Hodge Theory

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Contents

Chapter 1

Introduction

Our objective in this exposition is to state and prove the main theorems of Hodge theory.

In Chapter 2, we first describe a key motivation behind the Hodge theory for compact, closed, oriented Riemannian manifolds: the observation that the differential forms that satisfy certain partial differential equations depending on the choice of Riemannian metric (forms in the kernel of the associated Laplacian operator, or harmonic forms) turn out to be precisely the norm-minimizing representatives of the de Rham cohomology classes. This naturally leads to the statement our first main theorem, the Hodge decomposition—for a given compact, closed, oriented Riemannian manifold—of the space of smooth k-forms into the image of the Laplacian and its kernel, the subspace of harmonic forms. We then develop the analytic machinery—specifically, Sobolev spaces and the theory of elliptic differential operators—that we use to prove the aforementioned decomposition, which immediately yields as a corollary the phenomenon of Poincaré duality. We have consulted the exposition [1, §1] based on [2] and the exposition [4] based on [6] for the material in this chapter.

In Chapter 3, we appeal to the analytic machinery developed in the previous chapter to prove the Hodge decomposition for compact, closed Kähler manifolds, a canonical decomposition of each de Rham cohomology space with complex coefficients into Dolbeault cohomology spaces. We then conclude the exposition by showing that Hodge theory can be used to give elegant proofs of the Lefschetz decomposition of de Rham cohomology spaces into primitive components, the hard Lefschetz theorem and the Hodge index theorem. We have consulted [3] and the exposition [1, §4] based on [5] for the material in this chapter.

Chapter 2

Hodge Theory of Compact Oriented Riemannian Manifolds

2.1 Hodge star operator

Let (M, g) be a Riemannian n-manifold. We can consider g as an element of $TM^* \otimes TM^*$, and in particular, as a canonical bundle isomorphism $TM \to TM^*$ by evaluating one of the tensor factors of g pointwise at a given tangent vector. Thus, g defines a canonical metric g^* on TM^*. Furthermore, consider the canonical inner product $(g^*)^{\otimes p}$ on $(TM^*)^{\otimes p}$. Note that when viewed as a fiberwise inner product, $(g^*)^{\otimes p}$ is invariant under the natural group action of the symmetric group S_p on the fiber V_x over an arbitrary point $x \in M$. So, $(g^*)^{\otimes p}$ descends to a metric on the S_p-subrepresentation $\bigwedge^\bullet V_x$. It is clear that the above construction applied fiberwise over every $x \in M$ gives a functorial bilinear bundle morphism

$$g^* : C^\infty(M, \Omega^p) \times C^\infty(M, \Omega^p) \to C^\infty(M, \mathbb{R}), \tag{2.1}$$

where we have reused notation and will use g^* to denote the above map for varying values of p.

Next, we define an operator that is, as its name will suggest, central to Hodge theory. Let V be an n-dimensional Euclidean space with a choice of orientation. With the Euclidean inner product, V has a canonical volume form $\mathrm{vol} \in \bigwedge^n V$. Then, since the exterior product

$\wedge : \bigwedge^p V \times \bigwedge^{n-p} V \to \bigwedge^n V$ is a nondegenerate pairing, we can define the Hodge star operator $* : \bigwedge^p V \to \bigwedge^{n-p} V$ by the requirement that $\beta \in \bigwedge^p V$ is mapped to the unique form $*\beta$ satisfying

$$\alpha \wedge *\beta = g^*(\alpha, \beta) \operatorname{vol}$$

for all $\alpha \in \bigwedge^p V$. Similarly to before, we will abuse notation to denote by $*$ Hodge star operators for different values of p.

Take a positively oriented orthonormal basis $\{e_1, \ldots, e_n\}$ of V. Consider the associated 1-forms $\{de_1, \ldots, de_n\}$ and the forms $\{de^I\}_{I \subset \{1,\ldots,n\}}$ defined by

$$de^I := de_{i_1} \wedge \cdots de_{i_{|I|}}, \tag{2.2}$$

where $i_1, \ldots, i_{|I|}$ are the indices of the elements of I in increasing order. We can check that

$$e^I = \epsilon(I) de^{I^{\mathsf{c}}}, \tag{2.3}$$

where $\epsilon(I)$ denotes the signature of the permutation given by the elements of I in increasing order of index followed by the elements of its complement, I^{c}, in increasing order of index. In particular, we have $* \operatorname{vol} = 1$, $*1 = \operatorname{vol}$, and

$$*e_m = (-1)^{m-1} e_1 \wedge \cdots \wedge \widehat{e_m} \wedge \cdots \wedge e^n \qquad \text{for } 1 \leq m \leq n,$$

where the hat denotes omission. It is also evident that $** : \bigwedge^p V \to \bigwedge^p V$ acts as multiplication by $(-1)^{n(n-p)}$.

Now, suppose M is closed, compact, orientable, and fix an orientation. We can define an inner product on the sections of $\Omega^p M$ by

$$\langle \omega, \eta \rangle := \int_M \overline{g^*}(\omega, \eta) \operatorname{vol}^g = \int_M \omega \wedge *\eta, \tag{2.4}$$

where vol^g is the canonical volume form associated to g and our choice of orientation. For smooth differential forms $\omega \in C^\infty(M, \Omega^p)$ and $\eta \in C^\infty(M, \Omega^{p+1})$, we have

$$\langle d\omega, \eta \rangle = \int_M d\omega \wedge *\eta = (-1)^{p-1} \int_M \omega \wedge d(*\eta) = (-1)^{(p-1)+n(n-p)} \int_M \omega \wedge *(*d*)(\eta),$$

where we have used Stokes' theorem. This demonstrates that the formal adjoint to the exterior derivative d is given by

$$d^* := (-1)^{n(p+1)+1} * d* : \Omega^p \to \Omega^{p-1}. \tag{2.5}$$

Example 2.1. Fix a local orthonormal basis $\partial_1, \dots, \partial_n$ of TM at a point $x \in M$, and let $de_1, \dots, de_n$ denote the corresponding local dual basis of TM^*. Let $\omega = \sum_{i=1}^n f_i de_i$ denote a smooth 1-form on the local trivialization. Note then that we have

$$\begin{aligned} d^*\omega &= (-1)^{n(1+1)+1} * d * \omega \\ &= (-1) * d \sum_{i=1}^n (-1)^{i-1} f_i de_1 \wedge \cdots \wedge \widehat{de_i} \wedge \cdots \wedge de_n \\ &= (-1) * \sum_{i=1}^n \partial_i f_i de_1 \wedge \cdots \wedge de_n \\ &= -\sum_{i=1}^n \partial_i f_i. \end{aligned}$$

A key motivation behind Hodge theory is to find harmonic representatives of de Rham cohomology classes. Specifically, let $L^2(M, \Omega^p M)$ denote the Hilbert space arising from completing $C^\infty(M, \Omega^p)$, and consider the de Rham cohomology class $[\omega]$ of a given closed p-form ω. Then, its closure, given by

$$\overline{[\omega]} = \{\omega + \eta : \eta \in \overline{d(C^\infty(M, \Omega^{p+1})},$$

is a closed affine subspace of the Hilbert space $L^2(M, \Omega^p)$, and thus has a unique element ω_0 of minimal norm, which can equivalently be described as the condition of being perpendicular to the subspace. In other words, ω_0 is the unique elements of $\overline{[\omega]}$ that satisfies $\langle \omega_0, d\eta \rangle = 0$ for all exact p-forms $d\eta$. Note that this can be rewritten in terms of the formal adjoint of d, yielding the condition that $\langle d^*\omega_0, \eta\rangle = 0$ for all exact p-forms $d\eta$. Thus, we have that ω_0 is a solution to the differential equations $d\omega_0 = 0$ and $d^*\omega_0 = 0$.

Note that the above discussion also holds analogously when we switch the roles of d and d^*, in that the closure of a cohomology class defined with respect to d^* rather than d contains a unique element ω_0 of minimal norm that satisfies $d^*\omega_0 = 0$ and $d\omega_0 = 0$. Moreover, since d and d^* are formal adjoints, we have $\operatorname{im} d^* \perp \ker d$ and $\operatorname{im} d \perp \ker d^*$. These facts suggest an orthogonal decomposition of the form

$$C^\infty(M, \Omega^p) = (\ker d \cap \ker d^*) \oplus d(C^\infty(M, \Omega^{p-1})) \oplus d^*(C^\infty(M, \Omega^{p+1})). \tag{2.6}$$

Note that $(\ker d \cap \ker d^*) \oplus d(C^\infty(M, \Omega^{p-1}))$ comprise the closed forms, where each de Rham cohomology class is represented by a unique representative in $\ker d \cap \ker d^*$. Similarly, $(\ker d \cap \ker d^*) \oplus d^*(C^\infty(M, \Omega^{p+1}))$ comprise the coclosed forms, and each d^*-de Rham cohomology class (defined with respect to d^* rather than d, and denoted by $H^p_{d^*,dR}(M)$) is represented by a unique representative in $\ker d \cap \ker d^*$.

2.2 The main theorem

The composition $dd^* : C^\infty(M, \Omega^p) \to C^\infty(M, \Omega^p)$ sends all summands other than $d(C^\infty(M, \Omega^{p-1}))$ to 0. Consider dd^* as a map on this summand. Note that $d : C^\infty(M, \Omega^p) \to d(C^\infty(M, \Omega^p))$ is surjective, which shows that its formal adjoint $d^* : d(C^\infty(M, \Omega^p)) \to C^\infty(M, \Omega^p)$ is injective. Likewise, analogous discussion shows that $d : d^*(C^\infty(M, \Omega^p)) \to C^\infty(M, \Omega^p)$ is injective. Thus, the composition

$$d(C^\infty(M, \Omega^{p-1})) \xrightarrow{d^*} d^*(C^\infty(M, \Omega^p)) \xrightarrow{d} d(C^\infty(M, \Omega^{p-1}))$$

is injective. An analogous discussion holds for the composition d^*d, which sends all direct summands of $C^\infty(M, \Omega^p)$ other than $d^*(C^\infty(M, \Omega^{p+1}))$ to 0.

Define the *Hodge–de Rham Laplacian on p-forms* by

$$\Delta := dd^* + d^*d : C^\infty(M, \Omega^p) \to C^\infty(M, \Omega^p),$$

again abusing notation by using Δ for different values of p. We can use our computations in Example 2.1 (whose notation we retain) to verify that as operators on smooth 0-forms, Δ coincides with the standard Laplacian on a local trivialization. First, note that d^* as an operator on $C^\infty(M, \Omega^0)$ is trivial, so $\Delta = d^*d$ in our case. Hence, for a smooth function f,

$$\Delta f = d^* df = d^* \sum_{i=1}^{n} \partial_i f de_i = -\sum_{i=1}^{n} \partial_i \partial_i f,$$

as expected. In fact, one can similarly compute that for a general smooth p-form

$$\omega = \sum_{1 \le j_1, \ldots, j_p \le n} f_{j_1, \ldots, j_p} de_{j_1} \wedge \cdots de_{j_p},$$

we have

$$\Delta \omega = -\sum_{k=1}^{n} \sum_{1 \le j_1, \ldots, j_p \le n} \partial_{x_k}^2 f_{j_1, \ldots, j_p} de_{j_1} \wedge \cdots de_{j_p}.$$

In other words, the Hodge–de Rham Laplacian on p-forms acts on ω by applying the usual Laplacian (the Hodge–de Rham Laplacian on 0-forms) on the coefficients with respect to the standard basis $de_{j_1} \wedge \cdots de_{j_p}$.

By our work in Section 2.1, we know that on $d(C^\infty(M, \Omega^{p-1}))$, Δ acts like the injective map dd^*, and on $d^*(C^\infty(M, \Omega^{p+1}))$, Δ acts like the injective map d^*d. Finally, Δ sends the remaining summand, $\ker d \cap \ker d^*$, to 0. Define a *harmonic form* to be a C^∞ form in the kernel of Δ. We see that $\omega \in C^\infty(M, \Omega^p)$ is harmonic if and only if $\omega \in \ker d \cap \ker d^*$. In particular, we see that the only harmonic 0-forms are the constant ones. Denote the subspace $\ker \Delta \subset C^\infty(M, \Omega^p)$ of harmonic p-forms by $\mathcal{H}^p(M)$, for which we have

$$\mathcal{H}^p(M) = \ker d \cap \ker d^*, \tag{2.7}$$

as shown above.

The following is the main theorem of this chapter, and fulfills our original motivation of finding harmonic representatives of de Rham cohomology classes.

Theorem 2.2 (Hodge)**.** *Let (M, g) be a compact, closed, orientable Riemannian manifold with a choice of orientation. The following are true:*

(i) $\mathcal{H}^p(M)$ *is finite-dimensional.*

(ii) We have an L^2 *orthogonal decomposition given by*

$$C^\infty(M, \Omega^p) = \mathcal{H}^p(M) \oplus \Delta(C^\infty(M, \Omega^p)). \tag{2.8}$$

Note that the above decomposition is equivalent to (2.6), with the identifications

$$\ker d = \mathcal{H}^p(M) \oplus d(C^\infty(M, \Omega^{p-1})), \tag{2.9}$$

$$\ker d^* = \mathcal{H}^p(M) \oplus d^*(C^\infty(M, \Omega^{p+1})). \tag{2.10}$$

In light of (2.9) and (2.10), we immediately have the following corollary.

Corollary 2.3. *Retain the hypotheses of Theorem 2.2. The natural maps (given by sending a harmonic p-form to its cohomology class, whose unique element of minimal norm is the original harmonic p-form)* $\mathcal{H}^p(M) \to H^p_{dR}(M)$ *and* $\mathcal{H}^p(M) \to H^p_{d^*,dR}(M)$ *are isomorphisms. In particular, the dimensions of these vector spaces are finite and do not depend on the choice of Riemannian metric.*

Moreover, the straightforward observation that Δ commutes with $*$ yields the following:

Corollary 2.4 (Poincaré duality)**.** *Retain the hypotheses of Theorem 2.2. The Hodge star operator induces an isomorphism* $\mathcal{H}^p(M) \to \mathcal{H}^{n-p}(M)$.

In combination with Corollary 2.3, we obtain the well-known Poincaré duality isomorphism $H^p_{dR}(M) \to H^{n-p}_{dR}(M)$.

We will later see that proving Theorem 2.2 reduces to establishing the ellipticity of the differential operator Δ, a property we will define and investigate in Section 2.4.

