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MAJORITY RULE WITH LOTTERIES ON ALTERNATIVES *

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Social decision procedures usually do not allow lotteries on alternatives to compete as potential social choices.[1] This paper looks at the implications in the familiar majority rule choice procedure if such a restriction is placed on the domain of potential choices. It demonstrates that unattractive social choices may result whenever lotteries are not allowed to compete. However, it also shows that whenever lotteries need to be considered as serious contenders, intransitivities in social choice will always arise. To avoid such difficulties, there must be a certain alternative that defeats not only all certain alternatives, but all lotteries as well. The conditions for which this will be the case are described in the closing pages of the paper.

On a most basic level, one might object to any majority rule procedure which excludes lotteries on alternatives as potential choices because it cannot take account of intensities of individuals' preferences. Lotteries can be employed to get some measure of these intensities, that is, to get cardinal scalings for individuals' utilities. Cardinal measurement for each individual need not imply any interpersonal comparison of preference intensities. In fact, the mere introduction of lotteries on alternatives in the majority rule choice procedure may not be sufficient to make interpersonal comparisons.[2]

It is quite possible that a decision-making group will be faced

* I am indebted to Professors Amartya Sen and Robert Dorfman for many helpful comments.

1. Procedures which impose this restriction are consistent with Professor Kenneth Arrow's General Impossibility Theorem. (Arrow calls this a General Possibility Theorem, but the title impossibility theorem has gained general acceptance.) Condition 3 of that theorem, usually called the Independence of Irrelevant Alternatives, rules out decision procedures that rely on more than individuals' ordinal rankings of alternatives. It specifically excludes decision procedures that rely on individuals' rankings of all possible lotteries on alternatives. It can be argued that in general Condition I, which restricts the form of the social welfare function, also rules out the use of more than ordinal rankings. (*Social Choice and Individual Values*, Second Edition, New York: John Wiley & Sons, Inc., 1964.)

2. Conversely, cardinal measurement of individuals' utilities may be acceptable even though interpersonal comparisons are prohibited.

with situations in which lotteries on alternatives are possible choices. Two parties forming a majority coalition in a mayoralty race might agree to a fifty-fifty lottery on Smith or Jones, rather than accept the victory of Brown, who would win in the absence of the coalition agreement on the lottery.

Refusal to entertain lotteries on alternatives can lead to outcomes that to many appear to be inequitable and perhaps even inefficient. For example, consider a three-alternative case. Assume for the moment that the number of voters is odd and that the alternatives can be arrayed so that all individuals' preference functions are single peaked. A pairwise majority choice procedure, one in which each alternative competes against each other in binary contests, will lead to the choice of the median alternative. It is quite possible that for all individuals the median alternative will be but microscopically preferred to the one which for them is worst. For all individuals but one, the man whose first choice is the median alternative, the chosen alternative may be exceedingly less attractive than the one which is their first choice. The overwhelming majority in such a case would prefer a lottery on the other two alternatives to the choice of the median alternative.

The 101 Club must choose a single form of entertainment for all club members. The membership rolls contain fifty football fanatics, fifty ballet *aficionados,* and a single lover of musical comedy. For the footballers the musical is almost as bad as the ballet. For the ballet enthusiasts the musical is little better than football. By majority rule, using pairwise comparisons without lotteries, the musical will be chosen. If lotteries were permitted, a fifty-fifty football-ballet lottery would defeat the musical by any required plurality up to 100 out of 101.

A political system, such as the one in the United States, which rules out lotteries might lead to dominance by the center, even if the right and left could put together a majority coalition that would prefer a lottery on the extremes to a middle outcome.

It might seem that we should merely expand our field of choice so that all lotteries on alternatives as well as all certain alternatives are possible choices. Unfortunately, difficulties arise. No lottery can be chosen that will meet the standards which we usually require of those certain alternatives that are chosen in conventional majority procedures. In pairwise majority contests, no lottery can win a majority over every certain alternative and over every lottery on certain alternatives.

I shall demonstrate this fact for the three alternative cases. It covers all essential complexities. Employing standard von Neumann-Morgenstern expected utility analysis, utilities are assigned to certain outcomes. Individuals rank lotteries in accord with their own expected utility values.

