J. Bradford De Long
Harvard University and National Bureau of Economic Research

Andrei Shleifer
University of Chicago and National Bureau of Economic Research

Lawrence H. Summers
Harvard University and National Bureau of Economic Research

Robert J. Waldmann
European University Institute

The Survival of Noise Traders in Financial Markets*

We present a model of portfolio allocation by noise traders with incorrect expectations about return variances. For such misperceptions, noise traders who do not affect prices can earn higher expected returns than rational investors with similar risk aversion. Moreover, such noise traders can come to dominate the market, in that the probability that they eventually have a high share of total wealth is close to one. Noise traders come to dominate despite their taking of excessive risk and their higher consumption. We conclude that the case against their long-run viability is not as clear-cut as is commonly supposed.

I. Introduction

Economists have long asked whether investors who misperceive asset returns can survive in a competitive asset market such as a stock or a currency market. The classic answer, given by Friedman (1953), is that they cannot. Friedman argued that mistaken investors buy high and sell low, as a result lose money to rational investors, and eventually lose all their wealth. In response, Figlewski (1979) pointed out that it might take irrational investors a very long time to lose their entire wealth, but he agreed that in the long run those who choose their portfolios irrationally are doomed. Advocates of the importance of traders with incorrect expectations—or "noise traders"—for the determination of asset prices (Shiller 1984; Kyle 1985; Black 1986; and Campbell

* We would like to thank the Alfred P. Sloan and Russell Sage Foundations for financial support. We are also grateful to David Cutler, James Poterba, and especially the referee from the *Journal of Business* for helpful comments.

(*Journal of Business*, 1991, vol. 64, no. 1)

0021-9398/91/6401-0004$01.50

and Kyle 1986) simply assume an outside source of new noise traders and do not deal with their performance over time.

In an earlier paper (De Long, Shleifer, Summers, and Waldmann 1990) we questioned the presumption that traders who misperceive returns do not survive. Since noise traders who are on average bullish bear more risk than do investors holding rational expectations, as long as the market rewards risk taking such noise traders can earn a higher expected return even though they buy high and sell low on average. The relevant risk need not even be fundamental: it could simply be the risk that noise traders' asset demands will become even more extreme tomorrow than they are today and bring losses to any investor betting against them. Because Friedman's argument does not take into account the possibility that some patterns of noise traders' misperceptions might lead them to take on more risk, it cannot be correct as stated.

But this objection to Friedman does not settle the matter, for expected returns are not an appropriate measure of long-run survival. Even when noise traders have a higher expected wealth because they take on more risk, they might end up bankrupt with high probability and extremely wealthy with low probability (Samuelson 1971, 1977). To adequately analyze whether noise traders are likely to persist in an asset market, one must describe the long-run distribution of their and rational investors' wealth, not just the level of expected returns.

In this article we take a first step in considering the long-run distribution of wealth and examine a model in which noise traders do not affect prices. If they did affect prices, the returns on assets would depend on the distribution of wealth between noise traders and rational investors. This added complication would make obtaining analytical solutions for our model very difficult. The assumption that noise traders do not affect prices enables us to deal with the implications of their misperceptions for the long-run distribution of their wealth rather than just for expected returns, but not with Friedman's concern that noise traders buy high and sell low. Strictly speaking, we provide comparative statics results for long-run wealth distributions taking prices as given.

To describe the long-run evolution of rational investor and noise trader wealth, we adopt the following definitions of "survival" and "dominance."

Survival. A given group of investors x "survives in the long run" if its share of the economy's total wealth does not approach zero almost surely as time passes, that is, there are $\varepsilon_1, \varepsilon_2 > 0$ such that for all times t:

$$\text{prob}\,\{\omega_t^x > \varepsilon_1\} > \varepsilon_2, \tag{1}$$

where ω_t^x is the share of the economy's total wealth at time t that belongs to investor group x.

Dominance. A given group of investors x "dominates" another group y if after sufficient time the probability that group x has a higher share of wealth than group y is greater than one-half. That is, no matter what the initial relative wealth levels ω_0^x and ω_0^y of the two groups, there is a t_0 such that for every time $t > t_0$:

$$\text{prob}\,\{\omega_t^x > \omega_t^y\} > \frac{1}{2}. \tag{2}$$

As long as the distribution of gross returns is the same across periods and entails no risk of losing all one's wealth, (2) implies (2′):

For every positive integer n, there is a t_n such that for every time $t > t_n$:

$$\text{prob}\,\{\omega_t^x > \omega_t^y\} > \frac{n-1}{n}. \tag{2′}$$

In the context of our model, (1) and (2) hold if and only if the expected rate of change of log wealth is higher for group x than for group y. Subject to the assumptions that return distributions are unchanging and do not allow for a negative 100% return, if group x survives, then it dominates. But the distinction between the two concepts is worth preserving for situations in which return distributions do change over time.

We analyze the evolution of the wealth of noise traders and rational investors using these definitions of "survival" and "dominance" in a model with infinitely lived investors. We allow for the possibility that excessive risk taking brings noise traders virtually certain ruin. We also take account of the fact that noise traders falsely believe that they can earn excess returns, as a result overestimate their wealth, and possibly consume too much, thereby reducing their survival prospects.

In our model, noise traders falsely believe that a particular asset is mispriced and take positions in it to exploit this perceived mispricing. Because the positions they assume do not properly hedge market risk, such noise traders wind up bearing more market risk than rational investors with the same wealth and degree of risk aversion. Since the extra market risk that noise traders bear is priced, they earn a higher rate of change of log wealth than rational investors as long as investors are more risk adverse than is implied by log utility. It is well known that the closer an investor's utility is to log, the higher will be his expected rate of change of log wealth. In our model, noise traders' misperceptions make them unwittingly hold portfolios closer to those that would be held by investors with log utility and so give them a higher geometric average rate of return.