2.3 Sobolev spaces

Let $\mathbb{T}^n$ denote the n-torus $(\mathbb{R}/2\pi\mathbb{Z})^n$, and let $L^2(\mathbb{T}^n, \mathbb{C}^m)$ denote the space of square-integrable functions $\mathbb{T}^n \to \mathbb{C}^m$. We introduce for $s \in \mathbb{R}$ the *Sobolev space* $W^s(\mathbb{T}^n, \mathbb{C}^m)$, defined as the Hilbert space given by the completion of $L^2(\mathbb{T}^n, \mathbb{C}^m)$ with respect to the inner product

$$\langle f, g\rangle_s := \left(\sum_{\xi\in\mathbb{Z}^n} (1+|\xi|^2)^s \overline{\hat{f}(\xi)}\hat{g}(\xi)\right)^{\frac{1}{2}} \tag{2.11}$$

and the corresponding norm $\|\cdot\|_s$. We note that since the subspace $C^\infty(\mathbb{T}^n, \mathbb{C}^m) \subset L^2(\mathbb{T}^n, \mathbb{C}^m)$ of smooth functions $\mathbb{T}^n \to \mathbb{C}^m$ is dense in $L^2(\mathbb{T}^n, \mathbb{C}^m)$, we can equivalently define $W^s(\mathbb{T}^n, \mathbb{C}^m)$ as the completion of $C^\infty(\mathbb{T}^n)$ rather than of $L^2(\mathbb{T}^n, \mathbb{C}^m)$.

One can ask for a more concrete description of the elements of $W^s(\mathbb{T}^n, \mathbb{C}^m)$. This can be obtained by observing in the frequency domain. Let $\ell^2_{\mathbb{C}^m}(\mathbb{Z}^n)$ denote the space of $\mathbb{C}^m$-valued square-summable sequences indexed by $\mathbb{Z}^n$. By the duality between $L^2(\mathbb{T}^n)$ and $\ell^2_{\mathbb{C}^m}(\mathbb{Z}^n)$ given by the Fourier transform, any element of $L^2(\mathbb{T}^n, \mathbb{C}^m)$ can be viewed as an element of $\ell^2_{\mathbb{C}^m}(\mathbb{Z}^n)$ without loss of information, and vice versa. Then, it is evident that the element of $L^2(\mathbb{T}^n, \mathbb{C}^m)$ corresponding to a given $\sigma = (\sigma_\xi)_{\xi\in\mathbb{Z}^n} \in \ell^2(\mathbb{Z}^n)$ is contained in $W^s(\mathbb{T}^n)$ if and only if

$$\sum_{\xi\in\mathbb{Z}^n} (1+|\xi^2|)^s |\sigma_\xi|^2 < \infty.$$

Throughout this exposition, we will abuse notation by abbreviating the notation for the Sobolev space $W^s(\mathbb{T}^n, \mathbb{C}^m)$ to W^s for a fixed m (we note that it is instructive to first consider the case $m = 1$ in order to conceptually understand the general case), as well as considering elements of W^s to be $\mathbb{C}^m$-valued square-summable sequences indexed by $\mathbb{Z}^n$, when convenient.

The fact that (2.11) is in fact an inner product is a special case of the following consequence of Cauchy–Schwartz:

$$\left|\sum_{\xi\in\mathbb{Z}^n} (1+|\xi|^2)^{\frac{s+t}{2}} \sigma\cdot\tau\right|^2 \le \left(\sum_{\xi\in\mathbb{Z}^n} (1+|\xi|^2)^s |\sigma_\xi|^2\right)^2 \left(\sum_{\xi\in\mathbb{Z}^n} (1+|\xi|^2)^t |\tau_\xi|^2\right)^2,$$

where $\cdot$ denotes the Hermitian inner product. The above can be restated as

$$|\langle\sigma,\tau\rangle_s| \le \|\sigma\|_{s+t}\|\tau\|_{s-t}.$$

Moreover, note that the definition (2.11) of $\langle \sigma, \tau \rangle_s$ is also well-defined for $\sigma \in W^t$ and $\tau \in W^u$ such that $s = (t+u)/2$.

A key fact about Sobolev spaces that if $\sigma \in W^s$ for sufficiently large s, then it in fact corresponds to a function whose derivatives up to some order exist, and this order directly correlates with the size of s. Often called Sobolev's lemma, this fact plays a crucial role in the phenomenon of elliptic regularity, in which for a generalized solution σ of a partial differential equation of a certain form, if σ belongs to W^s for sufficiently large s, then σ corresponds to an actual solution. Sobolev's lemma has the following base case.

Lemma 2.5 (Sobolev)**.** *Suppose $s > n/2$. Then, we have a continuous inclusion $W^s \hookrightarrow C^0(\mathbb{T}^n, \mathbb{C}^m)$.*

Proof. First, we need to show that $\sigma \in W^s$ actually corresponds to a function, or in other words, that

$$\sum_{\xi \in \mathbb{Z}^n} \sigma_\xi e^{ix\cdot\xi}$$

is uniformly convergent. It suffices to show absolute convergence, i.e., show that the sum

$$\sum_{\xi \in \mathbb{Z}^n} |\sigma_\xi|$$

is convergent. Let N be a positive integer. By applying Cauchy–Schwartz, we verify that

$$\begin{aligned} \sum_{|\xi|\le N} |\sigma_\xi| &= \sum_{|\xi|\le N} (1+|\xi|^2)^{-\frac{s}{2}}(1+|\xi|^2)^{\frac{s}{2}}|\sigma_\xi| \\ &\le \sum_{|\xi|\le N} \frac{1}{(1+|\xi|^2)^s} \sum_{|\xi|\le N} (1+|\xi|^2)^s |\sigma_\xi|^2 \\ &\le \left(\sum_{\xi\in\mathbb{Z}^n} \frac{1}{(1+|\xi|^2)^s} \right) \|\sigma\|_s^2. \end{aligned}$$

However, our hypothesis implies that

$$c := \sum_{\xi\in\mathbb{Z}^n} \frac{1}{(1+|\xi|^2)^s} < \infty,$$

which shows our desired claim, and thus demonstrates that we have an inclusion $W^s \hookrightarrow C^0(\mathbb{T}^n)$. In fact, the above constant c only depends on s, so the inclusion is continuous. □

A similar proof will show that if $s > k + n/2$, then we have a continuous inclusion $W^s \hookrightarrow C^k(\mathbb{T}^n, \mathbb{C}^m)$. Beforehand, we first define some terminologies. Define a *differential operator of order ℓ (on $\mathbb{C}^m$-valued C^∞ functions of $\mathbb{T}^n$)* to be a linear operator $L : C^\infty(\mathbb{T}^n, \mathbb{C}^m) \to C^\infty(\mathbb{T}^n, \mathbb{C}^m)$ that, with respect to the standard basis, is an $m \times m$ matrix L of the form

$$L_{ij} = \sum_{[\alpha] \leq \ell} a_{ij}^\alpha D^\alpha, \tag{2.12}$$

where $a_{ij}^\alpha \in C^\infty(\mathbb{T}^n, \mathbb{C})$ with at least one a_{ij}^α not identically zero for some i, j and α with $[\alpha] = \ell$, and

$$D^\alpha := (-i)^{[\alpha]} \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}. \tag{2.13}$$

As a matter of convention, we appended the factor of $(-i)^{[\alpha]}$ in (2.13) in order to make sure that the Fourier transform $\widehat{D^\alpha f}$ does not have the factor of $i^{[\alpha]}$, since

$$\widehat{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} f} = i^{[\alpha]} \hat{f}.$$

Note that for $x \in \mathbb{T}^n$, the $m \times m$ matrix $a^\alpha(x)$ whose entries are given by $a_{ij}^\alpha(x)$ represents an element of $\mathrm{End}(C^\infty(\mathbb{T}^n, \mathbb{C}^m)_x)$. Furthermore, since $C^\infty(\mathbb{T}^n, \mathbb{C}^m)$ is dense in any Sobolev space W^s, differential operators can be extended to W^s, and in this exposition, we will consider differential operators to be these extended operators when appropriate. Note that while we use the standard basis in the above definition, this is equivalent to the general definition of a differential operator on arbitrary bundles over a general M; the general definition necessitates that the operator has entries of the form described in (2.12) for any local trivialization. Note that L acts on $f = (f_1, \ldots, f_n) \in C^\infty(\mathbb{T}^n, \mathbb{C}^m)$ by

$$Lf = \left(\sum_{j=1}^m L_{1j} f_j, \ldots, \sum_{j=1}^m L_{mj} f_j \right),$$

and the formal adjoint L^* of L is given by

$$L_{ij}^* = \sum_{[\alpha] \leq \ell} D^\alpha a_{ji}^\alpha.$$

For a vector $\alpha = (\alpha_1, \dots, \alpha_n)$ with nonnegative integer entries, define

$$[\alpha] := \alpha_1 + \cdots + \alpha_n$$

and

$$\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$$

for $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{Z}^n$, with the convention $0^0 = 1$. Abusing notation, define the formal differentiation operator $D^\alpha : W^s \to W^{s-[\alpha]}$ by

$$(D^\alpha(\sigma))_\xi := (\xi)^\alpha \sigma_\xi.$$

This operator was defined so that it corresponds to D^α defined in (2.13) when acting on functions $f \in C^\infty(\mathbb{T}^n, \mathbb{C}^m)$. This operator allows us to define a useful norm equivalent to $\|\cdot\|_s$ on $f \in C^\infty(\mathbb{T}^n, \mathbb{C}^m)$ in terms of the L^2 norms of the order $\leq s$ partial derivatives of f. Note that by Parseval's identity, we have

$$\|\widehat{D^\alpha} f(\xi)\|_{L^2} = \left(\sum_{\xi \in \mathbb{Z}^n} \xi^{2\alpha} |\hat{f}(\xi)|^2 \right)^{\frac{1}{2}}.$$

We claim that the norm

$$f \mapsto \sum_{[\alpha] \leq s} \|\widehat{D^\alpha} f(\xi)\|_{L^2} \tag{2.14}$$

is equivalent to $\|\cdot\|_s$. In fact, for nonnegative real numbers $a_1, \dots, a_k$, we have

$$\frac{1}{k} \left(\sum_{j=1}^k a_j \right)^2 \leq \sum_{j=1}^k a_j^2 \leq \left(\sum_{j=1}^k a_j \right)^2,$$

so it is equivalent to show that the norm

$$f \mapsto \sum_{[\alpha] \leq s} \sum_{\xi \in \mathbb{Z}^n} |\widehat{D^\alpha} f(\xi)|^2$$

is equivalent to $\|\cdot\|_s$. Indeed,

$$\sum_{\xi\in\mathbb{Z}^n}\sum_{[\alpha]\leq s}|\widehat{D^\alpha f}(\xi)|^2 = \sum_{\xi\in\mathbb{Z}^n}|\hat{f}(\xi)|^2\sum_{[\alpha]\leq s}\xi^{2\alpha} < \sum_{\xi\in\mathbb{Z}^n}|\hat{f}(\xi)|^2(1+(|\xi_1|^2+\cdots+|\xi_n|^2))^s = \|f\|_s$$

and there exists a constant $c > 0$ depending only on n and s such that

$$\|f\|_s = \sum_{\xi\in\mathbb{Z}^n}|\hat{f}(\xi)|^2(1+|\xi|^2)^s < \sum_{\xi\in\mathbb{Z}^n}|\hat{f}(\xi)|^2\left(c\sum_{[\alpha]\leq s}\xi^{2\alpha}\right) = c\sum_{\xi\in\mathbb{Z}^n}\sum_{[\alpha]\leq s}|\widehat{D^\alpha f}(\xi)|^2,$$

which shows our claim. This gives us the description of $\|\cdot\|_s$ as the sum of the order $\leq s$ partial derivatives' L^2 norms.

We can show the following result about the effect of D^α on the Sobolev norm.

Lemma 2.6. *Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a vector with nonnegative integer entries. We have*

$$\|D^\alpha\|_{s-[\alpha]} \leq \|\sigma\|_s$$

for all $\sigma \in W^s$.

Proof. We check that

$$|(D^\alpha\sigma)_\xi|^2 = |\xi^\alpha\sigma_\xi|^2 \leq (1+|\xi|^2)^{[\alpha]}|\sigma_\xi|^2$$

and sum over all $\xi \in \mathbb{Z}^n$. □

This immediately allows to prove the aforementioned generalization of Sobolev's lemma.

Lemma 2.7 (Sobolev)**.** *Suppose $s > k + n/2$. Then, we have a continuous inclusion $W^s \hookrightarrow C^k(\mathbb{T}^n, \mathbb{C}^m)$.*

In addition, it is clear that for $s < t$, we have that $W^t \subset W^s$ and that the inclusion $W^t \hookrightarrow W^s$ is continuous. The following result, often called the Rellich Compactness Lemma, shows that this inclusion is also compact.

Lemma 2.8 (Rellich)**.** *Suppose $s < t$, and consider a sequence $(\sigma_j)_{j\in\mathbb{Z}_{\geq 0}}$ of elements in W^t with $\|\sigma_j\|_t \leq 1$. Then, $(\sigma_j)_{j\in\mathbb{Z}_{\geq 0}}$ has a subsequence that is convergent in W^s.*

Proof. For any fixed $\xi \in \mathbb{Z}^n$, the sequence $\big((1+|\xi|^2)^{s/2}(\sigma_j)_\xi\big)_{j\in\mathbb{Z}_{\geq 0}}$ is bounded and has a convergent subsequence. Thus, by a diagonalization argument, we can obtain a subsequence $(\sigma_{j_k})_{k\in\mathbb{Z}_{\geq 0}}$ such that for every $\xi \in \mathbb{Z}^n$, the sequence

$$\Big((1+|\xi|^2)^{s/2}(\sigma_{j_k})_\xi\Big)_{k\in\mathbb{Z}_{\geq 0}}$$

is convergent. We now show that $(\sigma_{j_k})_{k\in\mathbb{Z}_{\geq 0}}$ is a Cauchy sequence, and thus convergent in the complete space W^s. Let N be a positive integer to be specified later, and split the following sum accordingly:

$$\begin{aligned}
&\sum_{\xi\in\mathbb{Z}^n}(1+|\xi|^2)^s|(\sigma_{j_{k_1}})_\xi-(\sigma_{j_{k_2}})_\xi|^2\\
&=\sum_{|\xi|\leq N}(1+|\xi|^2)^s|(\sigma_{j_{k_1}})_\xi-(\sigma_{j_{k_2}})_\xi|^2+\sum_{|\xi|>N}(1+|\xi|^2)^s|(\sigma_{j_{k_1}})_\xi-(\sigma_{j_{k_2}})_\xi|^2\\
&\leq\sum_{|\xi|\leq N}(1+|\xi|^2)^s|(\sigma_{j_{k_1}})_\xi-(\sigma_{j_{k_2}})_\xi|^2\\
&\quad+\frac{1}{(1+N^2)^{t-s}}\sum_{|\xi|>N}(1+|\xi|^2)^t\left(|(\sigma_{j_{k_1}})_\xi|^2+2|(\sigma_{j_{k_1}})_\xi||(\sigma_{j_{k_2}})_\xi|+|(\sigma_{j_{k_2}})_\xi|^2\right)\\
&\leq\sum_{|\xi|\leq N}(1+|\xi|^2)^s|(\sigma_{j_{k_1}})_\xi-(\sigma_{j_{k_2}})_\xi|^2+\frac{4}{(1+N^2)^{t-s}},
\end{aligned}$$

where we have used the hypothesis that $\|\sigma_j\|_s \leq 1$ for all j. The last term $4/(1+N^2)^{t-s}$ can be made arbitrarily small by taking N to be sufficiently large, so $(\sigma_{j_k})_{k\in\mathbb{Z}_{\geq 0}}$ is indeed a Cauchy sequence in W^s. □

We will also need the following:

Proposition 2.9 (Peter–Paul inequality)**.** *Suppose $s < t < u$. For any $\varepsilon > 0$, there exists a constant c such that*

$$\|\sigma\|_t^2 < \varepsilon\|\sigma\|_u^2 + c\|\sigma\|_s^2$$

Proof. For sufficiently large N, we have

$$(1+|\xi|^2)^t < \varepsilon(1+|\xi|^2)^u$$

for all $|\xi| > N$. Then, c can be made sufficiently large so that $c\|\sigma\|_s^2$ is at least as large as the remaining terms corresponding to $|\xi| \le N$. □

Next, we define a useful operator that allows us to transmit information from one Sobolev space to another. Define the operator $K_t : W^s \to W^{s-2t}$ by

$$(K_t(\sigma))_\xi = (1 + |\xi|^2)^t \sigma_\xi$$

for $\sigma \in W^s$. Note that K_t is an isometry, and that

$$\langle \sigma, \tau \rangle_s = \langle \sigma, K_t \tau \rangle_{s-t}$$

It is straightforward to verify that if t is a nonnegative integer, then K_t corresponds to the differential operator of order $2t$ given by

$$\left(1 - \sum_{j=1}^{n} \partial_{x_i}^2\right)^t$$

when acting on functions in $C^\infty(\mathbb{T}^n, \mathbb{C}^m)$, and also that

$$\langle \sigma, \tau \rangle_s = \langle \sigma, K_t \tau \rangle_{s-t} = \langle K_t \sigma, \tau \rangle_{s-t}$$

for all $s, t \in \mathbb{R}$ and $\sigma, \tau \in W^s$. The operator K_t allows us to show the following result that will prove to be useful.

