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Resource allocation when projects have ranges of increasing returns

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Abstract A fixed budget must be allocated to a finite number of different projects with uncertain outputs. The expected marginal productivity of capital in a project first increases then decreases with the amount of capital invested. Such behavior is common when output is a probability (of escaping infection, succeeding with an R&D project...). When the total budget is below some threshold, it is invested in a single project. Above this cutoff, the share invested in a project can be discontinuous and non-monotone in the total budget. Above an upper cutoff, all projects receive more capital as the budget increases.

Keywords Capital allocation · Increasing returns · Probabilistic returns · Egalitarian allocation · Complete specialization

JEL Classification D24 · C60 · D84

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We address a common capital allocation problem in which an agent can implement various investment projects at different levels. Such projects usually have uncertain payoffs; hence the objective is to maximize total expected benefits. In the public realm, a federal agency may have to determine a vaccination strategy for a contagious disease, such as avian flu, with a limited national vaccination budget. Given that herd immunity develops, it is well known that the expected social benefit curve from a vaccination campaign is *S*-shaped in the proportion of those inoculated in a locale. A similar problem arises if a very limited amount of a drug is to be distributed to reduce new infections in a nation heavily afflicted with HIV. An egalitarian allocation might have little beneficial effect. This application to public health policy operates in an uncertain environment.

Our general model also applies to a range of allocation problems. A wide range of applications arises in the field of prevention, where an agency must determine where to allocate preventive efforts, given the fact that the probability of success of the preventive effort is usually *S*-shaped. Many efforts to market a new product in different areas under uncertainty have this character. The financing of R&D projects, where the probability of discovery would be the dependent variable, represents another branch of the literature that relates to our topic. In the field of productive efficiency, a credit-rationed entrepreneur may have the potential to invest in various independent projects, each of which offers increasing and then decreasing expected returns to invested capital. The industrial policy of a country is also faced with the same dilemma of determining the sectors that should be favoured. Weitzman (1979) considered the case where there is a finite number of different opportunities, each yielding an unknown reward. He proposed an algorithm that tells at each stage whether or not to continue searching and if so, which project to finance. This could apply to the optimal sequential search strategy for developing various uncertain technologies that meet the same or similar purpose. This model has been extended by Roberts and Weitzman (1981) to a more general framework. Our analysis approaches this one, except that the expected benefit function generated by each project is assumed to be known in our case.

Our analysis maximizes expected benefits.[1] The central ingredient of the general model is that the expected marginal benefit of each possible action is hump-shaped, i.e., the expected marginal benefit of an action reaches a maximum at some intermediate intensity of that action. The objective function, namely the sum of the expected benefits extracted from the different actions, is therefore not concave in the vector of decision variables. Given this assumption, it is appropriate to give up the idea of distributing resources to all projects,

[1] This objective function finesses risk aversion. There are three justifications: (1) In many instances the scale of the payoffs is small relative to aggregate payoffs, implying that risk aversion is not a major concern. (2) For many applications, e.g., health risks, variability is due to the number of people affected. For individuals considering their own risk, only their expected probability matters, not how many will suffer when they do. (3) To foster intuition and expositional clarity, it is best to deal with the simplifying assumption of risk neutrality.

because projects funded at a modest scale offer low productivity. Hereafter, when we talk about benefit or marginal benefit, the modifier “expected” should be understood.

The question of increasing returns in an economy has been widely studied, in a theoretical framework, notably in the seventies. Indeed, with increasing returns, a competitive market may lack an equilibrium. Many authors proposed solutions to avoid this problem (see for instance Rader 1970; Aoki 1971; Crémer 1977; Brown and Heal 1979 or Heal 1999). The optimal financing of a finite number of projects, each presenting an *S*-shape has been addressed in particular in an important early paper by Ginsberg (1974). He characterizes the solution in the general case and explains how the budget is usually shared among an increasing number of projects when the benefit functions are identical. He typically uses the average benefit function to solve the problem. We adopt a different approach. We focus on plausible shapes for the benefit function and we find that they yield sensible solutions. To do so, we introduce some families of functions defined by interesting and fairly general properties for the marginal benefit function. We also highlight the features of the aggregate benefit function.

Section 1 motivates the paper by giving examples where the benefit function is *S*-shaped. Section 2 states some general properties in the case of a low budget level. Section 3 analyzes higher budget levels for the case of identical benefit functions. Section 4 tackles the case of heterogeneous benefit functions in the context of a high budget level. Section 5 describes the properties of the aggregate benefit function, and Section 6 concludes.

Much of our analysis relates outcomes to the shape of benefit functions, and two distinctive properties for shapes are identified. When the total budget is below a lower cutoff, all of it is invested in a single project. Above an upper cutoff, all projects receive more capital as the budget increases. In between cutoffs, for some plausible expected benefit functions, namely those like Fig. 7, the results are tractable. The optimal investment strategy goes from full specialization (a single project) to equal division among supported projects. However, when benefit functions resemble those in Fig. 6, as the total allocation increases, positive but unequal allocations will be experienced. Furthermore, optimality will require that some projects experience a reduction in budget over some range. When benefit functions differ across projects, matters are more complex still.

1 *S*-shaped productivity in various domains

This section argues that a total productivity curve that is *S*-shaped is found across a broad array of areas. The shape usually arises because three elements are at work: (1) Small investments accomplish little. Thus, $100,000 has virtually no chance to produce a sophisticated new invention, nor dent the national consciousness in a media campaign for a new product. (2) Over an intermediate range of investment, significant output starts to be realized.

(3) Beyond a certain level of investment, decreasing returns set in. Hence, productivity first rises with expenditure at an increasing rate, and then the rate decreases. The *S*-shaped curve, as is shown in Fig. 1, emerges.

The concept of herd immunity is well known in epidemiology. Each individual who gets immunized against a communicable disease within a closed population conveys a positive externality. Since he can no longer get the disease, he can no longer communicate it to others. The first few immunizations yield little external benefit, since there remain so many other individuals who can still convey infection. However, once a significant proportion of individuals has been vaccinated, the whole population is substantially protected, which leads to the label herd immunity. Beyond a certain point, additional vaccinations therefore yield little additional protection (see for instance Fine 1993). Once more, all these considerations apply in an uncertain environment: infection only arises with a certain probability.

Efforts to produce inventions have long been recognized to exhibit an *S* -shape in the function that relates probability of success to level of investment. The empirical evidence relating the probability of non-zero patenting to the level of R&D spending shows a clear *S*-shape (see Scherer 1983 Fig. 1, whose study looks across 4,274 individuals lines of business in firms). With low expenditures, the probability of getting a patent is very low but once a sufficiently high level of expenditures has been devoted to research, a given increase in the patenting probability is more difficult to achieve. *S*-shaped curves for product performance are a driving concept behind Utterback's (1994) (pp. 158–160) analysis of radical innovations, and Christensen's (1997) (pp. 39–41) model of disruptive technologies. Successor (radical or disruptive) technologies come along when the first technology is operating beyond its

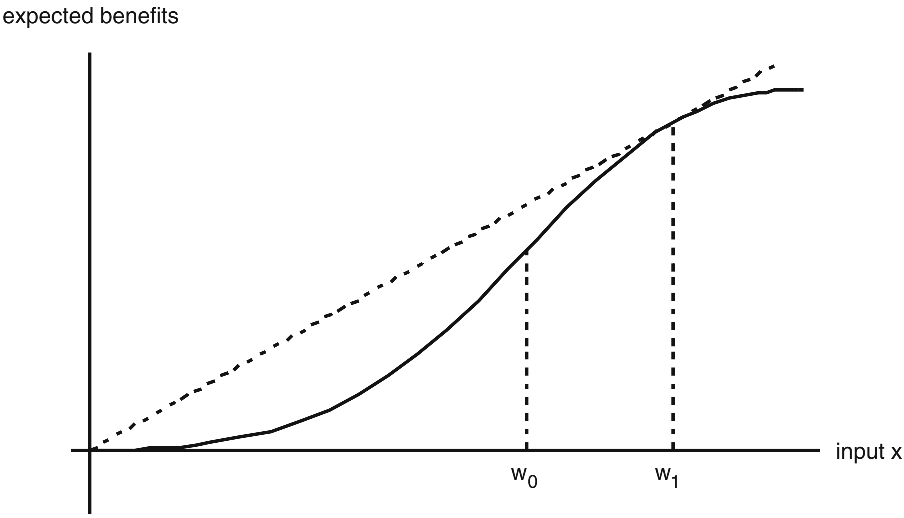

Fig. 1 The expected benefit function as a function of the amount invested in the specific project

inflection point. Kuznets (1967) (pp. 31–33) noted the same S-shape phenomenon for an industry as a whole, which might be relevant say for government R&D and tax policies that seek to push various industries forward.

Little (1979) provides an overview look at the returns to aggregate advertising of various products, drawing on the work of others. He identifies S-shaped responses, e.g., of sales/capita in response to advertising/capita, though he also alerts readers to more complex patterns. He concludes (p. 639) "that advertising models should accommodate S-shaped curves".

In general, in any investment arena where there is a range of increasing expected returns, we should expect to find S-shaped response curves. That is because we know that decreasing returns set in at some point. Except where natural resources are involved, we do not see one product, or one firm, or one industry dominating a major economy. When two or more entities must compete for investment, and where those entities each experience S-shaped returns, the lessons of this paper apply.

In many instances, the success of a project depends upon whether demanded capacity exceeds installed capacity. This is the case for example for an electricity transportation network (which breaks down if overloaded), or for the number of emergency units built to face a crisis for which the demand for medical assistance is uncertain. In such cases, assuming risk neutrality, the cumulative distribution of the demand determines an ex ante benefit function for the quantity of emergency capacity. It is often the case that this cumulative distribution is S-shaped. The logistic distribution is a notable case of an S-shaped distribution. Balakrishnan (1991) in his handbook of the logistic distribution proposes a range of applications. It was first applied to model population growth. But Oliver (1969) used the logistic distribution to model the spread of innovation, and more precisely the thousands of agricultural tractors in Great Britain from 1950 to 1965. In economics, the logistic distribution has been used to model both income distribution (Fisk 1961) and agricultural production (Oliver 1964). It is also widely employed in public health. For instance, the ratio of disease incidence among those exposed versus those not exposed to the risk factor of interest (it is called the relative risk) may be estimated with a logistic function. Plackett (1959) was the first to use the logistic function in the analysis of survival data. He developed a model that applies to operations on cancer patients, but also to labour turnover, business failures or animal experiments. Therefore, this article applies to a large number of risky events whose distribution is S-shaped. However, in the rest of the article, we are going to speak of an investor who has a given budget to invest in different projects without specifying their nature.

2 Low budget levels: general case

We first state the problem in the general case where a finite number of projects presenting different benefit functions $b_1,..., b_n$ are available to the investor. We assume that $b_i(0) = 0\ \forall i$, and that each function b_i is increasing in its argument.

Finally, we assume, as shown in Fig. 1, that there exists a critical investment level w_0 such that each function b_i is locally convex in $[0, w_0[$, and that it is locally concave in $]w_0, +\infty[$. The investor is endowed with a budget w to finance these projects, and he finances each project b_i with an amount x_i. In this case, the problem he has to solve reads

$$B(w) = \max_{x_1 \dots x_n} b_1(x_1) + \dots + b_{n-1}(x_{n-1}) + b_n(x_n), \tag{1}$$

subject to

$$\begin{array}{l} x_i \geq 0 \quad \forall i = 1 \dots n, \\ \sum_{i=1}^{n} x_i = w. \end{array}$$

The budget constraint is binding since each function b_i is increasing. Following Ginsberg (1974), we focus on the average benefit function to obtain the following result whose proof is straightforward and thus omitted.

Lemma 1 *Suppose there are n benefit functions $b_1, \cdots, b_n$ each characterized by $w_i^* = \arg\max b_i(x)/x$. If $w \leq \min_i w_i^*$, then the entire budget w goes to the project with the highest benefit $b_i(w)$.*

When the total budget is less than $\min_i w_i^*$, it is optimal to invest the entire budget in the project offering the highest benefit because of the increasing returns to scale at low intensities. The following parts study the optimal allocation in the case of higher budget levels. We solve this problem in two steps, focusing first on the case of identical benefit functions, and studying then different benefit functions.

