Gamma}(\bar{\tau} + \rho\lambda) + \bar{Z} = (\bar{\Gamma} - \Gamma)\bar{\tau} + (\bar{Z} - Z) + \bar{\Gamma}\rho\lambda + (\Gamma\bar{\tau} + Z) \geq 0,$$

so $\tau_1^{lp}(\Gamma, Z) + \varepsilon \geq \tau_1^{lp}(\bar{\Gamma}, \bar{Z})$. Similarly, $\tau_1^{lp}(\bar{\Gamma}, \bar{Z}) + \varepsilon \geq \tau_1^{lp}(\Gamma, Z)$, and the result follows with $\eta = \delta_b/2$. □

**Lemma A2** *Let $\hat{\tau}_1 = \min\{\tau_1 : \underline{\Gamma}_0\tau + Z \geq 0\}$ where $Z \sim N(\mu, \Sigma)$. $\hat{\tau}_1$ has a continuous distribution (continuous c.d.f.). Also, $\mathbb{R}$ is the support of $\hat{\tau}_1$; in particular, for any $q \in \mathbb{R}$, $\Pr(\hat{\tau}_1 \leq q) \in (0, 1)$.*

Proof: Let $F$ denote the c.d.f. of $\hat{\tau}_1$. Suppose $F$ is discontinuous at some $t$. Let $\epsilon = \Pr(\hat{\tau}_1 = t)$. Then $\epsilon > 0$. Let $A_t = \{Z : \tau_1^{lp}(\underline{\Gamma}_0, Z) = t\}$. Then, $\epsilon = \Pr(Z \in A_t)$. For $c > 0$, define $E_c = \{Z : Z = Z' + \underline{\Gamma}_0 c\lambda \text{ for } Z' \in A_t\}$. For $c$ small enough, $\Pr(Z \in E_c) \geq \epsilon/2$.

Take any $Z \in A_t$. Let $\bar{\tau}$ be a solution to $\min\{\tau_1 : \underline{\Gamma}_0\tau + Z \geq 0\}$, so $t = \bar{\tau}_1$. Let $\bar{\bar{\tau}}$ be a solution to $\min\{\tau_1 : \underline{\Gamma}_0\tau + (Z + \underline{\Gamma}_0 c\lambda) \geq 0\}$. Since $0 \leq \underline{\Gamma}_0\bar{\tau} + Z = \underline{\Gamma}_0(\bar{\tau} - c\lambda) + (Z + \underline{\Gamma}_0 c\lambda)$, $\bar{\bar{\tau}}_1 \leq \bar{\tau}_1 - c\lambda_1$. Also, $0 \leq \underline{\Gamma}_0\bar{\bar{\tau}} + (Z + \underline{\Gamma}_0 c\lambda) = \underline{\Gamma}_0(\bar{\bar{\tau}} + c\lambda) + Z$, so $\bar{\tau}_1 \leq \bar{\bar{\tau}}_1 + c\lambda_1$. Hence, $\bar{\bar{\tau}}_1 = t - c\lambda_1$. So, $\Pr(\hat{\tau}_1 = t - c\lambda_1) \geq \Pr(Z \in E_c) \geq \epsilon/2$. We can pick an infinite number of small $c$'s yielding such mass points, which yields a contradiction. Hence, $F$ is continuous everywhere.

Now take $q \in \mathbb{R}$. Choose $c$ such that $c\lambda_1 = q$. Then,

$$0 < \Pr(Z \geq -\underline{\Gamma}_0 c\lambda) = \Pr(\underline{\Gamma}_0 c\lambda + Z \geq 0) \leq \Pr(\hat{\tau}_1 \leq c\lambda_1) = \Pr(\hat{\tau}_1 \leq q).$$

And, similarly, we can show $\Pr(q \leq \hat{\tau}_1) > 0$. □

Define $q_\alpha^{\Gamma,\mu,\Sigma} = \inf\{q : \Pr^*(\tau_1^{lp}(\Gamma, Z^*) \leq q) \geq \alpha\}$ where $Z^* \sim N(\mu, \Sigma)$.

**Lemma A3** *Given $\alpha \in (0, 1)$, $q_\alpha^{\Gamma,\mu,\Sigma}$ is continuous in $(\Gamma, \mu, \Sigma)$ at $(\underline{\Gamma}_0, \underline{\mu}, \underline{\Sigma}_0)$ for any finite $\underline{\mu}$.*

Proof: Given $\eta > 0$, show that there exists $\delta > 0$ such that $|q_\alpha^{\Gamma,\mu,\Sigma} - q_\alpha^{\underline{\Gamma}_0,\underline{\mu},\underline{\Sigma}_0}| \leq \eta$ for $(\Gamma, \mu, \Sigma) \in \mathcal{N}_\delta^{\underline{\Gamma}_0,\underline{\mu},\underline{\Sigma}_0}$.