Proposition 2.10. *Let ϕ be a smooth, complex-valued function on $\mathbb{T}^n$. Then, for $s = 0$, we have*

$$\langle \phi\sigma, \tau \rangle_0 = \langle \phi\sigma, \tau \rangle_0.$$

For a general integer s, there exists a constant $c > 0$ only depending only on s, n and ϕ such that

$$|\langle \phi\sigma, \tau \rangle_s - \langle \sigma, \bar{\phi}\tau \rangle_s| < c\|\sigma\|_s \|\tau\|_{s-1}$$

for all $\sigma, \tau \in W^s$.

Proof. The first claim immediately follows from the fact that the Sobolev norm $\|\cdot\|_0$ coincides with the L^2 norm. To prove the second claim, we consider separately the cases $s > 0$ and $s < 0$. In the case of $s < 0$, check that

$$\begin{aligned}\langle \phi\sigma, \tau\rangle_s = \langle \phi K_{-s} K_s \sigma, K_s \tau\rangle_0 = \langle K_{-s} K_s \sigma, \bar{\phi} K_s \tau\rangle_0 &= \langle K_s \sigma, K_{-s} \bar{\phi} K_s \tau\rangle_0 \\ &= \langle \sigma, \bar{\phi}\tau\rangle_s + \langle K_s\sigma, [K_{-s}, \bar{\phi}] K_s \tau\rangle_0\end{aligned}$$

where $[K_{-s}, \bar{\phi}]$ denotes the commutator of the two operators (the right entry denotes multiplication by ϕ). However, we note a fact that we will use several times: for differential operators A of order a and B of order b, the commutator $[A, B]$ is itself a differential operator of order $\leq a + b - 1$ (the $\leq$ sign is used to include the case when the commutator vanishes). In particular, $[K_{-s}, \bar{\phi}]$ is a differential operator of order $\leq -2s + 1$, so writing it in terms of D^α for $[\alpha] \leq -2s + 1$, we obtain

$$\begin{aligned}|\langle K_s\sigma, [K_{-s}, \bar{\phi}] K_s \tau\rangle_0| &= O\left(\sum_{[\alpha]\leq -2s+1} |\langle K_s\sigma, D^\alpha K_s \tau\rangle_0|\right) \\ &= O\left(\sum_{[\alpha]\leq -2s+1} \|K_s\sigma\|_{-s} \|D^\alpha K_s \tau\|_s\right) \\ &= O\left(\sum_{[\alpha]\leq -2s+1} \|K_s\sigma\|_{-s} \|K_s \tau\|_{s+[\alpha]}\right) \\ &= O\left(\|K_s\sigma\|_{-s} \|K_s\tau\|_{-s-1}\right) \\ &= O\left(\|\sigma\|_s \|\tau\|_{s-1}\right),\end{aligned}$$

which proves our claim.

The case of $s > 0$ is proven analogously. □

We can also deduce the following result about how multiplying an element of W^s by a complex-valued function affects the Sobolev norm.

Proposition 2.11. *Let ϕ be a smooth, complex-valued function on $\mathbb{T}^n$ and s be an integer. There exist constants $c_1, c_2 > 0$ depending only on s, n and ϕ, such that for all*

$f \in C^\infty(\mathbb{T}^n, \mathbb{C}^m)$, we have

$$\|\phi f\|_s \leq c_1 \|f\|_{s-1} + c_2 \|\phi\|_\infty \|f\|_s.$$

In particular, there exists a constant $c > 0$ depending only on s, n, and the derivatives of f up to order s, for which

$$\|\phi f\|_s \leq c \|f\|_s.$$

Proof. First, consider the case of $s \geq 0$. Recall the equivalence of (2.14) with $\|\cdot\|_s$, which shows that

$$\|\phi f\|_s = O\left(\sum_{[\alpha] \leq s} \|D^\alpha(\phi f)\|_{L^2}\right).$$

However, $[D^\alpha, \phi]$ is a differential operator of order $\leq [\alpha] - 1$. Thus, we have

$$\begin{aligned}
\sum_{[\alpha] \leq s} \|D^\alpha(\phi f)\|_{L^2} &\leq \sum_{[\alpha] \leq s} \|D^\alpha(\phi f) - \phi D^\alpha(f)\|_{L^2} + \sum_{[\alpha] \leq s} \|\phi D^\alpha(f)\|_{L^2} \\
&= O\left(\sum_{[\alpha] \leq s-1} \|D^\alpha(f)\|_{L^2} + \|\phi\|_\infty \sum_{[\alpha] \leq s} \|D^\alpha f\|\right) \\
&= O\left(\|f\|_{s-1} + \|\phi\|_\infty \|f\|_s\right),
\end{aligned}$$

thereby showing that $\|\phi f\|_s$ is also $O\left(\|f\|_{s-1} + \|\phi\|_\infty \|f\|_s\right)$.

For the remaining case $s < 0$, we use the operator K_t to reduce to the former case. Specifically, we have

$$\begin{aligned}
\|\phi f\|_s^2 &= \langle \phi K_s f, K_s \phi f \rangle_0 \\
&= \langle K_{-s} \phi K_s f, K_s \phi f \rangle_0 + \langle [\phi, K_{-s}] K_s f, K_s \phi f \rangle_0 \\
&= |\langle K_{-s} \phi K_s f, K_s \phi f \rangle_0| + O\left(\sum_{[\alpha] \leq -2s-1} |\langle D^\alpha K_s f, K_s \phi f \rangle_0|\right),
\end{aligned}$$

where we have used that $[\phi, K_{-s}]$ is a differential operator of order $\leq -2s + 1$. The first term can be bounded as follows:

$$\begin{aligned}|\langle K_{-s}\phi K_s f, K_s\phi f\rangle_0| = |\langle \phi K_s f, K_s\phi f\rangle_{-s}| &\leq \|\phi K_s f\|_{-s}\|K_s\phi f\|_{-s}\\ &= O\left(\|\phi\|_\infty\|K_s f\|_{-s} + \|K_s f\|_{-s-1}\right)\|K_s\phi f\|_{-s}\\ &= O\left(\|\phi\|_\infty\|f\|_s + \|f\|_{s-1}\right)\|\phi f\|_s,\end{aligned}$$

where we have used Cauchy–Schwartz. The second term can be bounded as follows:

$$\begin{aligned}\sum_{[\alpha]\leq -2s-1}|\langle D^\alpha K_s f, K_s\phi f\rangle_0| &= O\left(\sum_{[\alpha]\leq -2s-1}\|D^\alpha K_s\phi\|_s\|K_s\phi f\|_{-s}\right)\\ &= O\left(\|K_s\phi f\|_{-s}\sum_{[\alpha]\leq -2s-1}\|K_s f\|_{s+[\alpha]}\right)\\ &= O\left(\|K_s\phi f\|_{-s}\|K_s f\|_{-s-1}\right)\\ &= O\left(\|\phi f\|_s\|f\|_{s-1}\right).\end{aligned}$$

So, we overall have

$$\|\phi f\|_s^2 = O\left(\|\phi\|_\infty\|f\|_s + \|f\|_{s-1}\right)\|\phi f\|_s,$$

and dividing by $\|\phi f\|_s$ (if this quantity is zero, the claim is already trivially true), we obtain the claim for $s < 0$. □

We end this section with the following generalization of Proposition 2.11 describing the effect of general differential operators on the Sobolev norm of $f \in C^\infty(\mathbb{T}^n, \mathbb{C}^m)$.

Proposition 2.12. *Let L be a differential operator of order ℓ and s be an integer. There exists $c_1 > 0$ depending only on n, m, s, ℓ and $c_2 > 0$ depending only on n, m, s, ℓ and the order $\leq \ell$ partial derivatives of the coefficients of L satisfying*

$$\|Lf\|_s \leq c_1 M\|f\|_{s+\ell} + c_2\|f\|_{s+\ell-1}$$

for all $f \in C^\infty(\mathbb{T}^n, \mathbb{C}^m)$, where

$$M := \max_{[\alpha]=\ell}|a_{ij}^\alpha|.$$

In particular, there exists $c > 0$ depending only on n, m, s, t and the order $\leq \ell$ partial derivatives of the coefficients of L satisfying

$$\|Lf\|_s \leq c\|f\|_{s+\ell}$$

for all $f \in C^\infty(\mathbb{T}^n, \mathbb{C}^m)$, and thus L extends to a continuous operator $W^{s+\ell} \to W^s$.

Proof. The case $m = 1$ follows immediately from Proposition 2.11, and the case for general m follows from the case of $m = 1$ combined with the bound

$$\|Lf\|_s = O\left(\sum_{1\leq i,j\leq m} \|L_{ij} f_j\|_s\right),$$

where the f_j denote the entries of f and the implied constant depends only on m. □

2.4 Elliptic theory

Let L, whose entries are denoted by

$$L_{ij} = \sum_{[\alpha]\leq\ell} a_{ij}^\alpha D^\alpha, \tag{2.15}$$

be a differential operator of order ℓ on $C^\infty(\mathbb{T}^n, \mathbb{C}^m)$. The fact that

$$((D^\alpha)\, \sigma)_\xi = \xi^\alpha \sigma_\xi$$

suggests the utility of the multilinear form defined on $\xi \in T_x^*(\mathbb{T}^n) \cong \mathbb{R}^n$ (for a given $x \in \mathbb{T}^n$) obtained by replacing D^α in (2.15) with ξ^α, when investigating a differential operator L of order ℓ. Motivated by this, we define the *principal symbol* $S_L(x, \cdot) : T_x^*(\mathbb{T}^n) \to \mathrm{End}(C^\infty(\mathbb{T}^n, \mathbb{C}^m)_x)$ of L by

$$S_L(x, \xi)_{ij} := \sum_{[\alpha]=\ell} a_{ij}^\alpha(x)\xi^\alpha.$$

Note that while we have used the standard basis to trivialize $T_x^*(\mathbb{T}^n) \cong \mathbb{R}^n$, this is equivalent to the general definition of the principal symbol for differential operators between arbitrary bundles over a general M. The general definition is analogously defined for an arbitrary local trivialization about x, but can be shown to be well-defined.

We say that L is *elliptic at* $x \in \mathbb{T}^n$ if for every nonzero $\xi \in T_x^*(\mathbb{T}^n)$, the principal symbol $S_L(x, \xi)$ is nonsingular. We say that L is *elliptic* if it is elliptic at all $x \in \mathbb{T}^n$.

A key fact about elliptic operators we will later use is the following result that is sometimes called the fundamental inequality for elliptic operators.

Theorem 2.13 (Fundamental inequality)**.** *Let L be an elliptic operator of order ℓ, and let s be an integer. Then, there exists a constant $c > 0$ such that*

$$\|\sigma\|_{s+\ell} \leq c(\|L\sigma\|_s + \|\sigma\|_s) \tag{2.16}$$

for all $\sigma \in W^{s+1}$.

Proof. It suffices to prove the claim for σ corresponding to $f \in C^\infty(\mathbb{T}^n, \mathbb{C}^m)$. The proof is comprised of several steps. First, we prove the claim under the assumption that L has constant coefficients and has only nonzero terms of order p, or in other words, that L is homogeneous. This means that the following discussion holds uniformly for any $x \in \mathbb{T}^n$. By ellipticity, we have that for any nonzero $\xi \in T_x^*\mathbb{T}^n$ and nonzero $v \in C^\infty(\mathbb{T}^n, \mathbb{C}^m)_x \cong \mathbb{C}^m$, the quantity $|L(\xi)v|$ is strictly positive. By the compactness of the unit sphere of $\mathbb{C}^m$, there exists a constant $C > 0$ such that

$$|L(\xi)v|^2 \geq C$$

for v and ξ taken to be unit vectors, or in other words,

$$|L(\xi)v|^2 \geq C|\xi|^{2\ell}|v|^2$$

for v and ξ arbitrary. Applying this lower bound, we obtain

$$\sum_{\xi \in \mathbb{Z}^n} |\xi|^{2\ell} |\sigma_\xi|^2 (1 + |\xi|^2)^s = O\left(\sum_{\xi \in \mathbb{Z}^n} |L(\xi)\sigma_\xi|^2 (1 + |\xi|^2)^s\right) = O\left(\|L\sigma\|_s^2\right).$$

It follows that

$$\|\sigma\|_{s+\ell}^2 = O\left(\sum_{\xi\in\mathbb{Z}^n} |\sigma_\xi|^2(1+|\xi|^2)^{s+\ell}\right) = O\left(\sum_{\xi\in\mathbb{Z}^n} |\sigma_\xi|^2(1+|\xi|^2)^{s}(1+|\xi|^{2\ell})\right)$$
$$= O\left(\|L\sigma\|_s^2 + \|\sigma\|_s^2\right)$$
$$= O\left((\|L\sigma\|_s + \|\sigma\|_s)^2\right),$$

as needed.