Moreover, we show that noise traders as a group might survive and come to dominate rational investors in wealth even when on average a rational investor dominates any noise trader of a fixed type in wealth. Excess consumption, excess bearing of market risk because of a failure to properly hedge, and excess bearing of idiosyncratic risk associated with individual securities that noise traders favor together impart a downward drift to each individual trader's wealth relative to that of an average rational investor. But the wealth of noise traders as a group relative to that of rational investors as a group need not tend toward zero, for the downward drift imparted by idiosyncratic risk does not affect noise traders' collective wealth. If idiosyncratic risk is large, each individual noise trader with high probability fails to survive in the market, but noise traders as a whole can nevertheless survive. Evolution may leave an ever-shrinking army of ever-richer fools who collectively dominate the market.

Since the rate at which individuals' wealth grows depends on how close their coefficient of relative risk aversion is to one—how close their preferences are to log utility—noise traders can only exhibit a faster rate of wealth accumulation if their misperceptions cause them to mimic rational investors with relative risk aversion closer to one. If, as we believe is likely, rational investors are more risk averse than log utility, noise traders can exhibit faster rates of wealth accumulation only if their misperceptions of returns lead them to hold portfolios corresponding to a greater risk tolerance. We show in this article that there is a large class of plausible misperceptions that would lead to this result.

Our model considers the long-run evolution of relative wealth in an environment in which noise traders do not affect prices. But it can also be interpreted in a context where noise traders do exert pressure on prices and thus, as Friedman indicates, buy high and sell low. As the noise trader share of wealth drops, the price pressure they exert and the degree to which they buy high and sell low drop also. For these reasons, the conditions necessary for the dominance of noise traders when they do not affect prices translate into conditions necessary for their survival (but not dominance) when they do.

Section II motivates our assumptions about noise traders' misperceptions of returns by discussing the misperceptions of subjects of psychological experiments. Section III lays out a 1-period model and calculates the expected returns earned by noise traders and rational investors. It also shows that the utility cost of being a noise trader is small. Section IV considers a dynamic model of wealth accumulation with infinitely lived rational investors and noise traders and explores noise traders' chances for long-run survival in the market. Section V reinterprets our conditions for the dominance of noise traders in the case where their trades do not affect prices as conditions for the long-run survival in the marketplace of noise traders when their trades do affect prices. Section VI concludes.

II. The Plausibility of Misperceptions

In this article we assume that noise traders are poor assessors of probability distributions, especially of variances. Moreover we assume that the misperceptions of different noise traders about a particular asset are correlated, for if all traders confused about the returns on a stock have different misperceptions, their trades will cancel out.[1] We justify this assumption by summarizing some psychological evidence on systematic judgment errors made by experimental subjects.

Experiments reveal that individuals are consistently poor assessors of probabilities. They use a variety of heuristics to estimate probabilities that can lead to biases (Tversky and Kahneman 1974) that are not random but instead correlated across subjects. People agree which particular player has a "hot hand" (Gilovich, Valone, and Tversky 1985), and they see the same nonexistent trends and patterns in artificially generated as in real stock price series (Andreassen and Kraus 1987).

We focus on one of the best documented biases: the tendency to underestimate variances and to be overconfident (Alpert and Raiffa 1959; Einhorn and Hogarth 1978; Lichtenstein, Fischoff, and Phillips 1982). Experts and novices alike are too certain about their predictions given the true odds of being wrong. Alpert and Raiffa's (1959) original finding that business school students are overconfident has been confirmed for many different populations using a variety of questions on which respondents had varying degrees of expertise (Tversky and Kahneman 1971; Slovic 1982). Central Intelligence Agency (CIA) analysts, experienced psychologists, and physicians are all overconfident. Overconfidence in the precision of one's estimate does not arise from lack of concern by experimental subjects for the accuracy of their distributions: students were more overconfident when their performance was linked to grades than when it was not. Moreover, overconfidence gets worse, not better, when the difficulty of the task increases (Langer 1975).

In addition, overconfidence in the precision of one's estimate is likely to become more extreme over time as those who succeed attribute their success to their own skill and judgment. In Langer's words, "Heads I win, tails it's chance." In asset markets, the richest individuals may well be those who placed large bets on very risky gambles and won. Their success would naturally tend to reinforce their confidence in their own hunches whether or not such confidence is justified.

This psychological literature provides suggestive hints of how noise traders might tend to behave. First, perceptions of risks and opportunities might well be strongly correlated across agents, and might depend on past patterns of prices and volume in not very rational ways. Second, noise traders might fail to accurately assess expected returns—although it is hard to predict in what direction any systematic bias

1. Pagano (1989) studies thin markets where, even though noise traders' misperceptions are uncorrelated, their trades need not cancel out. We assume that markets are thick enough that the law of large numbers applies.

might lie. Third, and most important, no matter what return they expect, many investors are likely to be overconfident. They are likely both to have hunches and to underestimate the risk that they are assuming when they choose portfolios based on these hunches. The following two sections demonstrate that "overconfidence" in this sense is likely to make investors bear more systematic risk than they desire and to give them a higher geometric mean rate of return.

III. A One-Period Model

This section develops a 1-period model that serves as the basis for the multiperiod, infinite-horizon model considered in Section IV. We first present our assumptions about noise traders' beliefs. We then compute the distributions of both rational investors' and noise traders' wealth as a function of each type's perceptions of asset returns. We show that noise traders earn higher expected returns than rational investors for a large set of possible misperceptions.

