AM108 1:00pm

Goals for the day:

1. How does a second order differential equation relate to a 2D system?
2. What is the relationship between eigenvalues/eigenvectors and analytic solutions of a 2D linear system?
3. Given the eigenvalues and eigenvectors of a matrix associated with a 2D linear dynamical system, construct the *phase portrait*.
4. Use a phase portrait to identify all of the possible long term behaviors of a particular 2D linear system.

Team problems:

1. (Solutions to second order linear constant-coefficient equations)
   Let
   $$\ddot{x} + \dot{x} - 2x = 0.$$
   (a) Consider a possible solution $x(t) = Ae^{rt}$. Using the method of substitution, determine r so that this is a solution.

   (b) You should have found two possible values of r. Do those values correspond to solutions showing exponential growth or exponential decay?

   (c) The general solution of the differential equation is written as an arbitrary linear combination of $e^{r_1 t}$ and $e^{r_2 t}$. Write down the general solution.

   (d) Find $\dot{x}(t)$, the derivative with respect to time of the general solution (this will be useful in the next question).

2. (Rewriting a higher order equation as a first order system)
   In this class, we will work with systems of first order equations because the bifurcation analysis methods we are developing apply to first order systems. To convert from a second order (or higher) equation to a first order system, we use dummy variables. The 2nd order equation above can be rewritten using a first order system. The system should have solutions that match those of the equation. We will check that this is the case by constructing solutions for the system using the solution we already found for the equation.

   (a) First we'll rewrite the second order equation as a first order system. Define the dummy variable $y = \dot{x}$. Using the equation above, express $\dot{y}$ in terms of x and y. Next, rewrite the second order equation as a first order system:
   $$\dot{x} =$$
   $$\dot{y} =$$

   (b) The first order system is equivalent to the second order equation, and so should the same solution functions. For the second order equation, the solution is a single value $x(t)$, while for the first order system, the solution is a vector $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, where $x(t)$ is the same as for the second order equation, and $y(t) = \dot{x}(t)$.
   You have $x(t)$, the solution from Q1. You also have $y(t)$ from the last part of the first question. Putting these together in a vector should give a solution to the first order system you've constructed, so use these to write a general solution to the matrix equation of the form
   $$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = .$$
   Once you have done this, manipulate it to rewrite the solution in the form $c_1 e^{r_1 t}\mathbf{v}_1 + c_2 e^{r_2 t}\mathbf{v}_2$. To show that this is a general solution to the first order system, show that $\mathbf{v}_1$ and $\mathbf{v_2}$ are eigenvectors of the matrix in the first order system, and that they have the expected eigenvalues.

(c) On the xy plane, sketch the solutions corresponding to the eigenvectors (these are straight lines in the plane). Denote the direction of time along the solution trajectories using an arrow.

(d) Use continuity and local vector field information to fill in the rest of the phase portrait.

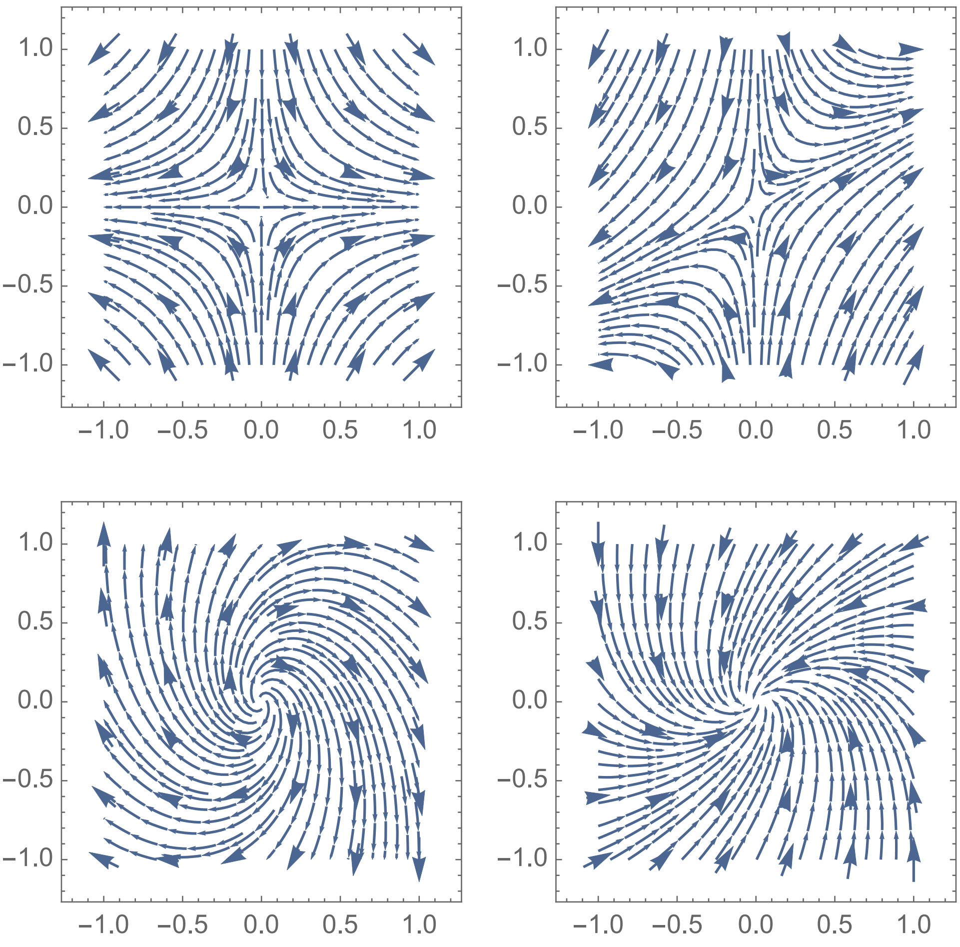
(e) What type of fixed point is the fixed point at $(0,0)$?

(f) Consider the third order equation $\dddot{x} + \ddot{x} + x^2 = 0$. Using dummy variables $y = \dot{x}$ and $z = \dot{y}$, rewrite this equation as a first order system

$$\begin{aligned} \dot{x} &= \\ \dot{y} &= \\ \dot{z} &= \end{aligned}$$

3. Use trace and determinant to match the following systems to one of the phase portraits below. If necessary, use eigenvectors as well.

$$\begin{aligned} \dot{x} &= \ x \\ \dot{y} &= \ x - y \end{aligned} \qquad \begin{aligned} \dot{x} &= \ -x - y \\ \dot{y} &= \ x - 2y \end{aligned} \qquad \begin{aligned} \dot{x} &= \ x \\ \dot{y} &= \ -y \end{aligned} \qquad \begin{aligned} \dot{x} &= \ x + y \\ \dot{y} &= \ -2x + y \end{aligned}$$



4. (extra, if you like probability 5.3.14) Suppose we pick a linear system at random. What's the probability the origin will be stable, unstable, or a saddle?
To be more specific about what we mean by "random", consider the system $\dot{\underline{x}} = A\underline{x}$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Suppose we pick the entries a, b, c, d independently and at random from a uniform distribution on the interval $[-1, 1]$.