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Equilibrium with agglomeration economies

John H. Lindsey II, John W. Pratt, Richard J. Zeckhauser*

Kennedy School of Government, Harvard University, 79 JFK Street, Cambridge, MA 02138, USA

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Abstract

We represent the desires of agents to be close to one another, and to such features as harbors or museums, by disutility of distance functions. Each individual (or firm) adds disutilities, then selects the location that minimizes the sum.

We find the spatial (Nash) equilibrium, first on a line, and then for higher dimensional spaces. Under reasonable conditions on disutility functions, individuals will have a unique optimal choice in reaction to the choices of others. Moreover, the equilibrium will be unique, though not Pareto optimal. Surprisingly, some Pareto-superior outcomes are actually less dense than the equilibrium.

Keywords: Agglomeration; Spatial equilibrium; Location theory; Location model; Proximity

JEL classification: R10; R13; R12

0. Introduction

The clustering of firms and individuals fosters economic activity. By facilitating the exchange of information, thickening markets, and reducing transportation costs, concentrating economic activity proves beneficial. Concentration also permits residents to capitalize on economies of scale in consumption, to enjoy such facilities as theaters and museums, specialized retail establishments, a diversity of restaurants, and the like. In the classic

* Corresponding author. Phone: (617) 495-1174. Fax: (617) 496-3783. E-mail: richardz@ksg1.harvard.edu.

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words of Jane Jacobs (1969, p. 6), "Cities have long been acknowledged as primary organs of cultural development; that is, of the vast and intricate collections of ideas and institutions called civilization."

Our discussion employs the metaphor of physical location decisions. The analysis, however, is perfectly general, and applies to any space of attributes where decision makers have a personal preferred position which is balanced against the desire to be close to others. For example, individuals selecting political attitudes, modes of behavior, or approaches to scientific problems might be concerned with how closely aligned they are to their colleagues and friends, yet might be pulled as well toward their own underlying beliefs and philosophies. Our model is one of pure location choice; rents, if any, are exogenous.

The new growth theory in economics – see the pioneering works of Romer (1986) and Lucas (1988) – pays fundamental attention to the gains from clustering. Romer (1986, p. 1003) stresses the role of knowledge in fostering growth, and the natural externality associated with its production. "The creation of new knowledge by one firm is assumed to have a positive external effect on the production possibilities of other firms because knowledge cannot be perfectly patented or kept secret." Presumably, the more closely the firms are clustered, the greater the spillovers. Lucas, addressing the puzzle of why rates of economic development do not converge across nations, also stresses the central role played by another externality, the external benefits that accretions of human capital have on the productivity of all factors of production. Lucas makes comparisons based on national data, the most readily available source. The external effects he discusses would apply on whatever geographic scale factors interact, such as cities or regions. Intra-nation migration may spread the benefits that originally derive from agglomeration on a more local basis.

There is direct evidence of agglomeration effects at the city level.[1] In assessing the economic factors that account for the existence of urban areas, Nakamura (1985, p. 108) concludes that: "Agglomeration economies are the most important in explaining modern cities." Following Isard (1956), he divides agglomeration benefits into two categories: localization and urbanization economies. The former relate to external economies in a city among firms within an industry. Urbanization economies refer to city-specific external effects among firms in different industries. Looking at Japanese

[1] The city as public good and the city as production center are two well established functions in the urban economics literature. Papageorgiou (1979) adds the third role of city as service center, where cities reduce aggregate production costs, one possible interpretation of our formulation below.

urban manufacturing firms, he finds that firms in light industries experience greater urbanization economies in production; for heavy industries, localization economies are more significant.

Segal (1976, p. 347), in work using U.S. data for 1967, found an agglomeration effect "makes units of labor and capital 8% more productive" in the largest cities. Fogarty and Garofalo (1988, p. 69) examine agglomeration economies in Standard Metropolitan Statistical Areas (SMSAs) in the United States. They find clear evidence of agglomeration economies. Of the virtues of density, they conclude that: "declining central densities and flattening of density gradients may have reduced productivity growth over the period 1957–77."