Assumptions of the Model

Investment opportunities consist of one safe asset paying a known gross return, $(1 + r)$, and a continuum of risky assets indexed by i in the interval [0, 1]. The return on the risky asset i is

$$R_i = \rho + \eta + \varepsilon_i, \tag{3}$$

where ρ is the average dividend paid on all risky assets, and η and ε_i are uncorrelated mean-zero random variables satisfying $E(\eta) = E(\varepsilon_i) = 0$, $E(\eta^2) = \sigma_\eta{}^2$, and $E(\varepsilon_i^2) = \sigma_i^2$. Under these assumptions all assets have a market β of one. This simplifies the algebra without loss of generality. Returns are assumed to be exogenous, with no investor having an effect on the price of any risky asset. The supply of assets is thus assumed to be infinitely elastic.

In this section, we focus on a single type of noise trader who misperceives the return distribution of a single risky asset i. In Section IV we consider a continuum of types of noise traders, with each type misperceiving the return distribution on only one of the continuum of risky assets. We index noise trader types by the same i that indexes risky assets. Noise traders of type i correctly perceive the distribution of returns of every asset except i, but they falsely believe that the distribution of asset i's net returns is given not by (3) but by

$$(\hat{R})_i = r + \mu(\rho - r) + \tau(\eta + \varepsilon_i), \tag{4}$$

for some parameters μ and τ. A caret (^) above a variable denotes the noise traders' perception of the variable.

The parameters μ and τ allow noise traders to have different misperceptions of the mean and variance of the returns on asset i. A noise trader's μ describes his opinion about the mean return on asset i. If $\mu \neq 1$, then noise traders of type i misperceive the expected return on asset i:

$$E(\hat{R})_i = \mu(\rho - r) + r \neq E(R_i) = \rho. \tag{5}$$

If μ is greater (less) than one, then noise traders overestimate (underestimate) asset i's expected return. The parameter τ describes the opinion about the standard deviation of the return on asset i. If $\tau \neq 1$, then noise traders misperceive both asset i's idiosyncratic variance and its market β:

$$(\hat{\sigma})_i^2 = \tau^2\sigma_i^2 \neq \sigma_i^2, \tag{6}$$

$$(\hat{\beta})_i = \tau \neq 1. \tag{7}$$

Note that noise traders have the same misperception of each component of the variance of the return on asset i.

Given his own perception of the distribution of returns, each investor maximizes:

$$E(U) = E(W_1) - \frac{\gamma}{2W_0(1 + r)}\sigma_w^2, \tag{8}$$

where W_1 is the wealth of the investor at the end of the period, σ_w^2 is its variance, W_0 is the investor's initial wealth, γ is the coefficient of relative risk aversion, and expectations are taken using each investor's own beliefs. The investor's local degree of absolute risk aversion is inversely proportional to his expected end-of-period wealth that appears in the denominator of (8). In continuous time, maximizing (8) is equivalent to maximizing a constant relative risk aversion utility function. As long as both mean excess returns $(\rho - r)$ and variances are small, and excess returns are not large relative to variances, (8) is a good approximation to constant relative risk aversion utility.

Because noise traders affect asset quantities but not prices, we can calculate the equilibrium portfolio allocations of noise traders and rational investors separately. Rational investors maximizing (8) hold equal infinitesimal amounts of each risky asset to avoid idiosyncratic risk. They therefore invest a share of their wealth $\alpha(1 + \mathrm{r})$ in the equally weighted market portfolio of risky assets, where

$$\alpha = \frac{\rho - r}{\gamma\sigma_\eta^2}. \tag{9}$$

Rational investors invest the rest of their wealth in the riskless asset.

Noise traders do not confine their investments to positions in the riskless asset and the diversified equally weighted risky market portfolio. They also perceive an additional investment opportunity in asset i. Because noise traders believe that asset i is mispriced, they choose to hold it in a proportion different from its infinitesimal share of the risky market portfolio. This perceived mispricing of asset i does not, however, make noise traders wish to hold a different amount of the common risk factor η. Noise traders hedge their holdings of i using the market portfolio so that they (falsely) believe that their additional investment in i has no effect on their exposure to aggregate market risk.

The net result is that noise traders' portfolios are made up of three pieces. The first is their investment of $\alpha(1 + r)W_0$ in the equally weighted risky market portfolio, which is identical to the risky market holdings of rational investors. The second is their holding of the riskless asset. The third is their investment in the perceived zero-β portfolio (henceforth PZBP) for asset i. A unit of this PZBP consists of a position long one unit of asset i and short τ units of the market. This unit has a net cost of $(1 - \tau)$, carries what noise traders believe to be no exposure to market risk, carries in fact unit exposure and in noise traders' opinion τ exposure to the idiosyncratic risk ϵ_i, and has in noise traders' estimation a nonzero expected return.

Since its true β is not zero, the PZBP actually earns an excess expected return relative to the riskless rate:

$$R_i^z - r = (1 - \tau)(\rho - r + \eta) + \epsilon_i, \tag{10}$$

But noise traders (falsely) believe that this PZBP has a different excess return that arises not from its covariance with the market but from their false perception that asset i is mispriced. Noise traders believe this excess return on the PZBP will on average be

$$(\hat{R})_i - r = (\mu - \tau)(\rho - r) + \tau\epsilon_i. \tag{11}$$

If $\mu - \tau > 0$, then noise traders believe that asset i is underpriced and so they go long its PZBP, and its PZBP has a positive β if $\tau < 1$. If $\mu - \tau < 0$, then noise traders think that asset i is overpriced and they sell short its PZBP, and its PZBP has a negative β if $\tau > 1$. Noise traders (falsely) believe that the PZBP has idiosyncratic but no market risk, and so they hold a quantity $\lambda_i(1 + r)W_0$ of the PZBP, where

$$\lambda_i = \frac{(\mu - \tau)(\rho - r)}{\gamma\tau^2\sigma_i^2}. \tag{12}$$

The difference between noise traders' and rational investors' share of wealth held in the riskless asset is $(1 - \tau)(1 + r)\lambda_i$.

