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THE J-SHAPE OF PERFORMANCE PERSISTENCE GIVEN SURVIVORSHIP BIAS

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Abstract—Performance may enhance survival probability. When it does, the induced lack of randomness challenges robust and unbiased inference. If survivors are sorted into two groups based on past performance, spurious persistence has been demonstrated if variance in performance is heterogeneous. However, as we show both theoretically and with simulations, if performance is categorized finely, the spurious persistence will be J-shaped; that is, at the bottom better performance in one period ''predicts'' worse performance for another period. We propose a simple *t*-test applied to the quadratic coefficient in a regression to distinguish between a spurious J-shape and monotonic patterns. Mutual funds, our example, exhibit the monotonically increasing pattern produced by true performance persistence.

I. Introduction

SOCIAL scientists must generally base their inferences on observations of nonexperimental settings, rather than the more comprehensible environment of controlled experiments. The challenge to unbiased robust inference from real-world data is often compounded by the nonrandomness of the available sample. For example, competition often culls weaker members of a population, leaving a survivorship-biased subset to be observed. Survival is in question for employees subject to evaluations and possible dismissal, forecasting services whose future depends on the accuracy of their predictions, and open-end mutual funds whose redemptions are linked to their recent investment returns.[1]

The special difficulties posed by such truncated samples have long been recognized (Tobin (1958) and Heckman (1976, 1990)), but their implications for evaluating time-series dependencies has been less well studied. Recently Brown et al. (1992), hereafter BGIR, explored the problems of survivorship bias in the context of an assessment of the ability of mutual funds to deliver superior performance. Mutual funds go out of business if they perform poorly relative to their peers, and researchers who fail to include (or do not have) data on such terminated funds will have a survivorship-biased sample. Suppose they sort them into superior performers and inferior performers in period 1, and analyze the relative performance of the two groups in period 2. BGIR show that if funds differ in the variances of their returns, then survivorship-biased samples will display spurious performance persistence; that is, better performers in period 1 will perform better in period 2, even though there is no true performance predictability. BGIR's analyses raise doubts about the validity of the performance persistence reported in recent studies of mutual funds—for example, Hendricks et al. (1993), hereafter HPZ, or Goetzmann and Ibbotson (1994); the patterns they report may be mere artifacts of survivorship-biased samples of populations characterized by heterogeneity in the variance of performance. For concreteness, we discuss mutual funds, but generalization to other applications is straightforward.

BGIR sort mutual funds into two performance groups. As we have learned from game theory and information economics, analyses employing binary classification can miss interesting phenomena that arise with multiple (or continuous) players or groups. In this paper, we study the effect of survivorship bias when actors are sorted into more than two performance groups. Since, as Heckman and Honoré (1990) note, few reliable inferences can be drawn from truncated samples unless sufficient prior information about the population distributions is available, we make distributional assumptions to obtain clear analytic results. Fortunately our assumptions may be reasonable in practice. For example, the performance distribution of equity mutual funds can be reliably inferred from a knowledge of the funds' management fees and the returns distribution of the stocks that comprise their investment portfolios.[2] We consider groups of participants; naturally, the results are also valid for ''groups'' with one member.

The interesting case arises when the performance variances of the members of the population from which the sample is drawn are heterogeneous, as in BGIR. Consider sorting a sample of survivors into enough groups such that at least the two lowest ranked groups exhibit an in-sample

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[1] The performance of macroeconomic forecasters has been evaluated, for instance, by McNees and Ries (1983) and Steckler (1987). Patel et al. (1994) provide evidence on the link between performance and flows in and out of open-end mutual funds.

