Supplementary Material: Possibly Final Offers

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Abstract

This document contains supplemental material for "Possibly Final Offers," including proofs of Propositions 5, 7 and 8, supporting calculations for Section 3.4, and a discussion of the model in Section 3.1 when the outside buyer's willingness-to-pay is endogenous.

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1 Additional Proofs

Proof of Proposition 5. First, note that Proposition 1 implies that whenever HIGH is risk averse, it is optimal to use a PFO when $\mu = \frac{p_l}{p_h}$. We prove Proposition 5a through a series of claims. Let $b(\mu)$ be the value function for the seller's optimization problem:

$$b(\mu) = \max_y \mu p_1(y) + (1-y)(1-\mu) r_l,$$

and note that $b(0) = r_l$ and $b(1) = r_h$. Further, by the Theorem of the Maximum, $b(\mu)$ is continuous on $[0,1]$. Let $y(\mu)$ be the optimal second-offer probability for μ. Figure 3 illustrates the argument.

Claim 1: $b(\mu)$ is convex.

Proof of Claim 1: Let $\mu^t = t\mu' + (1-t)\mu''$, $y' = y(\mu')$, $y'' = y(\mu'')$, and $y^t = y(t\mu' + (1-t)\mu'')$.

$$\begin{aligned}
b\left(\mu^t\right) &= \mu^t p_1\left(y^t\right) + \left(1-y^t\right)\left(1-\mu^t\right) r_l \\
&= t\left[\mu' p_1\left(y^t\right) + \left(1-y^t\right)\left(1-\mu'\right) r_l\right] \\
&\quad + (1-t)\left[\mu'' p_1\left(y^t\right) + \left(1-y^t\right)\left(1-\mu''\right) r_l\right] \\
&\leq t\left[\mu' p_1\left(y'\right) + \left(1-y\left(\mu'\right)\right)\left(1-\mu'\right) r_l\right] \\
&\quad + (1-t)\left[\mu'' p_1\left(y''\right) + \left(1-y\left(\mu''\right)\right)\left(1-\mu''\right) r_l\right] \\
&= tb\left(\mu'\right) + (1-t)\, b\left(\mu''\right).
\end{aligned}$$

Claim 2: For μ sufficiently high or sufficiently low, it is not optimal to use a PFO.

Proof of Claim 2: Note that $\mu p_1(y) + (1-y)(1-\mu) r_l$ is concave in y, and $p_1'(y) > 0$, where at $y = 0$ and $y = 1$ we refer to the properly defined one-sided derivatives. The first derivative of the objective function with respect to y is:

$$\mu p_1'(y) - (1-\mu) r_l.$$

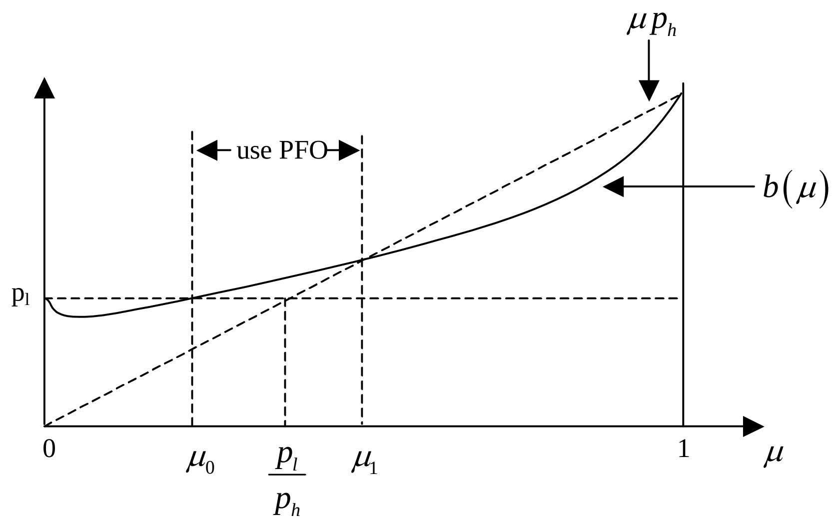

Figure 3: The region over which PFOs are optimal.