Next, we prove the general case. Fix $x \in \mathbb{T}^n$. We will show that there exists an open neighborhood $U_x \ni x$ such that (2.16) holds for all σ corresponding to all $f \in C^\infty(\mathbb{T}^n, \mathbb{C}^m)$ supported on U. Evaluating the coordinate functions c_{ij}^α of L at x for each α with $[\alpha] = \ell$, we obtain a homogeneous differential operator L_0 of order ℓ that agrees with the order ℓ part of L evaluated on x. Using the previous case, observe that

$$\|\sigma\|_{s+\ell} = O\left(\|L_0\sigma\|_s + \|\sigma\|_s\right) = O\left(\|L\sigma\|_s + \|(L-L_0)\sigma\|_s + \|\sigma\|_s\right).$$

Let $c_1 > 0$ denote the implied constant in $\|\sigma\|_{s+\ell} = O\left(\|L\sigma\|_s + \|(L-L_0)\sigma\|_s + \|\sigma\|_s\right)$. Fix a positive $\varepsilon < 1/(2c_1c_2)$, where c_2 denotes the implied constant in (2.12). On a sufficiently small open neighborhood of x, the coefficients of the highest-order part of $L - L_0$ have absolute value less than ε. Let $\tilde{L}$ be a differential operator agreeing with $L - L_0$ on some smaller open neighborhood $U_x \ni x$ such that the coefficients of the order p part are all uniformly less than ε in absolute value. Then, for σ whose corresponding f is supported on U_x, we may deduce from Proposition 2.12 and the choice of ε that

$$\|\sigma\|_{s+\ell} = O\left(\|L\sigma\|_s + \|\tilde{L}\sigma\|_s + \|\sigma\|_s\right)$$
$$\leq \frac{1}{2}\|\sigma\|_{s+\ell} + O\left(\|L\sigma\|_s + \|\sigma\|_{s+\ell-1} + \|\sigma\|_s\right).$$

Applying the Peter–Paul inequality, we further have

$$\|\sigma\|_{s+\ell} \leq \frac{3}{4}\|\sigma\|_{s+\ell} + O\left(\|L\sigma\|_s + \|\sigma\|_s\right),$$

which proves our desired claim.

Consider the open cover of $\mathbb{T}^n$ by the collection of U_x indexed by $x \in \mathbb{T}^n$. Since $\mathbb{T}^n$ is compact, we can find a finite subcover $U_1, \dots, U_k$. We take a partition of unity $\rho_1, \dots, \rho_k$ associated to this subcover, satisfying the additional condition

$$\sum_{j=1}^{k} \rho_j^2 = 1.$$

Now, let σ corresponding to $f \in C^\infty(\mathbb{T}^n, \mathbb{C}^m)$ be arbitrary. Then, applying the aforementioned condition, Proposition 2.10 and Proposition 2.11, we have

$$\|\sigma\|_{s+\ell}^2 = \langle \sigma, \sigma \rangle_{s+\ell} = \left\langle \sum_{j=1}^{k} \rho_j, \sigma \right\rangle_{s+\ell} = \sum_{j=1}^{k} \langle \rho_j f, \rho_j f \rangle + O\left(\|f\|_{s+\ell} \|f\|_{s+\ell-1} \right).$$

Since $\rho_j f$ is supported on one of the finitely many, specially chosen open sets $U_1, \dots, U_k$, we can continue the above computation in the following way:

$$\begin{aligned} \|\sigma\|_{s+\ell}^2 &= \sum_{j=1}^{k} \langle \rho_j f, \rho_j f \rangle + O\left(\|f\|_{s+\ell} \|f\|_{s+\ell-1} \right) \\ &= O\left(\|f\|_s^2 + \|f\|_{s+\ell} \|f\|_{s+\ell-1} + \sum_{j=1}^{k} \|L \rho_j f\|_s^2 \right). \end{aligned}$$

We find a useful bound for $\|L\rho_j f\|_s^2$. Observe that

$$\begin{aligned} |\|L\rho_j f\|_s^2 - \langle L\rho_j^2 f, Lf \rangle_s| &\leq |\langle L\rho_j f, L\rho_j f \rangle_s - \langle \rho_j L \rho_j f, Lf \rangle_s| + |\langle \rho_j L \rho_j f, Lf \rangle_s - \langle L\rho_j \rho_j f, Lf \rangle_s| \\ &= |\langle L\rho_j f, L\rho_j f - \rho_j L f \rangle_s| + |\langle Lf, \rho_j L \rho_j f - L\rho_j \rho_j f \rangle_s|, \end{aligned}$$

but both $[L, \rho_j]$ and $[\rho_j, L\rho_j]$ are differential operators of order $\leq \ell - 1$. Thus, we can apply Proposition 2.12 and Cauchy–Schwartz to obtain

$$\|L\rho_j f\|_s^2 \leq \langle L\rho_j^2 f, Lf \rangle_s + |\|L\rho_j f\|_s^2 - \langle L\rho_j^2 f, Lf \rangle_s| = \langle L\rho_j^2 f, Lf \rangle_s + O\left(\|f\|_{s+\ell} \|f\|_{s+\ell-1} \right)$$

This permits us to continue our computation:

$$\begin{aligned}
\|\sigma\|_{s+\ell}^2 &= O\left(\|f\|_s^2 + \|f\|_{s+\ell}\|f\|_{s+\ell-1} + \sum_{j=1}^{k} \|L\rho_j f\|_s^2\right) \\
&= O\left(\|f\|_s^2 + \|f\|_{s+\ell}\|f\|_{s+\ell-1} + \sum_{j=1}^{k} \langle L\rho_j^2 f, Lf\rangle_s\right) \\
&= O\left(\|f\|_s^2 + \|f\|_{s+\ell}\|f\|_{s+\ell-1} + \|Lf\|_s^2\right) \\
&= \frac{1}{2}\|f\|_s^2 + O\left(\|f\|_s^2 + \|f\|_{s+\ell-1}^2 + \|Lf\|_s^2\right) \\
&= \frac{3}{4}\|f\|_s^2 + O\left(\|f\|_s^2 + \|Lf\|_s^2\right),
\end{aligned}$$

where we have used the arithmetic mean-geometric mean inequality followed by the Peter–Paul inequality in the last two lines. This concludes our proof. □

A bit of additional work would yield a proof of elliptic regularity for $\mathbb{T}^n$, from which the general case follows. While the following theorem represents an essential characteristic of elliptic operators, we will not prove it as the information of elliptic regularity already contained in the fundamental inequality will suffice for our purposes.

Theorem 2.14 (Elliptic regularity)**.** *Let L be an elliptic operator of order ℓ. Suppose that $\sigma \in W^s$ and $\tau \in W^t$ satisfy*

$$L\sigma = \tau.$$

Then, σ is in $W^{t+\ell}$.

Example 2.15. The Cauchy–Riemann operator $\partial_x + i\partial_y$ is clearly elliptic. Thus, elliptic regularity tells us that under the relatively weak condition that σ satisfying $(\partial_x + i\partial_y)\sigma = 0$ is in W^s for some s, we have that σ is in W^t for all t, and thus is smooth by a local application of Sobolev's lemma. In particular, a holomorphic function must be smooth.

2.5 Proof of the main theorem

We now prove the Hodge decomposition for a compact, closed, orientable Riemannian manifold M with a choice of orientation.

First, we port our results on Sobolev spaces and elliptic operators on $\mathbb{T}^n$ globally to a general M. Using the compactness of M, fix an open cover of M with coordinate charts $U_1, \ldots, U_k$ and the corresponding homeomorphisms $\varphi_j : U_j \to V_j \subset \mathbb{T}^n$, to open sets V_j contained in $\mathbb{T}^n$. Also, fix a partition of unity $\rho_1, \ldots, \rho_k$ associated to this cover. On each coordinate chart, we can consider smooth p-forms to be functions in $C^\infty(\mathbb{R}^n, \mathbb{R}^m) \subset C^\infty(\mathbb{R}^n, \mathbb{C}^m)$ for $m = \binom{n}{p}$, where the inner product on $\mathbb{R}^m \subset \mathbb{C}^m$ is defined by the inner product (2.4) on p-forms. Reusing notation, for $\omega, \eta \in C^\infty(M, \Omega^p)$, we define the inner product $\langle \cdot, \cdot \rangle_s$ on $C^\infty(M, \Omega^p)$ by

$$\langle \varphi_1, \varphi_2 \rangle_s = \left(\sum_{j=1}^{k} \langle \rho_j \omega \circ \varphi_j, \rho_j \eta \circ \varphi_j \rangle_s^2 \right)^{\frac{1}{2}},$$

where $\langle \cdot, \cdot \rangle_s$ in the left-hand side defined in terms of the earlier defined $\langle \cdot, \cdot \rangle_s$ in the right-hand side. The associated norm is

$$\|\varphi\|_s = \left(\sum_{j=1}^{k} \|\rho_j \varphi \circ \varphi_j\|_s^2 \right)^{\frac{1}{2}},$$

again with the notation of the right-hand side corresponding to the earlier definition. It is straightforward to check that this inner product does not depend on the choice of open cover and partition of unity. Define the Sobolev space $W^s(M, \Omega^p)$ to be the completion of $C^\infty(M, \Omega^p)$ with respect to this inner product.

Also, analogous to the $\mathbb{T}^n$ case, define a differential operator of order ℓ for M by a linear operator $L : C^\infty(M, \Omega^p) \to C^\infty(M, \Omega^p)$ that, in any local trivialization over some coordinate chart, is a $m \times m$ matrix with entries of the form

$$L_{ij} = \sum_{[\alpha] \leq \ell} a_{ij}^\alpha D^\alpha,$$

where $a_{ij}^\alpha \in C^\infty(M, \mathbb{C})$ with at least one a_{ij}^α not identically zero for some i, j and α with $[\alpha] = \ell$. Note again that for $x \in M$, the $m \times m$ matrix $a^\alpha(x)$ whose entries are given by $a_{ij}^\alpha(x)$ represents an element of $\mathrm{End}(C^\infty(M, \Omega^p)_x)$. Moreover, such differential operators can be extended to $W^s(M, \Omega^p)$, and we will consider differential operators to be these extended operators when suitable. We define the principal symbol $S_L(x, \cdot) : T_x^*(M) \to \mathrm{End}(C^\infty(M, \Omega^p)_x)$ of L similarly to before. Likewise, we say that L is elliptic at $x \in M$ if for every nonzero $\xi \in T_x^*(M) \cong M$, the principal symbol $S_L(x, \xi)$ is nonsingular, and that L is elliptic if it is elliptic at all M. One can see that this is equivalent to L being elliptic on each coordinate chart. Furthermore, one can show an equivalent, coordinate-free definition of ellipticity at x, given by the condition that

$$L(\phi^\ell \omega)(x) \neq 0$$

for every smooth p-form ω such that $\omega(x) \neq 0$ and every smooth function ϕ on M such that $\phi(x) = 0$ and $d\phi(x) \neq 0$.

It is straightforward to verify that our previous results about Sobolev spaces, differential operators, and elliptic operators—in particular, Rellich's lemma and the fundamental inequality—remain true in this general case.

In order to utilize the nice properties of elliptic operators, we actually have to prove the following long-awaited fact.

Proposition 2.16. *The Hodge–de Rham Laplacian* Δ *is elliptic.*

Proof. By our discussion earlier, it is equivalent to show that for every $x \in M$,

$$L(\phi^2 \omega)(x) \neq 0 \tag{2.17}$$

for every smooth p-form ω such that $\omega(x) \neq 0$ and every smooth function ϕ on M such that $\phi(x) = 0$ but $d\phi(x) \neq 0$. By our computation of d^* in Section 2.1, we have

$$\Delta = (-1)^{n(p+1)+1} d * d * + (-1)^{np+1} * d * d.$$

Let ξ denote the nonzero vector $d\phi(x) \in T_x^* M$. To compute the left-hand side of (2.17), we work out

$$d{*}d{*}(\phi^2\omega)(x) = (d{*}d(\phi^2\omega)(x) = (2d{*}\phi(d\phi){*}\omega)(x) = (2(d\phi){*}(d\phi){*}\omega)(x) = 2\xi{*}\xi{*}(\omega(x)),$$

and similarly,

$$*d*d*(\phi^2\omega)(x) = 2*\xi*\xi(\omega(x)),$$

so that overall,

$$\Delta(\phi^2\omega)(x) = -2\left((-1)^{np}*(\xi\wedge\cdot)*(\xi\wedge\cdot)+(-1)^{n(p-1)}(\xi\wedge\cdot)*(\xi\wedge\cdot)*\right)(\omega(x)). \quad (2.18)$$

We have to show that the above is nonzero. To do so, we appeal to the fact that for an exact sequence

$$U \xrightarrow{A} V \xrightarrow{B} W$$

of finite-dimensional inner-product spaces, the self-map $B^*B + AA^* : V \to V$ is an automorphism. Indeed, for nonzero $v \in V$,

$$\langle (B^*B + AA^*)v, v\rangle = \langle Bv, Bv\rangle + \langle A^*v, A^*v\rangle.$$

If $Bv \neq 0$, then the above is nonzero so that $(B^*B + AA^*)v \neq 0$. On the other hand, if $Bv = 0$, then $v \in \operatorname{im} A$ by exactness, but A^* is injective on $\operatorname{im} A$. This shows that $A^*v \neq 0$ and as in the above, this implies that $(B^*B + AA^*)v \neq 0$.

We apply the above observation to the setting

$$\bigwedge^{p-1}(T_x^*M) \xrightarrow{\xi\wedge\cdot} \bigwedge^{p}(T_x^*M) \xrightarrow{\xi\wedge\cdot} \bigwedge^{p+1}(T_x^*M),$$

where the vector spaces are equipped with the inner product

$$\langle \omega, \eta\rangle = *(\omega \wedge *\eta).$$

Indeed, this sequence is exact, and that the adjoint of $\xi\wedge\cdot : \bigwedge^p(T_x^*M) \to \bigwedge^{p+1}(T_x^*M)$ is $(-1)^{np}*\xi*$. Thus,

$$(-1)^{np}*(\xi\wedge\cdot)*(\xi\wedge\cdot)+(-1)^{n(p-1)}(\xi\wedge\cdot)*(\xi\wedge\cdot)*$$

is an automorphism of $\bigwedge^p(T_x^*M)$, which shows our desired claim that (2.18) is nonzero. □

The ellipticity of Δ finally allows us to prove the Hodge decomposition for M.

Proof of Theorem 2.2. By the fundamental inequality, we have

$$\|\omega\|_2 = O\left(\|\omega\|_0 + \|\Delta\omega\|_0\right) = O\left(\|\omega\|_0\right) = O(1)$$

uniformly for all $\omega \in \ker \Delta$. Thus, in the diagram of identity maps

$$(\ker \Delta, L^2) \to (\ker \Delta, W^2) \to (\ker \Delta, L^2),$$

the first map is continuous. The second map is compact by Rellich's lemma, and the composition is clearly compact. Consequently, the closed unit ball of $\ker \Delta$ is compact, which proves that $\ker \Delta$ is finite-dimensional.