[2] Zhou (1993) and references therein provide evidence for the distribution of equity portfolio returns being fat-tailed and nongaussian but contained within the well-behaved class of elliptical distributions such as the *t*-distribution with 5 to 8 degrees of freedom or the mixture-normal distribution.

performance lower than the population mean. Under conditions we show to be plausible, the relative ranks of these two groups can be expected to reverse between the first period and the second period. An intuitive understanding of this result flows from jointly considering that (1) the group that performed worst in the first period is further below the mean and thus is more likely to have come from the population with the higher variance; and (2) conditional on survival, the mean performance in any subperiod within the sample is higher for a group drawn from the population with the higher variance. This intuition suggests further that for groups with performances above the population mean, the relative ranks will be positively correlated across subperiods. Thus considering all groups within the survivorship-biased sample, we are able to demonstrate that a spurious J-shaped relation arises between first-period performance and second-period performance.[3]

We explore a simple regression-based approach to discriminate between a spurious J-shaped pattern of persistence performance and a true monotonic persistence in performance. This tool is simple to use and appears quite effective in simulations we conducted.

In the next section we sketch the theoretical basis for our claim for general populations with unequal variances and survivorship biases. Then, employing parameters identical to those in the BGIR simulations, we demonstrate that the J-shaped pattern arises with octile groupings of mutual funds. Finally, we propose a simple t-test of a quadratic regression coefficient that proves useful in distinguishing between performance persistence that is J-shaped and persistence that is increasing monotonically.

II. Theory

A. *Preliminaries*

Consider two random variables A and B with probability density functions f and g. Truncate both distributions from below at k; the survivors' sample consists of draws above k. For many carefully constructed samples, such as samples of mutual funds or continuing employees at a firm, k is typically considerably below the mean of the untruncated distributions. Denote the truncated random variables by A^+ and B^+, respectively. Let f^+ denote the truncated probability density function associated with A^+, that is,

$$f^+(x) = \begin{cases} f(x), & \text{if } x \geq k \\ 0, & \text{if } x < k. \end{cases}$$

Define $f^-(x) - f(x) - f^+(x)$, and g^+ and g^- similarly. We focus on the case where k produces a significant truncation—at least one of f^- and g^- is not zero almost everywhere.

Suppose a researcher observes two independent observations x and y that are draws of either A^+ or B^+. The truncation point k is known and fixed, and the prior odds ratio that the draws are taken from A^+ rather than from B^+ is $R(k)$. Our main interest is the conditional expectation of y given x,

$$E(y|x;k) = p(A^+|x;k)\mu_{A^+} + p(B^+|x;k)\mu_{B^+} \tag{1}$$

where $p(\cdot)$ denotes the probability of the event in parentheses, and μ_{A^+} and μ_{B^+} are the means of A^+ and B^+, respectively.[4] Hereafter we suppress the notation indicating conditioning on k to simplify exposition. We explore how $E(y|x)$ varies with x.

Define the likelihood ratio $l(x) \equiv f(x)/g(x)$, where x is in the range of the support of g. Define $l^+(x)$ as the likelihood ratio of the truncated distributions. For $x > k$, $l^+(x) = \theta(k)l(x)$, where $\theta(k) = \int_k^\infty g(x)\,dx \Big/ \int_k^\infty f(x)\,dx$. We recast equation (1) using Bayes' formula, so that $E(y|x)$ is related to the means of the truncated distributions and the likelihood ratio

$$E(y|x) = \mu_{A^+} + \frac{\mu_{B^+} - \mu_{A^+}}{R(k)\theta(k)l(x) + 1}. \tag{2}$$

Since very little can be inferred from samples with arbitrary underlying distributions that are subjected to truncation and selection biases (Heckman and Honoré (1990)), we must make some assumptions about $l(x)$ to derive useful results from equation (2).

ASSUMPTION: For some λ, the likelihood ratio $l(x)$ increases over $x < \lambda$ and decreases over $x > \lambda$.

Note that λ is the critical point of the likelihood ratio, which corresponds to the global maximum. The assumption is met if, for instance, A and B are gaussian random variates with $\mu_A = \mu_B$ and $\sigma_A < \sigma_B$. With suitable parameter values, f and g can more generally be members of commonly used regular unimodal distributions. This assumption enables us to obtain the following theorem about the relation between the means of the two truncated distributions.

THEOREM: Suppose that the means of the untruncated distributions are equal, i.e., $\mu_A = \mu_B$.