International experience is suggestive on the relationship between density and productivity. Japan and the Gang of Four (Korea, Hong Kong, Taiwan, and Singapore), exemplar nations for economic development in the past few decades, are all geographically small countries with highly concentrated populations. In both the developed and developing world, the increasing concentration of populations in cities and their environs has been a salient phenomenon. (The decline of many American cities may have come about because such factors as racial tension, pollution, congestion, and crime, more than counterbalanced agglomeration benefits.)

Assuming that people do wish to live in close proximity to each other, and that there are certain existing features – harbors, highways, universities – that are also attractors, what sort of spatial equilibrium will result? We represent preferences for proximity by an additive disutility of distance function.[2] We examine the spatial equilibrium assuming each individual maximizes for himself, ignoring the well-being and response of others. That is, we seek a Nash equilibrium. See Fujita's (1990) excellent overview of this subject area. He refers to situations where the agent chooses location freely, taking the locations of others as fixed, as the general-location equilibrium problem.

We shall be interested in three primary questions. Does an equilibrium

[2] Earlier analyses have also employed disutility of distance functions, though usually in a more specialized form. Ogawa and Fujita (1980), for example, develop a non-monocentric model of land use based on the assumption that transaction costs between any two firms are equal to a constant times the distance between the two.

Papageorgiou and Thisse (1985) consider firm and household locations on a line. Households are attracted to firms because of the opportunities they offer. Firms are attracted to households because of profit opportunities. Both households and firms are repelled by their own type, households because of congestion, firms because of intensified competition. Despite the repulsion elements, the spatial equilibrium is an agglomeration.

exist? If so, will it be unique? And will it be Pareto optimal? Section 1 briefly lays out the model. Section 2 addresses the existence and uniqueness questions for the one-dimensional case. Section 3 considers these issues in multiple dimensions. Section 4 shows that equilibria will in general be inefficient. Section 5 concludes.

1. The model

We find a general-location equilibrium, first on a line, and then for higher dimensional spaces. We show that under reasonable conditions on disutility functions – stronger in multiple dimensions – there will be a unique equilibrium.[3] Though our discussion is phrased in terms of individuals selecting locations, the analysis applies immediately if some or all of the individuals are in fact firms or government agencies.

There are two components to an individual's welfare: his distance from other individuals, and his distance from his ideal location in relation to existing physical features. In each case closer is better. Our formulation is presented in terms of disutility, where disutility is separable and additive in the two components. Disutility is assumed to be convex, i.e. to have increasing marginal costs of distance.

More formally, individual i chooses position x_i, $i = 1, 2, \ldots, n$. Let $f_{ij}(x_i - x_j)$ be the disutility to i of separation from j. Let $g_i(x_i)$ be the disutility to i of less than ideal physical location as regards features such as sports facilities or geographic amenities. If there is an exogenous rent function, it can be incorporated in each g_i.

[3] In effect, we solve for an equilibrium assuming either no rent gradient, or an exogenous rent gradient which is reflected in the disutility of distance function. The no-rent case is consistent with examples where one wants to be close to others in behavior or scientific beliefs, say, or any situation where space is not scarce, perhaps because 'residents' do not take up space. Future work should introduce rents into these models. When there is a preference for proximity, rents rise if as more individuals choose the same location it becomes increasingly expensive to accommodate them, for example because high-rise buildings cost more per square foot than low-rise. All residents at a location pay the marginal cost of the last entrant. The landlord reaps infra-marginal rents. Such rents may push a present resident away, inefficiently so if the benefits the departer's proximity yields to others exceeds the gap between the rental price and the departer's willingness-to-pay to reside at the location. If locations compete to become high-density areas, implying that they will yield rents to landlords, then the landlords will be willing to lose monies at the start-up stage. Subsidies to early committers represent an efficient form of early loss; newly constructed office buildings with a good chance to stand vacant are an inefficient form.

2. One-dimensional case

We invoke two assumptions.