If $\mu p_1' (1) - (1-\mu) r_l > 0$, $y^* = 1$, and it is not optimal to use a PFO. Since

$$\lim_{\mu->1} \mu p_1' (1) - (1-\mu) r_l > 0,$$

for μ sufficiently close to 1 it is not optimal to use a PFO. Similarly, if $\mu p_1' (0) - (1-\mu) r_l < 0$, $y^* = 0$, and it is not optimal to use a PFO. Since

$$\lim_{\mu->0} \mu p_1' (0) + (1-\mu) r_l < 0,$$

for μ sufficiently close to 0 it is not optimal to use a PFO.

Claim 3: Equation $b(\mu) = r_l$ has exactly two solutions, $\mu = 0$ and μ_0, where $0 < \mu_0 < 1$. Similarly, $b(\mu) = \mu r_h$ has exactly two solutions, $\mu = 1$ and μ_1, where $0 < \mu_1 < 1$.

Proof of Claim 3: Since using a PFO is not optimal for sufficiently small μ and $b(0) = r_l$, $b() < r_l$ for small μ. By convexity and continuity of $b()$ there is exactly one point where $b(\mu_0) = r_l$. Since $b(1) = r_h$, this point must be such that $0 < \mu_0 < 1$. Similarly, since using a PFO is not optimal for sufficiently large μ and $b(1) = r_h$, $b(\mu) < \mu r_h$ for large μ. By continuity and convexity of $b()$, there is exactly one point where $b(\mu_1) = \mu_1 r_h$. Since $b(0) = r_l$, this point must be such that $0 < \mu_1 < 1$.

Claim 4: A PFO strategy is optimal on the closed interval $[\mu_0, \mu_1]$.

Proof of Claim 4: A PFO is optimal when $b(\mu) \geq \max\{r_l, \mu r_h\}$, $b(\mu) \geq r_l$ on $[\mu_0, 1]$, and $b(\mu) \geq \mu r_h$ on $[0, \mu_1]$. The intersection of these two sets is $[\mu_0, \mu_1]$. Note: Claims 1 - 4 suffice for Proposition 5a, except for the fact that $\mu_0 < \mu_1$.

Claim 5: Increasing HIGH's risk aversion decreases μ_0 and increases μ_1.

Proof of Proposition 5b: Consider two utility functions $v_1()$ and $v_2()$ for HIGH and let $v_2()$ be more risk averse than $v_1()$. Denote the seller's value function when HIGH has utility function $v_1()$ by $b^1(\mu)$, and similarly let $b^2(\mu)$ be the seller's value function when HIGH has utility function $v_2()$, and let μ_0^t and μ_1^t solve $b^t(\mu_0^t) = r_l$ and $b^t(\mu_1^t) = \mu_1^t r_h$, respectively. By Proposition 4, increasing risk aversion strictly increases the seller's profit whenever a PFO is optimal. Hence $b^2(\mu) > b^1(\mu)$ for $\mu_0^1 \leq \mu \leq \mu_1^1$. By continuity, $\mu_0^2 < \mu_0^1$ and $\mu_1^2 > \mu_1^1$. Claim 5 suffices for Proposition 5b.

Finally, note that for any finite level of risk aversion, $b\left(\frac{r_l}{r_h}\right) > r_l$, and therefore by the continuity argument above, $\mu_0 < \frac{r_l}{r_h} < \mu_1$ for any finite level of risk aversion.■

Proof of Proposition 7. For the purposes of the proof, it is notationally simpler to work with $z_j = 1 - y_j$, the (conditional) probability of making a j^{th} offer after the seller's offer of p_{j-1} is rejected. Let $\zeta_{kt} = \left(\prod_{j=k+1}^{t} z_j\right)$. For each k, constraints (14) can be rewritten as:

$$\begin{aligned} \theta_k + v(w - p_k) &\geq \zeta_{kt}(\theta_k + v(w - p_t)), \text{ for } t = k+1, ..., n, \text{ or} \\ v(w - p_k) &\geq (1 - \zeta_{kt}) v(w - r_k) + \zeta_{kt} v(w - p_t), \text{ for } t = k+1, ..., n. \end{aligned}$$

Let $P_k(z_{k+1})$ satisfy:[1]

$$v(w - P_k(z_{k+1})) = (1 - z_{k+1}) v(w - r_k) + z_{k+1} v(w - r_n), \text{ for } k = 1, ..., n-1. \tag{26}$$

Let $z_1 = 1$, and let $P_n(z_{n+1}) = r_n$. Note that $r_k > P_k(z_{k+1}) > r_n = r_k$ if $0 < z_{k+1} < 1$.