Given the above assumption on the shape of the likelihood ratio, a truncation at k results in $\mu_{A^+} \leq \mu_{B^+}$ with equality obtained only if f^+ and g^+ are constant multiples of each other.

Proof: See the appendix, which also allows the case of $\mu_A < \mu_B$.

[3] We thank a referee for noting that the pattern looks like the letter J.

[4] BGIR study Pr $(y > x)$ instead of $E(y|x)$. Given our assumptions, the two approaches lead to similar insights.

B. Application

We show that equation (2) and the theorem together imply that the relation of $E(y|x)$ to x will be J-shaped whenever $k < \lambda$. The null hypothesis is that the expected performances of the funds in the population are equal within every interval. To evaluate the null hypothesis, a sample of mutual funds' performances over some period is examined. The sample only includes funds whose performance in each subperiod exceeded k.

For example, suppose that funds whose performance falls below k in any subperiod (say a quarter) immediately face significant redemptions and are forced to close or merge.[5] Commercial organizations that serve investors may prune them from their databases since their clients would only need to know of survivors for ongoing portfolio choices. An analyst who acquires such a database to study a past period will have a survivorship-biased sample. Other assumptions are that the untruncated fund performances are independent across quarters, and that the conditions of the theorem and equation (2) are satisfied.

Truncation above the Likelihood Ratio's Critical Point λ: We begin with the case where the truncation point $k \geq \lambda$. Since $l(x)$ decreases with x in this range by assumption, and the theorem tells us that $\mu_{B^+} > \mu_{A^+}$, we see from equation (2) that $E(y|x)$ increases with x. Suppose fund 1 outperformed fund 2 in a particular quarter within the sample period. Then, conditional on this information, the mathematical expectation for fund 1's performance will be greater than that for fund 2's for any other subperiod. If we sort funds from such a sample into many groups based on their performance in one quarter, then we expect the groups' performances in quarters other than the sample period to be positively related to their rank in the formation quarter. BGIR note this possibility and question whether the persistence inferences drawn by studies of mutual funds that appear to ignore survivorship bias are valid.

Truncation below the Likelihood Ratio's Critical Point λ: In practice, firms, employees, mutual funds, forecasting services, and other entities are likely to survive as long as they achieve some minimum level of performance k, which may be considerably below λ. (Note that if A and B were well-behaved symmetric distributions like the gaussian distribution, the critical point of the likelihood ratio would equal the mean, and $k \geq \lambda = \mu$ indicates an extremely high threshold. In practice, we rarely observe the high culling rates implies by $k \geq \mu$.) Since, by assumption, $l(x)$ increases with x below the critical point λ, we immediately obtain from equation (2) that $E(y|x)$ decreases with increasing x in the range $\lambda > x > k$. Thus, if in a quarter two funds' excess returns x_1 and x_2 satisfy $\lambda > x_1 > x_2 \geq k$, fund 1 would be expected to perform *worse* than fund 2 in any other period from the sample, that is, their relative performance should reverse.

When $x_1 > x_2 \geq \lambda \geq k$, the discussion of the preceding subsection applies. We conclude that the relation of $E(y|x)$ to x will be J-shaped when $k < \lambda$. (Another way to see this result is to recognize the inverse relation between $E(y|x)$ and $l(x)$ in equation (2) and the inverted J-shape of $l(x)$ by assumption.) So if we sort the mutual funds into rank groups so that at least two groups' performance ceilings are below λ and at least two groups' performance floors are above λ, then we expect a J-shaped pattern relating group performance to rank and not a monotone pattern.

