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Investment Flexibility and the Acceptance of Risk*

Christian Gollier

GREMAQ and IDEI, Université de Toulouse, Place Anatole France, 31042 Toulouse Cedex, France

and

John Lindsey and Richard J. Zeckhauser

Kennedy School of Government, Harvard University, Cambridge, Massachusetts 02138

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The hypothesis examined in this paper is that the greater the investor's flexibility, the easier it is for him to change his portfolio depending on his results, the more willing he will be to accept risks. When the investor has no control on the size of the risky investment, but can choose between one risky and one riskless asset, this conjecture is shown to be correct. However, if there is more than one risky asset each period, counterexamples demonstrate that flexibility rarely ensures greater risk taking. For the standard portfolio problem in which investors are free to determine the size of their investment in a risky asset, flexibility always raises the demand for the risky asset if constant relative risk aversion is less than unity. But counterexamples can always be found when the constant relative risk aversion is larger than unity. *Journal of Economic Literature* Classification Numbers: G11, D81, D84.

1. INTRODUCTION

Common wisdom suggests that the greater one's ability to shift one's investment after intermediary shocks to wealth, the riskier should be one's portfolio. For example, an investor with decreasing risk aversion should be more likely to take a risky investment if he can exit should returns plummet, that is when the costs of risk escalate.

This analysis examines the relationship between flexibility—we label its opposite rigidity—and the risk of portfolios optimized to maximize expected utility. The strong form of rigidity arises when the investor is not able

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to modify his exposure to risk from period to period. In this paper, we examine a somewhat weaker form of rigidity, i.e., the investor can modify his risk exposure from one period to another, but he is forced to fix it a priori, i.e., before observing realizations of the early returns on his investment. We simplify the problem by considering a two-period model. The individual invests at the beginning or period 1, receives investment returns, and then must invest in period 2. In the rigid economy, period 2 investment decisions must be made before period 1 results are learned.

In Section 2, we consider a model in which the investor can only invest in indivisible investment projects. Our main result is that, except for a handful utility functions, it is always possible to build a counterexample in which less risk will be taken in the flexible economy than in the rigid one. We exclude the case of constant absolute risk aversion, for which wealth has no effect on risky investment choice, hence flexibility has no effect on the risk of the optimal portfolio.

Section 3 is devoted to the standard portfolio problem. We assume that one can invest in two assets, one being risk free. The returns of the risky asset are not serially correlated. In this case, the two forms of rigidities coincide, since the investor who is forced to determine today the size of his investment in the risky asset tomorrow will invest the same amount in the risky asset in each of the two periods. The impact of flexibility upon risk has important consequences in the context of portfolio management. In the United States, for example, retirement funds are more flexible than ordinary investments because capital gains taxes need not be paid when retirement investments are shifted from one security to another. This suggest a fundamental maxim due to flexibility: an individual with both ordinary savings and a retirement account should emphasize (riskier) assets in the latter.[1]

2. THE CASE OF INDIVISIBLE INVESTMENT PROJECTS

2.1. *The Model*

We consider a simple two-period model. The investor is endowed with wealth w at the beginning of the first period. He maximizes the expected utility of his final wealth. The utility function u is assumed to be twice differentiable, increasing and concave. $A(z) = -u''(z)/u'(z)$ denotes absolute risk aversion. In each period $t = 1$ or 2, he must choose one of two indivisible investment projects. The net payoffs of the two projects available

[1] This parallels the well known result that the tax-protected retirement funds should disproportionately be allocated higher current yield assets.

at period t are denoted respectively $\tilde{x}_t$ and $\tilde{y}_t$. Random variables $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{y}_1$, $\tilde{y}_2$ are independently distributed.

We compare two economies which differ on the timing of decisions. In what we call the "rigid" economy, the investor has to decide at the beginning of the first period about which project to undertake at each period. It is a "rigid" economy in the sense that the investor cannot reexamine his second period risk exposure due to first period results, perhaps large losses or gains. The problem of the investor in the rigid economy is to find the pair (δ_1^r, δ_2^r) that solves the following program:

$$\max_{\substack{\delta_1 \in \{0,1\} \\ \delta_2 \in \{0,1\}}} Eu\left(w + \sum_{t=1}^{2} (\delta_t \tilde{y}_t + (1-\delta_t)\tilde{x}_t)\right). \tag{1}$$

It is a simple maximization at the beginning of period 1. Choice variable δ_t takes value 1 if project $\tilde{y}_t$ is undertaken in period t, otherwise $\delta_t = 0$ and project $\tilde{x}_t$ is undertaken in period t.

We compare this optimal risk-taking decision with the optimal decision in the so-called "flexible" economy. In the flexible economy, the investor can delay the decision on which project to undertake in period 2 until he has learned (observed) the return of his first period investment. In this economy, the dynamic problem of the investor is solved by using backward induction:

$$\delta_1^f \in \arg \max_{\delta_1 \in \{0,1\}} Ev(w + \delta_1 \tilde{y}_1 + (1-\delta_1)\tilde{x}_1). \tag{2}$$

Function v is the value function which is obtained by solving the second period problem:

$$v(z) = \max_{\delta_2 \in \{0,1\}} Eu(z + \delta_2 \tilde{y}_2 + (1-\delta_2)\tilde{x}_2). \tag{3}$$

Some comments are in order here. First, observe that program (1) can be seen as a constrained version of the dynamic problem, represented by equations (2) and (3), faced by the investor in the flexible economy. The constraint in the rigid economy is that δ_2 must be independent of z, the wealth level at the beginning of the second period. The investor is strictly better-off in the flexible economy. There is one exception to this rule, that is when the optimal decision of the investor in period 2 is independent of his wealth z. This will happen if $\tilde{x}_2$ and $\tilde{y}_2$ are identically distributed, if one set of gambles stochastically dominates another, or if all outcomes lie in a constant absolute risk aversion (CARA) portion of u. In the CARA case, decisions are not affected by wealth; hence learning period 1 results before deciding on period 2 investments offers no advantage.

It is worth noting that the value function v used by the investor in the flexible economy to solve his first period need not be concave. This was first noticed by Bell [1].[2] The existence of a discrete choice in period 2, as here, can induce a risk-loving behavior in period 1. As it is intuitively sensible, "an entrepreneur with an idea she believes will work, but without financial backers who agree, would be justified trying to raise the necessary capital in Atlantic City" (Bell [1]).

2.2. *A Positive Result when one of the Two Projects is Certain*

We first consider a simple special case of the problem presented above. The relationship between flexibility and risk is most easily understood when the investor must choose between a risky and a risk-free position. We assume here that one of the two projects available in period 1 is risk-free. Without loss of generality, we assume that $\tilde{x}_1$ is zero with probability 1. The investor's problem in period 1 is just to accept or to reject a risky investment project $\tilde{y}_1$.

Common wisdom suggests that more flexibility in risk-taking decisions induces people to be more willing to accept risks. This logic would mean that the first period investment is undertaken in the flexible economy if it is undertaken in the rigid economy. This proves to be true, as shown in the following Proposition.

PROPOSITION 1. *If $\tilde{x}_1$ is zero with probability* 1, *then the risky project is always undertaken in the flexible economy if it is undertaken in the rigid one*: $\delta_1^r = 1 \Rightarrow \delta_1^f = 1$.

Intuition. Introducing flexibility is not about to make you switch to the safe project in period 1, since with a safe project flexibility is useless because you know what is going to happen in period 1.