We now prove the second claim. Recalling that Δ can be naturally extended to a self-map of $L^2(M, \Omega^p)$, we will let $\bar{\Delta}$ denoted this extended map. Since $L^2(M, \Omega^p)$ is a Hilbert space, we have $\ker \bar{\Delta} = (\operatorname{im} \bar{\Delta}^*)^\perp = (\operatorname{im} \bar{\Delta})^\perp$, rather than just an inclusion. Thus, we would like to show the decomposition

$$L^2(M, \Omega^p) = \ker \bar{\Delta} \oplus \operatorname{im} \bar{\Delta} = (\operatorname{im} \bar{\Delta})^\perp \oplus \operatorname{im} \bar{\Delta}. \tag{2.19}$$

Proving the above is equivalent to showing that $\operatorname{im} \bar{\Delta}$ is closed. Consider an arbitrary element $\eta = [(\eta_j)_{j \in \mathbb{Z}_{\geq 0}}] \in \operatorname{im} \bar{\Delta}$, expressed as the equivalence class of a Cauchy sequence $(\eta_j)_{j \in \mathbb{Z}_{\geq 0}}$ of smooth p-forms. This means that there exists $\nu = [(\nu_j)_{j \in \mathbb{Z}_{\geq 0}}] \in L^2(M, \Omega^p)$, where $(\nu_j)_{j \in \mathbb{Z}_{\geq 0}}$ is also a Cauchy sequence of smooth p-forms, such that $\Delta\nu_j - \eta_j$ converges to 0 in the L^2 norm. Without loss of generality, we can assume each ν_j is in $\ker \Delta$.

We claim that ν has finite $\|\cdot\|_2$ norm. Indeed, suppose the contrary. This means that any subsequence of $(\nu_j)_{j \in \mathbb{Z}_{\geq 0}}$ has undefined $\|\cdot\|_2$ norm, which implies that $\|\nu_j\|_2 \to \infty$ as $j \to \infty$. Thus, the sequence $(\gamma_j)_{j \in \mathbb{Z}_{\geq 0}}$, where $\gamma_j := \nu_j / \|\nu_j\|_2$, has finite $\|\cdot\|_2$ norm and thus defines an element $\gamma \in W^2(M, \Omega^p)$. By Rellich's lemma, a subsequence $(\gamma_{j_k})_{k \in \mathbb{Z}_{\geq 0}}$ converges in $L^2(M, \Omega^p)$. But the smooth p-forms γ_{j_k} are in $(\ker \Delta)^\perp$ even though $\Delta\gamma_{j_k} \to 0$ in $L^2(M, \Omega^p)$ as $k \to \infty$. This implies that γ_{j_k} must converge to 0 in the L^2 norm, which means $\gamma = 0$ in $L^2(M, \Omega^p)$, and thus also in $W^2(M, \Omega^p)$. But this contradicts the fact that$\|\gamma\|_2 = 1$. Thus, ν has finite $\|\cdot\|_2$ norm, so by the fundamental inequality, $(\nu_j)_{j \in \mathbb{Z}_{\geq 0}}$ has a subsequence that converges in the L^2 norm to some element ν. It follows that $\eta = \Delta\nu$ is in $\operatorname{im} \bar{\Delta}$, as needed.

Finally, we show that $\ker \Delta = \ker \bar{\Delta}$, so that restricting the orthogonal decomposition $L^2(M, \Omega^p) = \ker \Delta \oplus \operatorname{im} \bar{\Delta}$ to $C^\infty(M, \Omega^p)$ gives our desired Hodge decomposition. Our proof is rooted in the phenomenon of elliptic regularity. Consider an arbitrary element $\eta = [(\eta_j)_{j \in \mathbb{Z}_{\geq 0}}] \in \ker \bar{\Delta}$, or in other words, $\Delta \eta_j \to 0$ in the L^2 norm as $j \to \infty$. Recall that Δ is continuous as a map $L^2(M, \Omega^p) \to W^{-2}(M, \Omega^p)$. Thus, we have that $\Delta \eta_j \to 0$ in the $\|\cdot\|_{-2}$ norm as $j \to \infty$, or equivalently, that $\Delta \eta = 0$ in $W^{-2}(M, \Omega^p)$. However, the fundamental inequality allows us to conclude the finiteness of $\|\eta\|_{s+2}$ from the finiteness of $\|\eta\|_s$. By induction starting from the base case of $s = -2$, we conclude that η is in fact in $C^\infty(M, \Omega^p)$, and thus in $\ker \Delta$, as desired. This completes our proof of the Hodge decomposition for compact, closed, oriented Riemannian manifolds. $\square$

In fact, the only specific property of Δ we have used is its ellipticity and, perhaps less prominently, the fact that it is a self-map. It is not difficult to see that the above argument can be applied to prove the following generalization, the statement of which relates to elliptic operators between general vector bundles of the same finite rank over M.

Theorem 2.17. *Let M be a compact, closed, orientable Riemannian manifold with a choice of orientation; E and F, vector bundles over M of the same finite rank; and $L : \Gamma(E) \to \Gamma(F)$, an elliptic operator. The following are true:*

(i) $\ker L$ *is finite-dimensional.*

(ii) We have an L^2 orthogonal decomposition

$$C^\infty(M, F) = \ker L^* \oplus L(C^\infty(M, E)).$$

Chapter 3

Hodge Theory of Compact Kähler Manifolds

3.1 Differential operators on complex manifolds

Let (X, h) be a compact, closed complex manifold of dimension n, with h denoting its Hermitian metric. We can also consider X to be a $2n$-dimensional Riemannian manifold, which we denote by $X_{\mathbb{R}}$. Recall the concept of complexification, denoted by $(\cdot)_{\mathbb{C}}$, which is defined by applying $\otimes_{\mathbb{R}}\mathbb{C}$ to a given real vector space or bundle. In our case, we let $\Omega^k_{\mathbb{C}}$ denote the complexification $\Omega^k X_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Then, analogously to the construction of g^* in (2.1), we can define the sesquilinear bundle morphism $h^* : C^\infty(M, \Omega^k_{\mathbb{C}}) \times C^\infty(M, \Omega^k_{\mathbb{C}}) \to C^\infty(M, \mathbb{C})$.

We can extend the Hodge star operator defined in Chapter 2 to our new setting, as follows. Let V be an n-dimensional complex Euclidean space, and let $V_{\mathbb{R}}$ be V considered as a $2n$-dimensional real Euclidean space. With the standard inner product, there is a canonical volume form $\mathrm{vol} \in \bigwedge^{2n} V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Then, since the exterior product

$$\wedge : \bigwedge^k V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \times \bigwedge^{2n-k} V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \to \bigwedge^{2n} V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$$

is a nondegenerate pairing, we can define the Hodge star operator $* : \bigwedge^k V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \to \bigwedge^{2n-k} V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ by the requirement that $\beta \in \bigwedge^k V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ is mapped to the unique form $*\beta$

satisfying

$$\alpha \wedge *\overline{\beta} = h^*(\alpha, \beta) \operatorname{vol}^h$$

for all $\alpha \in \bigwedge^k V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. This allows us to, as done in Chapter 2, define an inner product on $C^\infty(X, \Omega^k_{\mathbb{C}})$ by

$$\langle \eta, \nu \rangle := \int_X \eta \wedge *\overline{\nu} = \int_X h^*(\eta, \nu) \operatorname{vol}. \tag{3.1}$$

Furthermore, the almost complex structure J on $V_{\mathbb{R}}$ induces the decomposition $V = V^{1,0} \oplus V^{0,1}$, where $V^{1,0}$ denotes the eigenspace on which J acts by multiplication by i and $V^{0,1}$, the eigenspace on which J acts by multiplication by $-i$. This allows us to define

$$\bigwedge^{p,q} V := \bigwedge^p V^{1,0} \otimes_{\mathbb{C}} \bigwedge^q V^{0,1},$$

and applying fiberwise to $\Omega^k_{\mathbb{C}}$, we obtain a complex vector bundle $\Omega^{p,q}$, whose sections we call *(p,q)-forms*.

We now investigate the action of the Hodge star operator on the complex vector bundles $\Omega^{p,q}$. For an arbitrary point $x \in X$, let $dz_1, \ldots, dz_n$ denote a local frame of holomorphic coordinates that are isometric at x. Using the convention given in (2.2), let

$$u = \sum_{|I_1|=p, |I_2|=q} u_{I_1,I_2} dz^{I_1} \wedge \overline{dz^{I_2}}$$

and

$$v = \sum_{|I_1|=p, |I_2|=q} v_{I_1,I_2} dz^{I_1} \wedge \overline{dz^{I_2}}$$

denote arbitrary forms in $C^\infty(M, \Omega^{p,q})$. Then, we have

$$\langle u, v \rangle_x = \sum_{|I_1|=p, |I_2|=q} u_{I_1,I_2} \overline{v_{I_1,I_2}}. \tag{3.2}$$

Moreover, one can check that

$$*\overline{v}(x) = \sum_{|I_1|=p, |I_2|=q} \epsilon'(I_1, I_2) \overline{v_{I_1,I_2}} dz^{I_1^c} \wedge \overline{dz^{I_2^c}}, \tag{3.3}$$

where $\epsilon'(I_1, I_2) := (-1)^{q(n-p)}\epsilon(I_1)\epsilon(I_2)$ for ϵ defined in (2.3).

Since $*\overline{v}$ is of type $(n-p, n-q)$, the definition $u \wedge *\overline{v} = \langle u, v\rangle \,\mathrm{vol}$ shows that the Hodge star operator yields a $\mathbb{C}$-linear isometry $* : \Omega^{p,q} \to \Omega^{n-q,n-p}$. It follows that the decomposition

$$C^\infty(X, \Omega^k_{\mathbb{C}}) = \bigoplus_{p+q=k} C^\infty(X, \Omega^{p+q}) \tag{3.4}$$

is L^2 orthogonal. Indeed, suppose v is not only of type (p, q), but also of type (p', q'), where necessarily $p + q = p' + q'$. Then, $u \wedge *\overline{v}$ is of type $(n - p' + p, n - q' + q)$ for any u of type (p, q), which forces v to be 0 unless $p = p'$ and $q = q'$.

The exterior derivative d and its formal adjoint d^* can be extended by complexification to $\Omega^k_{\mathbb{C}}$. Note that $d(C^\infty(X, \Omega^{p,q}))$ is contained in $C^\infty(X, \Omega^{p+1,q}) \oplus C^\infty(X, \Omega^{p,q+1})$. Thus, it is natural to consider two more differential operators, often called the *Dolbeault operators*:

$$\partial := \pi^{p+1,q} \circ d : C^\infty(X, \Omega^{p,q}) \to C^\infty(X, \Omega^{p+1,q})$$

and

$$\overline{\partial} := \pi^{p,q+1} \circ d : C^\infty(X, \Omega^{p,q}) \to C^\infty(X, \Omega^{p,q+1}),$$

where $\pi^{p,q}$ denotes the projection map to the summand $C^\infty(X, \Omega^{p,q})$ in (3.4).

Recall from (2.5) that the formal adjoint d^* is given by $-{*d*}$, since X is even-dimensional as a Riemannian manifold. We now show that the formal adjoints of our newly defined operators ∂ and ∂^* take similar forms.

Lemma 3.1. *The formal adjoints of ∂ and $\overline{\partial}$ are given by*

$$\partial^* = - * \overline{\partial} *$$

and

$$\overline{\partial}^* = - * \partial * .$$

Proof. We show the second claim. In light of the fact that $*$ is a real operator, we need to verify that

$$\langle \overline{\partial}\eta, \nu\rangle = \langle \eta, *\overline{\partial} * \nu\rangle$$

for all $\eta \in C^\infty(X, \Omega^{p,q})$ and $\nu \in C^\infty(X, \Omega^{p,q+1})$. Note that $\eta \wedge *\overline{\nu} \in C^\infty(X, \Omega^{n,n-1})$, so we have $d(\eta \wedge *\overline{\nu}) = \overline{\partial}(\eta \wedge *\overline{\nu})$. Further computing, we obtain

$$d(\eta \wedge *\overline{\nu}) = \overline{\partial}(\eta \wedge *\overline{\nu}) = \overline{\partial}\eta \wedge *\overline{\nu} + (-1)^{p+q}\eta \wedge \overline{\partial} * \overline{\nu}.$$

The left-hand side vanishes by Stokes' theorem, so we have

$$\begin{aligned}\langle \overline{\partial}\omega, \eta \rangle = \int_X \overline{\partial}\omega \wedge *\overline{\eta} = (-1)^{p+q+1} \int_X \omega \wedge \overline{\partial} * \overline{\eta} = (-1)^{p+q+1} \int_X \omega \wedge \overline{\partial * \eta} \\ = (-1)^{p+q+1} \int_X \omega \wedge \overline{** ^{-1}\partial * \eta}.\end{aligned}$$

Using the fact that $*^{-1}\gamma = (-1)^{(2n-k)k} * \gamma$ for any k-form γ, we further obtain

$$\int_X \overline{\partial}\eta \wedge *\overline{\nu} = (-1)^{p+q+1} \int_X \eta \wedge *(-1)^{(p+q)(2n-p-q)} \left(\overline{*\partial * \nu}\right) = -\int_X \eta \wedge *\overline{*\partial * \nu} = \langle \eta, -*\partial * \nu \rangle,$$

as needed. The proof of the first claim is analogous. □

We can derive a local expression of the formal adjoint in terms of vector fields. Conider an open set $U \subset \mathbb{C}^n$ and fix a choice of standard coordinates $z_1, \ldots, z_n$. We can consider the standard Hermitian metric $\sum_{j=1}^n dz_j \otimes d\overline{z_j}$ on the tangent bundle TU, so that $\partial_{z_1}, \ldots, \partial_{z_n}$ comprise an isometric holomorphic frame for TU. Then, we can compute the following.

Lemma 3.2. *Let u be a compactly supported smooth k-form on U. Then, in the local coordinates chosen above, we have*

$$\overline{\partial}^* u = -\sum_{j=1}^n \partial_{z_j} \lrcorner \partial_{z_j} u$$

Proof. For any compactly supported function f, we have

$$\int_{\mathbb{C}^n} \partial_{z_j} f \, \mathrm{vol}^h,$$

as seen from applying the fundamental theorem of calculus separately to the real and imaginary parts. Thus, for any compactly supported smooth $(k-1)$-form v, we have

$$\begin{aligned}
0 &= \int_{\mathbb{C}^n} \sum_{j=1}^{n} \partial_{z_j} \langle \partial_{z_j} \lrcorner u, v\rangle \operatorname{vol}^h \\
&= \int_{\mathbb{C}^n} \sum_{j=1}^{n} \langle \partial_{z_j} \lrcorner \partial_{z_j} u, v\rangle \operatorname{vol}^h + \int_{\mathbb{C}^n} \sum_{j=1}^{n} \langle \partial_{z_j} \lrcorner u, \partial_{\bar{z}_j} v\rangle \operatorname{vol}^h \\
&= \int_{\mathbb{C}^n} \sum_{j=1}^{n} \langle \partial_{z_j} \lrcorner \partial_{z_j} u, v\rangle \operatorname{vol}^h + \int_{\mathbb{C}^n} \sum_{j=1}^{n} \langle u, d\overline{z_j} \wedge \partial_{\bar{z}_j} v\rangle \operatorname{vol}^h \\
&= \sum_{j=1}^{n} \left(\langle \partial_{z_j} \lrcorner \partial_{z_j} u, v\rangle + \langle u, \overline{\partial} v\rangle \right),
\end{aligned}$$

as needed. □

We note that the computations for ∂^* are analogous.