III. A J-Shape in Simulations

We illustrate the theoretical result by replicating the simulation of BGIR (see section II) closely, employing the same parameters, which are based on estimates of actual equity portfolios. Annual returns over a 4-year period for 600 funds are constructed. The returns for fund i in year t, r_{it}, are generated using the market model for asset returns that is familiar in finance,

$$r_{it} = r_f + \alpha_{it} + \beta_i(r_{mt} - r_f) + \epsilon_{it} . \tag{3}$$

The risk-free return r_f is fixed at 0.07 for each year. The α's (called Jensen's α in the financial performance literature) indicate the ex ante superior performance for fund i in period t. In the construction of the simulated returns, all funds are similar in ex ante performance; the α's are set to zero. We choose parameter values that mimic those in the real mutual fund data employed in HPZ. The β's for the funds are drawn from a normal distribution $N(0.95, 0.25)$. Excess market returns $(r_{mt} - r_f)$ for each of the 4 years are drawn from $N(0.086, 0.208)$. The residual returns for fund i, ϵ_i, are drawn from $N(0, \sigma_i)$.

A. Basic Simulation

In our first simulation we assume that the residual returns ϵ_i for each fund are independently distributed. The standard deviation of ϵ is specified following the empirical characterization adopted by BGIR,

$$\sigma_i = \sqrt{0.05349(1 - \beta_i)^2} . \tag{4}$$

To simulate survivorship bias, the worst 10% of the funds in each year are dropped.[6] In the first year the 60 worst performing funds are removed, in the second the worst

[5] Performance-based survivorship among mutual funds also arises due to the selection rule employed by fund managers when they open private funds to the public. During each period, a set of mutual funds is managed for a closed group of investors; those that achieve a stellar record are opened to the public, and those with weaker track records tend to be discontinued. Thus funds that become open to the public have strong performance records.

[6] Strict compliance with our theoretical framework would require us to remove funds whose observed excess returns in any year fell below $-1.28^*\sigma_\epsilon$; this level would remove 10% of the funds on average each year. The actual removal rule simply follows BGIR, and is sufficiently similar that the theoretical discussion should apply very closely.

performing 54 funds are removed, and so on. After 4 years this leaves a survivorship-biased sample of 393 funds. We generated 5000 such 4-year simulations. When we sort the survivorship-biased samples into merely two groups, we find a strong spurious performance persistence that closely mimics the persistence reported by BGIR.

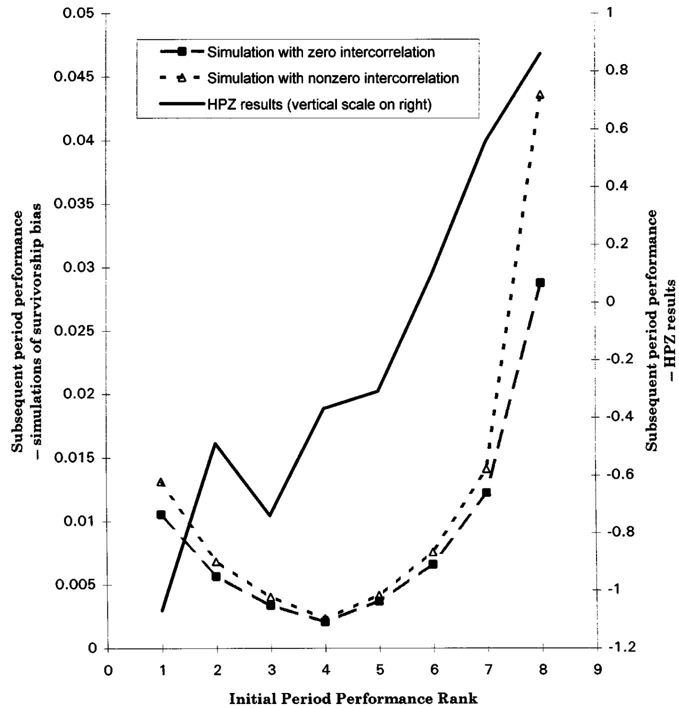
However, the result is very different when funds are sorted into eight groups. We sort the 393 surviving funds on the basis of performance (based on Jensen's α) in each year into eight groups. This choice follows HPZ and allows us to compare the simulation's results with the findings from a sample of real mutual funds. The simulated groups' performances are then computed for the following year. In order to calculate the α's for the octile groups, we keep track of the β's of the funds that are placed into each octile to compute the appropriate average β of each octile group for each of the 3 years. The mean of the Jensen's α is plotted against the initial period's rank for the different octile groups over the 5000 simulated data sets and is shown as the dashed line in figure 1. The pattern in the figure conforms to the J-shape that we predicted theoretically.

