AM108 1:00pm. Class 02

1. Consider the differential equation $\dot{x} = \cos x$.
   (a) On the whiteboard, sketch the velocity of $x(t)$, according to this equation, in the $x\dot{x}$ plane.
   (b) Identify all fixed points of the flow along the line. Do this graphically, and then again algebraically.
   (c) Where along the axis is $x(t)$ increasing? Add the corresponding arrows and then draw a phase portrait, indicating the stability of the fixed points with closed and open circles
   (d) Find the linearization of $\cos x$ for points near the fixed point $x^* = \pi$.
   ($g(x) \approx g(x^*) + (x - x^*)g'(x^*)$ for x near x^*). What information does this local linearization give us?
   (e) Is $g'(x^*)$ a derivative with respect to time?
   (f) Find the linearization at $x = 0$, which is not a fixed point. How does this differ from the linearization at a fixed point?

2. For each of the following, find the fixed points (*algebraically or graphically, whichever is easier*), sketch the vector field on the real line (this is the phase portrait), classify the stability of the fixed points, and sketch time series of $x(t)$ (solutions to the differential equations) for different initial conditions.
   (a) $\dot{x} = 4x^2 - 16$.
   (b) $\dot{x} = x - \cos x$. *Suggestion: use a plot of x and $\cos x$ verses x instead of plotting $\dot{x}$ itself.*
   (c) $\dot{x} = x/2 - \tanh x$.
   (d) $\dot{x} = \tanh x - x/2$.
   (e) $\dot{x} = x - x^3$.

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Plot[Tanh[x], {x, -4, 4}, AxesLabel -> {"x", "Tanh[x]"}]
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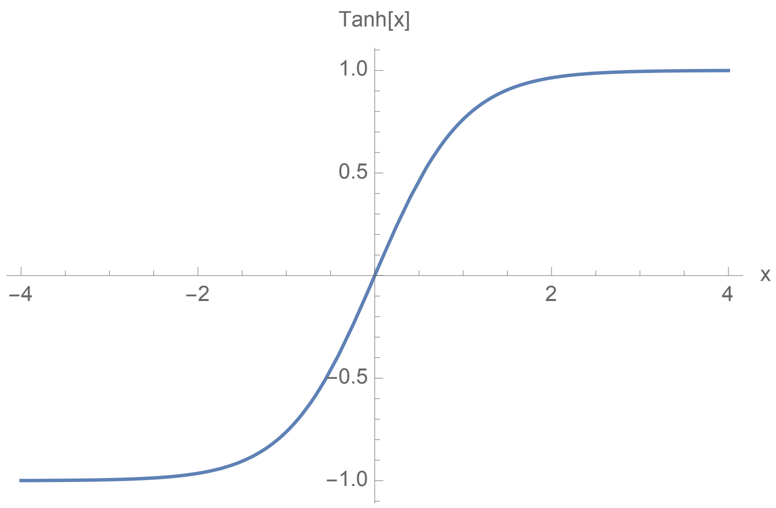

Plot included for your reference (made using Mathematica).

3. (Strogatz 2.2.10): For each of the following, find an equation $\dot{x} = f(x)$ with the stated properties, or if there are no examples, explain why not (assume $f(x)$ is smooth).
   (a) Every real number is a fixed point.
   (b) Every integer is a fixed point and there are no other fixed points.
   (c) There are precisely three fixed points, and all of them are stable.
   (d) There are no fixed points.
   (e) There are precisely 100 fixed points.

4. For the following differential equations, find the fixed points and classify their stability using linear stability analysis (an algebraic method). If linear stability analysis does not allow you to classify the point, then use a graphical argument.
   (a) Let $\dot{x} = x(3 - x)(1 - x)$. (See Strogatz 2.4.2)
   (b) Let $\dot{x} = 1 - e^{-x^2}$ (Strogatz 2.4.5)
   (c) Let $\dot{x} = rx - x^3$ where $r < 0$, $r = 0$, or $r > 0$. Discuss all three cases. (Strogatz 2.4.7)
5. (Strogatz 2.6.1) A simple harmonic oscillator, defined by $\ddot{x} = -\frac{k}{m}x$, has a solution $x(t) = A \sin \omega t + B \cos \omega t$ that oscillates on the x-axis.
   (a) Plug this expression for $x(t)$ into the differential equation to show that it is a solution for some ω and find that ω.
   (b) What happens to A and B?
   (c) We learned that oscillations are not possible in a one-dimensional system. This system is showing oscillations. Reconcile those two facts.

Extra problems if time permits:

1. Let $\dot{x} = r + x^2$. For $r = -2, -1/4, 0, 1$ find the fixed points and classify their stability (*do this graphically, not algebraically*). Now, finding the fixed points algebraically as a function of r, plot them in the rx-plane (so r is along the horizontal axis and the location of the fixed points is along the vertical). This is a *bifurcation diagram*, showing the location of fixed points as a parameter changes in the system. In a bifurcation diagram, stable fixed points are denoted with a solid line while unstable fixed points are denoted with a dashed line.

2. (Strogatz 3.2.3) Let $\dot{x} = x - rx(1 - x)$. Sketch each of the qualitatively different vector fields that occurs as r is varied. Sketch the bifurcation diagram of fixed points vs r in the rx-plane. Use solid and dashed lines to indicate the stability of the fixed points in your diagram.

3. Compare the populations models
$$\dot{N} = N(1 - N/K)$$
(logistic) and
$$\dot{N} = N(1 - N/K)(N/A - 1)$$
(strong Allee effect) where $0 < A < K$.
   (a) Based on the differential equation, what is the Allee effect?
   (b) Try to imagine a scenario where it is relevant (it was initially described in experiments on small fish).
   (c) Consider solutions, $N(t)$, to both equations. How, if at all, do solutions between the two equations differ qualitatively?
   (d) The term *basin of attraction* refers to the set of initial conditions that approach a particular fixed point. What is the basin of attraction of the extinction fixed point, $N^* = 0$, for each equation?