Step 1: Since $P_k(z_{k+1}) \leq r_k$, all participation constraints (12) are satisfied.

Step 2: For $0 < z_k < 1$, $k = 2, ..., n$, $P(z_{k+1})$ satisfy (14). For each $k = 1, ..., n$, and

[1] Functions $P_k(z)$ are similar to $p_1(y)$ from the two-type case except, being defined on z instead of y, $P_k(z)$ are decreasing and concave in z.

$t = k+1, ..., n$:

$$\begin{aligned} v\left(w - P_k\left(z_{k+1}\right)\right) &= \left(1 - z_{k+1}\right) v\left(w - r_k\right) + z_{k+1} v\left(w - r_n\right) \\ &\geq \left(1 - \zeta_{kt}\right) v\left(w - r_k\right) + \zeta_{kt} v\left(w - r_n\right) \\ &\geq \left(1 - \zeta_{kt}\right) v\left(w - r_k\right) + \zeta_{kt} v\left(w - P_t\left(z_{k+1}\right)\right) , \end{aligned}$$

where the first inequality follows from $\zeta_{kt} \leq z_{k+1}$ and $v\left(w - r_k\right) < v\left(w - r_n\right)$, and the second inequality follows from $P_t\left(z_{k+1}\right) \geq r_n$.

Step 3: Note that if $p_k \geq r_{k+1}$ for $k = 1, ..., n-1$, then (13) are satisfied.

Profit under $P_k\left(z_{k+1}\right)$ is given by

$$\sum_{k=1}^{n} \left(\prod_{j=1}^{k} z_j \right) \mu_k P_k\left(z_{k+1}\right).$$

As $v()$ becomes infinitely risk averse, $P_k\left(z_{k+1}\right) \to r_k$ for $z_{k+1} \in (0,1)$, and therefore $p_k > r_{k+1}$, satisfying (13). Thus, as buyers become infinitely risk averse, $\sum_{k=1}^{n} \left(\prod_{j=1}^{k} z_j \right) \mu_k P_k\left(z_{k+1}\right) \to \sum_{k=1}^{n} \left(\prod_{j=1}^{k} z_j \right) \mu_k r_k$. And, letting $z_k \to 1$, expected profit converges to $\sum_{k=1}^{n} \mu_k r_k$, the full information profit. Finally, note that every z_k must satisfy $0 < z_k < 1$ for this convergence to occur, and therefore that a PFO cascade is optimal.[2] ■

Proof of Proposition 8: Since $\pi^D < \pi^F$, and π^D is the largest profit the seller can earn without using a PFO, if we can show that $\pi^* \to \pi^F$ as HIGH becomes infinitely risk averse, this establishes the result. Consider the family of offers such that $x_1 = x_1^F$, $x_2 = x_2^F$, and $p_2 = p_2^F$. This satisfies LOW's participation constraint (18). Let $\bar{p}$ be the value of p_1 such that $u_l\left(x_1^F, w_l - \bar{p}\right) = 0$. Any value of p_1 such that $\bar{p} \leq p_1 \leq p^F$ satisfies (16) and (17). We will focus on the problem to one of choosing p_1 to maximize expected profit subject to $x_1 = x_1^F$, $x_2 = x_2^F$, $p_2 = p_2^F$, (15), and $\bar{p} \leq p_1 \leq p^F$. The solution to this problem is feasible but not necessarily optimal in the seller's original problem.

[2] If $z_k = 0$ for some k, then the game stops with probability 1, and profit can be no higher than $\sum_{j=1}^{k} \mu_k r_k$. If $z_k = 1$ for $k \neq 1$, then $p_k = p_{k+1} \leq r_{k+1}$, again bounding profit away from the full-information maximum.

Consider HIGH's incentive-compatibility constraint evaluated at x_1^F, x_2^F, and p_2^F.

$$u_h\left(x_1^F, w_h - p_1\right) \geq y u_h\left(x_2^F, w_h - p_2^F\right).$$

Since $u_h\left(x_1^F, w_h - p_1^F\right) = 0$, this can be rewritten as:

$$u_h\left(x_1^F, w_h - p_1\right) \geq (1-y)\, u_h\left(x_2^F, w_h - p_2^F\right) + y u_h\left(x_1^F, w_h - p_1^F\right),$$

and letting $\hat{p}$ be such that $u_h\left(x_1^F, w_h - \hat{p}\right) = u_h\left(x_2^F, w_h - p_2^F\right)$, this can again be rewritten as:

$$u_h\left(x_1^F, w_h - p_1\right) \geq (1-y)\, u_h\left(x_1^F, w_h - \hat{p}\right) + y u_h\left(x_1^F, w_h - p_1^F\right). \tag{27}$$

Note that since $u_h\left(x_2^F, p_2^F\right) > 0$, $\hat{p} < p_1^F$.