The J-shape that emerges from the simulations is starkly different from a monotonically increasing relation that would be predicted by a hypothesis of pure performance persistence. This difference allows us to evaluate the HPZ results. HPZ report that a portfolio of the lowest performing octile of funds, where performance is measured over one to eight quarters, has on average the lowest octile performance rank in the next quarter. A representative example, based on their table 3, panel C, is shown as the solid line without symbols in figure 1; it rises monotonically but for one short break. This real sample shows much stronger performance persistence than that from the survivorship-biased simulation results (the scale of the vertical axis on the right in the figure is a significantly compressed scale compared to the left axis used for the simulation patterns), and appears quite different from that arising on average from simulations of pure survivorship effects. More generally, across different evaluation periods and benchmark portfolios, HPZ report that relative performances of the octile portfolios in the evaluation period increase monotonically with their octile ranks in the formation period. (Spearman's statistics reported by HPZ are around 4 with p-values close to zero; in contrast, the Spearman's statistic corresponding to the spurious J-shape shown in figure 1 is 64, with a p-value of 33%.) HPZ's discovery of a monotonic pattern rather than a J-shape suggests that the performances of mutual funds are truly persistent in the short run. Section IV discusses a formal statistical test to distinguish a monotone pattern from a J-shape.

B. Simulations Incorporating Contemporaneous Correlation in Residuals

For a more realistic simulation of mutual fund performances, we must take account of the contemporaneous

FIGURE 1.—PATTERNS OF PERFORMANCE PERSISTENCE: SIMULATED SURVIVORSHIP BIAS AND HPZ RESULTS



correlations across the residual returns of funds. BGIR provide in their footnote 14 the following empirical characterization of the average correlation between fund performances:

$$\rho_{ij} = a - b(\beta_i + \beta_j) + c(\beta_i\beta_j); \quad a = 0.559,\ b = 0.732,\ c = 1.216. \tag{5}$$

BGIR point out that a two-factor model for ϵ's can deliver the correlation matrix corresponding to equation (5). In their implementation, they adopt a parameterization whose complexity increases exponentially with the number of funds and rapidly becomes unwieldy for samples of more than 30 funds. Fortunately we find a simple factor structure that should help other researchers who wish to deal with the intercorrelations of equation (5). Thus,

$$\epsilon_{it} = f_1(a, b, c, \beta_i)\varphi_{1t} + f_2(a, b, c, \beta_i)\varphi_{2t} + d\varphi_{it}. \tag{6}$$

Here the φ's are independent draws from a $N(0, 1)$ distribution; $f_1 = (r + s\beta_i)\sqrt{d/(1 - a + 2b\beta_i - c\beta_i^2)}$ and $f_2 = t\sqrt{d/(1 - a + 2b\beta_i - c\beta_i^2)}$, where $r = -b/\sqrt{c}$ and $s = \sqrt{c}$, $t = \sqrt{(ac - b^2)/c}$; and d is a free parameter to set the overall level of the residual variance.[7]

[7] BGIR observe in their footnote 14 that, given the structure of equation (5), "principal components will be an ineffective control for cross-sectional dependence." The factor representation in equation (6), which delivers idiosyncratic terms (φ_i) that are independent in the cross section, defuses the concern they raise.

The results of the simulation using equation (6) to generate the residuals are shown as the dashed line with triangles in figure 1. The J-shape is similar to that obtained with zero intercorrelations. In results not shown we find, not surprisingly, that adding intercorrelation between the residuals (ϵ's), while holding fixed the number of funds, increases the variance of the shape obtained across simulations; however, the J-shape generally continues to emerge.

IV. A Simple Test

A simple statistical test can help the researcher determine whether a sample displays true performance persistence or merely survivorship bias. Table 1 categorizes the competing possible patterns.

