AM108 1:00pm, Class 03

1. We'll start by revisiting **linear stability analysis**. Consider the differential equation

$$\dot{x} = x(1 + x).$$

This has two fixed points.

(a) Use geometric reasoning (i.e. draw a graph) to sketch the phase portrait.

(b) Find the Taylor approximation to second order about the fixed point at the origin. *This is a bit silly to do for a polynomial, but go ahead and take the derivatives.*

$$\text{The Taylor expansion about } x_0 \text{ is } f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_0) + \text{h.o.t.}$$

(c) In linear stability analysis we look at the sign of $f'(x^*)$ to understand the stability of a fixed point x^*, so we are using the $(x - x_0)f'(x_0)$ term and neglecting terms that are quadratic and above, include the $\frac{1}{2}(x - x_0)^2 f''(x_0)$ term. Fill out the following chart about the size of the two terms as $x - x_0$ gets smaller for the Taylor polynomial you found above:

$x - x_*$	$(x - x_*)f'(x_*)$	$\frac{1}{2}(x - x_*)^2 f''(x_*)$
1		
0.1		
0.01		
0.001		

When does it make sense to you to neglect terms that are nonlinear in $x - x_*$? What would change if $f'(x_0) = 0$?

2. Think of

$$\dot{x} = x(1 + x)$$

as being a member of the family of differential equations specified by

$$\dot{x} = x(r + x).$$

We'd like to understand all of the possible phase portraits for this family of equations. This is the purpose of a **bifurcation** diagram. For this problem, we consider r to be a parameter of the differential equation, t to be the independent variable, and x to be the dependent variable.

(a) Find the fixed points of the differential equation as a function of r. Does the number of fixed points vary with the value of r?

(b) Use linear stability analysis to identify the stability of one of the fixed points as a function of r.

(c) Create a bifurcation diagram showing the values of the fixed points vs r. Indicate the stability of the fixed points with solid lines for stable points and dashed lines for unstable fixed points.

(d) A bifurcation occurs when the phase portrait undergoes a qualitative change. Identify r_c, the critical value of the parameter at the bifurcation.

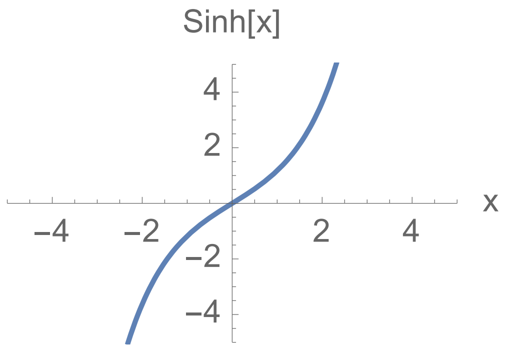
(e) What type of bifurcation is this?

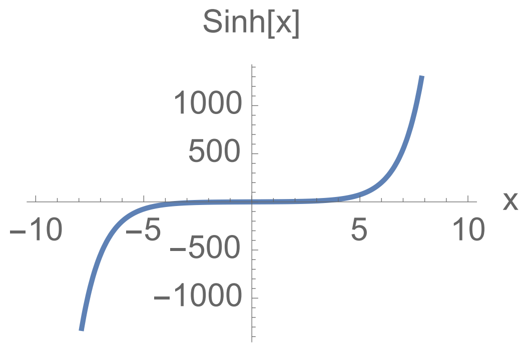
3. Now consider the differential equation

$$\dot{x} = rx - \sinh x$$

where $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$. Graphs of $\sinh x$ are given below.

```
GraphicsGrid[{{Plot[Sinh[x], {x, -5, 5}, AxesLabel -> {"x", "Sinh[x]"},
 PlotRange -> {{-5, 5}, {-5, 5}}],
   Plot[Sinh[x], {x, -10, 10}, AxesLabel -> {"x", "Sinh[x]"}]}}]
```





(a) Start by plotting rx and $\sinh x$ against x. Intersections of these curves give fixed points. To identify the stability of the fixed points you can use the sign of $rx - \sinh x$. The sign is determined by the relative position of the $\sinh x$ and rx curves. Sketch the qualitatively different vector fields (phase portraits) that occur as r is varied. *Perhaps start by thinking about $r = 0$. Adjusting r changes the slope of the line rx.*

(b) Argue that a bifurcation occurs and identify the type of bifurcation (including, if relevant, whether it is subcritical or supercritical).

(c) Use linear stability analysis to find r_c, the critical value of the parameter at the bifurcation.

(d) Sketch the bifurcation diagram.

4. (3.4.14) Consider the system $\dot{x} = rx + x^3 - x^5$.

(a) Find an algebraic expression for each of the fixed points as r varies. *You should be able to find a quadratic in x^2 to help you with this.*

(b) Calculate r_s, the parameter value at which the nonzero fixed points are born in a saddle-node bifurcation.

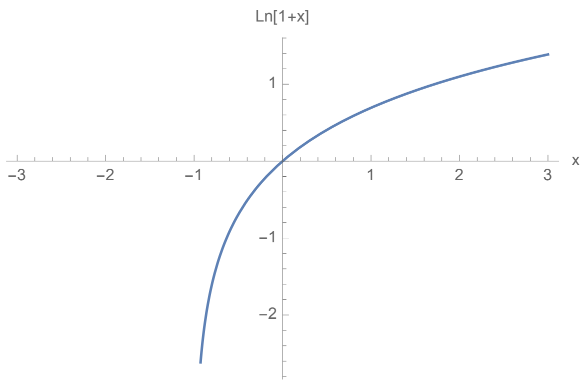
(c) Sketch the bifurcation diagram.

5. Use linear stability analysis to find r_c, the critical value of the parameter at the bifurcation. Sketch the bifurcation diagram.

(a) Let $\dot{x} = r + x - \ln(1 + x)$.

(b) Let $\dot{x} = rx - \ln(1 + x)$.

```
Plot[Log[1 + x], {x, -3, 3}, AxesLabel -> {"x", "Ln[1+x]"}]
```



6. Let $\dot{x} = x(r - x)(r^2 - x + 3)$. Draw a bifurcation diagram for this system, marking the location and type of any bifurcations. *Note: once you classify the stability of one solution, all the other stabilities should follow.*

A few bifurcation diagram shapes from the problems above (these diagrams do not include stability information):

```
GraphicsGrid[{{ContourPlot[x (r + x) == 0, {r, -4, 4}, {x, -3, 3},
    FrameLabel -> {"r", "x"}],
   ContourPlot[r x - Sinh[x] == 0, {r, -4, 4}, {x, -4, 4},
    FrameLabel -> {"r", "x"}],
   ContourPlot[r x + x^3 - x^5 == 0, {r, -1, 1}, {x, -2, 2},
    FrameLabel -> {"r", "x"}]}, {ContourPlot[
    r + x - Log[1 + x] == 0, {r, -3, 3}, {x, -3, 3},
    FrameLabel -> {"r", "x"}],
   ContourPlot[r x - Log[1 + x] == 0, {r, -1, 3}, {x, -2, 3},
    FrameLabel -> {"r", "x"}],
   ContourPlot[x (r - x) (r^2 - x + 3) == 0, {r, -5, 5}, {x, -5, 12},
    FrameLabel -> {"r", "x"}]}}]
```