AM108 1:00pm, Class 01

1. Introduce yourself to your teammates.

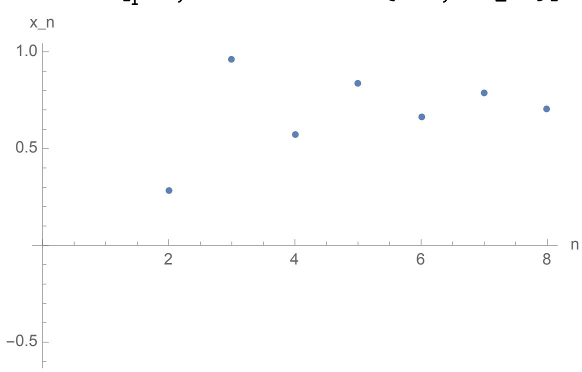
> **Solution:** NA. (Introduce yourself every time you're on a new team.)

2. Let $x \mapsto \cos x$. ($x \in \mathbb{R}$ defines the *state* of the system).

   (a) Select a starting point and try iterating this map. *You may use a calculator to do this exactly or a graph of* $\cos x$ *to do this approximately.* Plot a time series of the iterates. What happens?

Solution:

```
pts = {};
x1 = -5;
Do[AppendTo[pts, x0];
 x1 = Cos[x1], {8}]
ListPlot[pts, AxesLabel -> {"n", "x_n"}]
```



One set of iterates is shown (starting from $x_1 = -5$). These converge towards the value of x where $x = \cos x$:

```
FindRoot[x == Cos[x], {x, 0.75} ]
{x -> 0.739085}
```

   (b) How does your starting value of x matter?

Solution:
For any starting value $x \in \mathbb{R}$, the first iterate is $\cos(x) \in [-1, 1]$. Iterating again, we have $\cos(\cos(x)) \in [\cos(1) \approx 0.540302, 1]$. This means

$$\cos(\cos(\cos(x))) \in [\cos(1) \approx 0.540302, \cos(\cos(1)) \approx 0.857553].$$

With each iterate the range of possible values is narrowing. For all initial values $x \in \mathbb{R}$ we land in the range $[\approx 0.54, \approx 0.86]$ after just 3 iterates. Continuing with the process, they would all converge towards that same value of x. We haven't quite shown that, though.

1. To model population, let $x_{n+1} = ax_n$ where x_n is the population at time n and x_{n+1} is the population at time $n + 1$, one year later. This is the map

$$x \to ax.$$

Assume $a \in \mathbb{R}$ with $a > 1$.

- If $x_0 = b$ with $b \in (0, \infty)$, find a formula for x_n.
- What will happen to x_n at long times?

- Critique this as a population model. When could you imagine it might work and when would it not?
- Now change the constraint on a. Let $a \in \mathbb{R}$. What different behavior do you see as you change a? Describe all of the possibilities.
- Which values of a might be more or less appropriate for a population model? Justify your answer.

2. Now we will switch to a continuous population model. Instead of using discrete generations, let the population grow continuously at a rate α.

$$\frac{dN}{dt} = \alpha N$$

describes the change in population over time. Note: We will often write $\dot{N}$ in place of $\frac{dN}{dt}$.

- Describe this equation: linear/nonlinear, autonomous/nonautonomous, order of the equation.
- Show that $N(t) = N_0 e^{\alpha t}$ is a solution of this differential equation, and graph a time series of this solution for a few values of N_0 and α. *To show that an expression is a solution to an equation, you can plug the expression in and show that the equation then holds.*
- We can approximate this continuous model (with substantial error) by the discrete model $N_{n+1} = N_n + \alpha \Delta t N_n$. Relate the value of α in the continuous model to the value of a in our previous model.
- Describe how this discrete model approximates the continuous model.

Note: When we approximate the solution of a differential equation numerically via a computer we always work with a discrete approximation of the differential equation. The particularly approximation is called *forward Euler* and is rarely used.

3. The simple population model above had some limitations. Now consider

$$\frac{dN}{dt} = \alpha N(1 - N/K)$$

with $\alpha, K > 0$. This is called the *logistic equation*.

- Describe this equation: linear/nonlinear, autonomous/nonautonomous, order of the equation.
- Graph $\frac{dN}{dt}$ as a function of N. Mark K and α on your axes.
- When $N \in (0, K)$ is the population increasing over time or decreasing over time? How do you know?
- When $N > K$ is the population increasing over time or decreasing over time?
- What happens when $N = 0$ or when $N = K$?
- Assume the population starts at time 0 at $N = 2K$. Describe what happens to the population over time.
- Assume the population starts at time 0 at $N = K/2$. Describe what happens to the population over time.
- Describe what you think might be the general long term behavior of the population under this model.
- Critique this model as a population model. What do you like about it? What are some of its limitations?

4. Separation of variables for solving (nonlinear) differential equations.

- Consider $\frac{dx}{dt} = \frac{\cos x}{\sin x}$. To find solution functions $x(t)$:

$$\frac{\sin x}{\cos x}\frac{dx}{dt} = 1$$

$$\Rightarrow \int \frac{\sin x}{\cos x}\frac{dx}{dt}dt = \int dt$$

$$\Rightarrow \int \frac{\sin x}{\cos x}dx = t + C.$$

This method of solution, which results in an integral in terms of just x, is referred to as *separation of variables.* Use a u substitution to integrate, and thus to find a family of solutions to the differential equation.

- Consider $\frac{dx}{dt} = x(1-x)$. Use separation of variables and the method of partial fractions to find a family of solutions $x(t)$ to the differential equation.