To test H_0 versus H_1, the standard Spearman's rank correlation statistic is quite suitable (see HPZ for such tests). To test for H_1 when H_2 is possible, we propose a regression that fits a linear and quadratic relationship between the ranks of performance in periods 1 and 2.[8] For H_1 the transformation into ranks maps the monotone persistence relation into a simple linear relation.[9] For H_2 the transformation into ranks leads to a check-mark ($\surd$) shape, which is approximated by a quadratic relation. The regression is

$$\text{rank}\,[r_{i2}] = \beta_0 + \beta_1\,\text{rank}\,[r_{i1}] + \beta_2\,\text{rank}\,[r_{i1}]^2 + v_i\,. \tag{7}$$

Here subscripts 1 and 2 denote the two periods in which the performances are measured: rank $[\cdot]$ denotes the relative ranking of the performance within the period.

Under H_0, $\beta_1 = 0$ and $\beta_2 = 0$. Under H_1, $\beta_1 > 0$ and $\beta_2 = 0$. Under H_2, the critical point of the quadratic function, whose coordinate along r_{i1} is $\gamma = -\beta_1/2\beta_2$, lies in a range that corresponds to the bottom of the J-shape due to survivorship bias. Assuming that the survivorship level satisfies $k < \lambda$ (see second subsection in section IIB), we obtain $\gamma > 0$. Since $\beta_2 > 0$ for a check-mark shape, $\beta_1 < 0$ follows for H_2. Thus we can use the standard t-statistic for β_1, $t(\beta_1)$, to sort among H_0, H_1, and H_2: if $t(\beta_1)$ is insignificantly different from zero, then we cannot reject H_0; if $t(\beta_1)$ is significantly positive, the evidence favors H_1; if $t(\beta_1)$ is significantly negative, the evidence favors H_2.

The challenge is to distinguish between mild monotone persistence and mild survivorship bias. (When either is strong, then the power of the test based on $t(\beta_1)$ increases.)

TABLE 1.—SHAPES OF CURVES RELATING PERFORMANCE BETWEEN PERIODS

Survivorship Bias in Sample	Persistence in Performance: Negligible	Persistence in Performance: Significant
Negligible	H_0—Flat line, no relation	H_1—Monotone relation
Significant	H_2—J-shape, spurious relation	H_3—Complex shape possibilities

We assessed the effectiveness of the test based on $t(\beta_1)$ through simulations that parallel the previous section; only summary results are provided. We introduced "mild survivorship bias" by having a cull rate of 5% per year rather than 10%. Period 1 embraced the first 2 years, and period 2 years 3 and 4. We generated samples to produce 500 survivors. Regression (7) was fitted and the estimate of $t(\beta_1)$ saved. The distribution of $t(\beta_1)$ is shifted substantially to the left of zero (which is where it would be centered with a zero cull rate). Less than 5% of the t-values were above zero; in contrast, in simulations with true persistence, the substantial majority of the t-values proves to be positive.

We also explored the effects of culling that depends probabilistically on performance rather than deterministically. The qualitative results did not change, and the inferences appear robust to this variation. We can use the test approach based on regression (7) to assess whether the funds in HPZ's sample are mere survivors without true performance persistence. The value of $t(\beta_1)$ obtained with HPZ's sample is +3.06. Based on our simulations, we compute the probability of obtaining $t(\beta_1) \geq 3.06$ with a 5% cull rate per year and zero true persistence as close to zero. For the HPZ sample, we strongly reject pure survivorship bias with zero performance persistence.

V. Concluding Remarks

If most of a population survives and if a monotonically increasing relation is discovered between current and future performance at a disaggregated level, then observed performance persistence is real, not simply an artifact of survivorship bias. Survivorship bias by itself generates a distinctive J-shape relating performance in one period to that in another. Researchers interested in learning about performance persistence in a population—be it of basketball shooters, salespersons, or innovative firms—can estimate a simple quadratic regression in performance ranks between periods in order to sort between the J-shaped pattern attributable to survivorship bias and a monotone relation attributable to true and persistent differential performance.

