Oscillators:

1. (4.1.1) For which values of a does the equation $\dot{\theta} = \sin a\theta$ give a well-defined vector field on the circle?

   For $a = 3$, find and classify all the fixed points and sketch the phase portrait on the circle.

   See Activity17-02-06_bistability.nb for a plot of $\sin 3\theta$.

2. (4.3.3) For $\dot{\theta} = \mu \sin\theta - \sin 2\theta$, check that the vector field is well-defined on the circle. Draw the qualitatively different phase portraits that exist at different values of μ, classify the bifurcations that occur as μ varies, and find the bifurcation values of μ. For what values of μ is the system "oscillating"?

   See Activity17-02-06_bistability.nb for a plot of $\mu \sin\theta - \sin 2\theta = 0$, *which gives the shape of a bifurcation diagram, but not the stability information.*

   Bifurcations on the real line:

3. (3.6.2) Consider the system $\dot{x} = h + rx - x^2$. When $h = 0$ this system undergoes a transcritical bifurcation at $r = 0$. We will look at how the bifurcation diagram of x^* vs r is affected by the imperfection parameter h.

   (a) Plot the bifurcation diagram for $\dot{x} = h + rx - x^2$ for $h < 0$, $h = 0$, $h > 0$.

   See Activity17-02-06_bistability.nb for the fixed point and stability calculations as well as the shape of the bifurcation curves.

   (b) In the rh-plane, identify the regions that correspond to qualitatively different vector fields. Identify the number of fixed points in each region and label the bifurcations that occur on the boundary.

   Most groups got to this point

4. (3.7.4) Let

$$\dot{N} = RN(1 - N/K) - HN/(A + N), \quad H > 0, A > 0.$$

   This model is describing harvesting of fish. Nondimensionalizing and choosing $N_0 = K$ and $T_0 = 1/R$, the corresponding nondimensional equation is

$$\frac{dx}{d\tau} = x(1 - x) - h\frac{x}{a + x}.$$

   This is a two-parameter system. Our goal is to identify all of the qualitatively different phase portraits of the system, as well as where in parameter space they occur.

   Note: In the last class we learned that $\frac{x}{1+x}$ *is called a* ***Monod*** *function. Similar functions* $\frac{x^n}{1+x^n}$ *are called Hill functions. They switch more sharply with larger* n. *This Monod function is a switching function but the switching isn't too sharp.*

   (a) This is a harvesting model for a fish population where $HN/(A + N)$ is the harvesting term. Assume the harvesting is being done by humans. In this context, what are the meanings of H and A? Why might we choose to retain h and a as the nondimensional parameters?

   See Activity17-02-06_bistability.nb to plot the Monod function and use the "Manipulate" command to vary the values of a *and* h.

   (b) In Mathematica, find the fixed points of the dynamical system and use the result to find a curve in the ah-plane along which a bifurcation occurs. What type of bifurcation is it?

   In a population problem we are only interested in nonzero fixed points with $x^* > 0$. This creates a restriction on a so that $a < a_c$ is necessary for either of the non-zero fixed points to be non-negative. Identify a_c.

   (c) You have just explored the existence of positive fixed points. By copying and evaluating the appropriate two lines from previous Mathematica code, examine the stability of the $x = 0$ fixed point. Use the result to identify a curve in the ah-plane along which a bifurcation occurs. What type of bifurcation might it be?

(d) Use Mathematica to plot the two bifurcation curves you found above in the ah-plane.

Identify the number of fixed points in each region. Identify the number of fixed points in each region and label the bifurcations that occur on the boundaries between regions.

You might use the plot of fixed points as a function of h to help with this.

(e) Use the stability of the $x^* = 0$ fixed point along with your plot of the regions in the ah-plane to identify the number of non-negative stable fixed points in each region of the ah-plane. Label any region with bistability.

(f) Imagine we start with $a = 0.1$ and set harvesting at $h = 0.05$. Now we slowly ramp up harvesting, h, so that harvesting is $h(t) = 0.05 + 0.001t$. This is implemented in Mathematica using the NDSolve command. What happens to the number of fish in the system?

(g) What do you think should happen if we turn off harvesting after the population falls below a certain threshold?