AM108 1:00pm Class 04

Nondimensionalization:

1. Let $\dot{N} = rN(1 - N/K) - H$. This is a logistic model where there is a constant harvesting term impacting the population.
   (a) For each of the variables and each of the parameters, identify its dimension.
   (b) Create dimensional constants, N_0 and T_0, and use them to create nondimensional variables $x = N/N_0$ and $\tau = t/T_0$. Substitute x and τ in the equation and then simplify.
   (c) Identify all nondimensional groups.
   (d) Make choices for values of the constants T_0 and N_0 so that the equation is as simple as possible.
   (e) Define a new nondimensional parameter, and rewrite your equation as a nondimensional one.
   (f) How many parameters are there in the nondimensional system? How does this compare to the number in the dimensional version?
   (g) *Buckingham's π theorem states that, in most circumstances, nondimensionalizing reduces the number of parameters by the number of dimensions that were present in the original system.* Does the theorem appear to hold in this case?

2. Let $\dot{N} = rN(1 - N/K) - HN/(A + N)$. This is a slightly different harvesting case.
   (a) This harvesting term is in the form of a special function called a **Monod function**. How does the function behave as $N \to 0$ and how does it behave as $N \to \infty$? What do you think of that as a description of a harvesting process?
   (b) Identify the dimension of each of the variables and parameters. Once you nondimensionalize how many parameters do you expect to remain?
   (c) Nondimensionalize this equation. There are multiple good choices for N_0. What are some reasons to choose one or the other?

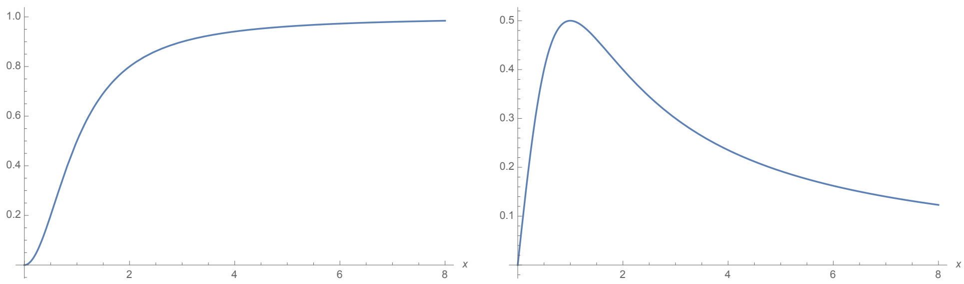
3. Let $\dot{N} = RN(1 - N/K) - BN^2/(A^2 + N^2)$. This model is describing spruce budworms growing on spruce trees and facing predation by birds. On the timescale of the budworm growth, the tree and bird populations are thought of as constant.
   (a) This is a lot of practice with very similar equations, but here goes: how is the predation term different from the harvesting term above?
   (b) Nondimensionalize, choosing N_0 and T_0 so that the nondimensional equation is

$$\frac{dx}{d\tau} = rx(1 - \frac{x}{k}) - \frac{x^2}{1 - x^2}.$$

Back to bifurcation:

4. (3.7.3) Consider the system $\dot{x} = x(1-x) - h$, which is a dimensionless model of a fish population under harvesting.
   (a) Show that a bifurcation occurs at some value of the parameter, h_c, and classify the bifurcation.
   (b) What happens at long times to the fish population for $h < h_c$, and for $h > h_c$?

5. Consider the spruce budworm model $\dot{x} = rx(1 - x/k) - x^2/(1-x^2)$.



   (a)

   Figure 1: On the left is a plot of $x^2/(1-x^2)$ while on the right is a plot of $x/(1-x^2)$.

   Use the plots to argue that the system can have one, two, or three fixed points, depending on values of r and k.
   (b) Identify what kinds of bifurcations occur in this system.
   (c) Sketch lines in the kr plane to show the parameter sets where there is an outbreak of budworms, the sets where the budworms remain at a low (termed *refuge*) level, and those where there is bistability between the two cases.

6. (3.7.4) Consider the system $\dot{x} = x(1-x) - hx/(a+x)$. This is a dimensionless model of a fish population under harvesting and is more sophisticated than the model above.
   (a) Analyze the dynamics near $x = 0$ and show that a bifurcation occurs when $h = a$. What kind of bifurcation is it?
   (b) Show that another bifurcation occurs when $h = \frac{1}{4}(a+1)^2$ for $a < a_c$ where a_c is to be determined. Classify this bifurcation.
   (c) Plot the stability diagram of the system in the ah-plane. Identify whether there are any regions where we expect hysteresis to occur.