[8] We initially explored a test based on an order statistic: specifically, consider the maximum Spearman's rank correlation coefficient by "flipping" the rank order around initial period performance ranks to accommodate the expected V-shape under H_2. This test statistic displayed very poor statistical power in rejecting the null hypothesis in favor of the alternate hypothesis when the latter in fact was true.

[9] The expected relation with finite samples will only be approximately linear, with a flattening of the relation at the extreme ranks (both high and low). Funds with extreme ranks are funds that either are genuinely superior or inferior or have very high performance variance. High performance variance induces the common regression-to-the-mean effect at the "edges" of the performance relation.

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APPENDIX

We begin by proving two lemmas that are used in the proof of our theorem. Define $m_f = \int f\,dx$ and $\mu_f = \int fx\,dx/m_f$, where f is a nonnegative function (satisfied for our purposes by the probability density function). For the specific letters f and g: $m_f, m_g > 0$.

LEMMA 1: (a) $m_{f+g}\mu_{f+g} = m_f\mu_f + m_g\mu_g$.
(b) If $m_f, m_g > 0$, then μ_{f+g} is between μ_f and μ_g, strictly if $\mu_f \neq \mu_g$.
(c) If $f(x) = 0$ a.e. (almost everywhere) on $(-\infty, t)$, then $\mu_f > t$.
(d) If $f(x) = 0$ a.e. on (t, ∞), then $\mu_f < t$.

Proof: (a) $m_{f+g}\mu_{f+g} = \int (f+g)x\,dx = \int fx\,dx + \int gx\,dx = m_f\mu_f + m_g\mu_g$.
(b) By symmetry assume $\mu_f \leq \mu_g$. Then $m_f\mu_f + m_g\mu_f \leq m_f\mu_f + m_g\mu_g \leq m_f\mu_g + m_g\mu_g$. The result follows from (a) after dividing by $m_{f+g} = m_f + m_g$.
(c) $m_f\mu_f - tm_f = \int_{-\infty}^{\infty} (x-t)f(x)\,dx = \int_t^{\infty} (x-t)f(x)\,dx > 0$.
(d) $m_f\mu_f - tm_f = \int_{-\infty}^{\infty} (x-t)f(x)\,dx = \int_{-\infty}^{t} (x-t)f(x)\,dx < 0$.

LEMMA 2: Define $f(x) = r(x)g(x)$ with $r(x)$ nonnegative and nondecreasing. If f and g are constant multiples of each other a.e., then $\mu_g = \mu_f$. Otherwise $\mu_g < \mu_f$. If $r(x)$ is decreasing, the inequality is reversed.

Proof: Replacing f by f/m_f and g by g/m_g we may assume $m_f = m_g = 1$. Suppose $r(x) \geq 1$ on $(-\infty, \infty)$. Then $0 \leq \int_{-\infty}^{\infty} [f(x) - g(x)]\,dx = m_f - m_g = 0$. This forces equality, and it forces the integrand to be zero a.e., so $f = g$ a.e. If $r(x) \leq 1$ on $(-\infty, \infty)$ the inequality is reversed, so we may assume that $r(x) - 1$ takes on both signs.

Let $t = \sup\{x \mid r(x) \leq 1\}$. Then $f \leq g$ on $(-\infty, t)$; $f \geq g$ on (t, ∞); and

$$
\begin{aligned}
0 \leq \int_{-\infty}^{\infty} [f(x) - g(x)](x-t)\,dx &= \int_{-\infty}^{\infty} [f(x) - g(x)]x\,dx \\
&\quad - t\int_{-\infty}^{\infty} [f(x) - g(x)]\,dx \\
&= m_f\mu_f - m_g\mu_g \\
&\quad - t(m_f - m_g) = \mu_f - \mu_g .
\end{aligned}
$$

Equality here forces $g = f$ a.e., and the conclusion of the theorem holds.
If $r(x)$ is decreasing, apply the above to $f(-x) = r(-x)g(-x)$.
For fixed real number k define

$$
f^+(x) = \begin{cases} f(x); & \text{if } x \geq k \\ 0; & \text{if } x < k \end{cases}
$$

$f^-(x) = f(x) - f^+(x)$, and g^+ and g^- similarly. Note that only $m_{f^+}, m_{g^+} > 0$ is interesting and need be considered.