Proof. If the risky project is not undertaken in period 1, the expected utility of the investor is the same in the two economies, since waiting to decide is not worth anything when one knows what is going to happen in period 1. The expected utility in this case equals

$$U^f(\delta_1 = 0) = U^r(\delta_1 = 0) = \max_{\delta_2 \in \{0,\, 1\}} Eu(w + \delta_2 \tilde{y}_2 + (1 - \delta_2)\, \tilde{x}_2). \qquad (4)$$

Also, because the information has a value, when the risky project is undertaken in period 1, the investor can obtain a larger expected utility in the flexible economy than in the rigid one, i.e., $U^f(\delta_1 = 1) \geqslant U^r(\delta_1 = 1)$. Finally, assume that it is optimal to undertake the risky project in period 1

[2] For additional insights on the concavity of the value function, see Pratt [9]. Gollier [5] analyzed the dynamic problem of the investor in this flexible economy when risks are i.i.d. over time.

in the rigid economy, i.e., $U^r(\delta_1=1) > U^r(\delta_1=0)$. Combining these conditions yields

$$U^f(\delta_1=1) \geqslant U^r(\delta_1=1) > U^r(\delta_1=0) = U^f(\delta_1=0). \quad \blacksquare \tag{5}$$

In this simple framework, the value of the information in the flexible economy not only increases the expected utility of the investor, but it also makes him less risk-averse towards the first period project. This is not true in general, as the next section explains.

2.3. *A General Counterexample: Flexibility Need Not Increase Risk Taking*

Once the choice set is allowed to contain more than one risky project from period to period, it is less likely that flexibility will promote risk taking in all circumstances. In effect, we find that—except for utility functions that satisfy a very specialized, unintuitive condition—a counterexample will always exist to the assertion that flexibility-promotes-risk taking.

This result is particularly surprising for utility functions displaying monotonic risk aversion. With such utility functions, the riskier the first-period lottery, the greater will be the diversity in risk aversion (risks costs) depending on the first-period outcome. Hence, second-period investments that are attractive for some first-period outcomes will be unattractive for others. The more diverse is a utility function's relative risk aversion, it would seem, the greater the benefits reaped from flexibility in the choice of second-period lottery. Hence, we might conjecture that if the riskier first-stage lottery is optimal in the rigid economy, it would also be optimal in the flexible economy as in Proposition 1. Our Proposition 2 below shows, however, how to construct a counterexample to this conjecture for quite general utility functions, including rather general monotonic risk averse utility functions.

In the previous section, risk $\tilde{y}_1$ was obtained by adding some noise to risk $\tilde{x}_1$ which was degenerate at zero. Here we extend here this definition by allowing $\tilde{x}_1$ to be any random variable.

PROPOSITION 2. *Suppose that there exists a scalar z such that u'''' is continuous at z and $A''(z) \neq A'(z)\,A(z)$. Take $\tilde{y}_1$ by adding some noise $\tilde{\varepsilon}$ to some realization of $\tilde{x}_1$. Then, there exists $\tilde{x}_1$, $\tilde{\varepsilon}$, $\tilde{x}_2$, $\tilde{y}_2$ such that the investor prefers the first period lottery $\tilde{y}_1$ in the rigid economy, but the lottery $\tilde{x}_1$ in the flexible economy.*

Proof. See the Appendix.

In the remainder of this section, we provide some intuition for this result. Let us assume that $A''(z) > A'(z)\,A(z)$. The case $A''(z) > A'(z)\,A(z)$ corresponds to the notion of strict "local properness", as introduced by Pratt and Zeckhauser [10]. To illustrate, consider two lotteries with payoffs $\tilde{\varepsilon}_1$ and $\tilde{\varepsilon}_2$. These lotteries are small and their payoffs are independently distributed. Suppose that if you were offered either lottery, you would be indifferent whether or not to accept it. Might you accept the two lotteries if they were offered jointly? If your answer is no, as we might think intuitively to be appropriate, then your risk aversion is said to be strictly locally proper. Small risks are not complementary: taking on one small undesirable (or neutral) risk does not make another small undesirable (or neutral) risk desirable. It can be inferred from the work by Pratt and Zeckhauser that risk aversion is strictly locally proper if $A''(z) > A'(z)\, A(z)$.

Under strict local properness, one can build a counterexample as follows: normalize w_0 to zero, and consider $\tilde{x}_1$ which takes values z and $\hat{z}$ with some probabilities. Lottery $\tilde{y}_1$ is obtained from $\tilde{x}_1$ by adding some noise $\tilde{\varepsilon}$. Assume that $\tilde{\varepsilon}$ is zero with probability 1 if $\tilde{x}_1 = \hat{z}$, and $\tilde{\varepsilon}$ is some small risk $\tilde{\varepsilon}_1^z$ if $\tilde{x}_1 = z$. We assume that $Eu(z + \tilde{\varepsilon}_1^z) = u(z)$, i.e., a myopic agent would be indifferent between $\tilde{x}_1$ and $\tilde{y}_1$. In period 2, we have $\tilde{x}_2$ equal to zero, and $\tilde{y}_2$ is some small risk such that $Eu(z + \tilde{y}_2) = u(z)$. In short, at each period a myopic investor with wealth z would be indifferent between the two projects.

In the flexible economy, the investor selects project $\tilde{y}_2$ in period 2 if he got payoff z in period 1. Since we assumed that risk aversion is locally proper at z, it means that noise $\tilde{\varepsilon}_1^z$ which was considered to be neutral by the myopic agent is strictly disliked by the dynamic investor in the flexible economy. Since $\tilde{y}_1$ differs from $\tilde{x}_1$ only by the addition of noise $\tilde{\varepsilon}_1^z$ if $\tilde{x}_1 = z$, the option to invest in $\tilde{y}_2$ in period 2 makes the flexible investor prefer the safer $\tilde{x}_1$ in period 1. This illustrates the fact that a small neutral risk—here risk $\tilde{y}_2$—can substitute for any other small risk ($\tilde{\varepsilon}_1^z$).

Since risk aversion is locally proper at z, A may not be maximal at z. So, select $\hat{z}$ such that $A(\hat{z}) > A(z)$. In the rigid economy, the investor evaluates the desirability of the second-period lottery $\tilde{y}_2$ without knowing whether his wealth in period 2 will be $\hat{z}$, z, or $z + \tilde{\varepsilon}_1^z$. Recall that the investor is neutral to $\tilde{y}_2$ at z. Since $\tilde{\varepsilon}_1^z$ is small and A is continuous, this is also almost true at $z + \tilde{\varepsilon}_1^z$. But as $A(\hat{z}) > A(z)$, the investor strictly dislikes $\tilde{y}_2$ at wealth $\hat{z}$. Therefore, the investor in the rigid economy does not undertake the risky project in period 2. He behaves in period 1 like the myopic investor, i.e., he is indifferent between $\tilde{x}_1$ and $\tilde{y}_1$. A slight modification in the distribution of the noise can induce him to prefer the riskier project $\tilde{y}_1$. This is a paradox since the same investor prefers the less risky project $\tilde{x}_1$ in the flexible economy.

This discussion suggest that paradoxes may arise when investment rules are rigid, because risky positions may be transferred from one period to another. In the general counterexamples we construct for the case $A''(z) > A'(z)\, A(z)$, the investor in the rigid economy takes a riskier position in period 1, but accepts no risk at all in period 2. In the flexible economy, a less risky project is taken initially, but the loss in expected return is more than made up by sometimes taking a risky project in period 2.