Assumption I: Convex disutility. The domain of possible positions is an interval and each individual i has disutility

$$\sum_j f_{ij}(x_i - x_j) + g_i(x_i) \tag{1}$$

where all f_{ij} and g_i are non-negative, differentiable, convex functions, all $f_{ij}(0) = 0$, and each g_i achieves its minimum on a non-empty set B_i.[4]

The cost functions f_{ij} on separation between individuals are U-shaped with a minimum at 0; the g_i can have a minimum anywhere, including at one end of the interval of possible positions. None of these functions need be symmetric or strictly convex.

Thus for i it would be ideal to have everyone located at a single point in B_i, and outside B_i would be strictly worse; accordingly, we call B_i i's ideal set. It follows from the continuity and convexity of g_i that B_i is a closed interval. We allow g_i to be constant (B_i to be the entire space), as it would be for a purely social individual who did not care about proximity to geographic features. We also allow g_i to take infinite values, implying that locating in particular regions incurs infinite cost or is physically impossible. This makes it possible to restrict different individuals to different convex sets.

Too little strict convexity could lead to multiple equilibria. For example, a clique of individuals who want to be together, but don't care about outsiders or geographic location, could locate anywhere. This difficulty is obviated through a reasonable additional assumption, which we label no floating groups. Informally, every group has at least one member who is attracted either to someone outside the group or to a particular location.

Assumption II: No floating groups. Every non-empty set of individuals includes at least one individual i with either strictly convex g_i or strictly convex f_{ij} for some j not in the set.

Assumption II clearly holds if every g_i is strictly convex, in which case every individual has just one ideal point. It is easily checked recursively by calling i attached either if g_i is strictly convex, or if f_{ij} is strictly convex and j is already attached, repeating the latter step until no one more becomes

[4] See Fujita and Smith (1990) for a discussion of the additive-interactive formulation.

attached. Assumption II holds if everyone ultimately becomes attached. Otherwise the unattached individuals constitute a floating group.

Applying Assumption II to the set of all individuals, or using the attachment step based on g_i in the previous paragraph, shows that:

Proposition 1. Given Assumptions I and II, at least one ideal set is a single point.

Applying Assumption II to a one-person set gives an important result:

Proposition 2. (Unique optimal reactions). Given Assumptions I and II, for every i, at least one of the functions g_i or f_{ij} is strictly convex. Hence i's disutility (1) *is strictly convex in x_i, as is i's expected disutility for any probability distribution of the others' positions. Therefore i's optimal reaction function is single-valued, whether the others' strategies are pure or mixed.*

With these propositions in hand, we can proceed to prove the existence and uniqueness of an equilibrium.[5]

Theorem 1. In one dimension, under Assumptions I and II, there is a unique Nash equilibrium and all equilibrium strategies are pure. Moreover, the equilibrium positions x_i all lie in the shortest interval B containing at least one point of every ideal set B_i.[6]

Specifically, B is defined as follows. Recalling that B_i is closed, let $\underline{b}_i$ and $\bar{b}_i$ be its left and right end-points (possibly infinite). Let $\underline{b} = \max_i \underline{b}_i$ and $\bar{b} = \min_i \bar{b}_i$. Since at least one B_i is a single point (Proposition 1), it is easy to see that $\bar{b} \leq \underline{b}$ and the interval $B = [\bar{b}, \underline{b}]$ is the shortest interval containing at least one point of every B_i.

Proof of Theorem 1. Assumption I implies that i's payoff function (utility) is continuous in all x_j jointly and quasi-concave in x_i. Hence if the domain of possible positions is compact, a classic equilibrium theorem shows that a

[5] We need Assumption II and not merely the properties of Propositions 1 and 2 because all these properties hold without a unique equilibrium if one individual has a one-point ideal set and all other individuals form a clique of the type mentioned earlier. We also remark that if, contrary to the first sentence of Proposition 2, for some i, the function g_i and all the functions f_{ij} have linear portions, then for some positions x_j, individual i's disutility will have a linear portion. However, the slope there need not be 0, and i's optimal reaction function might still be single-valued.

[6] We thank a colleague for suggesting the shortest interval result for the case when each ideal set is a point.

pure-strategy equilibrium exists. (See Fudenberg and Tirole, 1991, p. 34.) We shall show that it is the only equilibrium and that it has all x_i in B. It follows immediately that the same is true when the domain is not compact.