3 Intermediate budget levels: the case of identical benefit functions

To secure intuition for the results, we first analyze the case of two identical benefit functions. We then use induction to extend the results to a finite number of benefit functions. As before, all benefits should be thought of as expectations.

3.1 Two identical projects

3.1.1 General properties

In the general case, we rapidly characterize the solution following Ginsberg (1974). If x and $w - x$ denote the budget invested in each project, the choice problem is

$$B(w) = \max_{0 \leq x \leq w} \quad b(x) + b(w - x). \tag{2}$$

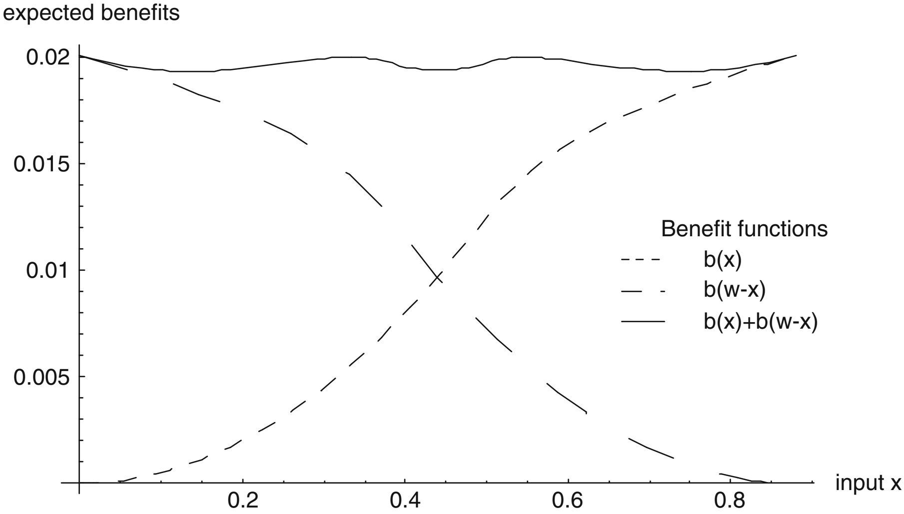

Fig. 2 The two expected benefit functions together with the objective function

In Fig. 2, the objective function $b(x) + b(w - x)$ is represented for a given value of w as well as the two benefit functions $b(x)$ and $b(w - x)$. This maximization program has three types of solutions as the following proposition tells us.[2]

Lemma 2 *The optimal solution of problem 2 belongs to one of the following three types*

1. *the full-specialization type* $x^* \in \{0, w\}$*,*
2. *the symmetric -or egalitarian- type* $x^* = w/2$*,*
3. *the asymmetric interior type* $x^* = \widehat{x}(w)$*, where* $\widehat{x}(w) < w_0 < w - \widehat{x}(w)$ *and* $b'(\widehat{x}(w)) = b'(w - \widehat{x}(w))$*.*

Notice that program 2 is symmetric relative to $w/2$. Therefore, in the rest of our discussion of the 2 identical-projects case, we are going to focus on solutions that are greater or equal to $w/2$. The full-specialization solution will designate w, the equal solution $w/2$ and the asymmetric interior solution $w - \widehat{x}(w)$. In the case of the full-specialization solution, the entire budget is devoted to only one project. In the case of the symmetric solution, both projects get exactly the same amount, and in the case of the asymmetric interior solution, the two projects get a different positive amount (as in Fig. 2).

According to Lemma 1, when w is less than $w_1 = \arg\max b(x)/x$, all the budget is devoted to a unique project. Moreover, given the convexity of function

[2]This result is already known and thus the proof is omitted. Any interested reader can contact the corresponding author to obtain more information on the proof.

b on $[0, w_0]$, when the budget level is less than $2w_0$, the objective function in program 2 is convex in the decision variable x in its domain, and the symmetric strategy can never be a maximum. The following lemma provides more insights about how the optimal strategy evolves as the budget level w increases.

Lemma 3 *Consider the case of two identical projects. The optimal investment strategy has the following characteristics:*

1. *For low budget levels, the full-specialization strategy is optimal;*
2. *Then, as the total budget level w increases, the optimal strategy can switch to an asymmetric interior solution, or directly to the symmetric allocation;*
3. *Once the symmetric allocation is selected, it remains optimal for all larger w.*

Proof See the Appendix. □

According to Lemma 3, once a strategy (an asymmetric interior solution or the equal strategy) dominates the full-specialization strategy for a given budget level w, the full-specialization strategy will not be optimal for any budget level that is higher than w. Moreover, if the symmetric strategy is optimal for a given budget level, it will remain optimal for any higher budget levels. This particular result has already been proved by Ginsberg (1974). Intuitively, when the budget level is low, the investor prefers to favour one project by investing the whole budget in it because of the low productivity at low budget levels (see Fig. 3 in which $w = 0.6$).

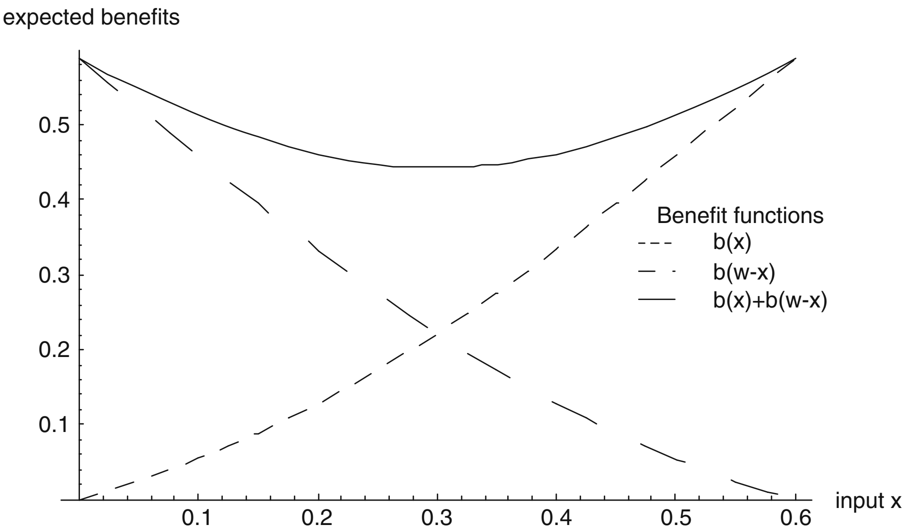

Fig. 3 The two expected benefit functions together with the objective function for a low value of w

On the contrary, when the total budget level is high enough, the investor prefers to share the budget equally between both projects because of the projects' decreasing productivity from w_0 on (see Fig. 4 in which $w = 1.3$).

In between, the investor wants to invest a strictly positive amount in each project but he still favours one project to the detriment of the other. It is not worthwhile to invest everything in a single project since, from the inflection point on, the marginal benefit of investing in a project is decreasing. In Fig. 2, which shows the objective function for w equal to 0.88, the solution is the asymmetric interior one.

To illustrate this result, consider an example where the benefit function is given by

$$b(x) = \frac{x^\gamma}{x^\gamma + k(1-x)^\gamma}, \tag{3}$$

with $\gamma = 2$ and $k = 2$. Observe that $w_0 = 0.613$ and $w_1 = 0.816$ in this numerical example. We have drawn the optimal strategy as a function of the total budget w in Fig. 5. When w is below 1, it is optimal to invest everything in one project. When w is between 1 and 1.225, the asymmetric interior solution is optimal. Finally, for larger w, the symmetric strategy is optimal.

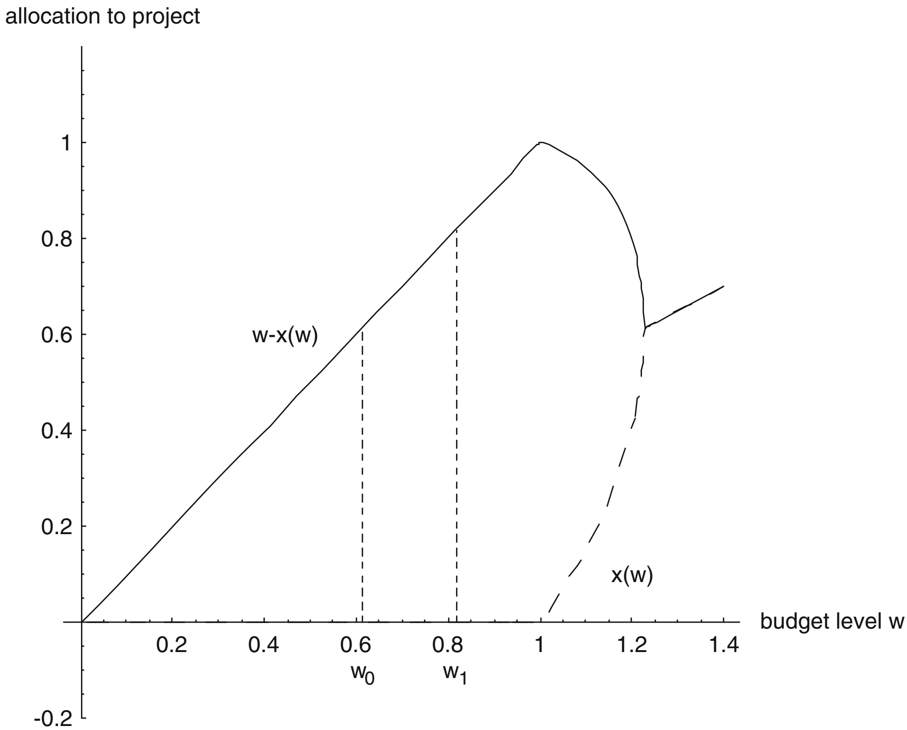

Fig. 4 The two expected benefit functions together with the objective function for a high value of w

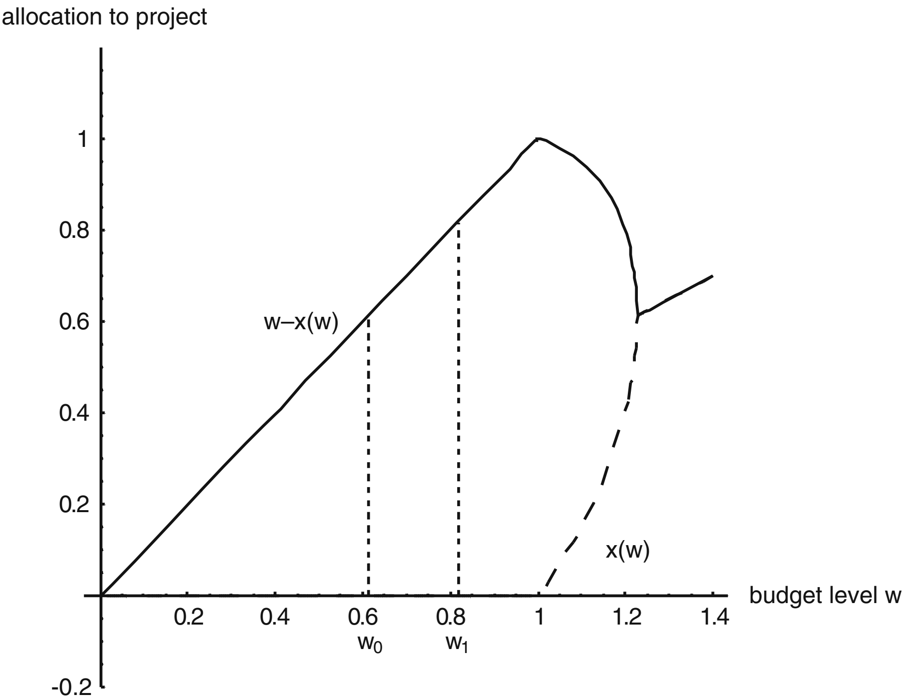

Fig. 5 The optimal investment in projects 1 and 2 as a function of the total budget

Concerning asymmetric interior solutions, observe that as wealth w increases, one of the two projects will get a *smaller* budget, as seen in Fig. 5.[3] This comes from the full differentiation of the first-order condition,[4] which yields

$$\frac{d\widehat{x}}{dw} = \frac{b''(w-\widehat{x})}{b''(\widehat{x})+b''(w-\widehat{x})} \text{ and } \frac{d\,(w-\widehat{x})}{dw} = \frac{b''(\widehat{x})}{b''(\widehat{x})+b''(w-\widehat{x})}. \qquad (4)$$

As $\widehat{x} < w_0 < w - \widehat{x}$, $b''(\widehat{x})$ is positive and $b''(w-\widehat{x})$ is negative, $d\widehat{x}/dw$ and $d(w-\widehat{x})/dw$ must have opposite signs.