The Difference in Expected Returns

Given these holdings, the expected end-of-period wealth of a noise trader of type i is

$$E(W_1^n) = W_0^n(1 + r)\left\{1 + \frac{(\rho - r)^2}{\gamma\sigma_\eta^2} + \frac{(1 - \tau)(\mu - \tau)(\rho - r)^2}{\gamma\tau^2\sigma_i^2}\right\}. \tag{13}$$

The expected end-of-period wealth of a rational investor is

$$E(W_1^s) = W_0^s(1 + r)\left\{ 1 + \frac{(\rho - r)^2}{\gamma\sigma_\eta^2} \right\}. \tag{14}$$

The first term inside the brackets in (13) and (14) captures the return all market participants would earn if the safe asset provided their sole investment opportunity. The second term captures the return everyone earns because they can invest in the risky market as well as in the riskless asset. The third term captures the difference in expected return that the noise traders earn because they misperceive the distribution of returns on asset i, take a nonzero position in asset i's PZBP, and so bear a different amount of market risk than they intend.

If $\tau = 1$—if noise traders perceive β_i correctly whether or not they misperceive the mean return on asset i—then noise traders earn the same expected return as do rational investors because the true expected excess return on the PZBP of asset i is zero. Noise traders do, however, bear a positive amount of asset i's idiosyncratic risk and as a result hold inefficient portfolios.

If $\mu = \tau$, then expected returns are again equal. Noise traders hold the same portfolio as rational investors because noise traders believe that asset i is correctly priced. Because their belief that it has a larger β offsets their perception of its higher excess return, they do not hold any of asset i's PZBP.

If $\mu = 1$ and $\tau \neq 1$—if noise traders correctly perceive the mean return on asset i but misperceive the variance—then they always earn a higher expected return. In this case noise traders necessarily hold portfolios that carry a larger degree of systematic risk than do rational investors. If noise traders underestimate β_i, they think that asset i is underpriced, go long its PZBP, and so hold more of the risky market than do rational investors because their underestimation of β_i gives the PZBP a positive covariance with the market. If noise traders overestimate β_i, they think asset i is overpriced, sell short its PZBP, and as a result hold more of the risky market than do rational investors because their overestimation of β_i gives their PZBP a negative covariance with the market.

PROPOSITION 1. A noise trader who misperceives only the variance of returns on a single risky asset earns higher expected returns than does a rational investor.

Proof. By inspection of equations (13) and (14).

Note that both overconfident and underconfident noise traders can earn higher expected returns. As we suggested in Section II, overconfidence, meaning $\tau < 1$, is likely to be the more important case empirically. In addition, if there are restrictions on short sales, then underconfident noise traders find themselves unable to hold their optimal portfolio since it involves selling asset i short. By contrast, overconfi-

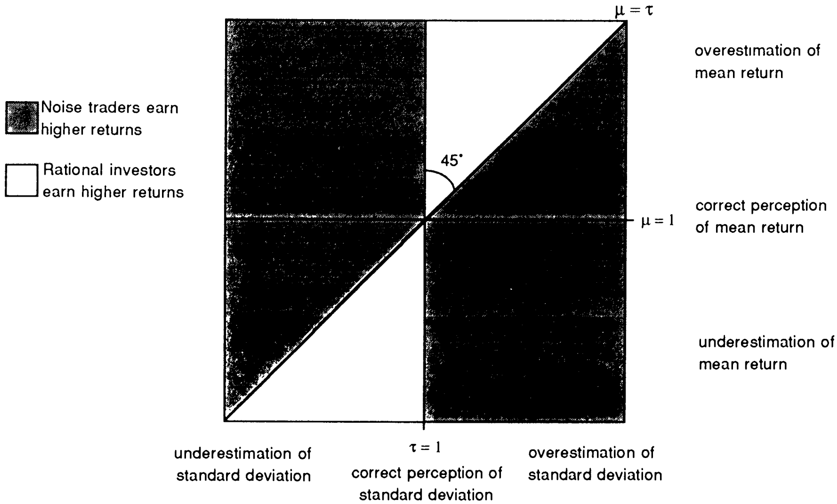

FIG. 1.—Area of (μ, τ) space where noise traders earn higher returns than rational investors.

dent investors need only to buy asset *i* and reduce their holdings of the market, without actually selling the market short for τ close to one.

Equations (13) and (14) reveal that noise traders earn lower expected returns than do rational investors only if $(1 - \tau)(\mu - \tau)$ is negative. This will hold only if the misperception of the mean return is in the same direction as, and greater than, the misperception of the standard deviation of returns. Misperceptions of mean returns associated with the same degree of misperceptions of standard deviations—if, say, investors overestimate the entire return distribution by a constant proportion—will not lead noise traders to receive lower expected returns.

Figure 1 suggests that noise traders may well earn higher expected returns, for they do so on three-fourths of the plane in (μ, τ) space. If noise traders' μ's and τ's are randomly and symmetrically distributed around (1, 1), then the probability that a given noise trader earns a higher expected return is three-fourths. The empirical finding of widespread overconfidence suggests that people are likely to underestimate standard deviations by more than they underestimate means.