Let us consider the following numerical example to illustrate this construction. Take $u(t) = \log(t)$, a locally proper utility function. Initial wealth is normalized to 1. Lottery $\tilde{x}_1$ is distributed as $(z = 0, 1/2;\ z = 2, 1/2)$, whereas lottery $\tilde{y}_1$ is obtained from lottery $\tilde{x}_1$ by adding risk $\tilde{\varepsilon}_1^z = (1.5, 1/2; -1, 1/2)$ to the good realization of $\tilde{x}_1$. In period 2, the decision-maker has an option to invest in $\tilde{y}_2 = (2, 1/2; -1, 1/2)$ or not. In the flexible economy, it is easy to check that this option is used in period 2 if wealth at that time is larger or equal to 2. In consequence, independent of whether or not the agent purchases risk $\tilde{\varepsilon}_1^z$ or not, the option to invest is used in period 2 if the realization of $\tilde{x}_1$ is 2. Two observations lead to the counter-example. First, given wealth $w_0 = 1$, the agent is indifferent between $\tilde{x}_1$ and $\tilde{y}_1$ when he has no option to take risk in the future. Second, it can be shown that the second-period option on $\tilde{y}_2$ *increases* local risk aversion with respect first-period risks in the interval $[2, 4 + 3\sqrt{2}]$ of first-period wealth; v is strictly more concave than u in this interval. Since $\tilde{y}_1$ is riskier than $\tilde{x}_1$ in this interval, and is the same elsewhere, the flexible investor now prefers $\tilde{x}_1$ due to the existence of the option on $\tilde{y}_2$. In short, the flexible investor selects the less risky $\tilde{x}_1$ in the first period, and he invests in $\tilde{y}_2$ in case of a success. In contrast, the rigid investor never invests in $\tilde{y}_2$, since he would otherwise finish with a zero wealth with some positive probability (1/4). Given this, he invests in $\tilde{y}_1$, the riskier investment.

This example illustrates the fact that the existence of counter-intuitive results are not limited to the case of adding extremely small noises to the initial gamble, nor to the case involving options on small risks in the future. The problem of larger risks is more difficult, as shown by Pratt and Zeckhauser [10]. Even global properness is not very useful for our problem, since global properness is the condition for any *undesirable* (future) risk not to make any other undesirable risk desirable. Since we consider options on future risks, only the effect of *desirable* risks matters. However, by continuity, if a neutral risk generates the result, it will also pertain for risks that are not too much desirable. This is the case for risks $\tilde{y}_2$ for z not too much larger than 2, as in the example. Indeed, the acceptance of $\tilde{y}_2$ increases local risk aversion for any wealth level between 2 and $4 + 3\sqrt{2}$, which is in fact "much larger" than 2.

In the case of $A''(z) < A'(z)\,A(z)$, we know that a small neutral risk always *reduces* the aversion to any other small risk: small risks are complementary. This counter-intuitive effect allows for a construction symmetric to the one presented above. The flexible investor selects the safer asset in period 1 and sometimes selects the safer asset in period 2, while the rigid investor commits to holding the riskier asset in both periods. The intuition is that bearing more risk in period 2 in the rigid economy induces the investor to accept more risk in period 1 by complementarity. A full description of how to build a counterexample in this case is presented in the Appendix. Building a numerical example in this case is not simple, since all familiar utility functions we know satisfy the local properness condition.

Such constructions might not be possible when $A''(z)=A'(z)\,A(z)$, i.e., if the absolute risk aversion is not equal to minus the absolute risk aversion of the absolute risk aversion. In this case, accepting a gamble on a small risk for which one is indifferent does not affect the willingness to gamble on another small risk. For such utility functions, one might have a 5th-order preference. Except for the CARA class, $-e^{z}$, z, the set of utility functions that satisfy $A''=A'A$ is rather exotic: It contains $u(z)=z^3$, $u(z)=z$, $u(z)=z-\sin(z)$, $u(z)=z-\sinh(z)$, $u(z)=z+\sinh(z)$, $u(z)=-e^{-Az}$, and all u obtained by replacing $u(z)$ by $a+bu(c+dz)$.

3. THE PORTFOLIO PROBLEM WITH CONSTANT RELATIVE RISK AVERSION

In this section, we examine a two-period portfolio problem with one riskfree asset offering zero return and one risky asset with returns $\tilde{y}_1$ and $\tilde{y}_2$ respectively in period 1 and 2. Moreover, we assume that $\tilde{y}_1$ and $\tilde{y}_2$ are i.i.d. and distributed as $\tilde{y}$. The expectation of $\tilde{y}$ is positive. As before, we examine the effect of flexibility on the optimal exposure to risk in period 1.

In the rigid economy, the investor must determine the structure of his portfolio at each period at the beginning of the first period. The problem of the investor in the rigid economy is written as[3]

$$(\alpha_1^r, \alpha_2^r) \in \arg\max_{\alpha_1, \alpha_2} Eu\left(w+\sum_{t=1}^{2} \alpha_t \tilde{y}_t\right). \tag{6}$$

An important difference from problem (1) is that α_t is not restricted to be zero or one. The size of the exposure to risk is now a decision variable. A simple diversification argument[4] implies that the solution to problem (6) must be symmetric. Suppose by contradiction that it is not. Then, we have

$$\alpha_1 \tilde{y}_1+\alpha_2 \tilde{y}_2=\frac{\alpha_1+\alpha_2}{2}(\tilde{y}_1+\tilde{y}_2)+\frac{\alpha_1-\alpha_2}{2}(\tilde{y}_1-\tilde{y}_2), \tag{7}$$

[3] Eeckhoudt, Gollier and Levasseur [4] compares α_1^r with the optimal solution for the one-period problem $\max_\alpha Eu(w+\alpha\tilde{y})$.

[4] See for example Rothschild and Stiglitz [11].

where $E[\tilde{y}_1 - \tilde{y}_2 \mid \tilde{y}_1 + \tilde{y}_2 = z]$ is obviously zero for any z. This means that $\alpha_1 \tilde{y}_1 + \alpha_2 \tilde{y}_2$ is a Rothschild–Stiglitz increase in risk of $((\alpha_1 + \alpha_2)/2)(\tilde{y}_1 + \tilde{y}_2)$ in comparison to a symmetric choice. Therefore, the optimal solution to (6) must be symmetric. The problem in the rigid economy is thus equivalent to

$$\alpha^r \in \arg\max_{\alpha} Eu(w + \alpha(\tilde{y}_1 + \tilde{y}_2)). \tag{8}$$

As before, we compare this optimal risk-taking decision with the optimal decision in the flexible economy. In such an economy, the dynamic problem of the investor is solved by using backward induction:

$$\alpha_1^f \in \arg\max_{\alpha_1} Ev(w + \alpha_1 \tilde{y}_1). \tag{9}$$

Function v is the value function which is obtained by solving the second period problem:

$$v(z) = \max_{\alpha_2} Eu(z + \alpha_2 \tilde{y}_2). \tag{10}$$

As is well-known in dynamic finance, the problem of the investor in the flexible economy is substantially simplified if we assume constant absolute risk aversion (CARA), or constant relative risk aversion (CRRA)[5, 6]. The problem is trivial under CARA, since we obviously get $\alpha_1^r = \alpha_1^f$. Under CRRA, myopia is optimal in the flexible economy. Myopia means that the investor considers each period in isolation to determine his portfolio. This is due to the fact that v inherits the same degree of concavity as u when u is CRRA. Therefore, we hereafter consider the case of $u'(z) = z^{-\gamma}$, $\gamma > 0$. Since u and v are equivalent, the first-order condition for program (9) can be written as

$$E[\tilde{y}_1 u'(w + \alpha_1^f \tilde{y}_1)] = 0. \tag{11}$$

We are interested in determining whether the demand for the risky asset in the rigid economy is less than the demand for the same asset in the flexible economy, i.e., whether α^r is less than α_1^f. Since the objective function in (8) is concave in the decision variable, this is true if and only if

$$E(\tilde{y}_1 + \tilde{y}_2)\, u'(w + \alpha_1^f (\tilde{y}_1 + \tilde{y}_2)) \leqslant 0. \tag{12}$$

[5] Gollier and Zeckhauser [7] consider other utility functions. They obtained the necessary and sufficient condition on the utility function to assure that α_1^f is larger than the optimal demand for the risky asset in the one-period/myopic problem, $\max_\alpha Eu(w + \alpha\tilde{y}_1)$.