To get more information on how the solution to the maximization program 2 evolves as the total budget level increases, we hereafter study three particular classes of functions: symmetric benefit functions, benefit functions that are "pulled down", and benefit functions that are "lifted up". This analysis focuses

[3]Observe that there is another reason for why the project-specific budgets do not increase monotonically the total budget. When the optimal strategy switches from full specialization to full diversification, the previously financed project gets a 50% reduction in its budget.

[4]$b'(x^*) - b'(w-x^*) \begin{cases} = 0 \text{ if } x^* < w, \\ \geq 0 \text{ if } x^* = w. \end{cases}$

on the shape of the marginal benefit function, and therefore goes beyond Ginsberg (1974). It allows us to characterize cases for which the outcome is very simple, and then complex.

3.1.2 Symmetric benefit functions

A symmetric benefit function can be seen as either having symmetric first order derivatives with respect to the inflection point w_0, or as a 180° rotated function from the part below w_0 to the part above w_0. In this case, the marginal benefit function is symmetric relative to the axis $w = w_0$. With such a benefit function, any asymmetric interior allocation $\widehat{x}$ is excluded. If we denote w_2 the unique critical wealth below which full-specialization dominates the symmetric solution, and above which the symmetric allocation is preferred to full-specialization, the following proposition tells us that w_2 equals $2w_0$.

Proposition 1 *Suppose that b' is symmetric in the sense that $b'(w_0 + \delta) = b'(w_0 - \delta)$ for all $\delta \in [0, w_0]$. Then, the fully specialized strategy is optimal if w is smaller than $2w_0$, whereas the symmetric strategy is optimal if w is larger than $2w_0$.*

Proof See the Appendix. □

When the benefit function is symmetric, the optimal strategy requires full-specialization when $w \leq 2w_0$, and is egalitarian otherwise. In other words, for any budget w below $2w_0$, one project gets all the budget w, otherwise the two projects get exactly the same amount $w/2$. This special case serves as a benchmark for the next two cases, where the benefit function is not symmetric.

The analysis proves simpler if it is conducted using the marginal benefit function. We consider two cases. In the first, beyond the inflection point both the total and marginal functions lie below their equivalent function for the hypothetical symmetric case.[5] We refer to this as having the benefit functions (both total and marginal) "pulled down". In the second case, both the total and marginal benefit functions lie above their symmetric counterparts. We call this the "lifted up" case.

3.1.3 "Pulled down" (PD) benefit functions

Let us first give the definition of a benefit function that is pulled down.

[5] When the total benefit function is symmetric, the shape of the marginal benefit function to the right of the inflection point is a mirror reflection of what is to the left. For the asymmetric case, if the marginal benefit function lies below its symmetric counterpart, then the corresponding total benefit function will also lie below its counterpart, but the reverse is not true.

Definition 1 A benefit function is said to be pulled down (PD) if $b'(w_0 + \delta) \leq b'(w_0 - \delta)$ for all $\delta \in [0, w_0]$.

If a benefit function is PD, beyond the inflection point w_0 the marginal benefit curve is pulled to the left so that it lies everywhere below the symmetric case curve. In Fig. 6, the marginal benefit function in the PD case is represented.

Given PD, once the maximal productivity $b'(w_0)$ has been reached, the increase in productivity is less than in the symmetric case. The following lemma allows us to characterize the solution.

Proposition 2 *Suppose that the benefit function b is PD, then $w_2 \leq 2w_0$. Moreover, the symmetric strategy is optimal whenever $w > 2w_0$.*

Proof See the Appendix. □

When the benefit function is PD and the budget level exceeds $2w_0$, it is optimal to share the budget equally between the two projects. Remember that we obtained the same result in the symmetric case. Because the increase in productivity from w_0 on is less rapid than in the symmetric case, the attractiveness of the specialized solution is weakened. Since it was already inferior to the egalitarian solution in the symmetric case, the PD case reinforces that result.

To illustrate, focus on the case of a logistic distribution function on [0, 1] whose cumulative distribution function is equal to $\frac{1}{2\tanh(1)}\left(\tanh\left(\frac{8x-4}{4}\right) + \tanh(1)\right)$. In order to introduce the concept of PD and LU benefit functions, we add a transformation and the benefit function equals

$$b(x) = \frac{1+\alpha x}{1+\alpha}\frac{1}{2\tanh(1)}\left(\tanh\left(\frac{8x-4}{4}\right) + \tanh(1)\right). \tag{5}$$

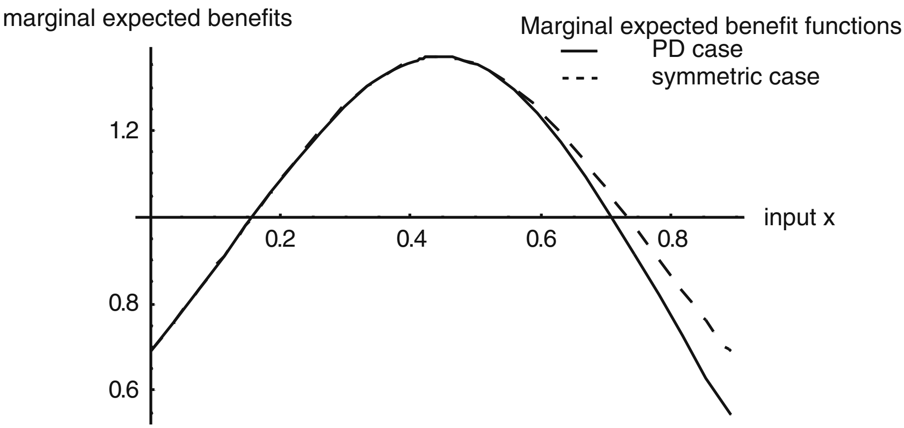

Fig. 6 Marginal expected benefit function in the PD case

If $\alpha = 0$, b is a symmetric function with respect to w_0. If α is negative, function b is PD, and if α is positive, function b is lifted up. We take α to be equal to -0.2. In this case, $w_0 = 0.445$ and $w_2 = 0.793 < 2w_0$. For $w > 2w_0 = 0.890$, the symmetric allocation is optimal. An asymmetric interior solution exists when $w \in [0.831, 0.890]$.

3.1.4 "Lifted up" (LU) benefit functions

A second important case arises when both the total and marginal benefit functions are "lifted up", so they lie above their hypothetical symmetric functions. We define this term more formally as:

Definition 2 A benefit function is said to be lifted up (LU) if $b'(w_0 + \delta) \geq b'(w_0 - \delta)$ for all $\delta \in [0, w_0]$.

If a benefit function is LU, beyond the inflection point w_0 the marginal benefit curve is pulled to the right so that it lies everywhere above the symmetric case curve.[6] Therefore, the increase in productivity beyond w_0 is more rapid than in the symmetric case. Intuitively, this reinforces the attractiveness of the more specialized strategies. In other words, unlike in the PD case, it should be more likely that one project receives a greater share of the total budget than the other even when $w > 2w_0$. Consider the quantity $x(\delta)$ defined by $b'(w_0 - \delta) = b'(x(\delta))$, with $x(\delta) > w_0$. It is defined for all $\delta \in [0, w_0]$. In fact, for each δ, there exists a $w(\delta)$ such that $x(\delta) = w(\delta) - (w_0 - \delta)$ and $w_0 - \delta$ is an asymmetric interior solution. We are interested in the quantity $z(\delta) = x(\delta) - (w_0 + \delta)$ (see Fig. 7).

It corresponds to the horizontal distance between the LU marginal benefit function and its symmetric equivalent. As b is LU, we know that $z(\delta) \geq 0$ for all $\delta \in [0, w_0]$. A condition on this function $z(.)$ allows us to characterize the shape of the optimal solution in the case of a LU benefit function.

Proposition 3 *Suppose that the benefit function b is LU, then $w_2 \geq 2w_0$. Moreover*

1. *If $w \leq 2w_0$, then the full-specialization strategy is optimal.*
2. *If $w > 2w_0$ and if $\delta \longmapsto z(\delta)$ is increasing, then the optimal strategy cannot be an asymmetric interior one.*

Proof See the Appendix. □

We see that, in the LU case, the egalitarian strategy will not in general be optimal even when $w \geq 2w_0$. If function z is increasing, we have a complete

[6] Note that the only way for a function to be simultaneously PD and LU is to be symmetric. If this is not the case, a LU benefit function cannot be PD and vice versa. However, these two notions are not mutually inclusive since a benefit function might be neither PD nor LU.

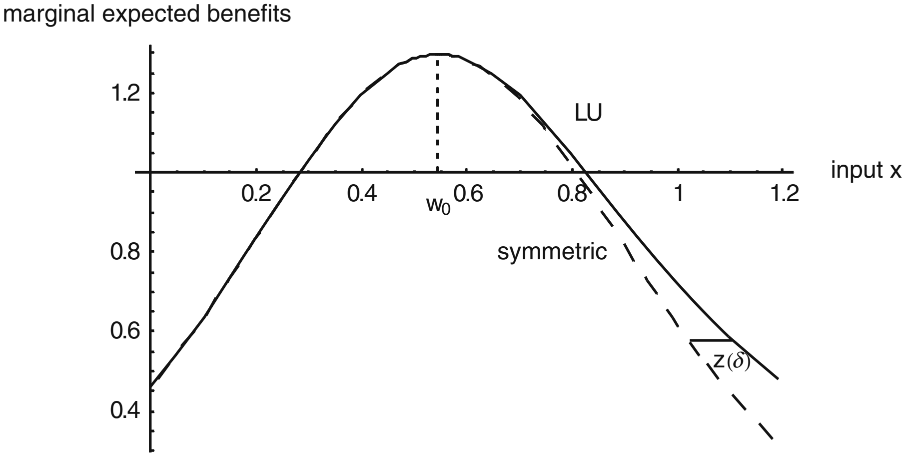

Fig. 7 A LU marginal expected benefit function together with function z

characterization of the solution: it employs the full-specialization strategy for $w \leq w_2$ and then switches to the equal-allocation strategy for $w > w_2$. Because $w_2 \geq 2w_0$, even when the total budget is higher than $2w_0$, it is still optimal to favour one project. This is because beyond w_0 productivity increases more rapidly than in the symmetric case. Given that z is an increasing function, the marginal productivity decreases less rapidly than in the symmetric case. A LU benefit function is thus all the more attractive. Thanks to this condition, an interior allocation satisfying the first order condition is a local minimum and should therefore not be taken into account for the search of the optimal solution (the second order condition is not satisfied). If the condition stated in Proposition 3 is not satisfied, the asymmetric solution may be a local maximum. Therefore, the optimal strategy may begin with the full-specialization strategy when w is very low, and then switch successively to an asymmetric interior solution as w increases and ultimately reaches the equal-allocation strategy. We now turn to the case of a finite number of identical benefit functions.

3.2 n identical projects

As already stated in Section 2, with n projects, the investor has to solve the maximization program

$$B(w) = \max_{x_1,\ldots,x_{n-1}} b(x_1) + \ldots + b(x_{n-1}) + b(w - x_1 - \ldots - x_{n-1}) \tag{6}$$

subject to

$$x_i \geq 0, \forall i = 1 \ldots n-1,$$
$$\sum_{i=1}^{n-1} x_i \leq w.$$

It is convenient to maximize this benefit function in two steps.