Just as we have done in Chapter 2, we can define the Laplacian operators associated to d, ∂ and $\overline{\partial}$, given by

$$\Delta_d := dd^* + d^*d$$

$$\Delta_\partial := \partial\partial^* + \partial^*\partial$$

$$\Delta_{\overline{\partial}} := \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$$

We say that $\eta \in C^\infty(X, \Omega^k_{\mathbb{C}})$ is *harmonic* (respectively, Δ_∂-*harmonic*; respectively,$\Delta_{\overline{\partial}}$-*harmonic*) if $\Delta_d \eta = 0$ (respectively, $\Delta_\partial \eta = 0$; respectively, $\Delta_{\overline{\partial}} \eta = 0$). A proof analogous to that of (2.7) shows that the harmonic forms (respectively, Δ_∂-harmonic forms; respectively, $\Delta_{\overline{\partial}}$-harmonic forms) are precisely the forms that are simultaneously d- and d^*-closed (respectively, ∂- and ∂^*-closed; respectively, $\overline{\partial}$- and $\overline{\partial}^*$-closed). It follows from Proposition 2.16 that Δ_d is elliptic, and it can similarly be proven that Δ_∂ and $\Delta_{\overline{\partial}}$ are elliptic. In particular, this allows us to apply Theorem 2.17 to $\Delta_{\overline{\partial}}$, yielding the following fact.

Theorem 3.3. *Let (X, h) be a compact, closed complex manifold with Hermitian metric h. Let $\mathscr{H}^{p,q}(X)$ denote the space of $\Delta_{\bar{\partial}}$-harmonic forms of type (p, q). We have that:*

(i) $\mathscr{H}^{p,q}(X)$ is finite-dimensional.

(ii) We have an L^2 orthogonal decomposition

$$C^\infty(X, \Omega^{p,q}) = \mathscr{H}^{p,q}(X) \oplus \Delta_{\overline{\partial}}(C^\infty(X, \Omega^{p,q})).$$

Just as we have for the Hodge decomposition in the Riemannian case, we deduce that the above decomposition is equivalent to

$$C^\infty(X, \Omega^{p,q}) = \mathscr{H}^{p,q}(X) \oplus \overline{\partial}(C^\infty(X, \Omega^{p,q-1})) \oplus \overline{\partial}^*(C^\infty(X, \Omega^{p,q+1})), \tag{3.5}$$

where

$$\ker \overline{\partial} = \mathscr{H}^{p,q}(X) \oplus \overline{\partial}(C^\infty(X, \Omega^{p,q+1})),$$
$$\ker \overline{\partial}^* = \mathscr{H}^{p,q}(X) \oplus \overline{\partial}^*(C^\infty(X, \Omega^{p,q+1})).$$

In particular, this implies the following fact about the natural map $\mathscr{H}^{p,q}(X) \to H^{p,q}(X)$, which is well-defined since a $\Delta_{\overline{\partial}}$-harmonic form must be $\overline{\partial}$-closed.

Corollary 3.4. *Retain the hypotheses of Theorem 3.3. Then, the natural map*

$$\mathscr{H}^{p,q}(X) \to H^{p,q}$$

is an isomorphism. In particular, $H^{p,q}(X)$ is finite-dimensional.

In other words, every Dolbeault cohomology class in $H^{p,q}(X)$ has a unique $\Delta_{\overline{\partial}}$-harmonic representative.

3.2 Differential operators on Kähler manifolds

For now, we remove the hypotheses of compactness on (X, h) stated in Section 3.1, but we suppose that X is a closed *Kähler* manifold. Let ω denote the Kähler form corresponding to h, so that the canonical volume form vol^h is given by $\mathrm{vol}_\omega := \omega^n/n$. The map $\eta \mapsto \omega \wedge \eta$ is a differential operator

$$L : C^\infty(X, \Omega^k_{\mathbb{C}}) \to C^\infty(X, \Omega^{k+2}_{\mathbb{C}})$$

of degree 0, which we name the *Lefschetz operator*. Let Λ denote its formal adjoint.

Proposition 3.5. *For*

$$\Lambda : C^\infty(X, \Omega_{\mathbb{C}}^k) \to C^\infty(X, \Omega_{\mathbb{C}}^{k-2}),$$

we have

$$\Lambda\nu = (-1)^k (*L*)\nu$$

for all $\nu \in C^\infty(X, \Omega_{\mathbb{C}}^k)$.

Proof. Since ω is a real differential form, it suffices to prove the claim for an arbitrary $\nu \in C^\infty(X, \Omega_{\mathbb{R}}^k)$. We need to show that

$$\langle L\eta, \nu\rangle = \langle \eta, (-1)^k (*L*)\nu\rangle$$

for all $\eta \in C^\infty(X, \Omega_{\mathbb{R}}^{k+2})$. Check that

$$L\eta \wedge *\nu = \omega \wedge \eta \wedge *\nu = \eta \wedge \omega \wedge *\nu = \eta \wedge * \left((-1)^k * (\omega \wedge *\nu)\right),$$

which suffices. ∎

Note also that since ω is a real differential form, the maps L and Λ are invariant under complex conjugation.

Observe that $[\partial, L] = [\overline{\partial}, L] = [\partial^*, \Lambda] = [\partial^*, \Lambda] = 0$. Indeed, these are all equivalent by applying complex conjugation or the formal adjoint property, and we can check that

$$[\partial, L]\eta = \partial(\omega \wedge \eta) - \omega \wedge \partial\eta = (\partial\omega) \wedge \eta = 0$$

for any $\eta \in C^\infty(X, \Omega_{\mathbb{C}}^k)$. We would like to compute the other commutators with one entry given by L or Λ and the other entry, by one of the Dolbeault operators and their formal adjoints. Motivated by these, we work out the following.

Proposition 3.6 (Kähler identities)**.** *We have*

$$[\overline{\partial}^*, L] = i\partial,$$

$$[\partial^*, L] = -i\overline{\partial},$$
$$[\Lambda, \overline{\partial}] = -i\partial^*,$$
$$[\Lambda, \partial] = i\overline{\partial}^*.$$

Proof. Just like before, it suffices to prove the first identity, since the other identities can then be deduced by applying complex conjugation or the formal adjoint property.

At a given $x \in X$, there exists a local choice of holomorphic coordinates so that h is given by
$$\sum_{j=1}^{n} dz_j \otimes d\overline{z_j}$$
plus possibly an error term of order 2. Thus, we can reduce to the case of $X = U \subset \mathbb{C}^n$, an open set equipped with the standard Kähler form

$$\omega = -\Im\left(\sum_{j=1}^{n} dz_j \otimes d\overline{z_j}\right) = -\Im\left(\sum_{j=1}^{n}(dx_j \otimes dx_j + -2idx_j \otimes dy_j - dy_j \otimes dy_j)\right)$$
$$= 2\sum_{j=1}^{n} dx_j \wedge dy_j = \frac{2}{2\cdot 2i}\sum_{j=1}^{n}(dz_j + d\overline{z_j}) \wedge (dz_j - d\overline{z_j}) = i\sum_{j=1}^{n} dz_j \wedge d\overline{z_j}.$$

Let $\eta \in C^\infty(X, \Omega^k_{\mathbb{C}})$ be arbitrary. Using Lemma 3.2, we compute

$$[\overline{\partial}^*, L]\eta = \overline{\partial}^*(\omega \wedge \eta) - \omega \wedge (\overline{\partial}^*\eta)$$
$$= -\sum_{j=1}^{n}\left(-\partial_{z_j} \lrcorner \partial_{z_j}\left(i\sum_{\ell=1}^{n} dz_\ell \wedge d\overline{z_\ell} \wedge \eta\right) + i\sum_{\ell=1}^{n} dz_\ell \wedge d\overline{z_\ell} \wedge (\partial_{z_j} \lrcorner \partial_{z_j}\eta)\right)$$
$$= -\sum_{j=1}^{n}\left(-\partial_{z_j} \lrcorner\left(i\sum_{\ell=1}^{n} dz_\ell \wedge d\overline{z_\ell} \wedge \partial_{z_j}\eta\right) + i\sum_{\ell=1}^{n} dz_\ell \wedge d\overline{z_\ell} \wedge (\partial_{z_j} \lrcorner \partial_{z_j}\eta)\right)$$
$$= \sum_{j=1}^{n}\left(-0 + idz_j \wedge \eta - i\sum_{\ell=1}^{n} dz_\ell \wedge d\overline{z_\ell} \wedge (\partial_{z_j} \lrcorner \partial_{z_j}\eta) + i\sum_{\ell=1}^{n} dz_\ell \wedge d\overline{z_\ell} \wedge (\partial_{z_j} \lrcorner \partial_{z_j}\eta)\right)$$
$$= i\partial\eta,$$

as needed. □

Also, we prove the following addenda to the Kähler identities that will be useful in our proof of the Lefschetz decomposition in Section 3.4.

Proposition 3.7. *We have*

$$[\Delta_\partial, L] = 0$$

and

$$[L, \Lambda]\eta = H\eta,$$

for all $\eta \in C^\infty(X, \Omega^k_{\mathbb{C}})$, where H denotes the counting operator that sends $\eta \in C^\infty(X, \Omega^k_{\mathbb{C}})$ to $(k-n)\eta$. More generally, for $0 \le k \le n$ and $0 \le r \le n-k$, we have

$$[L^r, \Lambda]\eta = (r(k-n) + r(r-1))L^{r-1}\eta$$

for all $\eta \in C^\infty(X, \Omega^k_{\mathbb{C}})$.

Proof. To show the first claim, we check that

$$\begin{aligned}
[\Delta_\partial, L] &= (\partial\partial^* + \partial^*\partial)L - L(\partial\partial^* + \partial^*\partial) \\
&= \partial(\partial^* L - L\partial^*) + \partial L\partial^* + \partial^*(\partial L - L\partial) + \partial^* L\partial \\
&\quad + (\partial L - L\partial)\partial^* - \partial L\partial^* + (\partial^* L - L\partial^*)\partial - \partial^* L\partial \\
&= \partial(\partial^* L - L\partial^*) + \partial^*(\partial L - L\partial) + (\partial L - L\partial)\partial^* + (\partial^* L - L\partial^*)\partial \\
&= \partial(i\partial) + \partial^* 0 + 0\partial^* + i\partial\partial = 0.
\end{aligned}$$

We now prove the second claim. Since it suffices to prove the claim fiberwise, we may reduce to the case that $X = V = \mathbb{C}^n$ with the standard Hermitian metric. Our proof is by induction on n. Consider the base case $n = 1$. With respect to the standard coordinate $z = x + iy$, we have that L acts as $1 \mapsto \omega$ on $\bigwedge^0 V^*$ and coincides with the zero map otherwise. Likewise, Λ acts as $\omega \mapsto 1$ on $\bigwedge^2 V^*$ and coincides with the zero map otherwise. Thus, $[L, \Lambda]$ acts as $-\Lambda L = -1$ on $\bigwedge^0 V^*$, as the zero map on $\bigwedge^1 V^*$ and as $L\Lambda = 1$ on $\bigwedge^2 V^*$, as needed.

Next, we prove the inductive step. Let $V = \mathbb{C}^n$ for $n > 1$, and assume the inductive hypothesis that $[L, \Lambda] = H$ for all $m < n$. Take a decomposition $X = W_1 \oplus W_2$, compatible with the Kähler structure, for positive-dimensional subspaces $W_1 \cong \mathbb{C}^{n_1}$ and $W_2 \cong \mathbb{C}^{n_2}$. The Kähler form ω on X decomposes as $\omega_1 \oplus \omega_2$, where ω_j denotes the Kähler form on W_j. Thus, L decomposes as the direct sum of the Lefschetz operators L_1 and L_2 of W_1 and W_2, respectively. Specifically, $L = L_1 \otimes 1 + 1 \otimes L_2$ as operators on

$$\bigwedge^{\bullet} V^* \cong \bigwedge^{\bullet} W_1^* \otimes \bigwedge^{\bullet} W_2^*$$

Let $\alpha, \beta \in \bigwedge^{\bullet} V^*$, where we can without loss of generality suppose that they are split, i.e., $\alpha = \alpha_1 \otimes \alpha_2$ and $\beta = \beta_1 \otimes \beta_2$ for $\alpha_j, \beta_j \in \bigwedge^{\bullet} W_j^*$. Then, $\langle \alpha, \beta \rangle = \langle \alpha_1, \beta_1 \rangle \langle \alpha_2, \beta_2 \rangle$, and thus we have

$$\begin{aligned}
\langle \alpha, L\beta \rangle &= \langle \alpha, L_1\beta_1 \otimes \beta_2 \rangle + \langle \alpha, \beta_1 \otimes L_2\beta_2 \rangle \\
&= \langle \alpha_1, L_1\beta_1 \rangle \langle \alpha_2, \beta_2 \rangle + \langle \alpha_1, \beta_1 \rangle \langle \alpha_1, L_2\beta_2 \rangle \\
&= \langle \Lambda_1\alpha_1, \beta_1 \rangle \langle \alpha_2, \beta_2 \rangle + \langle \alpha_1, \beta_1 \rangle \langle \Lambda_2\alpha_2, \beta_2 \rangle \\
&= \langle \Lambda_1\alpha_1 \otimes \alpha_2, \beta_1 \otimes \beta_2 \rangle + \langle \alpha_1 \otimes \Lambda_2\alpha_2, \beta_2 \rangle,
\end{aligned}$$

which shows that Λ also decomposes as the direct sum $\Lambda_1 \otimes 1 + 1 \otimes \Lambda_2$, where Λ_j is the formal adjoint of the Lefshetz operator of W_j. Consequently, we have

$$\begin{aligned}
[L, \Lambda](\alpha_1 \otimes \alpha_2) &= (L_1 \otimes 1 + L_2 \otimes 1)(\Lambda_1\alpha_1 \otimes \alpha_2 + \alpha_1 \otimes \Lambda_2\alpha_2) \\
&\quad - (\Lambda_1 \otimes 1 + 1 \otimes \Lambda_2)(L_1\alpha_1 \otimes \alpha_2 + \alpha_1 \otimes L_2\alpha_2) \\
&= [L_1, \Lambda_1]\alpha_1 \otimes \alpha_2 + \alpha_1 \otimes [L_2, \Lambda_2]\alpha_2 \\
&= (k_1 - n_1)\alpha_1 \otimes \alpha_2 + (k_2 - n_2)(\alpha_1 - \alpha_2) \\
&= (k_1 + k_2 - n_1 - n_2)\alpha_1 \otimes \alpha_2,
\end{aligned}$$

where k_j is such that $\alpha_j \in \bigwedge^{k_j} W_j^*$, and we have used the inductive hypothesis.