[6] Caballé and Pomansky [2] examined the properties of utility functions exhibiting mixed risk aversion, as is the case for CRRA functions. They discuss the implication of mixed risk aversion on portfolio selection, and on the management of multiple risk taking.

We begin with a positive result which holds for small risks.

PROPOSITION 3. *Let* $u'(z)=z^{-\gamma}$, $\gamma>0$. *If* $\tilde{y}$ *is small in the sense that* $|\tilde{y}E\tilde{y}/E\tilde{y}^2|$ *is small almost surely, how small depending on* γ, *then the demand for the risky asset is smaller in the rigid economy than in the flexible.*

Proof. See the Appendix.

The outline of the proof goes as follows: suppose that $\alpha_1^f\tilde{y}_1$ is small almost surely. Then condition (11) can be approximated by

$$E\tilde{y}_1[u'(w)+u''(w)\,\alpha_1^f\tilde{y}_1]\cong 0, \tag{13}$$

or $\alpha_1^f=(1/A(w))(E\tilde{y}/E\tilde{y}^2)$. Thus our assumption above holds by assuming that $\tilde{y}(E\tilde{y}/E\tilde{y}^2)$ is small. We may thus also approximate $u'(w+\alpha_s^f(\tilde{y}_1+\tilde{y}_2))$ in (12) by a second order Taylor approximation around w. It yields

$$E(\tilde{y}_1+\tilde{y}_2)[u'(w)+u''(w)\,\alpha_1^f(\tilde{y}_1+\tilde{y}_2)]\leqslant 0. \tag{14}$$

This is equivalent to

$$2u'(w)\,E\tilde{y}+2u''(w)\,\alpha_1^f[E\tilde{y}^2+(E\tilde{y})^2]\leqslant 0. \tag{15}$$

By eliminating α_1^f, inequality (15) is equivalent to $-2u'(w)((E\tilde{y})^3/E\tilde{y}^2)$. This is negative.

This result does not hold for larger risks. As a counterexample, consider the CRRA utility function with relative risk aversion $\gamma=5$. Take $\tilde{y}=(-0.1, 4/10; 10, 6/10)$. After some computations, we get $\alpha_1^r=0.1707$, which is larger than $\alpha_1^f=0.1678$. Our next Proposition provides a necessary and sufficient condition for $\alpha_1^f\geqslant\alpha_1^r$ under CRRA.

PROPOSITION 4. *A necessary and sufficient condition for* $\alpha_1^f\geqslant\alpha_1^r$ *is that the constant relative risk aversion be less than or equal to unity. It is necessary in the sense that if* γ *is larger than unity, than one can always find an initial wealth* z *and a distribution* $\tilde{y}$ *such that* α_1^f *if less than* α_1^r.

Proof. See the Appendix for the proof of the general case. We prove here that $\alpha_1^f\geqslant\alpha_1^r$ for the logarithmic ($\gamma=1$) investor, since this case does not require using sophisticated techniques. We have to prove that for any w and any $\tilde{y}_1$, $\tilde{y}_2$,

$$E\frac{\tilde{y}_i}{w+\alpha_1^f\tilde{y}_i}=0\Rightarrow E\frac{\tilde{y}_1+\tilde{y}_2}{w+\alpha_1^f(\tilde{y}_1+\tilde{y}_2)}\leqslant 0. \tag{16}$$

Notice that, for all z, $w+z$, $z/(w+z)=w((1/w)-(1/w+z))$. Also, because $E\tilde{y}>0$ by assumption, α_1^f is positive. Thus, the right condition in (16) is equivalent to

$$E\left[\frac{1}{w}-\frac{1}{w+\alpha_1^f(\tilde{y}_1+\tilde{y}_2)}\right]\leqslant 0. \tag{17}$$

The left condition in (16) can be rewritten in a similar way, implying that it is equivalent to

$$Eh(w+\alpha_1^f\tilde{y}_i)=h(w)\Rightarrow Eh(w+\alpha_1^f\tilde{y}_1+\alpha_1^f\tilde{y}_2)\leqslant h(w). \tag{18}$$

where $h(z)=-1/z$. Observe that h can be seen as a CRRA utility function. Condition (18) is interpreted as whether two lotteries $\alpha_1^f\tilde{y}_1$ and $\alpha_1^f\tilde{y}_2$ for which individual h is indifferent when considered in isolation, are jointly weakly undesirable. We know that this is true, since function h is proper (see Pratt and Zeckhauser [10]). Though successful for the logarithmic case, we were unable to extend this trick to other CRRA functions. ▮

4. THE CONCEPT OF FLEXIBILITY PREMIUM

In this section, we propose a way to quantify the effect of rigidity on welfare. We consider an economy with several assets and potentially several periods, with either flexibility or rigidity in the decision process. We compare this economy with a "dual" economy with only one asset, which is risk-free. The Certainty Equivalent Interest Rate (CEIR) is the per-period rate of return of the riskfree asset in the dual economy which makes the investor indifferent between living in the reference economy or in its dual one.

Let us consider the case of an economy with a single period and two assets: one is riskfree with a zero return, and the other is risky with return -10% or $+20\%$ with equal probability. Investor's utility is logarithmic. After some computations, it appears that his optimal strategy is to invest 250% of his wealth in the risky asset. Normalizing wealth to unity,[7] the maximum expected utility of the investor equals 0.0588. The CEIR in this case is 6.07%, since $0.0588=\log(1+0.0607)$. The investor is indifferent between being an expected-utility-maximizer in the two-asset economy and living in the dual economy with a risk-free rate of 6.07%.

Consider now two economies, f and r. The investor is more constrained in economy r than in economy f. We define the "constraint premium" as

[7] In general, the CEIR will be a function of the investor's wealth. This is not case here, since relative risk aversion is constant.

the difference between the certainty equivalent interest rate of economy f and the certainty equivalent interest rate of economy r. It is necessarily nonnegative. It measures the loss in welfare due to adding the constraint in the economy. The flexibility premium is an example of this, in which the constraint is on the timing of the decisions.

Consider now the example above with 2 periods. In economy f, the CEIR is still 6.07%. This results from the constancy of relative risk aversion, which makes myopia rational. In economy r, one can verify that the investor invests only 230% of his wealth in the risky position, yielding a CEIR of 5.79%. The flexibility premium is thus 0.28% per year. For three, four and five years, we get flexibilty premia of respectively 0.56%, 0.85% and 1.14% per year. In the case of the "square root" utility function, we get flexibility premia of respectively 1.83%, 3.3%, 4.5% and 5.65% per year, for respectively two, three, four and five years horizons.

Technically, in the case of CRRA utility functions, the CEIR in the flexible economy is obtained by solving the static problem of the investor, since myopia is optimal,

$$u(1+\mathrm{CEIR}_f)=\max_{\alpha} Eu(1+R+\alpha\tilde{x}), \tag{19}$$

where R is the risk-free rate and $\tilde{x}$ is the excess return of the risky asset. The computation of the CEIR in the flexible economy with utility functions which do not exhibit linear risk tolerance would be much more difficult. The CEIR in the rigid economy with n periods is implicitly defined by

$$u((1+\mathrm{CEIR}_r(n))^n)=\max_{\alpha} u(1+R+\alpha(\tilde{x}_1+\tilde{x}_2+\cdots+\tilde{x}_n)), \tag{20}$$

with $\tilde{x}_1,\ldots, \tilde{x}_n$ independent and identically distributed as $\tilde{x}$. Finally, the flexible premium in an economy with n periods of investment equals $\mathrm{CEIR}_f-\mathrm{CEIR}_r(n)$.