1. A first maximization

$$\max_{x_1,\dots,x_{n-2}} b(x_1) + \dots + b(x_{n-1}) + b(w - x_1 - \dots - x_{n-1}) \tag{7}$$

   finds the optimal allocation between $n-1$ projects when the total budget level that is available is equal to $w - x_{n-1}$. The solutions to this maximization are denoted $x_1^*(x_{n-1}, w), \dots, x_{n-2}^*(x_{n-1}, w)$.
2. Then, there remains to solve

$$\max_{x_{n-1}} b\big(x_1^*(x_{n-1}, w)\big) + \dots + b\big(x_{n-2}^*(x_{n-1}, w)\big) + b(x_{n-1}) + \\ + b\big(w - x_1^*(x_{n-1}, w) - \dots - x_{n-2}^*(x_{n-1}, w) - x_{n-1}\big). \tag{8}$$

This kind of problem has to be solved using induction arguments. It is difficult to extend the results concerning PD benefit functions in the case of n different projects. Indeed, recall that in the two projects case, we did not find a condition assuming that no asymmetric interior solution exists. Therefore, in this study of the n projects case, we generalize the 2 projects case we focus on to the case of lifted up benefit functions.[7]

Proposition 4 *Suppose b is a LU benefit function and $\delta \mapsto z(\delta)$ is an increasing function. Then, the optimal strategy is to share equally the budget between all financed projects. Moreover, as w increases, the number of financed projects increases until being equal to n.*

Proof See the Appendix. □

If the benefit function is LU, the decrease in productivity for wealth levels higher than or equal to w_0 is less rapid than in the symmetric case. Therefore, it is optimal to increase the number of financed projects as the total budget w increases, and to share it equally between all financed projects. We present the result in the case of a LU benefit function and of 3 projects in Fig. 8 for a new functions' family

$$b(x) = \frac{1+\alpha x}{1+\alpha}\frac{1}{\pi}\left(\arctan(2x-1) + \frac{\pi}{4}\right) \tag{9}$$

with $\alpha = 0.2$.

In this part, we managed to give the shape of the optimal allocation when the budget level increases in the case of a finite number of identical benefit functions. Now, we determine how these results generalize to the case of heterogeneous benefit functions.

[7]Note that this result also holds for n symmetric benefit functions since a symmetric benefit function is a special case of a LU benefit function where function z is equal to 0.

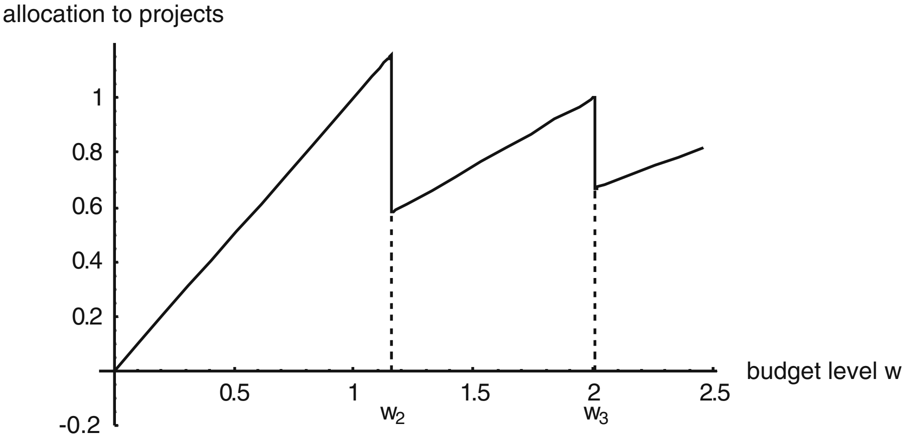

Fig. 8 Optimal strategy for a LU expected benefit function with three identical projects

4 Intermediate budget levels: the case of different benefit functions

Heterogeneity makes the problem much trickier. We consider thus a special case of two different projects, the second operating at a much larger scale than the first one. Their benefit functions are

$$b(x) \text{ and } c(x) = kb(x/j) \text{ with } 1 < k < j. \tag{10}$$

They are represented in Fig. 9.

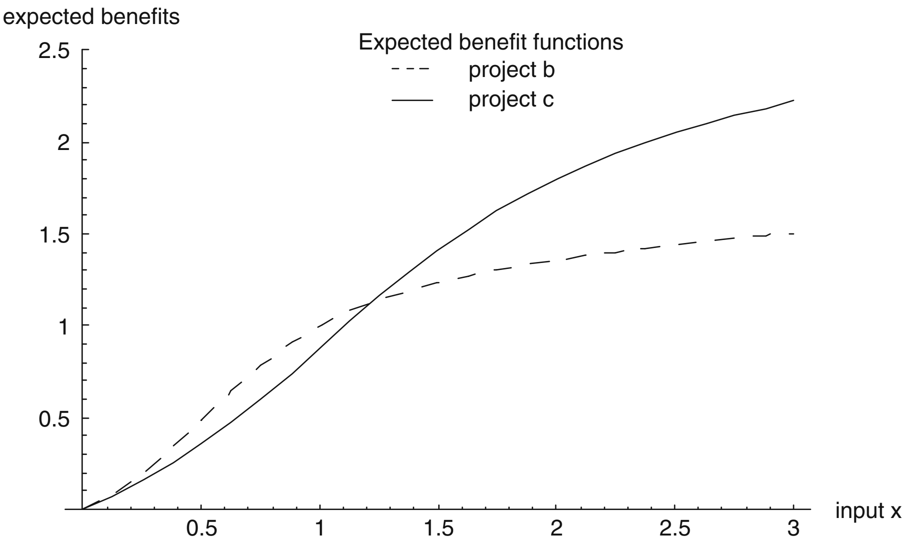

Fig. 9 Expected benefit functions b and c

Denote $w_1^* = \arg\max b(x)/x$ (resp. $w_2^* = \arg\max c(x)/x$). It can easily be shown that $w_2^* > w_1^*$ and that $\forall x \leq w_1^*, b(x) > c(x)$. According to Lemma 1, if the total budget level w is less than w_1^*, then project b gets the entire budget w and project c gets nothing. Indeed, for low budget levels, project b is more profitable than project c. Before delving further into the study of the optimal allocation, we describe the potential solutions to the investor's maximization program

$$\max_{0 \leq x \leq w} b(w-x) + c(x). \tag{11}$$

Lemma 4 *Suppose b is a benefit function and $c(x) = kb(x/j)$ with $1 < k < j$. The optimal solution of program 11 belongs to one of the following five types:*

1. $x(w) = 0$: *the whole budget goes to project b,*
2. $x(w) = w$: *the whole budget goes to project c,*
3. $\widehat{x}^1(w)$ *with* $b'\left(w - \widehat{x}^1(w)\right) = \frac{k}{j} b'\left(\frac{\widehat{x}^1(w)}{j}\right)$ *and* $w_0 < \frac{\widehat{x}^1(w)}{j} < w - \widehat{x}^1(w)$. *This solution will be called "interior solution 1",*
4. $\widehat{x}^2(w)$ *with* $b'\left(w - \widehat{x}^2(w)\right) = \frac{k}{j} b'\left(\frac{\widehat{x}^2(w)}{j}\right)$, $\frac{\widehat{x}^2(w)}{j} < w_0 < w - \widehat{x}^2(w)$, *and* $\frac{k}{j^2} b''\left(\frac{x}{j}\right) + b''(w-x) \leq 0$. *This solution will be called "interior solution 2",*
5. $\widehat{x}^3(w)$ *with* $b'\left(w - \widehat{x}^3(w)\right) = \frac{k}{j} b'\left(\frac{\widehat{x}^3(w)}{j}\right)$, $w - \widehat{x}^3(w) < w_0 < \frac{\widehat{x}^3(w)}{j}$, *and* $\frac{k}{j^2} b''\left(\frac{x}{j}\right) + b''(w-x) \leq 0$. *This solution will be called "interior solution 3".*

Proof See the Appendix. □

In this case, there exist three interior solutions. Taking the derivative of the first order condition, $b'(w - x(w)) = \frac{k}{j} b'\left(\frac{x(w)}{j}\right)$, with respect to w leads to

$$\frac{d\widehat{x}^i(w)}{dw} = \frac{b''\left(w - \widehat{x}^i(w)\right)}{\frac{j}{k^2} b''\left(\frac{\widehat{x}^i(w)}{j}\right) + b''\left(w - \widehat{x}^i(w)\right)} \text{ and} \tag{12}$$

$$\frac{d\left(w - \widehat{x}^i(w)\right)}{dw} = \frac{\frac{j}{k^2} b''\left(\frac{\widehat{x}^i(w)}{j}\right)}{\frac{j}{k^2} b''\left(\frac{\widehat{x}^i(w)}{j}\right) + b''\left(w - \widehat{x}^i(w)\right)}. \tag{13}$$

The ranking of $\frac{\widehat{x}^i(w)}{j}$ and $w - \widehat{x}^i(w)$, $i = 1, 2, 3$ relative to w_0 allows us to note that $\widehat{x}^1(w)$ and $\widehat{x}^2(w)$ are increasing functions of w, $\widehat{x}^3(w)$ is a decreasing function of w. $w - \widehat{x}^1(w)$ and $w - \widehat{x}^3(w)$ are increasing functions of w, but $w - \widehat{x}^2(w)$ is a decreasing function of w. When the interior solution 2 solves the maximization program 11, project c gets an increasing amount of the total budget whereas project b gets a decreasing amount of the total budget. This result is similar to the case of homogenous benefit functions. Quite the opposite happens with the interior solution 3: project b gets an increasing amount of the total budget whereas project c gets a decreasing amount of the total budget. This is not the case anymore with the interior solution 1, where the two optimal solutions are increasing functions of the total budget w: as the budget increases, each project gets more financing. The following lemma characterizes the optimal allocation.[8]

Lemma 5 *Suppose b is a benefit function and $c(x) = kb(x/j)$ with $1 < k < j$. The optimal solution to the maximization program 11 has the following characteristics:*

1. *When $x \leq w_1^*$, project b gets the whole budget,*
2. *When interior solution 1, $\widehat{x}^1(w)$, is the optimal allocation, it will remain so for any higher budget level.*

Proof See the Appendix. □

The path of the optimal allocation between the two projects as a function of the budget is quite different from the case where the benefit functions were identical. No general result indeed holds on the way the different allocations link together. However, there are two similarities. First, when the budget level is very low, only one project, project b, is financed. Second, once the interior solution 1 is reached, the funding of each project increases with w. But contrary to the identical benefit functions case, the two projects are not financed at the same level. Between the allocation that gives all the budget to project b and the interior solution 1, virtually anything may happen. In particular, either project can have an allocation that is an increasing function of the total budget whereas the allocation of the other project is a decreasing function of the total budget (interior solutions 2 and 3). Moreover, it can be the case that one project stops being financed (when project c gets all the budget), implying that the solutions may be not continuous.[9] It can also happen that the three allocations mix together. In order to illustrate this discussion, we consider the functions' family introduced in the previous section

$$b(x) = \frac{1+\alpha x}{1+\alpha}\frac{1}{\pi}\left(\arctan(2x-1) + \frac{\pi}{4}\right). \tag{14}$$

[8]Edward Shpiz helped with this lemma and with Lemma 4.

[9]Note that also in the 2 identical benefit functions case, there exist a lot of cases where the optimal solution is not continuous in the total available amount. This is the case if for instance $b'(x) = \begin{cases} kx & \text{if } x \leq w_0, \\ \frac{a}{x^6} + d & \text{if } x > w_0, \end{cases}$ where $k = 0.1$, $d = 0.001$ and c and d are chosen such that b and b' are continuous.

with $\alpha = 0.05$, $k = 1.8$ and $j = 2$. With these parameters, the inflection point, w_0, is equal to 0.5122, and $w_1^* = 0.74783$. The numerical resolution of this example gives the following results (which are depicted in Fig. 10):

- If $w < 1.1306$, then project b gets the entire budget,
- If $1.1306 < w < 1.2407$, then the optimal solution is the interior solution 2. The two projects are financed, but as w increases, project b is less financed whereas project c is more financed,
- If $1.2407 < w < 1.6296$, then project c gets the whole budget w,
- If $1.6296 < w < 1.7073$, the interior solution 2 is once again the optimal solution,
- If $w > 1.7073$, the interior solution 1 is optimal, meaning that the funding of each project increases with w.