If any one of the three alternatives is the first choice of a majority, there is no problem; no lottery can defeat it. Consider then the complementary case in which no alternative is a majority first choice. Whichever alternative wins in the pairwise contest between the two alternatives that compose a lottery will defeat the lottery in a majority contest.

Any three-alternative lottery can be broken down into a lottery on two components, one of the components being a certain alternative, the other a lottery on two alternatives. The component which is a lottery on two alternatives will be defeated by one of its component alternatives. This means that any three-alternative lottery will be defeated by a two-alternative lottery, which in turn will be defeated by a certain alternative.

If some individuals are indifferent between pairs of alternatives, or if society as a whole is indifferent because of even numbers problems, there may be complications, but the result is the same. If a lottery is not to be defeated by the third alternative or by any lottery including the third alternative, then neither of its component alternatives can be defeatable by the third alternative. Otherwise, the third alternative could be substituted for the alternative it defeats to create a new lottery that defeats the old one.

Only in the highly restricted case in which there are two (or more) undefeatable certain alternatives is it possible that a lottery on two certain alternatives will be undefeatable. But in this special case the lottery can do no better in pairwise comparisons than does either of its component alternatives. There would seem to be little reason to choose a lottery over one of its undefeatable component alternatives that it cannot outperform in pairwise comparisons with other alternatives.

The results with lotteries are not encouraging. We might wish to choose a lottery when there is no undefeatable certain alternative. In every such instance, all lotteries will be defeatable as well. Even in the case in which there is a certain alternative that can beat all certain alternatives, it may not be able to defeat all lotteries, as the 101 Club example clearly demonstrates. This presents us with a difficulty. We may be unhappy accepting a certain alternative as a group choice if it can be handily defeated by a lottery. But whenever a lottery is a victor in this manner, it will be defeatable;

it will fail to satisfy the criteria of acceptability that we would require of a chosen certain alternative. We might wish to investigate the conditions under which we can find an acceptable certain alternative, one that is defeated by neither a lottery nor a certain alternative.

It is well known that single peakedness of individual preference functions is sufficient to guarantee that one certain alternative will be able to defeat all others in pairwise majority votes (in the case in which the number of voters is odd). Our problem then is to discover what further characteristics of single-peaked preference functions are needed to prevent a lottery from defeating the alternative that is universal victor in pairwise comparisons between certain alternatives.

If either extreme alternative is universal victor, then it must be the first choice of a majority and no problems arise. The case of interest is the one in which the median alternative is victor over the two certain alternatives, but is not the first choice of a majority. Any lottery including the median will be defeated by the median. The median and the extreme alternative not included in the lottery are together the first choices of a majority. All the members of this majority will prefer the median over the lottery connecting it with the other extreme alternative. The median can lose only to a lottery on the two extreme alternatives.

Consider first the three-person case. The array of preference orderings consistent with our conditions has one individual, L, who perfers the left to the middle to the right alternative; a second individual, R, with the reverse preference ordering; and the third, M, who prefers the median alternative with the other two in either order. Each individual assigns von Neumann-Morgenstern utility values to the certain outcomes, arbitrarily giving the values 100 and 0 to his most and least favored alternatives. The necessary and sufficient condition that the lottery on the extremes does not defeat the median alternative is that the sum of the three individual von Neumann-Morgenstern utility values for the median alternative must be greater than 200. This is equivalent to the requirement that the two individuals whose first choices are extreme alternatives assign to the median choice *VN-M utility values* that sum to more than 100.

This result is intuitively plausible. If L assigns a utility value of, let us say, 70 to the median alternative, this means that he will prefer the median to any lottery on the extremes that offers less than a 70 per cent chance of the left alternative. If R has a utility value for the median somewhat greater than 30 (that is, 100 minus 70), he will require at least that probability of getting the right alternative before he will choose the lottery over the median. These two requirements cannot be met simultaneously. If their utility values for the median sum to more than 100, there will be no lottery that *L* and *R* simultaneously will prefer to the median. In such a situation, the choice of the median alternative gains some ethical support. It is not victor solely because the domain of choice is restricted to certain alternatives. Furthermore, the intensities of preference of those who favor the extremes do not allow them to agree on a lottery preferred to the median.