5. PARALLEL PORTFOLIOS

The analysis thus far has focused on properties of and comparisons between pure portfolios, all rigid or all flexible. Many real world investors, however, have both rigid and flexible investments. They have what we label "parallel portfolios." Given parallel portfolios, two questions come to mind. First, when will the flexible portfolio be relatively large enough that the individual can achieve the same outcome he would were he totally flexible? Second, is the maxim we mentioned at the outset satisfied: will the optimal holdings in the flexible portfolio be at least as risky as those in the rigid one?

We investigate these questions in a multi-period context assuming there is a safe asset, s, yielding a zero return, and a gambling asset, g, whose return can range from complete loss to arbitrarily large gains. We say the amount placed in the gambling asset is invested. You are not permitted to sell either asset short. (If you were, the fully flexible outcome could be trivially achieved.)

Let the net payoff for investing t be tx where $x > -1$ (you cannot lose more than your entire investment) and x has density $f(x)$. Let u be the utility function and w the initial wealth. Then the expected utility is $r(t) = \int_{-1}^{\infty} u(w + tx)\, f(x)\, dx$. As is well-known, the optimal t has the same sign as $E\tilde{x}$. Also, the demand for the gambling asset is proportional to wealth in the CRRA case, whereas the demand is independent of wealth in the CARA case. Finally, it can easily be shown that, for any amount d of the gambling asset, one can find distributions of returns of the gambling asset with an arbitrarily large return in some states for which d is the optimal demand, in the case of CRRA, or in the case of CARA.

Suppose that we have several periods of investment, and a flexible portfolio, f, where we can trade after each period and a rigid portfolio, r, whose holdings must be maintained. Let b_{ij} be the amount placed in portfolio i in the jth asset. Thus b_{fg} is the amount placed in the gambling asset in the flexible portfolio, and $b_g = b_{fg} + b_{rg}$ would be the total investment in the gambling asset. We shall investigate the nature of and potential for optimal solutions with the CARA utility function $u(x) = -e^{-kx}$ and the CRRA utility function $u(x) = x^{1-\gamma}/(1-\gamma)$, $\gamma > 0$.

5.1. *CARA Utility*

In the CARA case, the optimal level of g is independent of wealth. If the investor places some of his monies in the gambling asset of his rigid portfolio, he faces the risk of a large increase in the value of this asset, thereby overinvesting in it in the future. Also, since one cannot be short on the safe asset, the investor should invest enough in the flexible portfolio in order to have enough wealth to invest in the gambling asset at each period, even in the worst case of repeated losses. We thus obtain the following simple result.

PROPOSITION 5. *You can ensure investing exactly t each period for n periods iff the initial $b_f \geq nt$.*

Proof. You cannot invest anything in the rigid portfolio since it might rise above t forcing you to be overinvested in the next period. Thus we must invest t each period in the flexible portfolio. The worst-case scenario (approachable anyway) is that the gambling investment goes to 0 each period. Then you need $b_f \geq nt$. ∎

5.2. *CRRA Utility*

In the CRRA case, the optimum ratio of monies allocated to the gambling and safe assets, a ratio we label a, is independent of wealth and time-horizon. The following lemmas and proposition relate to a utility function with CRRA.

In this section through Proposition 6 both assets could have been gambling and they need not be independent of each other. All that is needed is that each can greatly outperform the other sometimes (that is if value of one is multiplied by r and the other by s in a period, then r/s can be arbitrarily close to zero or infinity.). Even without this last condition the sufficient condition for being able to invest as well as we could have with complete flexibility still holds.

LEMMA 1. *Suppose the initial b_{rg} is forced to be 0. Let $f_n = (1+a)^n - 1$. One will be able to allocate a times as much to the gambling asset as safe asset for n periods iff $b_f \geqslant f_n b_r$.*

Proof. We use induction on n, so suppose the lemma holds for n. The first period portfolio must satisfy $b_{fg} = a(b_{fs} + b_r)$. By the induction hypothesis, we also need that what is left of b_f after the first period must be at least $f_n b_r$. The worst case scenario is that the gambling investment goes to 0 so we need $b_{fr} \geqslant f_n b_r$. Multiplying this by $a+1$ and adding to $b_{fg} = a(b_{fs} + b_r)$ yields $b_f \geqslant [(a+1) f_n + a]\, b_r = f_{n+1} b_r$. ∎

LEMMA 2. *Suppose the initial b_{rs} is forced to be 0. Let $g_n = (1 + 1/a)^n - 1$. Then one can assure allocation exactly a times as much in the gambling asset as in the safe for n periods iff $b_f \geqslant g_n b_r$.*

Proof. The gambling investment appreciating by a factor of z is equivalent to the safe investment being multiplied by a factor of $1/z$. Thus there is a symmetry beween safe and gambling investments that changes a for f_n to $1/a$ for g_n. ∎

LEMMA 3. *One can ensure investing exactly a times as much in the safe for n periods iff for the first period $b_g = ab_s$, $b_{fg} \geqslant g_{n-1} b_{rg}$, and $b_{fs} \geqslant f_{n-1} b_{rs}$.*

The middle inequality is forced by Lemma 2 when the safe asset drops to zero and the last is forced by Lemma 1 when the gambling investment drops to 0. Assume all three comparisons. Suppose that gambling investment goes up. Without any adjustment you would be overinvested in the second period. Ignoring the safe portions, if you adjusted the gambling portion of your portfolio so that it had the right fraction invested and b_{fg2} becomes b^*_{fg2} by the proof of Lemma 2, $b^*_{fg2} \geqslant g_{n-2} b_{rg2}$. However, now you would be underinvested because of the “ignored” safe portions. Thus you do not need to trade away this much gambling stock and this last inequality still holds. Then ensurance follows from an induction hypothesis.

Suppose the gambling investment goes down. This is equivalent to the safe portion going up and the symmetric argument applies.

PROPOSITION 6. *One can assure investing exactly a times as much in the gambling investment as in the safe for n periods iff for the first period*

$$b_f \geqslant \left[\frac{(1+a)^n}{1+a^n} - 1 \right] b_r. \tag{21}$$

Proof. Suppose some b_{**} assignments do assure the required outcome. Then multiplying the equations of Lemma 3 by $g_{n-1} - f_{n-1}$, $(1+a)(1+f_{n-1})$, and $(1+a)(1+g_{n-1})$, respectively yields the inequality of the proposition. Conversely, suppose that proposition inequality holds. By treating some of the flexible portfolio as rigid we may assume equality. Letting

$$b_{fg} = ag_{n-1}/(1+g_{n-1}), \qquad b_{rg} = a/(1+g_{n-1}),$$
$$b_{fs} = f_{n-1}/(1+f_{n-1}), \qquad b_{rs} = 1/(1+f_{n-1})$$

gives three equalities in Lemma 3 and therefore the following inequality, which is a linear combination:

$$(1+g_{n-1}+a+af_{n-1})\, b_f \geqslant (f_{n-1}+f_{n-1}g_{n-1}+af_{n-1}g_{n-1}+ag_{n-1})\, b_r.$$

Using the defintions of f_{n-1} and g_{n-1}, this is rewritten as (21). Multiplying these figures by the appropriate constant gives the correct b_f and b_r and ensurance follows from Lemma 3. This completes the proof. ∎

When $n=2$ this becomes $b_f \geqslant [2a/(1+a^2)]\, b_r$ so $b_f \geqslant b_r$ suffices. Suppose $b_f = b_r$. If $a>1$, the last inequality of Lemma 3 forces the flexible porfolio to be more safely invested than the rigid. If $a<1$, the middle inequality of Lemma 3 forces the flexible portfolio to be more riskily invested than the rigid. In sum, this shows that depending on conditions, the optimal rigid portfolio can be riskier (have a higher proportion in the gambling asset) or less risky than the optimal flexible portfolio. Our conjectured maxim fails.