When the budget level is low, only project b gets financing. But once the budget level w passes the inflexion point, b becomes less profitable, whereas project c still presents increasing marginal productivity. Therefore, project c begins to be funded and project b gets a lower share of the total budget before ultimately being totally abandoned. There is a range of budget levels for which project c gets the whole budget: indeed, for these values of w, both marginal productivities are decreasing, but project c still presents a higher marginal productivity. But on [1.6296, 1.7073], project b comes back to life. The two decreasing marginal productivities get closer, but the dominance of project c relative to project b makes b lose funding as the budget increases. Once the last threshold 1.7073 is crossed, both projects get increasing funding as w gets larger. Indeed, the two marginal productivities, while decreasing, converge and the two projects both get financed with a strictly positive budget share. The

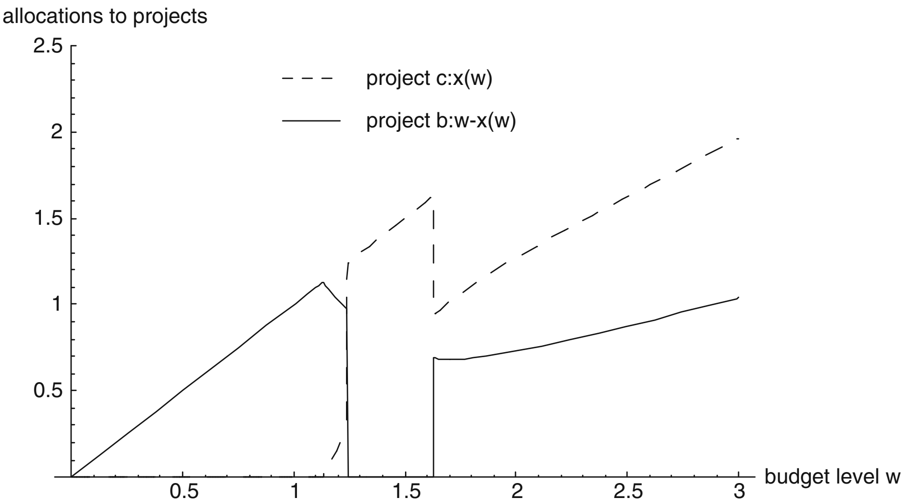

Fig. 10 Optimal allocations with heterogenous expected benefit functions

optimal allocation is thus much more complex than in the homogenous case; indeed the financing of each project may not be monotone with the budget level w. Moreover, the number of financed projects is not increasing with the total budget that is available. It is difficult to get a more precise description of the optimal allocation, but the main result is that after a succession of financing and non-financing of the different projects, they both end up being financed in an increasing way as the total funding increases.

5 Properties of the aggregate benefit function

The object of this section is to study function B defined by

$$B(w) = \max_{x_1 \ldots x_{n-1}} b_1(x_1) + \ldots + b_{n-1}(x_{n-1}) + b_n(w - x_1 - \ldots - x_{n-1}) \tag{15}$$

subject to

$$x_i \geq 0 \quad \forall i = 1 \ldots n-1, \qquad \sum_{i=1}^{n-1} x_i \leq w.$$

We can first state a simple property of function B.

Lemma 6 *Function B is increasing in the budget level w.*

Proof This is an application of the envelope theorem. □

From now on, our analysis will narrow to the case of identical benefit functions. Indeed, we have noted in the previous sections that it is very difficult to obtain general properties in the case where benefit functions differ. We begin with the case where the characterization of the optimal solution is straightforward, i.e., when the benefit function is LU.

5.1 The case of LU benefit functions[10]

In Section 3, we proved that under some condition, the optimal strategy is to finance equally an increasing number of projects as the budget increases. How does this result affect the aggregate benefit function?

Proposition 5 *Suppose that b is a LU benefit function and that function z is increasing. Then, function B is convex on* $[0, w_0]$ *and concave on* $[w_0, +\infty[$.

Proof See the Appendix. □

[10]The results of this section also apply to the case of symmetric benefit functions.

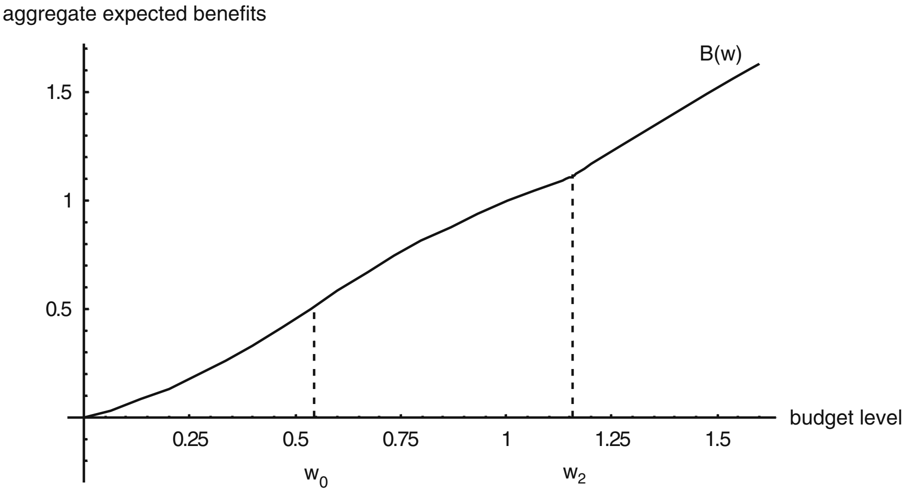

Fig. 11 Aggregate expected benefit function when the expected benefit functions are LU

In Fig. 11, we illustrate this result with the LU benefit functions' family we already used.

Note that it is not correct to say that the aggregate benefit function is S-shaped although it is successively convex and concave. Indeed, as the first derivative of function B is not continuous[11], there is a kink around w_2 and B can be seen as locally convex around this threshold (the derivative $b'(w)$ is lower than $b'(w/2)$). It is straightforward to extend this result to the case of a symmetric benefit function.

5.2 The general case

Recall that in the general case the characterization of the optimal strategy is tricky. Therefore, we concentrate on the case of two identical benefit functions. According to Proposition 3, we know how the solution evolves as the budget increases and we can state the following proposition.

Proposition 6 *In the case of two identical benefit functions, if there exist values of w for which the interior solution is optimal, the function B is successively convex, concave, convex to end up concave as the budget w increases.*

Proof See the Appendix. □

We illustrate this proposition with the benefit function $b(x) = \frac{x^2}{x^2+2(1-x)^2}$ in Fig. 12. B is convex on $[0, w_0]$, concave on $\left[w_0, \underline{w}\right]$, convex on $\left[\underline{w}, \overline{w}\right]$ and

[11]There is indeed no reason that $b'(w) = b'\left(\frac{w}{2}\right)$ at $w = w_2$.

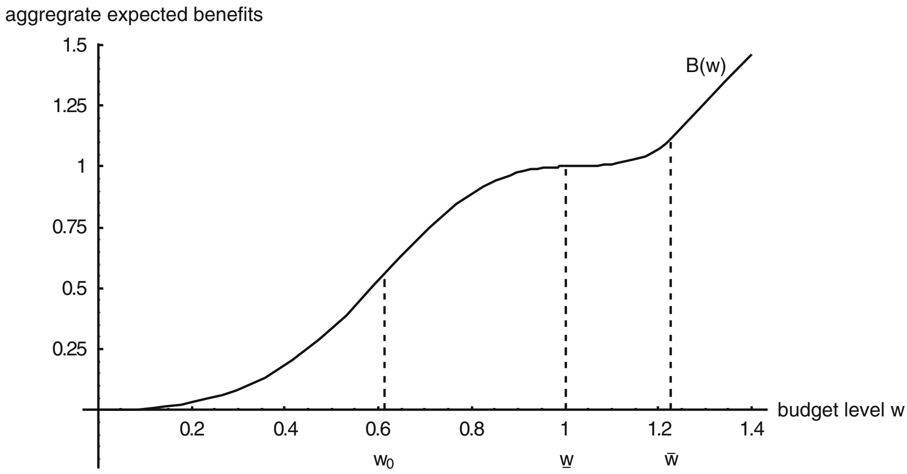

Fig. 12 Aggregate expected benefit function in the general case with two projects

concave on $[\overline{w}, +\infty[$. In fact, when the asymmetric interior solution is optimal, B is convex, meaning that the marginal aggregate benefit function is increasing. Indeed, in this case, the project that begins to be financed has a greater weight than the other one in terms of the second derivatives of the aggregate benefit function (the increase in the marginal benefit function is more important than the decrease). But once this property is no longer satisfied, we turn to the equal allocation and the aggregate marginal benefit function is concave. Note that in this example, the first derivative of function B is continuous. This comes from the continuity of function x in the special example we treated. However, as we discussed in footnote 9, this is not always the case and function B may present a kink as in the case of LU benefit functions.

6 Concluding remarks

We study the investment decision of an investor with multiple available projects, each presenting a range of increasing returns before returns decline. Such decisions are common across a great range of fields, such as allocating R&D investment, advertising budgets, or inoculations for communicable diseases, and are particularly prevalent when outcomes are uncertain or indexed probabilities. With n identical projects, when budget levels are low, the investor favours one project by investing the whole budget in it. Once he decides to invest a strictly positive amount in each project for a given budget level, he will keep on investing strictly positive amounts in each project.

The properties of the optimal allocation are most easily seen with just two projects. As the budget increases, allocations may be unequal though positive, and a project may actually experience a reduction in budget over some range.

When the total budget level is high enough, the investor shares the budget equally between the two projects, and this equal strategy remains optimal for any higher budget level. When the benefit function has a plausible shape, what we label lifted up, the optimal investment strategy goes from full specialization to equal division without passing through a range with positive but unequal division. These results extend immediately to the case of a finite number of projects.

Matters are more complex when the benefit for the projects may differ. Qualitatively, however, the same local and global marginal efficiency requirements must be satisfied, and the prime features of efficient allocations are maintained. Thus, first one project gets all resources. Then there is an intermediate range where multiple projects get funding, and the funding for some may be non-monotonic with the total budget. Finally, when the budget is large, all projects get funded, and the funding for each increases as the budget grows further.

The aggregate benefit function for the lifted up case is first convex and then concave. More generally, the aggregate benefit function is successively convex, concave, convex... to end up concave in the budget level. In short, an apparently straightforward and commonly encountered resource allocation problem, one that is particularly common when outputs are uncertain, turns out to have an intriguingly complex solution, despite perfectly intuitive efficiency conditions.

Appendix

Proof of Lemma 3

We are going to show the following three assertions:

1. As w increases, one can never switch from the fully diversified solution to the full-specialization one.
2. As w increases, one can never switch from an asymmetric interior solution to the full-specialization one.
3. As w increases, one can never switch from the fully diversified solution to an asymmetric interior one.

The first property is straightforward to prove and thus omitted. It is sufficient to prove that the function $b(w) - 2b(w/2)$ has a unique zero and that the derivative is positive at this zero.

Let us focus first on the second result. Consider a range of w for which an asymmetric interior solution $(\widehat{x}, w - \widehat{x})$ exists, where $\widehat{x}(w)$ is defined by the asymmetric solution to equation $b'(\widehat{x}) = b'(w - \widehat{x})$. We know from Proposition 2 that $\widehat{x} < w_0 < w - \widehat{x}$. Let us study the function $w \longmapsto g(w) = b(w) - \left[b(\widehat{x}(w)) + b(w - \widehat{x}(w))\right]$. Consider any solution $w = \overline{w}$ of equation $g(w) = 0$. We show that this implies that $g'(\overline{w})$ be nonpositive. Indeed, by the envelope theorem, we have that

$$g'(\overline{w}) = b'(\overline{w}) - b'(\overline{w} - \widehat{x}(\overline{w})).$$

Because $w_0 < \overline{w} - \widehat{x}(\overline{w}) \le \overline{w}$, we have that b' is decreasing between $\overline{w} - \widehat{x}(\overline{w})$ and $\overline{w}$. It implies that $b'(\overline{w} - \widehat{x}(\overline{w}))$ is larger than $b'(\overline{w})$, or equivalently, that $g'(\overline{w})$ is nonpositive. It implies that if one switches between the fully specialized solution and the asymmetric interior solution when wealth increases, it can only be from the former to the latter.