IV. A Multiperiod Model

We assume that both noise traders and rational investors have infinite-horizon constant relative risk aversion utility functions and optimally choose their consumption and investment plans given their beliefs. We assume that noise traders of type *i* continue to misperceive the returns on asset *i* by the same amount in every period: they do not learn from their mistakes. We assume for simplicity that noise traders correctly perceive the means of all return distributions ($\mu = 1$). This assumption greatly simplifies the algebra and reflects our lack of evidence on the sign of μ. In this framework we consider the evolution of the wealth of a continuum of noise traders, where each noise trader misperceives the return distribution on a different asset i.

Even if noise traders earn higher expected returns in every single period, they might not come to dominate the market with high probability in the long run. Three factors keep higher expected returns from translating immediately into a higher share of long-run wealth. First, noise traders who (falsely) believe they have a profit-making trading opportunity overestimate their permanent income and as a result consume too much. This slows down their wealth accumulation.

Second, having a higher period-by-period expected return is not identical to long-run dominance in wealth. As the time horizon increases, the distribution of the average per-period gross return earned by an investor who places constant wealth shares in different assets approaches log normal and is thus highly skewed. With a high probability noise traders might then become poorer than rational investors, but with a low probability they might become vastly richer. Noise traders' wealth share might asymptotically approach zero with probability one—they might fail to "survive" in the market on our definition—even if they have a higher expected wealth (Samuelson 1971).

Last, each individual type of noise trader holds an inefficient portfolio. Noise traders of type i bear a finite amount of idiosyncratic risk of asset i, and so their portfolios have more variance than necessary to attain their actual level of expected returns. This risk further increases the variance of noise traders' returns and so leaves them with an even smaller probability of having a high relative wealth share. We analyze the evolution of noise traders' wealth taking into account these three factors by first showing that idiosyncratic risk reduces the survival probabilities of individual noise traders but not of noise traders as a whole, then embedding the 1-period model of the previous section in an infinite-period context and considering how the skewness of the distribution of expected returns affects noise traders' survival prospects, and, last, analyzing how excess consumption impedes noise traders' wealth accumulation. We thus arrive at conditions for the long-run survival and dominance of noise traders.

Noise Traders' Individual and Aggregate Wealth

The extra risk imparted by the inefficiency of noise traders' portfolios is eliminated if we examine the total wealth of a group of noise traders with misperceptions distributed over different stocks. If noise traders of each type i misperceive the variance of stock i by the same τ, then noise traders as a whole bear no idiosyncratic risk and hold an efficient portfolio.

The proportional (prop.) 1-period variance of noise trader i's wealth is affected by his exposure to the idiosyncratic risk of asset i. Approximating expected end-of-period wealth by $(1 + r) W_0$, where W_0 is beginning-of-period wealth:

$$\begin{aligned}\text{prop. var}_i^n = {} & \frac{(\rho - r)^2}{\gamma^2 \sigma_\eta^2} + \frac{2(1 - \tau)^2 (\rho - r)^2}{\gamma^2 \tau^2 \sigma_i^2} \\ & + \frac{(1 - \tau)^4 (\rho - r)^2 \sigma_\eta^2}{\gamma^2 \tau^4 (\sigma_i^2)^2} + \frac{(1 - \tau)^2 (\rho - r)^2}{\gamma^2 \tau^4 \sigma_i^2} .\end{aligned} \tag{15}$$

The first term on the right-hand side of (15) is the variance arising from the systematic risk that rational investors bear and that noise traders believe that they themselves bear. The second and third terms arise from the added systematic risk borne by noise traders because the PZBP they hold does not actually have zero β. The fourth term arises from the idiosyncratic risk of stock i borne by noise traders.

The last term, however, disappears from the expression for the variance of returns earned by noise traders as a whole. As far as the wealth of noise traders as a group is concerned, the "consumption" and "systematic variance" effects are the only ones that drive a wedge between having a higher expected return and coming to dominate the market. Noise traders as a group might then survive in the marketplace, although the wealth share of a randomly selected noise trader type eventually falls with probability one, for the wealth of a small fraction of the noise trader population is increasing fast enough to give them a rising aggregate share of the company's wealth.

Distinguishing between High Expected Returns and Dominance

For the moment, we neglect consumption and consider only the returns on noise traders' and rational investors' portfolios. Assume that investors live forever, face an unchanging distribution of period-by-period returns, and exhibit constant relative risk aversion. Such investors devote the same portfolio share to a given asset each period. Their wealth is multiplied by an independently and identically distributed (i.i.d.) random variable $(1 + R_t)$ each period. Taking logs, the random variable $\ln(1 + R_t)$ is added to the *log* of wealth each period. The law of large numbers tells us that the average expected rate of change, g, of log wealth is

$$g = E[\ln(1 + R_t)] \approx \ln(1 + r) + E(R_t - r) - \frac{V(R_t)}{2} , \tag{16}$$

taking a second-order Taylor approximation around $1 + r$, and where $V(R_t)$ is the 1-period proportional variance of wealth as in (15).

To evaluate the relative survival chances of noise traders and rational investors, we therefore consider the difference in their expected rates of change of log wealth, which is approximately equal to

$$E(R^s - R^n) - \frac{V^s - V^n}{2}, \tag{17}$$

where V is the 1-period-ahead proportional systematic variance of each type's holdings. The second "drift" term reflects the likelihood that agents whose returns have a higher variance end up with lower wealth. Occasional large negative realization of returns decrease such investors' capital bases and reduce the future absolute change in their wealth so much that they might eventually have lower total wealth even if they earn higher period-by-period expected returns. As a result, investors for whom a larger drift outweighs their advantage of a higher expected return neither survive nor dominate the market in terms of wealth. If examined after a sufficiently long time interval, in an overwhelming proportion of cases investors with a higher expected rate of change of log wealth are richer.