5.3. *Cases Close to CRRA*

We may also be interested in the behavior of portfolios when utility is not pure CRRA. Such utilities are easily produced by mixing a small component of CARA with CRRA. Thus we consider the utility function $u(x) = \log(x) - de^{-kx}$, for small (absolute) values of d. With d positive (negative), the fraction of the portfolio invested in the gambling asset tends to decline (increase) with wealth.

Suppose the inequality of Proposition 6 is satisfied strictly. Then we may split off an auxiliary portion of the flexible portfolio so that the rest of it gives equality. Suppose the auxiliary is zw where w is the total wealth. Let $p = z \min(t, 1-t)$ where $t = a/(1+a)$.

With $u(x) = \log(x)$, $r(ws) = \log w + S(s)$ where $S(s) = \int_{-1}^{\infty} \log(1+sx) f(x)\, dx$ is maximized at $s = t$. Let $|d|$ be less than $\min(0.5, S(t) - S(t+p), S(t) - S(t-p))$. Let v be the optimal investment for $u(x) = -e^{-kx}$. Let $u(x) = \log x - de^{-kx}$. Since $d \geqslant -0.5$, u is concave. For all wealths the optimal investment fraction must be in $I = [t-p, t+p]$, since otherwise log loses more that $|d|$ from taking the fraction t, and the second summand of u cannot make this up.

Let there be two periods and start by investing t of the auxiliary and follow the scheme of the Proposition 6 for the rest of your wealth. Scale the original wealth so that v is the total investment. By Proposition 6, after the first period we can adjust the non-auxiliary part of our wealth so that we are fraction t invested. Since we invested the same way in the auxiliary as the rest of our wealth the auxiliary is still fraction z of our wealth. If we invest t of the auxiliary in the gambling investment we would be fraction t invested, with the proposition adjustment on the rest. We can adjust the auxiliary fraction by at least $\min(t, 1-t)$ either way and this adjusts the fraction of the total wealth fraction by at least p either way and this allows us to make it the optimum fraction which lies in I.

In the following we can get by (do as well as with wealth flexible) with equality in Proposition 6, implying that no auxiliary portfolio is available. Suppose $t = 0.5$ for the optimal investment fraction for $u(x) = \log x$. Let $u(x) = \log x - de^{-kx}$ with $-0.5 < d < 0$. As above take $|d|$ sufficiently small that the optimal investment fraction must be in $[0.25, 0.75]$ and let v be the optimal investment in the gambling asset for $u_2(x) = -e^{-kx}$. Let $b_f = b_r = v$ and invest half of each in the gambling asset in the first of two periods. Suppose the gambling investment goes up in the first period leaving us with wealth at least $2v$. By concavity $r_2'(0.5x) \leqslant r_2'(v) = 0$ and $r'(0.5w) \geqslant 0$. Thus our optimal second period investment lies in $[0.5w, 0.75w]$. Since b_{rs} is at most $0.25w$ we can adjust b_g to investing at least $0.75w$. Thus we can invest optimally in the second period. The case that the gambling investment goes down in the first period is similar. Conditions for more than two periods will, of course, be more difficult to satisfy.

6. CONCLUSION

Economists have long recognized that risky investments drive the economy, and they have praised those who take the risks. Much of modern finance, both in theory and in the marketplace, seeks to facilitate investors' ability to diversify their portfolios and make them more liquid, hence flexible. Though finance theory has constructed beautiful models to address diversification, relatively little attention has been paid to flexibility.

The hypothesis examined here is that the greater the investor's flexibility, the easier it is for him to change his portfolio depending on his results, the more willing he will be to accept risks. When the investor has no control on the size of the risky investment, but can choose between one risky and one riskless asset, this conjecture proves correct. However, if there is more than one risky asset each period, as would be the case in the real world, counterexamples demonstrate that flexibility rarely ensures greater risk taking. For the standard portfolio problem, counterexamples can even be found when the constant relative risk aversion is larger than unity.

Our focus here has been on the behavior of the individual decision maker; however, society, which sets the rules, may have a concern for the aggregate results of individuals' decisions in the macroeconomics of financial decision making. For example, in evaluating the tax treatment of retirement funds, policy makers look at the effects on aggregate savings rate. Presumably they are also concerned about effects on risk taking, the primary subject of this paper, and financial market stability. Our finding of a quite general counterexample to the maxim that flexibility promotes risk taking suggests that seemingly minor financial market constraints, or their relaxation, may have unexpected effects on securities prices. Moreover, our experience with sunspot equilibria (Shell [12], Jackson and Peck [8]) and related phenomena suggest that these effects may prove both substantial and unpredictable.

Factors that promote investment rigidity, such as trading costs, taxation, and exit charges, can be thought of as fulfilling the opposite function of innovative securities. In fact, many new derivative securities are designed to overcome rigidities, for example to hedge an appreciated position without requiring a disposition that would incur taxes. (See Duffie and Rahi [3] for a useful review on financial market innovations.) Our analysis suggests that in theory at least, new securities will have an unpredictable effect on risk taking.

Many developing nations limit capital ouflows, thereby reducing the flexibility of investments, and hence their attractiveness, but possibly dampening the type of speculative activity that often plagues securities markets of such nations.[8] We have shown how the microeconomics of risk taking

[8] Chile prevents foreign investors in its securities markets withdrawing their money in less than one year. Eduardo Engel observes that this rigidifying factor may have saved his nation from a financial collapse that Mexico recently suffered. Woo-Chan Kim, South Korean Ministry of Finance, tells us that his nation is concerned about opening its securities markets further to foreign investors because they may respond excessively to events between North and South Korea. (Personal communications, February 1996).

depends upon the flexibility of investments. Its macroeconomics, addressing which investments society undertakes and how they perform, will be no less dependent.

APPENDIX

Proof of Proposition 2. On a small enough open interval containing z, u', $A'' - A'A$ have constant signs and u'' is continuous. Restrict everything to this interval. Let $A(\hat{z}) \neq A(z)$. Let $\alpha = u'(z)(A(z)\,A'(z) - A''(z))$, $\beta = u'(z)(A(\hat{z}) - (A(z))$. Let B be the gamble paying h, $0.5(1+bh)$ of the time, otherwise, $-h$. Let D be the gamble paying h, $0.5(1+dh)$ of the time, otherwise $-h$. Let E denote expected utility with initial wealth z. Define $f(h) = E(B+D) - E(B) - E(D) + E(0)$. Then $f(0) = f'(0) = f''(0) = f'''(0) = 0$ and by Taylor's Theorem $f(h) = f''''(k)\,h^4/24$ for some $0 < k < h$. Now $f''''(0) = 6u''''(z) + 12(b+d)\,u'''(z) + 24bdu''(z)$ and when we substitute $c = 0.5A(z)$ for b and d, this becomes 6α. Since $f''''(k)$ is continuous in k, b, d at $k=0$, $b=d=c$, there exists $\varepsilon > 0$ such that $f''''(k)$ will have the same sign as α if h, b, d are within ε of 0, c, c respectively.

Let $g(h) = E(B) - E(0) = 0.5(1+bh)\,u(z+h) + 0.5(1-bh)\,u(z-h) - u(z)$. Then $g(0) = g'(0) = 0$ and by Taylor's Theorem $g(h) = 0.5g''(k)\,h^2$ for some $0 < k < h$. Since u is strictly monotone, we can and do take b so that $g(h) = 0$ and then $0 = g''(k) = 0.5(1+bk)\,u''(z+k) + bu'(z+k) + 0.5(1-bk)\,u''(z-k) + bu'(z-k)$. Solving for b:

$$b = \frac{-0.5(u''(z+k) + u''(z-k))}{u'(z+k) + u'(z-k) + 0.5k(u''(z+k) - u''(z-k))}.$$

As h and k approach 0, b approaches $c = 0.5A(z)$. Take h small enough that b is withing ε of c. Take d between $0.5A(z)$ and $0.5A(\hat{z})$ and within ε of c. Then we can and do take h small enough that $E(B+D) - E(D) = f(h)$ has the same sign as α.