To prove the third result, consider a range of w for which an asymmetric interior solution $(\widehat{x}, w - \widehat{x})$ exists, where $\widehat{x}(w)$ is defined by the asymmetric solution to equation $b'(\widehat{x}) = b'(w - \widehat{x})$. Let us study the function $w \longmapsto h(w) = b(\widehat{x}(w)) + b(w - \widehat{x}(w)) - 2b(w/2)$. Consider any solution $w = \overline{w}$ of equation $h(w) = 0$. We show that this implies that $h'(\overline{w})$ be nonpositive. Indeed, by the envelope theorem, we have that

$$h'(\overline{w}) = b'(\overline{w} - \widehat{x}(\overline{w})) - b'(\overline{w}/2).$$

We know from Proposition 2 that $\widehat{x} < w_0 < w - \widehat{x}$. We also know that b' is increasing and then decreasing in interval $[\widehat{x}(\overline{w}), \overline{w} - \widehat{x}(\overline{w})]$, and that the values of b' are the same at the boundaries of this interval. Because $\overline{w}/2$ belongs to this interval, we have that $b'(\overline{w}/2)$ is larger than $b'(\overline{w} - \widehat{x}(\overline{w}))$, or equivalently, that $h'(\overline{w})$ is nonpositive. It implies that if one switches between the asymmetric interior solution and the equal solution when wealth increases, it can only be from the former to the latter. □

Proof of Proposition 1

We first prove that the first-order condition $b'(x) = b'(w - x)$ may have only one root at $x = w/2$ when $w \neq 2w_0$. Suppose by contradiction that there exists $\widehat{x} \neq w/2$ such that $b'(\widehat{x}) = b'(w - \widehat{x})$. By symmetry, this can be true only if $\widehat{x} = w - \widehat{x}$, or if $w_0 - (\widehat{x} - w_0) = w - \widehat{x}$. The first case is equivalent to $\widehat{x} = w/2$, a contradiction. The second case is equivalent to $w = 2w_0$, also a contradiction. Thus, $x = w/2$ is the only candidate for an interior optimum.

We then show that $x = w/2$ is a minimum of the objective function when w is smaller than $2w_0$. To show this, we prove that $b'(x) \le b'(w - x)$ for all x smaller than $w/2$. Two cases must be considered depending upon whether $w - x$ is smaller or larger than w_0. When $w - x < w_0$, both x and $w - x$ are smaller than w_0. Because b' is increasing in this range, we indeed obtain that $b'(x) \le b'(w - x)$ if $x \le w - x$, which is true. When $w - x > w_0$, x and $w - x$ are on opposite sides of w_0. But $b'(w - x) = b'(w_0 + (w - x - w_0))$ is by symmetry equal to $b'(w_0 - (w - x - w_0))$, whose argument is smaller than w_0. Because b' is increasing in this range, it implies that $b'(x) \le b'(x + (2w_0 - w))$ if $w \le 2w_0$, which is also true.

A parallel proof can be written when w is larger than $2w_0$. □

Proof of Proposition 2

Suppose b is PD. We are going to prove that $b(2w_0) < 2b(w_0)$.

$$\begin{aligned} b(2w_0) &= \int_0^{w_0} b'(x)\,dx + \int_{w_0}^{2w_0} b'(x)\,dx \\ &= \int_0^{w_0} b'(w_0-\delta)\,d\delta + \int_0^{w_0} b'(w_0+\delta)\,d\delta \\ &\leq 2\int_0^{w_0} b'(w_0-\delta)\,d\delta \\ &= 2b(w_0)\,. \end{aligned}$$

Therefore, $b(2w_0) \leq 2b(w_0)$ and $w_2 \leq 2w_0$.

We turn to the second part of the Proposition and suppose that there exist $\overline{w}$ and $\widehat{x}$ such that $b'(\widehat{x}) = b'(\overline{w}-\widehat{x})$ with $\widehat{x} < w_0 < \overline{w}-\widehat{x}$. Define $\widehat{\delta} \in [0, w_0]$ such that $\widehat{x} = w_0 - \widehat{\delta}$. We have that

$$\begin{aligned} b'(\widehat{x}) &= b'\left(w_0-\widehat{\delta}\right), \\ &= b'(\overline{w}-\widehat{x}), \\ &= b'\left(\overline{w}-\left(w_0-\widehat{\delta}\right)\right), \\ &\geq b'\left(w_0+\widehat{\delta}\right). \end{aligned}$$

As b' is decreasing for $x \geq x_{0,}$ the above equality thus implies that $\overline{w} - \left(w_0-\widehat{\delta}\right) \leq w_0+\widehat{\delta}$, hence $\overline{w} \leq 2w_0$. Therefore, if $\overline{w}$ and $\widehat{x}$ exist, we must have that $\overline{w} \leq 2w_0$. Thus, if $w > 2w_0$, no interior asymmetric interior solution exists and the solution belongs to $\{w, w/2\}$. But we know that for a PD benefit function $2w_0 > w_2$. Therefore, if $w > 2w_0$, $2b(w/2) > b(w)$ and the allocation $\{w/2\}$ is the solution for $w > 2w_0$. □

Proof of Proposition 3

Suppose b is LU. We are going to prove that $b(2w_0) > 2b(w_0)$.

$$\begin{aligned} b(2w_0) &= \int_0^{w_0} b'(x)\,dx + \int_{w_0}^{2w_0} b'(x)\,dx, \\ &= \int_0^{w_0} b'(w_0-\delta)\,d\delta + \int_0^{w_0} b'(w_0+\delta)\,d\delta, \\ &\geq 2\int_0^{w_0} b'(w_0-\delta)\,d\delta, \\ &= 2b(w_0)\,. \end{aligned}$$

Therefore, $b(2w_0) \geq 2b(w_0)$ and $w_2 \geq 2w_0$.

Suppose now there exist $\overline{w}$ and $\widehat{x}$ such that $b'(\widehat{x}) = b'(\overline{w}-\widehat{x})$ with $\widehat{x} < w_0 < \overline{w}-\widehat{x}$. Define $\delta \in [0, w_0]$ such that $\widehat{x} = w_0 - \delta$. We have that

$$\begin{aligned} b'(\widehat{x}) &= b'(w_0-\delta), \\ &= b'(\overline{w}-\widehat{x}), \\ &= b'(\overline{w}-(w_0-\delta)), \\ &\leq b'(w_0+\delta). \end{aligned}$$

As b' is decreasing for $x \geq x_{0,}$ the above equality thus implies that $\overline{w} - (w_0-\delta) \geq w_0+\delta$, hence $\overline{w} \geq 2w_0$. Therefore, if $\overline{w}$ and $\widehat{x}$ exist, we must have that $\overline{w} \geq 2w_0$. Thus, if $w \leq 2w_0$, no interior asymmetric interior solution exists and the solution belongs to $\{w, w/2\}$. However, in this case, $w/2$ is a local minimum. Therefore, if $w \leq 2w_0$, the solution is the full-specialization strategy $\{0, w\}$.

We now focus on the case where $w > 2w_0$, as $\widehat{x} = w_0 - \delta$, the condition on function z implies that $w(\delta) - \widehat{x}(\delta) - (w_0 + \delta)$ is an increasing function, where $w(\delta)$ is defined by $b'(\widehat{x}(\delta)) = b'(w(\delta) - \widehat{x}(\delta))$. Therefore, $w'(\delta) \geq 0$. Differentiating $b'(\widehat{x}(\delta)) = b'(w(\delta) - \widehat{x}(\delta))$ with respect to δ and recalling that $\widehat{x}(\delta) = w_0 - \delta$ lead to $\left(w'(\delta) + 1\right) b''(w(\delta) - \widehat{x}(\delta)) = -b''(\widehat{x}(\delta))$. As $w'(\delta) \geq 0$, this implies that $b''(w(\delta) - \widehat{x}(\delta)) + b''(\widehat{x}(\delta)) \geq 0$ and the asymmetric interior solution is a local minimum and is not a potential candidate for the optimal allocation. The solution to program 2 belongs thus to $\{w, w/2\}$. □

%vspace*9.5pt

Proof of Proposition 4

Before proving the lemma, we introduce some notations and give a preliminary result. Let us first introduce the thresholds $w_2, ..., w_i, ..., w_n$ defined by

$$(i-1)\, b\left(\frac{w_i}{i-1}\right) = ib\left(\frac{w_i}{i}\right), \forall i = 2...n-1. \tag{16}$$

Note that we recover the definition of w_2. In the following lemma, we prove the uniqueness of these thresholds, and we rank them.

Lemma 7 *The thresholds $w_2, ..., w_n$ are uniquely defined by* Eq. 16 *and satisfy*

$$w_0 < w_2 < ... < w_i < ... < w_n.$$

Moreover, $\forall i = 2, ...n-1$,

$$\begin{array}{l}(i-1)\, b\left(\frac{w}{i-1}\right) > ib\left(\frac{w}{i}\right) \;\; \forall w < w_i, \\ (i-1)\, b\left(\frac{w}{i-1}\right) < ib\left(\frac{w}{i}\right) \;\; \forall w > w_i.\end{array}$$

We are going to prove this lemma in three steps. First, since b is a convex function on $[0, w_0]$, $b\left(\frac{w_0}{2}\right) < \frac{1}{2}b(w_0)$, meaning that $w_0 < w_2$.

Concerning the uniqueness of the thresholds defined by Eq. 16, let us consider function $f_i(x) = (i-1)\, b\left(\frac{x}{i-1}\right) - ib\left(\frac{x}{i}\right)$. The first derivative, $f_i'(x) = b'\left(\frac{x}{i-1}\right) - b'\left(\frac{x}{i}\right)$ is strictly negative when $x > iw_0$ (since b' is decreasing on $[w_0, +\infty[$) and is strictly positive when $x < (i-1)\, w_0$ (since b' is increasing on $[0, w_0]$). We hereafter show that there exists a unique a such that $f_i'(a) = 0$. Suppose by contradiction that there exist a and c, with $(i-1)\, w_0 < a < c < iw_0$ such that $b'\left(\frac{a}{i-1}\right) = b'\left(\frac{a}{i}\right)$ and $b'\left(\frac{c}{i-1}\right) = b'\left(\frac{c}{i}\right)$. $w_0 < \frac{a}{i-1} < \frac{c}{i-1} < \frac{iw_0}{i-1}$ implies that $b'(w_0) > b'\left(\frac{a}{i-1}\right) > b'\left(\frac{c}{i-1}\right) > b'\left(\frac{iw_0}{i-1}\right)$, and $\frac{(i-1)w_0}{i} < \frac{a}{i} < \frac{c}{i} < w_0$ implies that $b'\left(\frac{(i-1)w_0}{i}\right) < b'\left(\frac{a}{i}\right) < b'\left(\frac{c}{i}\right) < b'(w_0)$. This leads to a contradiction since $b'\left(\frac{a}{i-1}\right) = b'\left(\frac{a}{i}\right)$ and $b'\left(\frac{c}{i-1}\right) = b'\left(\frac{c}{i}\right)$. Therefore, a is unique and f_i is increasing on $[0, a]$ and decreasing on $[a, +\infty[$. As $f_i(0) = 0$, if a positive zero w_i exists to f_i, it is unique. f_i changes sign only once, from positive to negative.

Now, we prove that $w_i < w_{i+1}, \forall i = 2...n-1$. According to Eq. 16, w_{i+1} is such that $ib\left(\frac{w_{i+1}}{i}\right) = (i+1)\, b\left(\frac{w_{i+1}}{i+1}\right)$, or by dividing each member by x, $\frac{i}{w_{i+1}} b\left(\frac{w_{i+1}}{i}\right) = \frac{i+1}{w_{i+1}} b\left(\frac{w_{i+1}}{i+1}\right)$. We have already proved that function $x \mapsto b(x)/x$ is single peaked, increasing on $[0, w_1]$ and then decreasing on $[w_1, +\infty[$. Therefore, in order $\frac{i}{w_{i+1}} b\left(\frac{w_{i+1}}{i}\right) = \frac{i+1}{w_{i+1}} b\left(\frac{w_{i+1}}{i+1}\right)$ to hold, it must be the case that $\frac{w_{i+1}}{i+1} < w_1$ and $\frac{w_{i+1}}{i} > w_1$, or $iw_1 < w_{i+1} < (i+1)\, w_1$. To compare w_i and w_{i+1}, let us compute $ib\left(\frac{w_{i+1}}{i}\right) - (i-1)\, b\left(\frac{w_{i+1}}{i-1}\right)$. As $w_{i+1} > iw_1$, $w_1 < \frac{w_{i+1}}{i} < \frac{w_{i+1}}{i-1}$, $x \mapsto b(x)/x$ is decreasing and therefore $\frac{i}{w_{i+1}} b\left(\frac{w_{i+1}}{i}\right) > \frac{i-1}{w_{i+1}} b\left(\frac{w_{i+1}}{i-1}\right)$, meaning that $w_{i+1} > w_i$.