The difference between the expected changes of log wealth for noise traders and rational investors considered as groups is

$$\frac{(1 - \tau)^2 (\rho - r)^2}{\gamma \tau^2 \sigma_i^2} \left\{ 1 - \frac{1}{\gamma} - \frac{(1 - \tau)^2 \sigma_\eta^2}{2\gamma \tau^2 \sigma_i^2} \right\}. \tag{18}$$

Because the leading common factor is always positive, the sign of the difference in expected log wealth changes depends on the terms inside the brackets. The leading "1" inside the brackets reflects the greater expected returns that noise traders earn because of their unwittingly greater exposure to systematic risk. The second and third terms capture the increase in aggregate return variance that this exposure entails.

The third term is small if τ is close to one. In this case as long as investors are more risk averse than investors with logarithmic utility (have $\gamma > 1$), considered as a group noise traders who misperceive variances survive and come to dominate the market. For each $\gamma > 1$ there is some $\delta > 0$ such that, if $|\tau - 1| < \delta$, then noise traders as a group have a higher expected change in log wealth. Such noise traders are confused about variances, but their confusion is sufficiently small so that the higher expected return more than outweighs the larger drift induced by the greater variance.

Note, however, that, if $\gamma \leq 1$, there is no misperception of the variance of the return distribution that delivers a higher expected change in log wealth for noise traders. This point is equivalent to the observation that an investor wishing to maximize the long-run average rate of return earned on his portfolio should choose portfolio shares as if he had logarithmic utility, that is, $\gamma = 1$ (Samuelson 1971). An investor with $\gamma > 1$ does not choose such a portfolio because he is sufficiently averse to low wealth realizations to forgo at least some long-run expected return in order to reduce risk. This implies that all investors who take a position that bears marginally more systematic risk—even by mistake, as in the case of noise traders—have a higher expected change in log wealth and therefore come to dominate rational investors in wealth. Another implication of this is that no type of noise trader can dominate a rational investor with logarithmic utility.

Conversely, an investor with $\gamma < 1$ bears too much risk to maximize the long-run average rate of return on his portfolio. He values the occasional high realization of wealth enough to accept a substantial chance of having low wealth generated by a risky portfolio. Such an investor could increase his long-run average rate of return and improve his survival potential by reducing his holdings of the risky asset. We find the case $\gamma < 1$ unattractive because investors less risk averse than log utility fall victim to the St. Petersburg paradox (Samuelson 1977). Such investors are willing to pay an infinite amount for a gamble that pays zero with a probability arbitrarily close to one and pays finite amounts in every state of the world. As we discuss below, γ less than one is not an empirically plausible case.

Consumption

We now turn to the effects of noise traders' misperceptions on their consumption. If investors live forever, maximize the same approximation to a constant relative risk aversion utility function as in Section III, and face an unchanging distribution of returns, then their consumption is given by

$$c_t = W_t \left\{ \frac{\gamma - 1}{\gamma} \left[E(R_{t+1}) - \frac{\gamma V(R_{t+1})}{2} \right] + \frac{\delta}{\gamma} \right\}, \tag{19}$$

where expectations are taken with respect to the perceived distribution of returns (Merton 1969), and where δ is the subjective rate of time preference. Since all noise traders consume the same fraction of wealth, aggregation causes no problems.

Noise traders in this case consume more than do rational investors. They (falsely) believe that their portfolios have a risk-adjusted net rate of return higher than those of rational investors by

$$\lambda_i^2 \tau^2 \sigma_i^2 = \frac{(1 - \tau)^2 (\rho - r)^2}{2\gamma\tau^2\sigma_i^2}. \tag{20}$$

Noise traders' misperceptions lead them to consume a fraction of their wealth higher than that consumed by rational investors by

$$\frac{c^n}{W^n} - \frac{c^s}{W^s} = \frac{(\gamma - 1)(1 - \tau)^2(\rho - r)^2}{2\gamma^2\tau^2\sigma_i^2}. \tag{21}$$

Conditions for the Long-Run Survival of Noise Traders

Combining (21) with (18) gives an expression for the difference of the expected changes of noise traders' and rational investors' log aggregate wealth. The set of investors with the higher geometric mean growth rate both survives in the market and comes to dominate in wealth. Noise traders come to dominate the market in the long run if

$$\frac{(1-\tau)^2(\rho-r)^2}{\gamma\tau^2\sigma_i^2}\left\{1-\left[\frac{1}{\gamma}+\frac{(1-\tau)^2}{2\gamma\tau^2}\left(\frac{\sigma_\eta^2}{\sigma_i^2}\right)\right]-\frac{\gamma-1}{2\gamma}\right\}>0. \quad (22)$$

Leaving aside the positive common factor, this expression consists of three pieces. The leading "1" reflects the higher expected returns earned by noise traders. The final piece arises from noise traders' excess consumption. Note that the additional consumption effect is always outweighed by the extra return effect. Even though noise traders consume a higher fraction of their wealth than do rational investors, the average rate of wealth accumulation of noise traders who misperceive variances alone is always higher than that of rational investors.