With initial wealth $w = z$ or $w = z \pm h$, $E(D) - E(0) = 0.5g''(k)\,h^2$ for some $0 < k < h$ where

$$g''(k) = 0.5(1+dk)\,u''(w+k) + du'(w+k) + 0.5(1-dk)\,u''(w-k) + du'(w-k).$$

As h approaches 0, $g''(k)$ approaches $u''(w) + 2du'(w) = 2(d - 0.5A(w))\,u'(w)$. Taking h small enough, $g''(k)$ has the same sign as $(d-c)u'$, $0.5(A(\hat{z}) - A(z))u'$, and β. When $w = \hat{z}$, $d - 0.5A(w)$ reverses sign and $g''(k)$ has the same sign as $-\beta$ for small enough h. Here g and k change as the base wealth changes, but we make h small enough to guarantee that all k's are sufficiently small.

To summarize, D is strictly desirable at $\hat{z}$ when $-\beta$ is positive and strictly undesirable otherwise, the opposite at z, $z \pm h$.

Adjust b slightly enough that $E(B+D) - E(D)$ retains the same sign as α but $E(B) - E(0)$ takes on the same sign as $-\alpha$. Define the first period lottery $\tilde{x}_1$ to have outcome $\hat{z}$ with probability p and z with probability $q = 1 - p$. Let the other first period lottery $\tilde{y}_1$ be the same, except we throw in the lottery $\tilde{\varepsilon} = B$ when z arises. Let the second period lotteries $\tilde{x}_2$ and $\tilde{y}_2$ be 0 and D, respectively. Take p large enough that in the inflexible world we chose D exactly when D is desirable at $\hat{z}$. Define $ES(X)$ to be the expected utility after the second period when we start with X in the first period and, given this start and possible flexibility constraints, choose the second period optimally. Since $\hat{z}$ always comes up with the same probability and we always choose the same second period lottery when it does come up, we can ignore the $\hat{z}$-contribution to $ES(\tilde{y}_1) - ES(\tilde{x}_1)$. When we use D for the second period lottery at z, $z \pm h$, which is optimal given the first period lottery when $-\beta > 0$ and the economy is inflexible, or $-\beta < 0$ and the world is flexible, $ES(\tilde{y}_1) - ES(\tilde{x}_1) = q(E(B+D) - E(D))$ which has the same sign as α. When we use 0 for the second period lottery at z, $z \pm h$, which is optimal given whichever the first period lottery when $-\beta > 0$ and the economy is flexible or $-\beta < 0$ and the world is inflexible, $ES(\tilde{y}_1) - ES(\tilde{x}_1) = q(E(B) - E(0))$ which has the same sign as $-\alpha$. Thus in the flexible (resp. inflexible) world the sign of $ES(\tilde{y}_1) - ES(\tilde{x}_1)$ is the same as the sign of $\alpha\beta$ (resp. $-\alpha\beta$). Thus by choosing $A(\hat{z})$ on the sign $(-\alpha u'(z))$ side of $A(z)$, $\tilde{x}_1$ is strictly optimal in the flexible world and $\tilde{y}_1$ is strictly optimal in the inflexible world. We can find such a $\hat{z}$ else A has the inappropriate maximum or minimum at z and $A'(z) = 0$. But then $A(\hat{z}) - A(z)$ has the same sign as $A''(z)$ and $-\alpha u'(z)$ for $\hat{z}$ in some deleted neighborhood of z. ▮

Alternatively, suppose $A''(z) = A'(z)\,A(z)$, $A'(z) > 0$, and u'''' is continuous at z. Then the example still works: Let $\hat{z}$ be slightly larger than z so that $A(\hat{z}) - A(z)$ has the same sign as $A'(z)$. Let B be neutral at z and D at $z + h$. By the formula for b it is $0.5A(z)$ within second order terms in h. Similarly, $d = 0.5A(z+h) = 0.5A(z) + 0.5hA'(z)$ within second order terms. Since $f(h) = E(B+D) - E(B) - E(D) + E(0)$ is even in h we may use Taylor's formula with just the term $f''''(z)h^4/24$ and a sixth order error term. Then for small h the $0.5hA'(z)$ part of d dominates and is multiplied by $12u'''(z) + 24bu''(z) \approx 12u'''(z) + 24(-0.5u''(z)/u'(z))\,u''(z) = -12u'(z)\,A'(z)$ within second order terms. Thus $f(h)$ has the same sign $0.5hA'(z)(-12u'(z)\,A'(z))$ and $-u'(z)$ and the example works.

Now suppose u'''' is continuous on an interval I. Assume $A'' - A'A$ is zero on I since otherwise the theorem guarantees an example. Solving this for A and then solving the definition for A for u, the solutions for u, within the affine equivalence of u to $a+bu(c+dz)$, are e^z, z, z^3, $z-\sin z$, $z-\sinh z$, and $z+\sinh z$. The first two are CARA and there is no example since the preference between 0 and D is independent of wealth. The next three satisfy $A'>0$ and the last paragraph produces an example. Only for the last u do we not know whether an example exists.

Proof of Proposition 3. Since u is CRRA, we may assume $w=1$. Let us also assume without loss of generality that the supremum of $|\tilde{y}|$ equals 1. Let

$$\alpha' = \frac{1}{A(1)} \frac{Ey}{Ey^2} = \frac{1}{\gamma} \frac{Ey}{Ey^2}$$

and $0<|c-1|<1$. Then by the definition of u'', $|c\alpha' y| \leqslant 2\alpha'$ can be assumed small enough that $-|yc\alpha' y(c-1)\,\gamma/c| \leqslant yu'(1+c\alpha' y) - yu'(1) - yc\alpha' yu''(1) \leqslant |yc\alpha' y(c-1)\,\gamma/c|$. Taking expectations in this inequality yields

$$\begin{aligned} -\gamma\alpha'\,|c-1|\,Ey^2 &\leqslant Eyu'(1+c\alpha' y) - Ey + c\alpha'\gamma Ey^2 \\ &= Eyu'(1+c\alpha' y) + (c-1)\,\alpha'\gamma Ey^2 \\ &\leqslant \gamma\alpha'\,|c-1|\,Ey^2. \end{aligned}$$

When $c>1$ $(c<1)$ the right-hand (left-hand) inequality shows $Eyu'(1+c\alpha' y) \leqslant 0$ $(\geqslant 0)$ and $\alpha \leqslant c\alpha'$ $(\alpha \geqslant c\alpha')$. This shows we may make α relatively close (ratio close to 1) to α' by making Ey/Ey^2 small. In particular assume $0.5\alpha' < \alpha < 2\alpha' < 0.1$.