Now that Lemma 7 is proved, we focus on the proof of Proposition 4 that we are going to lead using induction arguments.

First of all with two benefit functions, we know according to Proposition 3 that when $w < w_2$, the optimal allocation is $x^*(w) = \{w, 0\}$ and when $w \geq w_2$, the optimal allocation is $x^*(w) = \{w/2, w/2\}$.

Now, we suppose that the result holds when the investor has the choice between $n-1$ projects. Let us prove that it then holds when the investor has n projects. According to the previous discussion, we maximize the investor's program in two steps. First of all, we solve

$$\max_{x_1,\ldots,x_{n-2}} b(x_1) + \ldots + b(x_{n-1}) + b(w - x_1 - \ldots - x_{n-1}).$$

As the result holds when the investor has the choice between $n-1$ projects, we know how to solve this program.

$$x_1^*(x_{n-1}, w) = \begin{cases} w - x_{n-1} & \text{if } w - x_{n-1} \leq w_2, \\ \frac{w - x_{n-1}}{2} & \text{if } w_2 < w - x_{n-1} \leq w_{3,} \\ \ldots & \\ \frac{w - x_{n-1}}{n-1} & \text{if } w - x_{n-1} > w_{n-1} \end{cases},$$

$$x_i^*(x_{n-1}, w) = \begin{cases} 0 & \text{if } w - x_{n-1} \leq w_2, \\ \ldots & \\ \frac{w - x_{n-1}}{i} & \text{if } w_i < w - x_{n-1} \leq x_{i+1,} \quad \forall i = 2, \ldots n-2. \\ \ldots & \\ \frac{w - x_{n-1}}{n-1} & \text{if } w - x_{n-1} > w_{n-1}, \end{cases}$$

There remains to solve the second step. Suppose that $w_i < w - x_{n-1} \leq w_{i+1}$. Therefore, the maximization comes down to

$$\max_{x_{n-1}} b(x_{n-1}) + ib\left(\frac{w - x_{n-1}}{i}\right),$$

subject to

$$\begin{aligned} x_{n-1} &< w - w_i, \\ x_{n-1} &\geq w - w_{i+1}. \end{aligned} \tag{17}$$

The first order conditions, $b'(x_{n-1}) = b'((w - x_{n-1})/i)$, lead to the following candidate solutions

1. $x_{n-1} = (w - x_{n-1})/i$, then $x_{n-1} = \frac{w}{i+1}$ and condition 17 leads to $w > \frac{i+1}{i} w_i$,
2. x_{n-1}^1 such that $b'\left(x_{n-1}^1\right) = b'\left(\left(w - x_{n-1}^1\right)/i\right)$ and $x_{n-1}^1 < w_0 < \left(w - x_{n-1}^1\right)/i$. We call x_{n-1}^1 the asymmetric interior solution 1,
3. x_{n-1}^2 such that $b'\left(x_{n-1}^2\right) = b'\left(\left(w - x_{n-1}^2\right)/i\right)$ and $\left(w - x_{n-1}^2\right)/i < w_0 < x_{n-1}^2$ (maximization problem 8 is not symmetric anymore). In this case, condition 17 leads to $w > w_0 + w_i$. We call x_{n-1}^2 the asymmetric interior solution 1. There are also the two corner solutions,
4. $x_{n-1} = w$ but this can be eliminated because condition 17 leads to $w_i < 0$,
5. $x_{n-1} = 0$.

We are going to prove that the two asymmetric interior solutions x_{n-1}^1 and x_{n-1}^2 do not exist. We first focus on x_{n-1}^1 . If we define $\delta_1 \in [0, w_0]$ such that $x_{n-1}^1 = w_0 - \delta_1$, $z(\delta_1) = (w(\delta_1) - (i+1) w_0 - (i-1)\delta_1)/i$. As it is increasing by assumption, $w'(\delta_1) \geq i - 1$. But $w(\delta_1)$ is defined by $b'(w_0 - \delta_1) = b'((w(\delta_1) - w_0 + \delta_1)/i)$. Differentiating this expression with respect to δ_1 leads to

$$-b''(w_0 - \delta_1) = \frac{w'(\delta_1) + 1}{i} b''((w - w_0 + \delta_1)/i).$$

As $w'(\delta_1) \geq i - 1$, $\left(w'(\delta_1) + 1\right)/i \geq 1$, and the following inequalities hold:

$$\begin{aligned} b''(w_0 - \delta_1) &= -\tfrac{w'(\delta_1)+1}{i} b''((w - w_0 + \delta_1)/i), \\ &\geq -b''((w - w_0 + \delta_1)/i), \\ &\geq -(1/i)\, b''((w - w_0 + \delta_1)/i). \end{aligned}$$

Therefore $b''(w_0 - \delta_1) + \frac{1}{i} b''((w - w_0 + \delta_1)/i) \geq 0$ and the asymmetric interior solution 1, if it exists, is unique and is a local minimum.

Before studying x_{n-1}^2, let us prove an intermediate result, that is $w_i > i w_0$, $\forall i \geq 2$. To do so, we compute $ib(w_0) - (i-1)\, b\left(\frac{i}{i-1} w_0\right)$.

$$\begin{aligned} ib(w_0) - (i-1)\, b\left(\frac{i}{i-1} w_0\right) &= \int_0^{w_0} b'(w_0 - \delta)\, d\delta - (i-1) \int_0^{\frac{w_0}{i-1}} b'(w_0 + \delta)\, d\delta, \\ &\leq \int_0^{w_0} b'(w_0 - \delta)\, d\delta - \int_0^{\frac{w_0}{i-1}} b'(w_0 - \delta)\, d\delta, \\ &\quad - (i-2) \int_0^{\frac{w_0}{i-1}} b'(w_0 + \delta)\, d\delta, \\ &= \int_{\frac{w_0}{i-1}}^{w_0} b'(w_0 - \delta)\, d\delta - (i-2) \int_0^{\frac{w_0}{i-1}} b'(w_0 + \delta)\, d\delta, \\ &= -b\left(w_0 - \frac{w_0}{i-1}\right) - (i-2) \\ &\quad \times \left(b\left(w_0 + \frac{w_0}{i-1}\right) - b(w_0)\right), < 0. \end{aligned}$$

Therefore, $w_i > iw_0$, $\forall i \geq 2$. We thereafter focus on the solution x_{n-1}^2 such that $b'\left(x_{n-1}^2\right) = b'\left(\left(w - x_{n-1}^2\right)/2\right)$ and $\left(w - x_{n-1}^2\right)/2 < w_0 < x_{n-1}^2$. There exists $\delta_2 > 0$ such that $x_{n-1}^2 = w_0 + \delta_2$. In this case,

$$z\left(\delta_2\right) = \left(w\left(\delta_2\right) + (i-1)\,\delta_2 - (i+1)\,w_0\right)/i.$$

Condition 17 implies that $w - x_{n-1}^2 > w_i$. Let us define $F\left(\delta_2\right) = w\left(\delta_2\right) - w_0 - \delta_2 - w_i$. Condition 17 implies that $F\left(\delta_2\right) > 0, \forall \delta_2 > 0$. $F'\left(\delta_2\right) = w'\left(\delta_2\right) - 1$. Recall that $w\left(\delta_2\right)$ is defined by $b'\left(w_0 + \delta_2\right) = b'\left(\left(w\left(\delta_2\right) - w_0 - \delta_2\right)/i\right)$. Taking the derivative of this expression with respect to δ_2, this leads to $b''\left(w_0 + \delta_2\right) = 1/i\left(w'\left(\delta_2\right) - 1\right) b''\left(\left(w\left(\delta_2\right) - w_0 - \delta_2\right)/i\right)$.

This last equality holds if and only if $w'\left(\delta_2\right) - 1 < 0$. Therefore $F\left(\delta_2\right)$ is strictly decreasing. $F(0) = w(0) - w_0 - w_i$ and $w(0) = (i+1)\,w_0$ for a LU benefit function. Therefore, $F(0) < 0$ and condition 17 is violated and this asymmetric interior solution 2 cannot exist for a LU benefit function. The two asymmetric interior solutions have been eliminated, thus $x_{n-1} = \frac{w}{i+1}$ or $x_{n-1} = 0$ and the proposition is proved. □

Proof of Lemma 4

The first two candidate solutions are the two corner solutions. Let us now focus on interior solutions characterized by the first order conditions $b'\left(w - \widehat{x}\right) = \frac{k}{j} b'\left(\frac{\widehat{x}}{j}\right)$. As $1 < k < j$, it follows that $b'\left(\frac{\widehat{x}}{j}\right) > b'\left(w - \widehat{x}\right)$. There are four candidate solutions to this inequality:

1. $b'\left(\frac{\widehat{x}}{j}\right) > b'\left(w - \widehat{x}\right)$ with $\frac{\widehat{x}}{j} > w_0$ and $w - \widehat{x} > w_0$. As b' is decreasing $\forall x > w_0$, it is the case if and only if $w_0 < \frac{\widehat{x}}{j} < w - \widehat{x}$. The second order condition, $\frac{k}{j^2} b''\left(\frac{\widehat{x}}{j}\right) + b''\left(w - \widehat{x}\right) \leq 0$, is satisfied because of the concavity of function b on $[w_0, +\infty[$. This candidate solution is therefore called the "interior solution 1".
2. $b'\left(\frac{\widehat{x}}{j}\right) > b'\left(w - \widehat{x}\right)$ with $\frac{\widehat{x}}{j} < w_0$ and $w - \widehat{x} < w_0$. In this case, $\frac{k}{j^2} b''\left(\frac{\widehat{x}}{j}\right) + b''\left(w - \widehat{x}\right) \geq 0$ and this solution is a local minimum. It can therefore be skipped.
3. $b'\left(\frac{\widehat{x}}{j}\right) > b'\left(w - \widehat{x}\right)$ with $\frac{\widehat{x}}{j} < w_0$ and $w - \widehat{x} > w_0$. This candidate turns out to be a potential solution if and only if the second order condition is satisfied, $\frac{k}{j^2} b''\left(\frac{\widehat{x}}{j}\right) + b''\left(w - \widehat{x}\right) \leq 0$.
4. $b'\left(\frac{\widehat{x}}{j}\right) > b'\left(w - \widehat{x}\right)$ with $\frac{\widehat{x}}{j} > w_0$ and $w - \widehat{x} < w_0$. This candidate turns out to be a potential solution if and only if the second order condition is satisfied, $\frac{k}{j^2} b''\left(\frac{\widehat{x}}{j}\right) + b''\left(w - \widehat{x}\right) \leq 0$. □

Proof of Lemma 5

The first result is an application of Lemma 1. Concerning the other results, they are similar to the results of Proposition 3. We are going to prove the following five results:

1. As w increases, one can never switch from the interior solution 2 to the allocation that gives the whole budget to project b,
2. As w increases, one can never switch from the interior solution 1 to the interior solution 2,
3. As w increases, one can never switch from the interior solution 1 to the allocation that gives the whole budget to project c,
4. As w increases, one can never switch from the interior solution 1 to the allocation that gives the whole budget to project b,
5. As w increases, one can never switch from the interior solution 1 to the interior solution 3.

We successively prove the five assertions.