Proposition 2 implies that all equilibrium strategies are pure. Furthermore, there cannot be two different pure Nash equilibria $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$. To see this, suppose there were two different equilibria with x and y labeled so that $\max_j y_j - x_j > 0$. Let S be the set of individuals who achieve this maximum, and let i satisfy Assumption II for the set S; then $y_i - x_i > 0$ and,

$$\text{for } j \overset{\notin}{\in} S,\ y_i - x_i \overset{>}{=} y_j - x_j, \text{ which implies } y_i - y_j \overset{>}{=} x_i - x_j; \tag{2}$$

$$\sum_j f'_{ij}(y_i - y_j) + g'_i(y_i) > \sum_j f'_{ij}(x_i - x_j) + g'_i(x_i)\,. \tag{3}$$

Now the two sides of (3) are agent i's marginal disutilities in the two equilibria. If both equilibria are interior, then both sides of (3) are equal to zero, which contradicts the strict inequality. In any case, either the left-hand side of (3) must be positive or the right-hand side must be negative. Therefore, either reducing y_i or increasing x_i would reduce i's cost in the corresponding equilibrium; since both actions are feasible, we still have a contradiction. If $\max_i y_i > \underline{b}$ in equilibrium, then choosing i to satisfy Assumption II among i maximizing y_i gives (3) with the right-hand side replaced by 0, again leading to a contradiction. Another similar argument or a reversal of sign shows $\min_i y_i < \bar{b}$ is also impossible. Q.E.D.

2.1. Remarks on uniqueness

The proof that the equilibrium is unique obviously depends on strict convexity somewhere and on something drawing the individuals toward some one location rather than another; Assumption II appears to be the weakest simple condition of this type that suffices.

An example illustrates that uniqueness requires differentiability even with all f_{ij} and g_i strictly convex. Consider a two-person world with $f_{12}(x) = f_{21}(x) = |x| + x^2$ and $g_1(x) = g_2(x) = x^2$. In this case, every (x_1, x_2) with $-1/2 \leqslant x_1 = x_2 \leqslant 1/2$ is an equilibrium. That is, wherever individual 1 locates in the interval will be the preferred location for 2, and vice versa.

3. Multidimensional case

We assume that the functions f_{ij} and g_i are sums of functions defined on the dimensions separately, say

$$f_{ij}(\boldsymbol{x}) = \sum_k f_{kij}(x^k) \quad \text{and} \quad g_i(\boldsymbol{x}) = \sum_k g_{ki}(x^k) \tag{4}$$

where x^k is the kth element of the vector $\boldsymbol{x}$. In this case, convexity and differentiability of the components and of the sum are equivalent, while applying Assumption II to the components is actually weaker than applying it to the sum, since it allows the strict convexity to occur differently in different components.

For the multidimensional case, the shape and orientation of the domain of possible locations prove critical. Theorem 2 proves uniqueness given that each dimension is restricted to an interval as it was by Assumption I for Theorem 1. If this restriction is not imposed, problems may arise, including multiple or nonexistent equilibria. We give an example illustrating the possibility of multiple equilibria on the boundary. Theorem 3 then shows that there cannot be multiple interior equilibria. Theorem 4 addresses the important special case of squared distance.

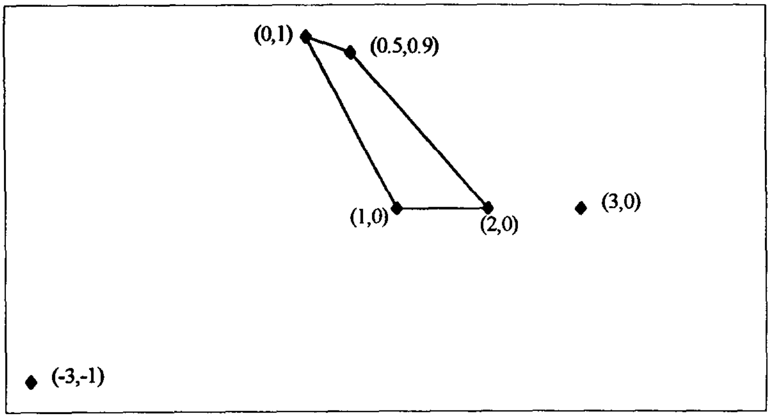
Theorem 2. Under (4), *if the* f_{kij} *and* g_{ki} *satisfy Assumptions I and II, then there is a unique Nash equilibrium, all equilibrium strategies are pure, and in each dimension, the equilibrium positions* x_i^k *all lie in the shortest interval* B^k *containing at least one point of every* B_i^k.

