Class 09: 2D conservative systems

Goals for the day:

1. Find a conserved quantity.
2. Identify the implications (for fixed points and trajectories) of having a conserved quantity in a system.

Team problems:

1. (6.5.1) Consider the system $\ddot{x} = -\mu x + x^3$.
   (a) Start by analyzing this system using the methods of our class (fixed points, etc). You can use computational tools to compute eigenvalues and eigenvectors if it is helpful.
   (b) Some of the fixed points were centers according to their linearization, so before making a phase portrait you'll need to determine whether they are centers only in the linearization or whether they are genuinely nonlinear centers. Check for a conserved quantity to do this.

   Conserved quantity tips: *Find an expression for $\frac{dy}{dx}$ by taking $\frac{dy}{dt}/\frac{dx}{dt}$. When this results in a separable differential equation, you can solve the equation to find a conserved quantity. Use this method to find a conserved quantity for this system.*
   For systems of the form $m\ddot{x} + \frac{dV}{dx} = 0$, $\frac{1}{2}m\dot{x}^2 + V(x)$ is a conserved quantity. Show that it is conserved on trajectories.
   (c) Now sketch a phase portrait, and use it to identify the possible long term behaviors of the system.

   Identify any homoclinic or heteroclinic orbits, and find an equation for those trajectories.

   Identify any regions of the phase space where oscilliations will occur.
   (d) This system is a model of a nonlinear spring. Add a small amount of drag, so $\dot{y} = -\mu x + x^3 - \epsilon y$ with $\epsilon > 0$. What happens to the system? Attempt to sketch a new phase portrait.
2. Consider the system

$$\dot{x} = -\mu y + xy$$
$$\dot{y} = \mu x + \frac{1}{2}(x^2 - y^2).$$

   Assume $\mu > 0$.

   Using similar methods to above, analyze this system.