The Taylor series for u' about 1 converges uniformly on any symmetric closed interval about 1 not containing 0. As $|\alpha(y_1+y_2)| \leqslant 2\alpha'(|y_1|+|y_2|) \leqslant (0.1)2 = 0.2$ we may use this series on the LHS's of equations (11) and (12). Then the LHS of equations (12) minus twice equation (11) becomes

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{1}{k!} u^{(k+1)}(1)\,\alpha^k [E(y_1+y_2)^{k+1} - Ey_1^{k+1} - Ey_2^{k+1}] \\ &= \sum_{k=1}^{\infty} \frac{1}{k!} u^{(k+1)}(1)\,\alpha^k \sum_{i=1}^{k} \binom{k+1}{i} Ey^i Ey^{k+1-i} \end{aligned}$$

Now $Ey < Ey^2$ by assumption and for $j>1$, $|Ey^j| \leqslant E\,|y^2 y^{j-2}| \leqslant Ey^2 1^{j-2} = Ey^2$. Thus for the tail of our series

$$\left|\sum_{k=4}^{\infty}\frac{1}{k!}u^{k+1}(1)\,\alpha^k\sum_{i=1}^{k}\binom{k+1}{i}Ey^iE^{k+1-i}\right|$$
$$\leqslant\sum_{k=4}^{\infty}\left|\frac{1}{k!}u^{(k+1)}(1)\,\alpha^k 2^{k+1}(Ey^2)^2\right|$$
$$\leqslant\frac{2^5\alpha^4}{(0.5)^4}\sum_{k=4}^{\infty}\frac{1}{k!}u^{(k+1)}(1)(-0.5)^k\,(Ey^2)^2$$
$$\leqslant 2^9\alpha^4u'(1-0.5)(Ey^2)^2$$
$$=2^{\gamma+9}\alpha^4(Ey^2)^2.$$

The remaining terms are

$$u''(1)\,\alpha 2(Ey)^2+\tfrac{1}{2}u'''(1)\,\alpha^2 6Ey\,Ey^2+\tfrac{1}{6}u''''(1)\,\alpha^3[6(Ey^2)^2+8Ey\,Ey^3]$$
$$=-\gamma\alpha[2(Ey)^2-3(\gamma+1)\,\alpha Ey\,Ey^2+(\gamma+1)(\gamma+2)\,\alpha^2(Ey^2)^2]$$
$$-4\gamma(\gamma+1)(\gamma+2)\,\alpha^3Ey\,Ey^3/3.$$

For the last term $|\alpha^3Ey\,Ey^3|\leqslant\alpha^3\,|Ey||Ey^2|=\alpha^3\alpha'\gamma(Ey^2)^2\leqslant 2\gamma\alpha^4(Ey^2)^2$. If we substitute α' for α in the bracket we get

$$2(Ey)^2-3(\gamma+1)\,\alpha' Ey\,Ey^2+(\gamma+1)(\gamma+2)\,\alpha'^2(Ey^2)^2$$
$$=\{2\gamma^2-3(\gamma+1)\,\gamma+(\gamma+1)(\gamma+2)\}\,\alpha'^2(Ey^2)^2=2\alpha'^2(Ey^2)^2.$$

This substitution introduces small relative error in each term in the brace where it is small absolute error which may be made small enough that before the substitution the brace is at least 1 and the original bracket (before we substituted for α) is at least $(\alpha^2/4)(Ey^2)^2$. Then, with α sufficiently small $-\gamma\alpha$ times the bracket is still negative even after adding in the last term and the tail of the series. ∎

Proof of Proposition 4.

Without loss of generality, we hereafter normalize α_1^f to be equal to unity. We look at conditions on the utility function which guarantee that the following condition holds:

$$E\tilde{y}_1u'(w+\tilde{y}_1)=0 \qquad \text{and} \qquad E\tilde{y}_2u'(w+\tilde{y}_2)=0$$
$$\Rightarrow E(\tilde{y}_1+\tilde{y}_2)\,u'(w+\tilde{y}_1+\tilde{y}_2)\leqslant 0, \tag{22}$$

for any w, $\tilde{y}_1$ and $\tilde{y}_2$, and for $u'(z)=z^{-\gamma}$. To solve this problem, we use the "bivariate diffidence theorem" obtained by Gollier and Kimball [6]:

THEOREM 1. *Suppose that function $h(x, y)$ satisfies $h(x, 0) \equiv 0$ and $h(0, y) \equiv 0$. Then, as long as $f'_1(0) \neq 0$ and $f'_2(0) \neq 0$, the two following conditions are equivalent*:

- $\forall \tilde{y}_1, \tilde{y}_2$: $Ef_1(\tilde{y}_1) = f_1(0)$ *and* $Ef_2(\tilde{y}_2) = f_2(0) \Rightarrow Eh(\tilde{y}_1, \tilde{y}_2) \leqslant 0$.
- $h(y_1, y_2) - \delta_1(y_1)(\frac{\partial h}{\partial y_1})(0, y_2) - \delta_2(y_2)(\frac{\partial h}{\partial y_2})(y_1, 0) + \delta_1(y_1)\,\delta_2(y_2)(\frac{\partial^2 h}{\partial y_1\, \partial y_2})(0, 0) \leqslant 0$, *for all* y_1 *in the domain of* f_1, y_2 *in the domain of* f_2 *and* $y_1 + y_2$ *in the domain of* h, *with* $\delta_i(t) = (f_i(t) - f_i(0))/f'_i(0)$.

We apply this Theorem with $h(y_{1\text{w}}, y_2) = (y_1 + y_2)\, u'(w + y_1 + y_2) - y_1 u'(w + y_1) - y_2 u'(w + y_2)$, $f_1(y_1) = y_1 u'(w + y_1)$ and $f_2(y_2) = y_2 u'(w + y_2)$. Accordingly, condition (22) is equivalent to

$$H(\gamma, w, y_1, y_2) = (y_1 + y_2)\left[\frac{u'(w)\, u'(w + y_1 + y_2)}{u'(w + y_1)\, u'(w + y_2)} - 1\right] - y_1\, y_2[2A(w) - A(w + y_1) - A(w + y_2)] \leqslant 0, \quad (23)$$

for all w, y_1, and y_2 such that $w > 0$, $w + y_1 > 0$, $w + y_2 > 0$ and $w + y_1 + y_2 > 0$. Notice that $H \equiv 0$ under CARA. With $u'(z) = z^{-\gamma}$, condition (23) can be rewritten as

$$H(\gamma, w, y_1, y_2) = (y_1 + y_2)\left[\frac{(w + y_1)^{\gamma}\,(w + y_2)^{\gamma}}{w^{\gamma}(w + y_1 + y_2)^{\gamma}} - 1\right] - \frac{\gamma y_1\, y_2[w(y_1 + y_2) + 2y_1\, y_2]}{w(w + y_1)(w + y_2)} \leqslant 0. \quad (24)$$

The analysis of function H shows that it is nonpositive in the relevant domain if γ is less or equal to 1. Let us denote

$$r(\gamma) = \frac{H(\gamma, w, y_1, y_2)}{\gamma} = (y_1 + y_2)\, f(\gamma) - b,$$

with

- $f(\gamma) = (\exp(a\gamma) - 1)/\gamma$;
- $a = \log(w + y_1)(w + y_2)\, w^{-1}(w + y_1 + y_2)^{-1}$;
- $b = y_1\, y_2(w(y_1 + y_2) + 2y_1\, y_2)\, w^{-1}(w + y_1)^{-1}\,(w + y_2)^{-1}$.

It is easily proved that f is increasing and that

$$r(1) = -\frac{(y_1\, y_2)^2\,(w + (w + y_1 + y_2))}{w(w + y_1)(w + y_2)(w + y_1 + y_2)}$$

and

$$r(-1) = -\frac{2(y_1\,y_2)^2}{w(w+y_1)(w+y_2)} \leqslant 0.$$

Suppose that $0<\gamma\leqslant 1$. If y_1+y_2 is positive, then r is increasing in γ. Thus, $r(\gamma)\leqslant r(1)\leqslant 0$. If y_1+y_2 is negative, r is decreasing in γ. Thus, $r(\gamma)\leqslant r(-1)\leqslant 0$. This proves that H is nonpositive when γ is less than 1.

If γ is larger than 1, H is positive for some values of (w, y_1, y_2). Indeed, fix y_1, y_2 positive and let w approach 0. Then the term with w^γ in the denominator dominates and makes H positive. ▮

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