ASSUMPTION: Suppose for f and g there exist $l(x)$ and λ so that $l(x)$ is increasing to the left of λ and decreasing to the right and $f(x) = l(x)g(x)$. Assume $m_{f^+}, m_{g^+} > 0$. Further, assume that the truncation is significant so that we rule out $f = g = 0$ a.e. to the left of $k < \lambda$.

THEOREM: Suppose that $\mu_f \leq \mu_g$. Then, given the above assumption, we obtain for the truncated distributions that $\mu_{f^+} \leq \mu_{g^+}$ with equality only if f^+ and g^+ are constant multiples of each other.

Proof: Let f, g be a counterexample so $\mu_{f^+} \geq \mu_{g^+}$. Replacing f by f/m_f and g by g/m_g we may assume $m_f = m_g = 1$. If $k \geq \lambda$, the result follows from applying Lemma 2 to g^+ and f^+, so assume that $k < \lambda$. Suppose $m_{g^-} = 0$. Then $f^- \leq l(k)g^- \Rightarrow m_{f^-} = 0$, contrary to the assumption. Thus $m_{g^-} > 0$. Suppose $m_{f^-} = 0$. Then by parts (c) and (d) of Lemma 1, $\mu_{g^-} < k < \mu_{g^+}$ and by Lemma 1(b), $\mu_f = \mu_{f^+} \geq \mu_{g^+} > \mu_{g^- + g^+} = \mu_g$, a contradiction. Thus $m_{f^-}, m_{g^-} > 0$.

Suppose $m_{f^-} < m_{g^-}$. By Lemma 2, $\mu_{f^-} \geq \mu_{g^-}$. Since $m_{f^-} + m_{f^+} = m_f = 1 = m_{g^-} + m_{g^+}$, by Lemma 1(a) we get

$$
\begin{aligned}
\mu_f &= m_{f^-}\mu_{f^-} + m_{f^+}\mu_{f^+} = m_{g^-}\mu_{f^-} + m_{g^+}\mu_{f^+} + (m_{f^-} - m_{g^-})(\mu_{f^-} - \mu_{f^+}) \\
&> m_{g^-}\mu_{f^-} + m_{g^+}\mu_{f^+} \geq m_{g^-}\mu_{g^-} + m_{g^+}\mu_{g^+} = \mu_g
\end{aligned}
$$

which is a contradiction. Thus we only need consider the case of $m_{f^-} \geq m_{g^-}$.

For $m_{f^-} \geq m_{g^-}$, $m_{f^-} \leq l(k)m_{g^-}$ implies $l(k) \geq 1$. On $[k, \lambda]$, $l(x)/m_{f^+} \geq l(k)/m_{f^+} \geq 1/m_{f^+} \geq 1/m_{g^+}$. Let $t = \sup\{x \mid x > k \text{ and } l(x)/m_{f^+} \geq 1/m_{g^+}\}$. Then $t \geq \lambda$. Define $f^*(x) = f^+(x)/m_{f^+}$ and $g^*(x) = g^+(x)/m_{g^+}$. Then $m_{f^*} = m_{g^*} = 1$, $f^*(x) \geq g^*(x)$ on $[k, t)$, and $f^*(x) \leq g^*(x)$ on (t, ∞). Suppose $t = \infty$. Then $0 \leq \int_{-\infty}^{\infty} [f^*(x) - g^*(x)]\,dx = m_{f^*} - m_{g^*} = 0$. This forces equality, which forces $g^* = f^*$ a.e., and the conclusion of the theorem holds. Suppose $t < \infty$. Then $0 \leq \int_{-\infty}^{\infty} [g^*(x) - f^*(x)](x-t)\,dx = \mu_{g^*} - \mu_{f^*} = \mu_{g^+} - \mu_{f^+}$. Equality here forces $g^* = f^*$ a.e., and the conclusion of the theorem holds.