The middle two terms of (22) reflect the downward relative drift imposed on the geometric mean of noise traders' relative returns by the extra variance of their portfolios. Examination of (22) reveals that when $\gamma > 1$ noise traders with small misperceptions survive and come to dominate the market. We can further simplify (22) to

$$(\gamma-1)-\left(\frac{1-\tau}{\tau}\right)^2\left\{\frac{\sigma_\eta^2}{\sigma_i^2}\right\}>0. \quad (23)$$

By inspection, if $\gamma > 1$, equation (23) holds for τ sufficiently close to one. Moreover, if idiosyncratic risk is large relative to market risk, then (23) holds as long as τ is not too close to zero. Only noise traders whose misperceptions of returns are truly extraordinary would then fail to dominate the market. Equations (22) and (23) demonstrate that for all parameters of the return distribution, there are plausible misperceptions by noise traders that allow them to survive and to dominate the market.

PROPOSITION 2. For any parameters of the return distribution there exists a δ such that, if $|\tau - 1| < \delta$, then noise traders who misperceive variances with parameter t survive and come to dominate the market.

Proof. By inspection of equation (23).

Not only are noise traders who misperceive variances by a small amount and are more risk averse than log utility guaranteed to survive in the market, but there are many types of noise traders who misperceive variances by a large amount and yet exhibit a faster degree of wealth accumulation than rational investors.

A simple calibration exercise demonstrates the empirical plausibility of this result. Estimates of investors' degree of relative risk aversion rarely fall below two and are often above 10. The highest coefficients come from studies that try to reconcile the large risk premia observed on assets with the relatively smooth growth path of consumption (Mehra and Prescott 1985; Hall 1988; and Weil 1989). Weil in particular finds that representative investor models fit the observed equity premium only for coefficients of relative risk aversion of 20 or more.[2] Similarly, Hall (1988) finds that the coefficient of relative risk aversion, which in his model is equal to the intertemporal elasticity of substitution in consumption, is greater than 10. Estimates of relative risk aversion from individual data are smaller, but they are still greater than one. Friend and Blume (1975) find using individual household data on portfolio holdings that the degree of relative risk aversion is greater than two. Gertner (1990) cites a number of studies that generate estimates of the coefficient of relative risk aversion between 1.5 and 15.

In order to calibrate σ_i^2/σ_η^2, note that market-model regressions of individual security returns on market returns usually produce R^2's of 0.1 or so, and rarely produce R^2's greater than 0.2 (Roll 1988). An R^2 of 0.2 corresponds to a σ_i^2/σ_η^2 of 5.

Using these parameter values to calibrate equation (23) reveals that even noise traders who are grossly overconfident will as a group exhibit higher rates of wealth accumulation. For $\gamma = 3$ and $\sigma_i^2/\sigma_\eta^2 = 5$, equation (23) implies that noise traders who misperceive standard deviations and have any $\tau > 0.24$—believe that the variance of returns on the assets they favor are not less than 6% of the true variance—will exhibit higher rates of wealth accumulation. Even for $\gamma = 2$ and $\sigma_i/\sigma_\eta = 1$, noise traders will exhibit faster rates of wealth accumulation if they have a $\tau > 1/2$.

We conclude that there is, in fact, a presumption that overconfident investors—even grossly overconfident investors—will tend to control a higher proportion of the wealth invested in securities markets as time passes. This presumption is based on the empirical observations that (*a*) most investors appear to be more risk averse than log utility, and (*b*) idiosyncratic risk is large relative to systematic risk. Under these conditions, investors who are mistaken about the precision of their estimate of the returns expected from a particular stock will end up taking on more systematic risk. Taken as a group, these investors will exhibit faster rates of wealth accumulation than fully rational investors with risk aversion greater than that given by log utility.

2. Although, as he points out, such models are then unable to fit the relatively low realized safe real rate of return.

V. Invasion

We have interpreted our model as a model of the long-run survival and dominance of noise traders in an environment where they do not affect prices. Our results also apply, with a more restricted interpretation, to models in which noise traders exert price pressure and distort prices against themselves. Our conditions sufficient for the dominance of noise traders in a model in which they do not affect prices have another interpretation in a model in which noise traders do affect prices as conditions sufficient for noise traders to be able to successfully invade the economy, in the sense that a small group of noise traders introduced into the economy will find that their wealth share tends to grow, not shrink, over time. Our sufficient conditions can further be interpreted as conditions sufficient for noise traders to survive in the long run, in the sense of having a share of the economy's wealth that is with finite probability bounded away from zero for all time.

When noise traders have an infinitesimal share of wealth, they distort prices and returns only an infinitesimal amount away from fundamental values. Hence, if noise traders have a higher average rate of wealth accumulation, then they can "invade" even if they distort prices. If their wealth share is infinitesimal, noise traders will exert negligible price pressure and so their wealth share will tend to grow. Noise traders therefore survive, in the sense that their wealth share does not drop toward zero in the long run with probability one. Our analysis of the long-run tendency of the noise trader share in models where they do affect prices is limited to these statements about "invasion" and "survival." We can make no statements about the conditions for noise trader dominance in this context because as soon as they acquire a nontrivial share of wealth they begin to affect prices in a nontrivial way, and our model and analysis no longer apply.

These results imply that a population composed entirely of rational investors is not "evolutionarily stable" (Maynard Smith 1982). If a small number of noise traders are introduced into the population, their relative wealth tends to grow. Noise traders can successfully "invade" the population. In a world in which investors occasionally "mutated" and changed from noise trader to rational investor or vice versa, it would be surprising to find a population composed almost entirely of rational investors.

VI. Conclusion

We have presented a model of portfolio allocation by noise traders who form incorrect expectations chiefly about the variance of the return distribution of a particular asset. We showed that, for plausible misperceptions, such noise traders as a group can not only earn higher returns than do rational investors but also can survive and dominate the market in terms of wealth in the long run. Such long-run success of noise traders occurs despite their excessive risk taking and excessive consumption. The case against their long-run viability is by no means as clear-cut as is commonly supposed.