Another way of stating the summation requirement is that it must be possible to place the three alternatives along the horizontal axis so that, if we plot the utility values for the individuals and connect them to get a graph for each individual, none of the individuals' graphs will be convex to the horizontal axis. From the theory of decision-making under uncertainty, we have come to associate concave preference functions with risk aversion. That interpretation is a bit farfetched in this context. The horizontal spacing need not correspond to any cardinal difference on any scale. Nevertheless, the risk-averting notion is not absent. The fact that no lottery on the extremes defeats the median means that there is no lottery such that for both *L* and *R* the chance of getting their most favored alternative more than outweighs the complementary probability that their least favored alternative is chosen. For every possible lottery, at least one of them would rather stick with the sure median. The individuals are, in some sense, averse to taking the risk of getting their worst alternative.

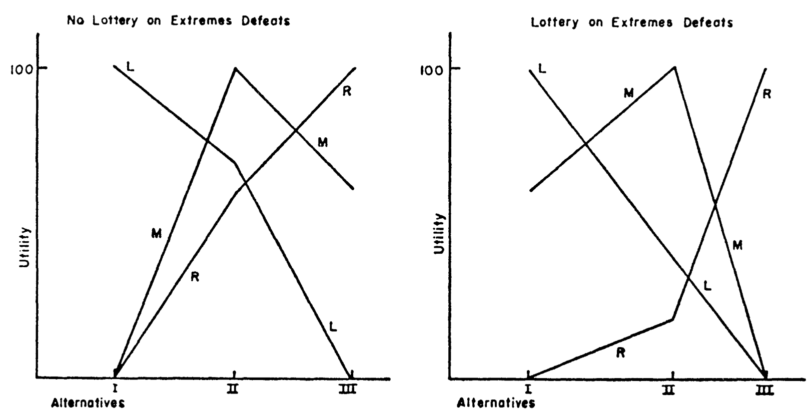

FIGURE I

TABLES OF UTILITY VALUES

	Alternatives I	II	III
L	100	70	0
M	0	100	60
R	0	60	100

	Alternatives I	II	III
L	100	40	0
M	60	100	0
R	0	20	100

In the right-hand graph the median alternative has been positioned so that *L*'s utility function is a straight line. *R*'s is convex to the horizontal axis. Any leftward shift in the placing of the median alternative will make *L*'s function convex, but a leftward shift is what is needed if *R*'s function is to lose its convex shape. Wherever the median is placed, the preference function of either *L* or *R* will be convex to the horizontal axis.

With many individuals the problem is more complex. It is no longer necessary, though it is still sufficient, that the alternatives be so placed that all individuals simultaneously have nonconvex functions. The required condition for no lottery to beat the median is still roughly the same. Individuals who prefer the extremes must be relatively unwilling to accept risks. To a certain degree, they must prefer a compromise on the median to a lottery on the extreme alternatives. The simplest way to show the exact condition, the extent of this certain degree, is with the aid of a graph (Figure II A).

Measure along the horizontal axis the probability that the lottery on the extremes chooses the left alternative. On the vertical axis plot the cumulative number of individuals whose first choice is the left alternative, who prefer the lottery to the median, the plot made as a function of the probability that the left alternative is chosen. This function will rise monotonically as we move to higher probabilities for the left alternative. Now do the same for those whose first choice is the right alternative. Invert the right preferrers' cumulative graph to get its mirror image. Now align the two graphs, that of the left preferrers and the inverted one of the right preferrers, so that their corresponding probability points are above one another and their horizontal axes are parallel at a distance that just represents a majority of the total population (including those whose first choice is the median).