Let $p_1(y)$ be defined as:

$$u_h\left(x_1^F, w_h - p_1(y)\right) \equiv (1-y)\, u_h\left(x_1^F, w_h - \hat{p}\right) + y u_h\left(x_1^F, w_h - p_1^F\right),$$

i.e., the maximum price that satisfies HIGH's (27). Since $\frac{\partial^2 u_h(x,w)}{\partial w^2} < 0$ for all (x,w), $p_1(1) = p_1^F$, $p_1(0) = \hat{p}$, and $p_1(y)$ is increasing and concave on the interval $(0,1)$.

Since we have written the problem as a one dimensional monetary lottery, an increase in risk aversion is equivalent to a increase in $p_1(y)$ for all y, and as HIGH becomes infinitely risk averse, $p_1(y)$ converges to p_1^F for $0 \leq y < 1$. Let $p_1^n(y)$ be a sequence of functions corresponding to increasingly risk averse versions of HIGH. Let $p_1^n(1) = p_1^F$, $p_1^n(0) = \hat{p}$, and $p_1^n(y)$ is increasing in y and concave on the interval $(0,1)$ for each n. Further, let $p_1^{n+1}(y) > p_1^n(y)$ for all $0 < y < 1$, and $\lim_{n\to\infty} p_1^n(y) = p_1^F$ for $0 < y < 1$.

$$\begin{aligned}
\lim_{y\longrightarrow 0}\lim_{n\longrightarrow\infty} \pi^n(y) &= \lim_{y\longrightarrow 0}\lim_{n\longrightarrow\infty} \left(\mu\left(p_1^n(y) - c\left(x_1^F\right)\right) + (1-y)(1-\mu)\left(p_2^F - c\left(x_2^F\right)\right)\right) \\
&= \lim_{y\longrightarrow 0} \mu\left(p_1^F - c\left(x_1^F\right)\right) + (1-y)(1-\mu)\left(p_2^F - c\left(x_2^F\right)\right) \\
&= \mu\left(p_1^F - c\left(x_1^F\right)\right) + (1-\mu)\left(p_2^F - c\left(x_2^F\right)\right) = \pi^F.
\end{aligned}$$

Let $\pi^{**} = \max_y \left(\mu\left(p_1^n(y) - c\left(x_1^F\right)\right) + (1-y)(1-\mu)\left(p_2^F - c\left(x_2^F\right)\right)\right)$. Since $\pi^F > \pi^D$, $\pi^* \geq$

π^{**}, and $\pi^{**} \to \pi^F$, this establishes that PFOs outperform deterministic strategies when HIGH is sufficiently risk averse. The argument that the seller's expected profit is non-decreasing in HIGH's risk aversion is similar to the one in Proposition 4.■

2 Supporting computations for section 3.4

A change of variables simplifies the analysis. Let $s = f(q) = \sqrt{q}$ be the utility earned by HIGH from consuming q dollars worth of quality. Hence in terms of s, the utility functions of the buyers can be written as $u_H(s,p) = (s-p)^{\frac{1}{b}}$ and $u_L(s,p) = (as-p)^{\frac{1}{b}}$. Given that q is measured in dollars, the cost of producing utility-from-quality s is given by $c(s) = f^{-1}(s) = s^2$. Hence there is a one-to-one correspondence between an offer (q,p) and an offer (s,p) for s defined in this way. We will refer to s simply as quality, although it should be understood as the utility, measured in dollar terms, that quality yields.