Proof. Under the assumptions, a multidimensional equilibrium must be an equilibrium in each dimension and Theorem 1 applies to each dimension. The result follows. Q.E.D.

If the domain of possible positions is an arbitrary convex set, not necessarily a generalized rectangle, then a variety of problems can arise. For example, an individual's ideal in each dimension may not be jointly possible – the world is cruel. Equilibria may occur at the boundary, and if the domain is not compact they may therefore fail to exist. Furthermore, the proof of uniqueness fails because the moves after (3) may not be possible. Indeed there may be multiple equilibria on the boundary, as the following example illustrates. (The notation $\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|$ denotes the distance between $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$.)

3.1. Example of multiple equilibria on the boundary

Consider two people in two dimensions. The domain of possible positions is the convex hull of (0, 1), (1, 0), (2, 0), and (0.5, 0.9), as shown in Fig. 1. The disutility functions are indicated on the figure. The pairs $\boldsymbol{x}_1 = (0, 1)$, $\boldsymbol{x}_2 = (0.5, 0.9)$ and $\boldsymbol{x}_1 = (1, 0)$, $\boldsymbol{x}_2 = (2, 0)$ are both equilibria if the convex fs



$g_1(\mathbf{x}) = \|\mathbf{x}+(3,\ 1)\|^2$ $\qquad$ $g_2(\mathbf{x}) = \|\mathbf{x}-(3,\ 0)\|^2$

$f_{12}(z, w) = f_1(-z)$ $\qquad$ $f_{21}(z, w) = f_2(-w)$

$f_1'(0.5) < 2,\ 6 < f_1'(1) < 8,$ $\qquad$ $\frac{25}{3} < f_2'(0.1) + 1.8 < 25$ $\qquad$ $f_2'(0) < \frac{10}{3}$

Fig. 1. Multiple equilibria on the boundary.

satisfy the inequalities shown on the figure. It is reassuring that any difficulties with multiple equilibria are confined to the boundary, as Theorem 3 shows.

Theorem 3. Under (4), if the disutilities satisfy Assumption I but without restriction on the domain of possible positions, and if the f_{kij} and g_{ki} satisfy Assumption II for each k, then there is at most one interior Nash equilibrium and it is pure.

Proof. Any interior optimum response is a unique optimum, and hence pure as before. If two interior equilibria did exist, then on a dimension where they differ, (3) would hold with both sides equal to 0, a contradiction. Q.E.D.

3.2. The case of squared distance

For particular distance metrics additional results are possible. For example, if individual's disutilities have all components proportional to squared distance, either distance from others or from their personal ideal point $\boldsymbol{b}_i$, then they are of the form (4) and also invariant under rotation. This affords a stronger result:

Theorem 4. If the domain of possible positions is a closed convex set, if

$f_{ij}(\mathbf{x}) = a_{ij}\|\mathbf{x}\|^2$ with $a_{ij} \geq 0$ for all i, j, if $g_i(\mathbf{x}) = c_i\|\mathbf{x} - \mathbf{b}_i\|^2$ with $c_i \geq 0$ for all i, and if every non-empty set S of individuals includes at least one individual i with $\Sigma_{j \notin S}\, a_{ij} + c_i > 0$, then there is a unique Nash equilibrium and all equilibrium strategies are pure.