To show the final claim, note that

$$[L^r, \Lambda] = L[L^{r-1}, \Lambda] + [L, \Lambda]L^{r-1},$$

so we prove by induction. The base case $r = 0$ is trivial, and $r = 1$ has been shown. For the inductive step, we have

$$\begin{aligned}[L^r, \Lambda]\eta &= L((r-1)(k-n) + (r-1)(r-2))L^{r-2}\eta + (k + 2r - 2 - n)L^{r-1}\eta \\ &= (r(k-n) + r(r-1))L^{r-1}\eta.\end{aligned}$$

□

The Kähler identities allow us to prove the following fundamental relationship between $\Delta_d, \Delta_\partial$ and $\Delta_{\overline{\partial}}$.

Theorem 3.8. *We have*

$$\Delta_d = 2\Delta_\partial = 2\Delta_{\overline{\partial}}.$$

In particular, the conditions of a k-form being harmonic, Δ_∂-harmonic, and $\Delta_{\overline{\partial}}$-harmonic are equivalent.

Proof. We show the first equality. Since $d = \partial + \overline{\partial}$, we have

$$\Delta_d = (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) + (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}).$$

By Proposition 3.6, we have $\overline{\partial}^* = -i[\Lambda, \partial]$, which implies

$$(\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) = \partial\partial^* - i\partial\Lambda\partial + \overline{\partial}\partial^* - i\overline{\partial}\Lambda\partial + i\overline{\partial}\partial\Lambda$$

and

$$(\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}) = \partial^*\partial + i\partial\Lambda\partial + \partial^*\overline{\partial} + i\partial\Lambda\overline{\partial} - i\Lambda\partial\overline{\partial}.$$

Proposition 3.6 also gives us that $\partial^* = i[\Lambda, \overline{\partial}]$, so we have

$$\partial^*\overline{\partial} = -i\overline{\partial}\Lambda\overline{\partial} = -\overline{\partial}\partial^*. \tag{3.6}$$

We thus have

$$\begin{aligned}\Delta_d &= \partial\partial^* - i\overline{\partial}\Lambda\partial + i\overline{\partial}\partial\Lambda + \partial^*\partial + i\partial\Lambda\partial\overline{\partial} \\ &= \Delta_\partial - i\Lambda\partial\overline{\partial} - i\overline{\partial}\Lambda\partial + i\partial\Lambda\overline{\partial} + i\overline{\partial}\partial\Lambda \\ &= \Delta_\partial + i\Lambda\overline{\partial}\partial - i\overline{\partial}\Lambda\partial + i\partial\Lambda\overline{\partial} - i\partial\overline{\partial}\Lambda \\ &= \Delta_\partial + i(\Lambda\overline{\partial} - \overline{\partial}\Lambda)\partial + i\partial(\Lambda\overline{\partial} - \overline{\partial}\Lambda) \\ &= \Delta_\partial + \partial^*\partial + \partial\partial^* = 2\Delta_\partial,\end{aligned}$$

where we have used the fact that $\partial\overline{\partial} = -\overline{\partial}\partial$ and Proposition 3.6. The proof of the equality for $\Delta_{\overline{\partial}}$ is analogous. □

This relationship between the Laplacian operators of a Kähler manifold has a number of important consequences, one of which is the following corollary that will be key in proving the Hodge decomposition for Kähler manifolds.

Corollary 3.9. *Let $\eta \in C^\infty(X, \Omega^{p,q})$. Then, $\Delta_d\eta$ is also in $C^\infty(X, \Omega^{p,q})$.*

Proof. Immediately follows from $\Delta_d\eta = (\Delta_\partial\eta)/2$. □

This naturally leads us to the following fundamental result regarding the space $\mathscr{H}^k(X) := \mathcal{H}^k(X_\mathbb{R}) \otimes_\mathbb{R} \mathbb{C}$ of harmonic forms on Kähler manifolds.

Theorem 3.10. *Let $u \in C^\infty(X, \Omega^k_\mathbb{C})$, and denote its decomposition into components of type (p,q) by*

$$u = \sum_{p+q=k} u^{p,q}.$$

Then, u is harmonic if and only if each $u^{p,q}$ is harmonic. In particular, we have the decomposition

$$\mathscr{H}^k(X) = \bigoplus_{p+q=k} \mathscr{H}^{p,q}(X),$$

where $\mathscr{H}^{p,q}$ denotes the space of harmonic forms of type (p,q). Moreover, we have

$$\mathscr{H}^{p,q}(X) = \overline{\mathscr{H}^{q,p}(X)}.$$

Proof. Since Δ_d maps (p,q)-type components to (p,q)-type components by Corollary 3.9, u is harmonic if and only if each $u^{p,q}$ is harmonic, and we have the claimed decomposition. Furthermore, for a harmonic form η of type (p,q), we have that $\overline{\eta}$ is of type (q,p) and satisfies

$$\Delta_\partial \overline{\eta} = \overline{\Delta_{\overline{\partial}} \eta} = \overline{\Delta_\partial \eta} = 0$$

by Theorem 3.8, and thus is harmonic. □

In the case that X is compact, the above theorem combined with Corollary 2.3—which yields $\mathscr{H}^k(X) \cong H^k(X, \mathbb{C})$, where we have used de Rham's theorem to justify our notation $H^k(X, \mathbb{R}) := H^k_{dR}(X_\mathbb{R})$ and $H^k(X, \mathbb{C}) := H^k_{dR}(X_\mathbb{R}) \otimes_\mathbb{R} \mathbb{C}$ that we will use from now on—and Corollary 3.4—which gives the isomorphism $\mathscr{H}^{p,q} \cong H^{p,q}(X)$—gives us the main theorem of this section.

Theorem 3.11 (Hodge)**.** *Let (X, h) be a compact, closed Kähler manifold. Then, we have the Hodge decomposition*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X), \tag{3.7}$$

and Hodge duality

$$H^{q,p}(X) = \overline{H^{p,q}(X)}.$$

Proof. The proof of Hodge duality follows immediately from the remarks in the paragraph preceding the theorem statement. However, regarding the Hodge decomposition, these remarks only show the weaker statement that

$$H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X).$$

The stronger statement of (3.7) states that the isomorphism is canonical, in the sense that it does not depend on the choice of the Hermitian metric h, which determines the Hodge star operator and the Laplacian operators, and thus could *a priori* affect the isomorphism. The fact that the isomorphism is canonical will be proven in Section 3.3. □

The Hodge decomposition yields a wealth of information on the Betti numbers b_k and their analogue, $h_{p,q} := \dim_{\mathbb{C}} H^{p,q}(X)$, associated to X.

Corollary 3.12. *Retain the hypotheses of Theorem 3.11. We have*

$$b_k = \sum_{p+q=k} h^{p,q} \qquad \forall k \in \mathbb{Z}_{\geq 0},$$

$$h^{p,q} = h^{q,p} \qquad \forall p, q \in \mathbb{Z}_{\geq 0},$$

$$H^{k,k}(X) \ncong 0 \qquad \forall k \in \{1, \ldots, n\},$$

and b_k is even for odd k.

Proof. The first, second and fourth claims are immediately deduced from the Hodge decomposition and Hodge duality. The third claim follows from applying to the Hodge decomposition the fact that the de Rham cohomology class $[\omega^k]$ of the kth power of the Kähler form is a nonzero element in $H^{2k}(X_{\mathbb{R}}, \mathbb{R})$, and thus a complex-conjugation-invariant nonzero element in $H^{2k}(X, \mathbb{C})$. Indeed, suppose for the sake of a contradiction that ω^k is exact, say equal to $d\eta$. Then,

$$\int_X \omega^n = \int_X d(\eta \wedge \omega^{n-k}) = \int_{\partial X} \eta \wedge \omega^{n-k} = 0,$$

which is a contradiction, since ω^n is a nontrivial volume form. □

As an instructive example, let us obtain the Dolbeault cohomology groups of $\mathbb{P}^n$ using the Hodge decomposition.

Corollary 3.13. *We have*

$$H^{p,q}(\mathbb{P}^n) \cong \begin{cases} \mathbb{C} & \textit{if } 0 \leq p = q \leq n, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. It is well-known that

$$H^{2k}(\mathbb{P}^n, \mathbb{C}) \cong \begin{cases} \mathbb{C} & \text{if } 0 \leq k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

However, $\mathbb{P}^n$ is Kähler, so by Corollary 3.12, we have $H^{k,k}(\mathbb{P}^n) \ncong 0$ for $0 \leq k \leq n$. The claim then follows from the Hodge decomposition. □

3.3 Bott–Chern cohomology and the $\partial\overline{\partial}$-Lemma

As stated before, we will prove that the isomorphism in the Hodge decomposition for compact, closed Kähler manifolds does not depend on the choice of the Hermitian metric. To accomplish this, we define a new notion of cohomology that is independent of the choice of Hermitian metric. For a complex manifold X, define the *Bott–Chern cohomology groups of* X by

$$H^{p,q}_{BC}(X) := \{\eta \in C^\infty(X, \Omega^{p,q}) : d\eta = 0\}/\partial\overline{\partial}C^\infty(X, \Omega^{p-1,q-1}).$$

Since $d\partial\overline{\partial} = 0$, the natural map

$$\{\eta \in C^\infty(X, \Omega^{p,q}) : d\eta = 0\} \to H^{p+q}(X, \mathbb{C})$$

induces a canonical map

$$H^{p,q}_{BC}(X) \to H^{p+q}(X, \mathbb{C}).$$

Moreover, since a d-closed form is also $\overline{\partial}$-closed, we have a natural map

$$\{\eta \in C^\infty(X, \Omega^{p,q}) : d\eta = 0\} \to H^{p,q}(X),$$

and it follows from $\overline{\partial}\partial\overline{\partial} = 0$ that we have a canonical map

$$H^{p,q}_{BC}(X) \to H^{p,q}(X).$$

Under the hypotheses of Theorem 3.11, we will see that the de Rham/singular, Dolbeault and Bott–Chern cohomologies are interrelated. To show this, we will need the following lemma, often called the *$\partial\overline{\partial}$-lemma*.

Lemma 3.14 ($\partial\overline{\partial}$-lemma)**.** *Retain the hypotheses of Theorem 3.11. Let η be a d-closed form of type (p, q). If η is ∂- or $\overline{\partial}$-exact, then there exists a form β of type $(p-1, q-1)$*

such that $\eta = \partial\overline{\partial}\beta$.

Proof. We first prove the claim for η ∂-exact, say $\eta = \partial\nu$. Since η is d-closed, it is also ∂- and $\overline{\partial}$-closed. It follows from (3.5) that ν can be decomposed as

$$\nu = \alpha + \overline{\partial}\beta + \overline{\partial}^*\gamma$$

for $\alpha \in \mathscr{H}^{p,q-1}(X)$, $\beta \in C^\infty(X, \Omega^{p-1,q-1})$ and $\gamma \in C^\infty(X, \Omega^{p-1,q+1})$. Since $\Delta_\partial\alpha = \Delta_{\overline{\partial}}\alpha = 0$, it follows that α is ∂-closed, and thus

$$\eta = \partial\overline{\partial}\beta + \partial\overline{\partial}^*\gamma.$$

We have $\partial\overline{\partial} = -\overline{\partial}\partial$, and we can take complex conjugates in (3.6) to see that $\partial\overline{\partial}^* = -\overline{\partial}\partial^*$. Thus, we have

$$\eta = -\overline{\partial}\partial\beta - \overline{\partial}^*\partial\gamma.$$

In particular, taking $\overline{\partial}$ of both sides yields that $0 = -\overline{\partial}\overline{\partial}^*\partial\gamma$. Then, $\overline{\partial}^*\partial\gamma$ is both $\overline{\partial}$-closed and $\overline{\partial}^*$-exact, which necessitates that γ is $\Delta_{\overline{\partial}}$-harmonic, a contradiction unless $\gamma = 0$.

The proof of the claim under the hypothesis that η is $\overline{\partial}$-exact is similar, and follows from the analogue of Theorem 3.3—and consequently, (3.5)—for the operator ∂. □

Equipped with the $\partial\overline{\partial}$-lemma, we are now ready to prove our desired relationship between the cohomology theories.

Corollary 3.15. *Retain the hypotheses of Theorem 3.11. Then, the canonical maps*

$$H^{p,q}_{BC}(X) \to H^{p,q}(X)$$

and

$$\bigoplus_{p+q=k} H^{p,q}_{BC}(X) \to H^k(X, \mathbb{C})$$

are isomorphisms. In particular, the Hodge decomposition isomorphism of Theorem 3.11 is canonical.

Proof. Recall from Corollary 3.4 that $H^{p,q}(X) \cong \mathscr{H}^{p,q}(X)$. In other words, every Dolbeault cohomology class has a unique representative that is $\Delta_{\overline{\partial}}$-harmonic, or equivalently by Theorem 3.8, Δ-harmonic. Since Δ-harmonic forms are d-closed, we see that the map $H^{p,q}_{BC}(X) \to H^{p,q}(X)$ is surjective.

It remains to verify injectivity. Consider $\gamma \in H^{p,q}_{BC}(X)$ whose image in $H^{p,q}(X)$ is 0. Let $\eta \in C^\infty(X, \Omega^{p,q})$ be a d-closed representative of γ in $H^{p,q}_{BC}$. By our hypothesis, η is $\overline{\partial}$-exact. Thus, by the $\partial\overline{\partial}$-lemma, we have that η is in the image of $\partial\overline{\partial}$, and thus $\gamma = 0$.

We have verified the first claim the canonical map $H^{p,q}_{BC}(X) \to H^{p,q}(X)$ is an isomorphism. In light of Corollary 2.3 and Theorem 3.10 together implying that

$$H^k(X, \mathbb{C}) \cong \mathscr{H}^k(X) = \bigoplus_{p+q=k} \mathscr{H}^{p,q}(X),$$

we can apply the first claim to the above, which proves the second claim. □

3.4 Lefschetz decomposition and the Hodge index theorem

Remove the compactness hypothesis again, so we suppose X is a closed Kähler manifold of dimension n. The Lefschetz operator L defined in Section 3.2 can be thought of as a bundle morphism over X,

$$L : \Omega^k_{\mathbb{R}} \to \Omega^{k+2}_{\mathbb{R}},$$

since ω is a real-valued form. In this section, we wish to show that the map

$$L^{n-k} : \Omega^k_{\mathbb{R}} \to \Omega^{2n-k}_{\mathbb{R}}$$

is a bundle isomorphism. Since $\Omega^k_{\mathbb{R}}$ and $\Omega^{2n-k}_{\mathbb{R}}$ are vector bundles of the same finite rank, it suffices to show that the morphism is injective at a fiber over an arbitrary point $x \in X$. We will show this on the way to proving the Lefschetz decomposition for k-forms.