Under the assumptions we have made, the relevant constraints are HIGH's incentive compatibility constraint and LOW's participation constraint. The seller's maximization problem is thus written:

$$\begin{aligned} \max_{p_1,s_1,p_2,s_2} \quad & \mu\left(p_1-(s_1)^2\right)+(1-y)(1-\mu)\left(p_2-(s_2)^2\right), \\ s.t. \quad & (s_1-p_1)^{\frac{1}{b}} \geq (1-y)(s_2-p_2)^{\frac{1}{b}}, \text{ and} \\ & (as_2-p_2)^{\frac{1}{b}} \geq 0. \end{aligned}$$

Clearly, LOW's participation constraint binds. Hence $as_2 = p_2$. Substitute this into the problem.

$$\begin{aligned} \max_{p_1,s_1,s_2} \quad & \mu\left(p_1-(s_1)^2\right)+(1-y)(1-\mu)\left(as_2-(s_2)^2\right), \\ s.t.\ (s_1-p_1)^{\frac{1}{b}} \quad \geq \quad & \left((1-y)^b(1-a)s_2\right)^{\frac{1}{b}}. \end{aligned}$$

HIGH's incentive-compatibility constraint is equivalent to $s_1 - p_1 \geq (1-y)^b(1-a)s_2$. Further, since the right hand side of the constraint does not depend on s_1 or p_1, we can let $k = (1-y)^b(1-a)s_2$ and thus separate out the problem of choosing a contract for HIGH that maximizes profits, subject to the constraint that HIGH's utility under the contract is at least k:

$$\max_{p_1,s_1} \left(p_1 - (s_1)^2\right)$$
$$s.t. \quad : \quad s_1 - p_1 \geq k.$$

Again, the constraint clearly binds, and we can write this as $\max s_1 - k - (s_1)^2$. This is maximized at $s_1^* = \frac{1}{2}$, implying $p_1^* = \frac{1}{2} - (1-y)^b (1-a) s_2$.

Substituting these values into the objective function yields:

$$\max_{y,s_2} : \mu \left(\frac{1}{4} - (1-y)^b (1-a) s_2\right) + (1-y)(1-\mu)\left(as_2 - (s_2)^2\right). \tag{28}$$

Thus the seller's problem can be written as an unconstrained optimization problem in two variables, subject to the boundary conditions that $y \in [0,1]$ and $s_2 \geq 0$.

The seller's problem in the deterministic case is equivalent to (28) with y set equal to zero:

$$\mu \left(\frac{1}{2} - (1-a) s_2 - \frac{1}{4}\right) + (1-\mu)\left(as_2 - (s_2)^2\right). \tag{29}$$

Differentiating with respect to s_2 and setting the result equal to zero yields the optimal value of s_2 when the seller chooses to make offers (s_1, p_1) and (s_2, p_2) that are accepted by HIGH and LOW, respectively.

$$\begin{aligned} -\mu + a - 2s_2^D + 2\mu s_2^D &= 0, \\ s &= \frac{1}{2}\frac{a-\mu}{1-\mu} \text{ if } a \geq \mu. \end{aligned}$$

Since s_2 must be non-negative, whenever $a < \mu$ the seller will offer $s_2 = 0$, which is equivalent to contracting only with HIGH. In the deterministic problem, the optimal quality for HIGH is given by $s_1^D = \frac{1}{2}$, implying that $s_2^D < s_1^D$ (since $a < 1$). Hence the seller will never choose to offer the same contract twice.

Computing profits establishes the claims in Proposition 9. When $a \geq \mu$, profit is given by:

$$\mu\left(p_1 - (s_1)^2\right) + (1-\mu)\left(p_2 - (s_2)^2\right) = \frac{1}{4}\frac{\mu - 2\mu a + a^2}{1-\mu}.$$

When $a < \mu$, profit is $\frac{\mu}{4}$. ■

Proof of Proposition 10. The first derivatives of 28 with respect to s_2 and y are:

$$D_{s_2} = -\mu\left(1-y^*\right)^b (1-a) + \left(1-y^*\right)(1-\mu)\left(a - 2s_2^*\right) \begin{cases} \leq 0 & \text{at } s_2^* \text{ if } s_2^* = 0 \\ = 0 & \text{at } s_2^* \text{ if } s_2^* > 0 \end{cases}, \text{ and} \quad (30)$$

$$D_y = \mu b\left(1-y^*\right)^{b-1}(1-a)\, s_2^* - (1-\mu)\left(a s_2^* - \left(s_2^*\right)^2\right) \begin{cases} \leq 0 & \text{at } y^* \text{ if } y^* = 0 \\ = 0 & \text{at } y^* \text{ if } y^* \in (0,1) \\ \geq 0 & \text{at } y^* \text{ if } y^* = 1 \end{cases}. \quad (31)$$

Suppose $y^* = 1$. Since D_{s_2} equals zero, the first-order condition with respect to s_2 is satisfied for any value of s_2. Further, the first term of D_y equals zero. For any $a > 0$ there exists an $s_2(a)$ such that $a s_2(a) - (s_2(a))^2 > 0$. Since $D_y > 0$ when $s_2 = s_2(a)$ and $y^* = 1$, $y^* = 1$ is not optimal. The seller would always prefer to offer quality $s_2(a)$ with some positive probability rather than set $y^* = 1$. Since offer $(a s_2(a), s_2(a))$ is feasible and offers the seller a positive profit, the optimal second offer must also yield a positive profit. Hence $y^* < 1$ and $s_2^* > 0$.