Proof. For $S = \{i\}$, we have $\Sigma_j\, a_{ij} + c_i > 0$ and hence strictly convex utilities and unique optimal reactions. Therefore a Nash equilibrium exists and all equilibrium strategies are pure, as before. It remains to prove uniqueness. Generalizing the proof of Theorem 1, let i maximize $\|\mathbf{y}_j - \mathbf{x}_j\|$ and rotate axes so that $y_i^1 > x_i^1$ and $y_i^k = x_i^k$ for $k > 1$; then the argument at (2) and (3) applies to dimension 1. Q.E.D.

4. Inefficiency of the Nash equilibrium

Barring very exceptional circumstances, the Nash equilibrium will not be efficient.

Theorem 5. Under (4), if the disutilities f_{kij} and g_{ki} satisfy Assumption I and every f_{kij} is strictly convex in some neighborhood of 0, then a Nash equilibrium is Pareto optimal if and only if it is ideal for everyone, that is, everyone's position is the same and belongs to every ideal set B_i.

Proof. Consider first the one-dimensional case. Note that the convexity conditions imply $f'_{ij}(x) > (<) 0$ for $x > (<) 0$ and $g'_i(x) > (<) 0$ for x above (below) B_i. Consider an equilibrium in which the positions x_i do not coincide. Increase the smallest x_i and decrease the largest x_i by δ. Only the movers can be losers. We shall show that everyone gains if δ is sufficiently small. For i and j at the same extreme, $f_{ij}(x_i - x_j)$ is unchanged. For i at one extreme and j elsewhere, the change in x_j decreases by order δ, that is, by at least $\epsilon\delta$ for some $\epsilon > 0$ and all sufficiently small δ. By the equilibrium conditions, the change in x_i costs i an amount of smaller order than δ. Hence, for all sufficiently small δ, everyone gains at least $\epsilon\delta$ and loses less, a Pareto improvement. It follows that in a Pareto-optimal equilibrium, everyone's position must be the same. If the common position is not in B_i, then i's disutility (1) has derivative $g'_i(x_i) \neq 0$ at x_i and i can gain by moving slightly in the direction of B_i, contradicting equilibrium.

In the multidimensional case, the foregoing proof can be applied to each dimension separately. Q.E.D.

Given the attraction of individuals to each other, it might seem that outcomes that are both Pareto optimal and Pareto superior to the Nash equilibrium would be denser than that equilibrium.[7] This will often be the case, but it is not necessarily so. Consider individuals 1, 2, 3, and 4 with unique ideal points increasing with number. Say 1 and 2 only care about each other; similarly with 3 and 4. Pareto improvements will move 1 and 2 closer together; so too with 3 and 4. Assume that 1 and 4 have exceedingly concave g_i functions (marginal cost rapidly increases with distance from ideal point), with the other cost functions being rather linear. This implies that to secure a net benefit from moving in further from their ideal points, 1 and 4 will require substantial outward movement from their partners. Pareto-optimal, Pareto-superior outcomes may thus require that 2 and 3 move out more than 1 and 4 move in. If so, the change in density is ambiguous.

5. Concluding remarks

This analysis treats location decisions as if they were determined through a tatonnement process, where all final decisions are individually optimal. Preferences are assumed to exhibit increasing marginal costs of distance from other individuals or attractive locales. In such a world, a traditional world of equilibrium analyses, there exists a unique Nash equilibrium. It is not surprising that equilibrium is inefficient, given the externalities that each individual's location imposes on others. An immediate extension of our analysis shows that a unique Nash equilibrium will also exist if people must choose locations over time, assuming early 'settlers' know the f functions of those who will locate subsequently.

Future work should consider a range of related problems where decision makers choose locations, including nonphysical realms such as the choice of philosophies or behavior patterns. Location models should also assess the roles of decisions not taken by individuals choosing locations. Governments, for example, enact zoning ordinances and site facilities, and thereby affect the spatial distribution of individuals and firms. Our results show that regularity, though alas not optimality, is likely to emerge from the chaos of myriad decision makers choosing for themselves.

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[7] See Cheng and Shieh (1992) for a discussion of cooperative (i.e. Pareto optimal) location decisions. They address the two-player case, namely bilateral monopolists.

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