If, and only if, the two cumulative functions touch or cross will there be a lottery on the extremes that can defeat the median alternative. Lotteries that defeat the median will be all those with probability values for which the cumulative functions overlap or

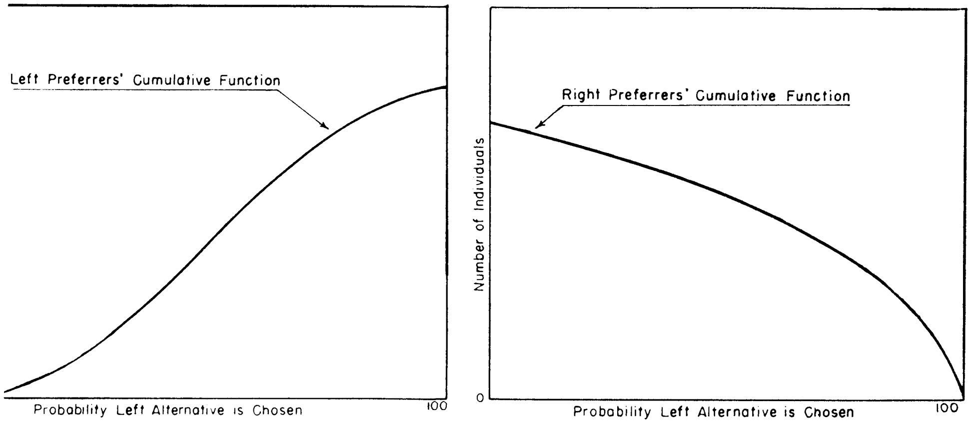

FIGURE II A

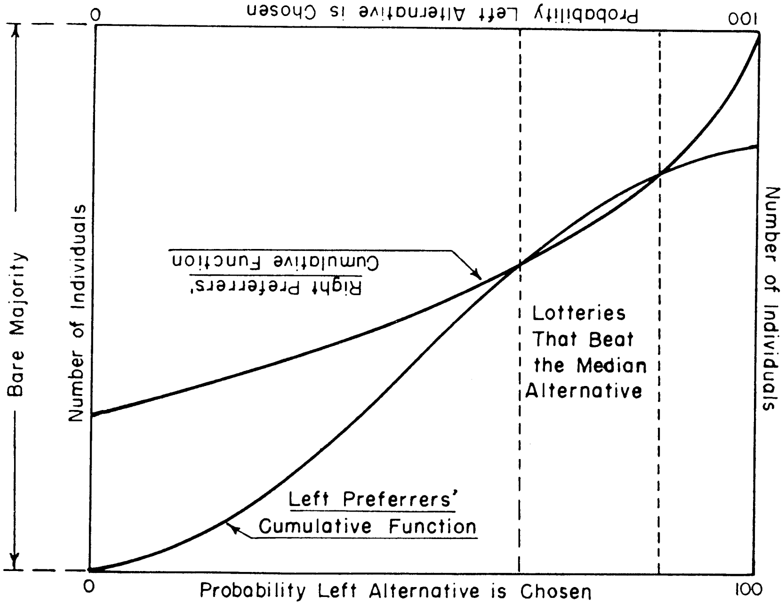

FIGURE II B

THE TWO GRAPHS ALIGNED

touch. Given such probabilities, a majority will prefer the lottery to the median alternative.

It is difficult to state the general condition that is required if lotteries on extremes are not to defeat the median. (In essence, it is the condition that lotteries on alternatives can be arrayed along the horizontal axis so that individuals' preferences are single peaked.) In intuitive terms the requirement is that people whose first choice is an extreme alternative are relatively unwilling to take their chances on their worst or best alternative, rather than take the median alternative for sure. Consider the earlier example. If the football rooters find the ballet a horror, but the musical comedy merely unexciting; if the ballet fans think that football is beastly, but the musical comedy just uninspiring; then no ballet-football lottery will exist that both ballet and football enthusiasts will prefer to the certain musical comedy. The 101 Club will not have to worry that a lottery will defeat its median alternative.

For preferences with but one peak
There will be a choice unique,
That gets a sure majority
O'er choices made with certainty.

With lotteries from which to choose
This unique choice will sometimes lose.
This raises once again, alas,
The nasty cyclic-choice morass.

No man or group should like to see
Such lack of transitivity.
We can avoid this dreaded curse
With voters mostly risk averse.

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