If $y^* \in (0,1)$, (30) and (31) hold with equality. These equations can easily be solved for y^* and s_2^*:

$$s_2^* = a\frac{b-1}{2b-1} \quad (32)$$

$$y^* = 1 - \left(\frac{a(1-\mu)}{(2b-1)\mu(1-a)}\right)^{\frac{1}{b-1}}. \quad (33)$$

Clearly $y^* < 1$, as established above. From (33), $y^* > 0$ whenever:

$$\begin{aligned}\frac{a(1-\mu)}{(2b-1)\mu(1-a)} &\leq 1, \text{ or} \\ b &\geq \frac{1}{2}\left(\frac{a(1-\mu)}{\mu(1-a)}+1\right).\end{aligned}$$

Hence a PFO is optimal whenever $b > b^* = \frac{1}{2}\left(\frac{a(1-\mu)}{\mu(1-a)}+1\right)$. The remainder of the optimal PFO is computed using (32) and (33).

Proof of Corollary 1. Consider (22) - (24). Note in (22) that $q_1^* = q_1^F$ for all b, and $p_1^* \to p_1^F$ as $b \to \infty$ since $\left(\frac{a(1-\mu)}{(2b-1)\mu(1-a)}\right) \to 0$. The convergence for (q_2^*, p_2^*) is obvious. Applying L'Hopital's rule to $\ln(1-y^*)$ shows that $y^* \to 0$.■

3 A Model with an Endogenous Outside Option

An alternative version of the model we discuss in Section 3.1 is that, instead of the outside buyer having known value p_0, selling to the outside buyer corresponds to drawing a new buyer whose reservation value has the same distribution as the current buyer, i.e., it is r_h with probability μ and r_l with probability $(1-\mu)$. In this case, the expected price from selling to a new buyer becomes endogenous, and the seller's problem can be approached as a dynamic programing problem. The seller begins by offering initial price p_1. If this price is rejected, the seller may make an immediate second offer to the current buyer, or alternatively request a new buyer. However, if a new buyer is requested, she arrives only after some delay. Let δ be the relevant discount factor, where $0 < \delta < 1$. If y is the probability that the seller will request a new buyer following a rejection, the highest initial price that a HIGH buyer will accept is once again given by $p_1(y)$. And, the seller's problem is stationary in the sense that the seller's expected value at the start of the game is the same as its expected value from choosing a new buyer following an initial rejection (conditional on that buyer's arrival). Thus, the seller's expected value as a function of y, which we denote $v(y)$, must satisfsy:

$$\begin{aligned}v(y) &= \mu p_1(y) + (1-\mu)(y * \delta * v(y) + (1-y) r_l), \text{ or} \\ v(y) &= \frac{\mu p_1(y) + (1-\mu)(1-y) r_l}{(1-(1-\mu) * y * \delta)}.\end{aligned}$$

In this setting, the seller's optimal y is found by maximizing $v(y)$.

Although a complete discussion of the solution to this problem falls beyond the scope of the paper, the two extreme cases of a very patient seller and a very impatient seller are worth brief comments. If the seller is extremely patient (i.e., δ is near one), then it can be shown that the seller's optimal strategy is to choose $y = 1$ and charge r_h. That is, the seller offers the current buyer price r_h, and if it is rejected he chooses a new buyer and again offers price r_h. Because there is no cost to drawing a new buyer, the seller is content to draw a new buyer until he finds one that is HIGH. Knowing this, HIGH buyers are willing to pay up to r_h to acquire the object.

On the other hand, when the seller is very impatient (i.e., δ is near zero), then the seller's problem looks very much like the one we considered above. The seller would like to avoid drawing a new buyer. However, a small threat of doing so induces a risk-averse HIGH buyer to raise her willingness to accept a high initial price. Thus, a PFO will still be optimal, provided that buyers are sufficiently risk averse and δ is sufficiently low.