It will be useful to note that on $\bigwedge^\bullet V^*$ (where $V \cong \mathbb{C}^n$ has the standard Hermitian structure, and in particular can denote a fiber of the tangent bundle of X), the operators L, Λ and H define a natural $\mathfrak{sl}(2)$-representation. Indeed, $\mathfrak{sl}(2)$ (over $\mathbb{R}$ or $\mathbb{C}$, depending on which field V is defined over) is the three-dimensional Lie algebra of 2×2 traceless matrices. A basis of $\mathfrak{sl}(2)$ is given by $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), f = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, with relations $[h, e] = 2e, [h, f] = -2f$, and $[e, f] = h$. Thus, the map given by $e \mapsto L, f \mapsto \Lambda$ and $h \mapsto H$ defines a Lie algebra morphism $\mathfrak{sl}(2) \to \mathrm{End}(\bigwedge^\bullet V^*)$, as needed.

Before we prove the main machinery behind this section below, we will need the following definition. We call a form $\alpha \in \bigwedge^k V^*$ *primitive* if $\Lambda\alpha = 0$. Let $P^k \subset \bigwedge^k V^*$ denote the subspace of primitive forms.

Proposition 3.16. *Let $V \cong \mathbb{C}^n$ have the standard Hermitian structure.*

(i) We have the Lefschetz decomposition

$$\bigwedge^k V^* = \bigoplus_{0 \leq j \leq \frac{k}{2}} L^j(P^{k-2j}),$$

which is orthogonal with respect to the inner product.

(ii) For $k > n$, we have $P^k = 0$.

(iii) For $k \leq n$, the map $L^{n-k} : \bigwedge^k V^* \to \bigwedge^{2n-k} V^*$ is injective on P^k.

(iv) For $k \leq n$, the map $L^{n-k} : \bigwedge^k V^* \to \bigwedge^{2n-k} V^*$ is bijective.

(v) For $k \leq n$, we have $P^k = \{\alpha \in \bigwedge^k V^* : L^{n-k+1}\alpha = 0\}$.

Proof. (i). We have shown that $\bigwedge^\bullet V^*$ is a finite-dimensional $\mathfrak{sl}(2)$-representation, and by the semisimplicity of $\mathfrak{sl}(2)$, we can write $\bigwedge^\bullet V^*$ as a direct sum of irreducible subrepresentations. Due to finite-dimensionality of $\bigwedge^\bullet V^*$, any subrepresentation has a primitive vector v, i.e., one that satisfies $\Lambda v = 0$. For any such primitive vector, Proposition 3.7 shows that the subspace spanned by $v, Lv, L^2v, \ldots$ defines a subrepresentation. It follows that this is the general form of the irreducible subrepresentations that comprise $\bigwedge^\bullet V^*$ by direct sum. This proves the Lefschetz decomposition, and the fact that it is orthogonal with respect to the inner product follows from Proposition 3.7.

(ii). Suppose for the sake of a contradiction that there exists nonzero $\alpha \in P^k$. Let $j > 0$

minimal such that $L^j\alpha = 0$. Then, by Proposition 3.7, we have

$$0 = [L^j, \Lambda]\alpha = j(k - n + j - 1)L^{j-1}\alpha. \tag{3.8}$$

In light of our hypothesis $k > n$ and the minimality of j, it follows that $j = 0$, which is a contradiction.

(iii). Consider a nonzero element $\alpha \in P^k$, and let $j > 0$ be minimal such that $L^j\alpha = 0$. The identity (3.8) still holds, and thus $k - n + j - 1 = 0$, i.e., $j = n - k + 1$. It follows that $L^{n-k}\alpha \neq 0$, as needed.

(iv). This follows from $(i), (ii)$ and (iii).

(v). In the proof of (iii), we have shown that $P^k \subseteq \ker(L^{n-k+1})$. We now show the converse. Let $\alpha \in \bigwedge^k V^*$ such that $L^{n-k+1}\alpha = 0$. We have

$$L^{n-k+2}\Lambda\alpha = L^{n-k+2}\Lambda\alpha - \Lambda L^{n-k+2}\alpha = (n - k + 2)L^{n-k+1}\alpha = 0.$$

However, we know from (iv) that the map L^{n-k+2} is injective, so $\Lambda\alpha = 0$, as needed. $\square$

Applying Proposition 3.16 fiberwise, we deduce that the bundle morphism

$$L^{n-k} : \Omega^k_{\mathbb{R}} \to \Omega^{2n-k}_{\mathbb{R}} \tag{3.9}$$

is in fact an isomorphism. This fact will, once we add back the compactness condition, allow us to prove a beautiful duality statement known as the Hard Lefschetz theorem. Before we do so, note that since ω is d-closed, the map

$$L : C^\infty(X, \Omega^k_{\mathbb{R}}) \to C^\infty(X, \Omega^{k+2}_{\mathbb{R}})$$

induces a map

$$L : H^k(X, \mathbb{R}) \to H^{k+2}(X, \mathbb{R}).$$

We prove the following result about this induced map.

Theorem 3.17 (Hard Lefschetz theorem)**.** *Let (X, h) be a compact, closed Kähler manifold* *of dimension n. Then, for $0 \leq k \leq n$, the map*

$$L^{n-k} : H^k(X, \mathbb{R}) \to H^{2n-k}(X, \mathbb{R})$$

is an isomorphism.

Proof. By Proposition 3.16 and Theorem 3.8, we know that Δ_d commutes with L, or equivalently, that L maps harmonic forms to harmonic forms. We can thus consider the induced map

$$L^{n-k} : H^k(X, \mathbb{R}) \to H^{2n-k}(X, \mathbb{R}) \tag{3.10}$$

By Corollary 2.3, it suffices to show that (3.10) is an isomorphism. By Theorem 2.2 and Corollary 2.4 (Poincaré duality), $\mathcal{H}^k(X)$ and $\mathcal{H}^{2n-k}(X)$ have the same finite dimension. Moreover, we have previously shown that (3.9) is a bundle isomorphism, and thus

$$L^{n-k} : C^\infty(X, \Omega^k_{\mathbb{R}}) \to C^\infty(X, \Omega^k_{\mathbb{R}})$$

is an isomorphism, and in particular injective. However, this shows in particular that L^{n-k} is injective on $\mathcal{H}^k(X)$, from which the statement follows by the fact that the domain and the range have equal finite dimension. $\square$

Retain the hypotheses of Theorem 3.17. We have seen that L (and consequently, Λ and H) commute with Δ_d, which means all three operators induce corresponding maps on $H^\bullet(X, \mathbb{R})$. Thus, we can define $[\eta] \in H^k(X, \mathbb{R}), 0 \leq k \leq n$ to be *primitive* if $\Lambda[\eta] = 0$, or equivalently (by Proposition 3.16(v)), $L^{n-k+1}[\eta] = 0$. Let $H^k(X, \mathbb{R})_{\text{prim}} \subset H^k(X, \mathbb{R})$ denote the subspace of primitive cohomology classes. Since the cohomology algebra $H^\bullet(X, \mathbb{R})$ is finite-dimensional, we can use a proof similar to that of Proposition 3.16(i) to deduce the following decomposition.

Theorem 3.18 (Lefschetz decomposition)**.** *Retain the hypotheses of Theorem 3.17. Then, we have the Lefschetz decomposition*

$$H^k(X, \mathbb{R}) = \bigoplus_{0 \leq j \leq \frac{k}{2}} L^j H^{k-2j}(X, \mathbb{R})_{\text{prim}}.$$

We note that every result we have obtained in this section can be done in the analogous setting for Dolbeault cohomology. Indeed, since the Kähler form ω is $\overline{\partial}$-closed, so the map

$$L : C^\infty(X, \Omega^{p,q}) \to C^\infty(X, \Omega^{p+1,q+1})$$

induces a map

$$L : H^{p,q}(X) \to H^{p+1,q+1}(X).$$

We can apply the Hodge decomposition and a proof analogous to that of Theorem 3.17 to obtain the following result over complex coefficients.

Theorem 3.19 (Hard Lefschetz theorem, Dolbeault-cohomology version)**.** *Retain the hypotheses of Theorem 3.17. Then, for all $p, q \geq 0$ such that $p + q \leq n$, the map*

$$L^{n-p-q} : H^{p,q}(X) \to H^{n-q,n-p}(X)$$

is an isomorphism.

An important application of the Lefchsetz decomposition is to the computation of the signature of the intersection form on $H^2(X, \mathbb{C})$ for a given compact Kähler surface X, a result known as the Hodge index theorem. To prepare the setting, recall from Corollary 2.4 (Poincaré duality) that we have a nondegenerate symmetric bilinear pairing

$$Q : H^2(X, \mathbb{R}) \times H^2(X, \mathbb{R}) \to \mathbb{R}$$

defined by $([\eta], [\nu]) \mapsto \int_X \eta \wedge \nu$. The sesquilinear form $\tilde{Q}([\eta], [\nu]) := Q([\eta], \overline{[\nu]})$ is in fact a Hermitian form on $H^2(X, \mathbb{C})$. The Lefschetz decomposition happens to be orthogonal with respect to this form.

Lemma 3.20. *Let X be a compact, closed Kähler manifold of dimension 2. We have the Lefschetz decomposition*

$$H^2(X, \mathbb{C}) = H^2(X, \mathbb{C})_{\text{prim}} \oplus \mathbb{C}[\omega],$$

which is orthogonal with respect to $\tilde{Q}$.

Note that the subspace $H^k(X,\mathbb{C})_{\text{prim}} \subset H^k(X,\mathbb{C})$ is defined analogously to $H^k(X,\mathbb{R})_{\text{prim}}$.

Proof. By base-changing the isomorphism in Theorem 3.18 to $\mathbb{C}$, we obtain the Lefschetz decomposition

$$H^2(X,\mathbb{C}) = H^2(X,\mathbb{C})_{\text{prim}} \oplus LH^0(X,\mathbb{C})_{\text{prim}} = H^2(X,\mathbb{C})_{\text{prim}} \oplus \mathbb{C}[\omega].$$

Let $[\eta] \in H^2(X,\mathbb{C})_{\text{prim}}$, or equivalently, $\omega \wedge \eta = 0$. We have

$$\tilde{Q}(\omega, \eta) = \int_X \omega \wedge \eta = 0,$$

which proves the second claim. □

The orthogonality with respect to $\tilde{Q}$ will be key in proving the following.

Theorem 3.21 (Hodge index theorem)**.** *Let X be a compact, closed Kähler manifold of dimension 2. Then, the signature of the intersection form Q on $H^2(X,\mathbb{R}) \cap H^{1,1}(X)$ is given by $(1, h^{1,1} - 1)$.*

Proof. Recall that Lemma 3.20 states that we have a Lefschetz decomposition

$$H^2(X,\mathbb{C})_{\text{prim}} \oplus \mathbb{C}[\omega]$$

that is orthogonal with respect to $\tilde{Q}$. In particular, the decomposition

$$H^2(X,\mathbb{R}) \cap H^{1,1}(X) = H^2(X,\mathbb{R}) \cap (H^{1,1}(X)_{\text{prim}} \oplus \mathbb{C}[\omega]) = (H^2(X,\mathbb{R}) \cap H^{1,1}(X)_{\text{prim}}) \oplus \mathbb{R}[\omega],$$

is orthogonal with respect to $\tilde{Q}|_{H^2(X,\mathbb{R})} = Q$, where we have used that $\mathbb{C}[\omega] \subset H^{1,1}(X)$. Since ω is a volume form, we have that $Q(\omega,\omega) > 0$. Thus, Q is positive definite on the 1-dimensional subspace $\mathbb{R}[\omega]$.

It remains to show that Q is negative definite on the orthogonal complement $H^2(X,\mathbb{R}) \cap H^{1,1}(X)_{\text{prim}}$. Consider a nonzero class $[\nu] \in H^{1,1}(X)_{\text{prim}}$ with unique harmonic representative η. We claim that $*\overline{\nu} = -\overline{\nu}$. By considering an arbitrary local trivialization, we can reduce to the case of an open set $U \subset \mathbb{C}^2$ with the standard metric $h = dz_1 \otimes d\overline{z_1} + dz_2 \otimes d\overline{z_2}$

and the associated Kähler form $\omega = i(dz_1 \wedge d\overline{z_1} + dz_2 \wedge d\overline{z_2})$. Denote ν in local coordinates by

$$\nu = \nu_{11} dz_1 \wedge d\overline{z_1} + \nu_{12} dz_1 \wedge d\overline{z_2} + \nu_{21} dz_2 \wedge d\overline{z_1} + \nu_{22} dz_2 \wedge d\overline{z_2}.$$

Then, by the formula (3.2), we have

$$\begin{aligned} 0 &= \omega \wedge \nu \\ &= i(dz_1 \wedge d\overline{z_1} + dz_2 \wedge d\overline{z_2}) \wedge (\nu_{11} dz_1 \wedge d\overline{z_1} + \nu_{12} dz_1 \wedge d\overline{z_2} + \nu_{21} dz_2 \wedge d\overline{z_1} + \nu_{22} dz_2 \wedge d\overline{z_2}) \\ &= i(\nu_{11} + \nu_{22}) dz_1 \otimes d\overline{z_1}, \end{aligned}$$

which shows that $\nu_{22} = -\nu_{11}$. Also, by the formula (3.3), we have

$$*\overline{\nu} = -\overline{\nu_{22}} dz_1 \wedge d\overline{z_1} + \overline{\nu_{21}} dz_1 \wedge d\overline{z_2} + \overline{\nu_{12}} dz_2 \wedge d\overline{z_1} - \overline{\nu_{11}} dz_2 \wedge d\overline{z_2}.$$

Comparing this to

$$\overline{\nu} = -\overline{\nu_{11}} dz_1 \wedge d\overline{z_1} - \overline{\eta_{21}} dz_1 \wedge d\overline{z_2} - \overline{\nu_{12}} dz_2 \wedge d\overline{z_1} - \overline{\eta_{22}} dz_2 \wedge d\overline{z_2},$$

we indeed obtain that $*\overline{\nu} = -\overline{\nu}$.

This allows us to observe that for any $[\eta] \in H^2(X, \mathbb{R}) \cap H^{1,1}(X)_{\text{prim}}$, we have

$$Q([\eta], [\eta]) = \int_X \eta \wedge \eta = -\int_X \eta \wedge *\eta = -\|\eta\|_{L^2} < 0$$

which shows that Q is indeed negative definite on the $(h^{1,1} - 1)$-dimensional subspace $H^2(X, \mathbb{R}) \cap H^{1,1}(X)_{\text{prim}}$, as needed. □

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