Take $\eta_a > 0$ such that $\eta_a \leq \eta/2$, $\max\{q_{\alpha+2\eta_a}^{\Gamma_0,\underline{\mu},\underline{\Sigma}_0} - q_{\alpha}^{\Gamma_0,\underline{\mu},\underline{\Sigma}_0}, q_{\alpha}^{\Gamma_0,\underline{\mu},\underline{\Sigma}_0} - q_{\alpha-2\eta_a}^{\Gamma_0,\underline{\mu},\underline{\Sigma}_0}\} \leq \eta/2$, and $\alpha \in (2\eta_a, 1-2\eta_a)$. From Lemma A1, choose $\delta_a > 0$ such that $\tau_1^{lp}(\Gamma, Z)$ is continuous at each $\Gamma \in \mathcal{N}_{\delta_a}^{\Gamma_0}$ and all $Z$. Choose $C_z$ such that $\Pr^*(\|Z^*\| \leq C_z) \geq 1-\eta_a$ for all $(\mu, \Sigma) \in \mathcal{N}_{\delta_a}^{\underline{\mu},\underline{\Sigma}_0}$ and $Z^* \sim N(\mu, \Sigma)$. Then $\tau_1^{lp}$ is uniformly continuous in $\Gamma$ on $\mathcal{N}_{\delta_a}^{\Gamma_0} \times \{Z : \|Z\| \leq C_z\}$. Choose $0 < \delta_b \leq \delta_a$ such that $\|\tau_1^{lp}(\Gamma, Z) - \tau_1^{lp}(\underline{\Gamma}_0, Z)\| \leq \eta_a$ for all $\Gamma \in \mathcal{N}_{\delta_b}^{\Gamma_0}$ and all $\|Z\| \leq C_z$.

Define $\mathcal{A}_\beta = \{Z : \tau_1^{lp}(\underline{\Gamma}_0, Z) \leq q_\beta^{\Gamma_0,\underline{\mu},\underline{\Sigma}_0}\}$. Now take $0 < \delta \leq \delta_b$ such that $\max\{|\Pr^*(Z^* \in A_{\alpha+2\eta_a}) - (\alpha + 2\eta_a)|, |\Pr^*(Z^* \in A_{\alpha-2\eta_a}) - (\alpha - 2\eta_a)|\} \leq \eta_a$ for $Z^* \sim N(\mu, \Sigma)$, and all $(\mu, \Sigma) \in \mathcal{N}_\delta^{\underline{\mu},\underline{\Sigma}_0}$.

Then, for $Z^* \sim N(\mu, \Sigma)$ with $(\Gamma, \mu, \Sigma) \in \mathcal{N}_\delta^{\Gamma_0,\underline{\mu},\underline{\Sigma}_0}$,

$$\begin{aligned}
\Pr^*(\tau_1^{lp}(\Gamma, Z^*) \leq q_{\alpha+2\eta_a}^{\Gamma_0,\underline{\mu},\underline{\Sigma}_0} + \eta_a) &\geq \Pr^*(\{\tau_1^{lp}(\underline{\Gamma}_0, Z^*) \leq q_{\alpha+2\eta_a}^{\Gamma_0,\underline{\mu},\underline{\Sigma}_0}\} \cap \{\|Z^*\| \leq C_z\}) \\
&\geq \Pr^*(Z^* \in A_{\alpha+2\eta_a}) + \Pr^*(\|Z^*\| \leq C_z) - 1 \\
&\geq (\alpha + 2\eta_a) - \eta_a + (1 - \eta_a) - 1 = \alpha.
\end{aligned}$$

Similarly, $\Pr^*(\tau_1^{lp}(\Gamma, Z^*) > q_{\alpha-2\eta_a}^{\Gamma_0,\underline{\mu},\underline{\Sigma}_0} - \eta_a) > 1 - \alpha$, so $\Pr^*(\tau_1^{lp}(\Gamma, Z^*) \leq q_{\alpha-2\eta_a}^{\Gamma_0,\underline{\mu},\underline{\Sigma}_0} - \eta_a) \leq \alpha$. So, $q_\alpha^{\Gamma_0,\underline{\mu},\underline{\Sigma}_0} - \eta \leq q_{\alpha-2\eta_a}^{\Gamma_0,\underline{\mu},\underline{\Sigma}_0} - \eta_a \leq q_\alpha^{\Gamma,\mu,\Sigma} \leq q_{\alpha+2\eta_a}^{\Gamma_0,\underline{\mu},\underline{\Sigma}_0} + \eta_a \leq q_\alpha^{\Gamma_0,\underline{\mu},\underline{\Sigma}_0} + \eta$ by the definition of $\eta_a$. The result follows. □