AM108 1:00pm

Class 08: 2D nonlinear systems

Goals for the day:

1. Use multivariate Taylor series to linearize and identify the behavior of trajectories very close to a fixed point.
2. Piece together a more global phase portrait using locally linear phase portraits.

Team problems:

1. (6.3.6) Consider the system

$$\begin{aligned}\dot{x} =& f(x,y) = xy - 1\\ \dot{y} =& g(x,y) = x - y^3\end{aligned}$$

   (a) Use Taylor expansion to find a quadratic approximation to the right hand side of each equation above around the point $(-1,-1)$. Recall that

$$\begin{aligned}f(x,y) \approx& f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b) + (x-a)^2\frac{f_{xx}(a,b)}{2}\\ &+ (x-a)(y-b)f_{xy}(a,b) + (y-b)^2\frac{f_{yy}(a,b)}{2} + h.o.t.\end{aligned}$$

   Use $u = x - (-1)$, $v = y - (-1)$ to simplify your expressions.

   (b) Sufficiently close to $(-1,-1)$, we have $|u|, |v| \ll 1$ and $u^2 \ll |u|, v^2 \ll |v|$, so drop quadratic order and higher terms to generate a linearization. Use your linearization to write a dynamical system of the form

$$\dot{\underline{u}} = A\underline{u},$$

   giving definitions for $\underline{u}$, A.

   Explain why the linearization only leads to this kind of matrix equation at a fixed point.

   (c) Create a linearized system about the fixed point $(1,1)$ as well.

   (d) Use the trace and determinant to classify your fixed points as **hyperbolic** (*no eigenvalues have zero real part*) or **nonhyperbolic** (*at least one eigenvalue has zero real part*) fixed points.

   If $\Delta = 0$ there is a zero eigenvalue. If $\tau = 0$ there may be a complex conjugate pair of eigenvalues with zero real part. What would the determinant be in that case? Stability information from the linearization can be used to classify hyperbolic fixed points. When a fixed point is nonhyperbolic the stability information from linearization is inconclusive.

   Identify the stability of hyperbolic fixed points.

   (e) Use eigenvalues and eigenvectors to sketch neighboring trajectories to the two fixed points. Try to fill in the rest of the phase portrait.

   What do you think the long term behavior would be for a trajectory starting at $(2,2)$? What about for one starting at $(1,2)$?

```
f[x_, y_] = x y - 1;
g[x_, y_] = x - y^3;
soln = Solve[ {f[x, y] == 0, g[x, y] == 0}, {x, y}]
fx = D[f[x, y], x];
fy = D[f[x, y], y];
gx = D[g[x, y], x];
gy = D[g[x, y], y];
A1 = {{fx, fy}, {gx, gy}} /. soln[[1]]
```

```
A2 = {{fx, fy}, {gx, gy}} /. soln[[4]]
x0 = x /. soln[[1]];
y0 = y /. soln[[1]];
sz = 0.5;
p1 = StreamPlot[
   A1.{{x - x0}, {y - y0}}, {x, x0 - sz, x0 + sz}, {y, y0 - sz,
    y0 + sz}, PlotRange -> {{-2, 2}, {-2, 2}}, StreamPoints -> 15];
x0 = x /. soln[[4]];
y0 = y /. soln[[4]];
p2 = StreamPlot[
   A2.{{x - x0}, {y - y0}}, {x, x0 - sz, x0 + sz}, {y, y0 - sz,
    y0 + sz}, PlotRange -> {{-2, 2}, {-2, 2}}, StreamPoints -> 15];
p3 = StreamPlot[{ f[x, y], g[x, y]}, {x, -2, 2}, {y, -2, 2},
  FrameLabel -> {"x", "y"}, StreamPoints -> 20]
Show[p1, p2, p3]
```