The main limitation of our model is that it does not allow noise traders to affect prices. This article therefore cannot address Friedman's main point: that noise traders buy high and sell low. In our earlier paper (De Long, Shleifer, Summers, and Waldmann 1990) we have shown that noise traders can earn higher *expected* returns than rational investors even when they buy high and sell low. But the model of our earlier paper could not deal with survival and dominance because the wealth of investors was held fixed. The next step in this literature, then, is to arrive at a tractable model in which noise traders affect prices and in which survival and dominance can be analyzed. The answers afforded by such a model would go a long way toward settling the question raised by Friedman (1953).

References

Alpert, M., and Raiffa, H. 1959. A progress report on the training of probability assessors. Reprinted in D. Kahneman, P. Slovic, and A. Tversky (eds.), *Judgment under Uncertainty: Heuristics and Biases*. Cambridge: Cambridge University Press, 1982.

Andreassen, P., and Kraus, S. 1987. Intuitive prediction by extrapolation: The stock market, exponential growth, divorce, and the salience of change. Mimeographed. Cambridge, Mass.: Harvard University.

Black, F. 1986. Noise. *Journal of Finance Proceedings* 41 (July): 529–43.

Campbell, J. Y., and Kyle, A. S. 1986. Smart money, noise trading, and stock price behavior. Mimeographed. Princeton, N.J.: Princeton University.

De Long, J. B.; Shleifer, A. M.; Summers, L. H.; and Waldmann, R. J. 1990. Noise trader risk in financial markets. *Journal of Political Economy* 98 (August): 703–38.

Einhorn, H., and Hogarth, R. 1978. Confidence in judgment: Persistence of the illusion of validity. *Psychological Review* 85 (September): 395–416.

Figlewski, S. 1979. Market "efficiency" in a market with heterogeneous information. *Journal of Political Economy* 86 (August): 581–97.

Friedman, M. 1953. The case for flexible exchange rates. In M. Friedman, *Essays in Positive Economics*. Chicago: University of Chicago Press.

Friend, I., and Blume, M. 1975. The demand for risky assets. *American Economic Review* 65 (December): 900–922.

Gertner, R. 1990. Game shows and economic behavior: Risk taking on *Card Sharks*. Mimeographed. Chicago: University of Chicago, Graduate School of Business.

Gilovich, T.; Valone, R.; and Tversky, A. 1985. The hot hand in basketball: On the misperception of random sequences. *Cognitive Psychology* 17 (July): 295–314.

Hall, R. 1988. Intertemporal substitution in consumption. *Journal of Political Economy* 96 (April): 339–57.

Kahneman, D.; Slovic, P.; and Tversky, A. (eds.). 1982. *Judgment under Uncertainty: Heuristics and Biases*. Cambridge: Cambridge University Press.

Kahneman, D., and Tversky, A. 1973. On the psychology of prediction. *Psychological Review* 80 (July): 237–51.

Kyle, A. S. 1985. Continuous auctions and insider trading. *Econometrica* 53 (November): 1315–36.

Langer, E. J. (1975). The illusion of control. *Journal of Personality and Social Psychology* 32 (August): 311–28.
Lease, R. C.; Lewellen, W. G.; and Schlarbaum, G. G. 1974. The individual investor: Attributes and attitudes. *Journal of Finance* 29 (May): 413–33.
Lichtenstein, S.; Fischhoff, B.; and Phillips, L. 1982. Calibration of probabilities: The state of the art as of 1980. In D. Kahneman, P. Slovic, and A. Tversky (eds.), *Judgment under Uncertainty: Heuristics and Biases*. Cambridge: Cambridge University Press, 1982.
Maynard Smith, J. 1982. *Evolution and the Theory of Games*. Cambridge: Cambridge University Press.
Mehra, R., and Prescott, E. 1985. The equity premium: A puzzle. *Journal of Monetary Economics* 15 (March): 145–61.
Merton, R. 1969. Optimal portfolio selection by dynamic stochastic programming. *Review of Economics and Statistics* 51 (August): 247–57.
Pagano, M. 1989. Trading volume and asset liquidity. *Quarterly Journal of Economics* 104 (May): 255–74.
Roll, R. 1988. R^2. *Journal of Finance Proceedings* 43 (July): 541–66.
Samuelson, P. A. 1971. The "fallacy" of maximizing the geometric mean in long sequences of investing or gambling. *Proceedings of the National Academy of Sciences* 68 (October): 2493–96.
Samuelson, P. A. 1977. St. Petersburg paradoxes. *Journal of Economic Literature* 15 (March): 24–55.
Shiller, R. 1984. Stock prices and social dynamics. *Brookings Papers on Economic Activity*, no. 2 (Fall), pp. 457–98.
Slovic, P. 1982. Toward understanding and improving decisions. In W. Howell (ed.), *Human Performance and Productivity*. Hillsdale, N.J.: Erlbaum Press.
Stein, J. 1987. Informational externalities and welfare-reducing speculation. *Journal of Political Economy* 95 (December): 1123–45.
Tversky, A., and Kahneman, D. 1971. The belief in the "law of small numbers." *Psychological Bulletin* 76 (August): 105–10.
Tversky, A., and Kahneman, D. 1974. Judgment under uncertainty: Heuristics and biases. *Science* 185 (September 27): 1124–31.
Weil, P. 1989. The equity premium puzzle and the risk-free rate puzzle. *Journal of Monetary Economics* 24 (November): 401–21.