1. Consider a range of w for which an interior solution 2 $(\widehat{x}^2, w-\widehat{x}^2)$ exists, where $\widehat{x}^2(w)$ is defined by $b'\left(w-\widehat{x}^2(w)\right)=\frac{k}{j}b'\left(\frac{\widehat{x}^2(w)}{j}\right)$ with $\frac{\widehat{x}^2(w)}{j}<w_0<w-\widehat{x}^2(w)$. Let us study the function $w\longmapsto g_1(w)=b(w)-\left[kb(\frac{\widehat{x}^2(w)}{j})+b(w-\widehat{x}^2(w))\right]$. Consider any solution $w=\overline{w}$ of equation $g_1(w)=0$. We show that this implies that $g_1'(\overline{w})$ be nonpositive. Indeed, by the envelope theorem, we have that
$$g_1'(\overline{w})=b'(\overline{w})-b'(\overline{w}-\widehat{x}^2(\overline{w})).$$
As $\overline{w}>w-\widehat{x}^2(w)>w_0$, b' is decreasing and $b'(\overline{w})<b'\left(w-\widehat{x}^2(w)\right)$. Therefore, $g_1'(\overline{w})$ is nonpositive. It implies that if one switches between the allocation that gives the whole budget to project b to the interior solution 2, it can only be from the former to the latter.
2. Consider a range of w for which an interior solution 1 $(\widehat{x}^1, w-\widehat{x}^1)$ and an interior solution 2 $(\widehat{x}^2, w-\widehat{x}^2)$ exist, where $\widehat{x}^2(w)$ is defined by $b'\left(w-\widehat{x}^2(w)\right)=\frac{k}{j}b'\left(\frac{\widehat{x}^2(w)}{j}\right)$ with $\frac{\widehat{x}^2(w)}{j}<w_0<w-\widehat{x}^2(w)$ and where $\widehat{x}^1(w)$ is defined by $b'\left(w-\widehat{x}^1(w)\right)=\frac{k}{j}b'\left(\frac{\widehat{x}^1(w)}{j}\right)$ and $w_0<\frac{\widehat{x}^1(w)}{j}<w-\widehat{x}^1(w)$. Let us study the function $w\longmapsto g_2(w)=kb\left(\frac{\widehat{x}^2(w)}{j}\right)+b(w-\widehat{x}^2(w))-\left[kb\left(\frac{\widehat{x}^1(w)}{j}\right)+b(w-\widehat{x}^1(w))\right]$. Consider any solution $w=\overline{w}$ of equation $g_2(w)=0$. We show that this implies that $g_2'(\overline{w})$ be nonpositive. Indeed, by the envelope theorem, we have that
$$g_2'(\overline{w})=b'(\overline{w}-\widehat{x}^2(\overline{w}))-b'(\overline{w}-\widehat{x}^1(\overline{w})).$$
As $\widehat{x}^2(\overline{w})<jx_0<\widehat{x}^1(\overline{w})$, $\overline{w}-\widehat{x}^2(\overline{w})>\overline{w}-\widehat{x}^1(\overline{w})>w_0$, by assumption. Therefore, b' is decreasing and $b'\left(\overline{w}-\widehat{x}^2(\overline{w})\right)<b'\left(\overline{w}-\widehat{x}^1(\overline{w})\right)$. Therefore, $g_2'(\overline{w})$ is nonpositive. It implies that if one switches between the

interior solution 2 to the interior solution 1, it can only be from the former to the latter.

3. Consider a range of w for which an interior solution 1 $(\widehat{x}^1, w - \widehat{x}^1)$ exists, where $\widehat{x}^1(w)$ is defined by $b'\left(w - \widehat{x}^1(w)\right) = \frac{k}{j} b'\left(\frac{\widehat{x}^1(w)}{j}\right)$ and $w_0 < \frac{\widehat{x}^1(w)}{j} < w - \widehat{x}^1(w)$. Let us study the function $w \longmapsto g_3(w) = kb\left(\frac{w}{j}\right) - \left[kb(\frac{\widehat{x}^1(w)}{j}) + b(w - \widehat{x}^1(w))\right]$. Consider any solution $w = \overline{w}$ of equation $g_3(w) = 0$. We show that this implies that $g_3'(\overline{w})$ be nonpositive. Indeed, by the envelope theorem, we have that

$$g_3'(\overline{w}) = \frac{k}{j} b'\left(\frac{\overline{w}}{j}\right) - b'(\overline{w} - \widehat{x}^1(\overline{w})).$$

As $w_0 < \frac{\widehat{x}^1(\overline{w})}{j}$ and $\widehat{x}^1(\overline{w}) < w$, $b'\left(\frac{\overline{w}}{j}\right) < b'\left(\frac{\widehat{x}^1(\overline{w})}{j}\right)$ and $\frac{k}{j} b'\left(\frac{\overline{w}}{j}\right) < \frac{k}{j} b'\left(\frac{\widehat{x}^1(\overline{w})}{j}\right) = b'(\overline{w} - \widehat{x}^1(\overline{w}))$. Therefore, $g_3'(\overline{w})$ is nonpositive. It implies that if one switches between the allocation that gives the whole budget to project c to the interior solution 1, it can only be from the former to the latter.

4. Consider a range of w for which an interior solution 1 $(\widehat{x}^1, w - \widehat{x}^1)$ exists, where $\widehat{x}^1(w)$ is defined by $b'\left(w - \widehat{x}^1(w)\right) = \frac{k}{j} b'\left(\frac{\widehat{x}^1(w)}{j}\right)$ and $w_0 < \frac{\widehat{x}^1(w)}{j} < w - \widehat{x}^1(w)$. Let us study the function $w \longmapsto g_4(w) = b(w) - \left[kb(\frac{\widehat{x}^1(w)}{j}) + b(w - \widehat{x}^1(w))\right]$. Consider any solution $w = \overline{w}$ of equation $g_4(w) = 0$. We show that this implies that $g_4'(\overline{w})$ be nonpositive. Indeed, by the envelope theorem, we have that

$$g_4'(\overline{w}) = b'(\overline{w}) - b'(\overline{w} - \widehat{x}^1(\overline{w})).$$

As $\overline{w} > w - \widehat{x}^1(w) > w_0$, b' is decreasing and $b'(\overline{w}) < b'\left(w - \widehat{x}^1(w)\right)$. Therefore, $g_4'(\overline{w})$ is nonpositive. It implies that if one switches between the allocation that gives the whole budget to project b to the interior solution 1, it can only be from the former to the latter.

5. Consider a range of w for which an interior solution 1 $(\widehat{x}^1, w - \widehat{x}^1)$ and an interior solution 3 $(\widehat{x}^3, w - \widehat{x}^3)$ exist, where $\widehat{x}^3(w)$ is defined by $b'\left(w - \widehat{x}^3(w)\right) = \frac{k}{j} b'\left(\frac{\widehat{x}^3(w)}{j}\right)$ with $w - \widehat{x}^3(w) < w_0 < \frac{\widehat{x}^3(w)}{j}$ and where $\widehat{x}^1(w)$ is defined by $b'\left(w - \widehat{x}^1(w)\right) = \frac{k}{j} b'\left(\frac{\widehat{x}^1(w)}{j}\right)$ and $w_0 < \frac{\widehat{x}^1(w)}{j} < w - \widehat{x}^1(w)$. Let us study the function $w \longmapsto g_5(w) = kb\left(\frac{\widehat{x}^3(w)}{j}\right) + b(w - \widehat{x}^3(w)) - \left[kb\left(\frac{\widehat{x}^1(w)}{j}\right) + b(w - \widehat{x}^1(w))\right]$. Consider any solution $w = \overline{w}$ of equation $g_5(w) = 0$. We show that this implies that $g_5'(\overline{w})$ be nonpositive. Indeed, by the envelope theorem, we have that

$$g_5'(\overline{w}) = b'(\overline{w} - \widehat{x}^3(\overline{w})) - b'(\overline{w} - \widehat{x}^1(\overline{w})).$$

As $w-\widehat{x}^3(w)<w_0<\frac{\widehat{x}^3(w)}{j}$ and $w_0<\frac{\widehat{x}^1(w)}{j}<w-\widehat{x}^1(w)$, it follows that $w-\widehat{x}^3(w)<w_0<\frac{\widehat{x}^1(w)}{j}<\frac{\widehat{x}^3(w)}{j}<w-\widehat{x}^1(w)$. $g_5'(\overline{w})=b'(\overline{w}-\widehat{x}^3(\overline{w}))-b'(\overline{w}-\widehat{x}^1(\overline{w}))=\frac{k}{j}b'\left(\frac{\widehat{x}^3(\overline{w})}{j}\right)-\frac{k}{j}b'\left(\frac{\widehat{x}^1(\overline{w})}{j}\right)$. As $\frac{\widehat{x}^3(\overline{w})}{j}>\frac{\widehat{x}^1(\overline{w})}{j}>w_0$, b' is decreasing and $\frac{k}{j}b'\left(\frac{\widehat{x}^3(w)}{j}\right)<\frac{k}{j}b'\left(\frac{\widehat{x}^1(w)}{j}\right)$. Therefore, $g_5'(\overline{w})$ is nonpositive. It implies that if one switches between the interior solution 3 to the interior solution 1, it can only be from the former to the latter. □

Proof of Proposition 5

The first step is to prove that in this case, $w_i>iw_0\ \forall i=2...n$ where the w_i have been defined in the proof of Proposition 4. The comes down to proving that

$$(i-1)\,b\left(\frac{iw_0}{i-1}\right)\geq ib(w_0)\,.$$

We have:

$$\begin{aligned}(i-1)\,b\left(\tfrac{iw_0}{i-1}\right) &= (i-1)\int_0^{w_0}b'(w_0-\delta)\,d\delta+(i-1)\int_0^{\frac{w_0}{i-1}}b'(w_0+\delta)\,d\delta\\ &\geq (i-1)\int_0^{w_0}b'(w_0-\delta)\,d\delta+(i-1)\int_0^{\frac{w_0}{i-1}}b'(w_0-\delta)\,d\delta\\ &= ib(w_0)+(i-1)\left(b(w_0)-b\left(\tfrac{i-2}{i-1}w_0\right)\right)-b(w_0)\\ &\geq ib(w_0)\end{aligned}$$

The last inequality holds since $(i-1)\left(b(w_0)-b\left(\frac{i-2}{i-1}w_0\right)\right)-b(w_0)\geq 0$. Indeed, as $\frac{i-2}{i-1}w_0<w_0<w_1$, it follows that $\frac{b\left(\frac{i-2}{i-1}w_0\right)}{\frac{i-2}{i-1}w_0}\leq\frac{b(w_0)}{w_0}$.

Recall now that under the conditions stated in the proposition, the aggregate benefit function has the following expression:

$$B(w)=\begin{cases}b(w) & \text{if}\quad w<w_2\\ 2b\left(\frac{w}{2}\right) & \text{if}\quad w_2\leq w<w_3\\ \ldots & \\ ib\left(\frac{w}{i}\right) & \text{if}\quad w_i\leq w<w_{i+1}\\ \ldots & \\ nb\left(\frac{w}{n}\right) & \text{if}\quad w\geq w_n\end{cases}.$$

With the result we just proved, B is convex on $[0,w_0]$ and concave on $[w_0,+\infty[$. □

Proof of Proposition 6

If an asymmetric interior solution exists, B has the following shape:

$$B(w)=\begin{cases}b(w) & \text{if}\quad w<\underline{w}\\ b(\widehat{x}(w))+b(w-\widehat{x}(w)) & \text{if}\quad \underline{w}\leq w<\overline{w}\\ 2b\left(\frac{w}{2}\right) & \text{if}\quad w\geq\overline{w}\end{cases},$$

where $\underline{w} \geq w_1$ and $\overline{w} \geq w_2$. Let us focus on what happens on $\left[\underline{w}, \overline{w}\right]$. According to the envelope theorem, $B'(w) = b'(w - \widehat{x}(w))$ and thus $B''(w) = \left(1 - \frac{d\widehat{x}(w)}{dw}\right) b''(w - \widehat{x}(w))$. Recall that in this case $w - \widehat{x}(w) > w_0$, therefore $b''(w - \widehat{x}(w)) \leq 0$ and $\left(1 - \frac{d\widehat{x}(w)}{dw}\right) \leq 0$: B is thus convex in this case. It follows that B is convex on $[0, w_0]$, concave on $\left[w_0, \underline{w}\right]$, convex on $\left[\underline{w}, \overline{w}\right]$, concave on $[\overline{w}, +\infty[$. □

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