AM108 1:00pm

Class 12: limit cycles in 2D

Goals for the day:

1. Explain what is meant by *closed orbit* or *closed trajectory*.
2. Use polar coordinates to represent a 2D dynamical system.
3. Rule out or rule in closed orbits in 2D systems.

Team problems:

1. (Ruling out closed orbits) Let

$$\dot{x} = y$$
$$\dot{y} = x - x^3 - \mu y, \quad \mu \geq 0.$$

This system is known as the unforced Duffing oscillator. The μy term is a damping term. Use Dulac's criterion to show that the system has no closed orbits for $\mu > 0$.

2. (Back in 1D) Consider the (1D) dynamical system

$$\frac{dx}{dt} = x(1 - x^2)(4 - x^2)$$

with x restricted to $x \geq 0$. Find the fixed points, determine their stability, and draw the phase portrait.

3. (Connecting 1D to 2D polar) Consider the (2D polar) dynamical system

$$\dot{r} = r(1 - r^2)(4 - r^2)$$
$$\dot{\theta} = 1.$$

Sketch the phase portrait for this system using the information from the question above.

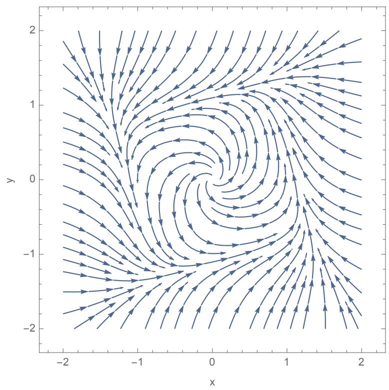
4. Consider the dynamical system

$$\dot{r} = r(1 - r^2)(4 - r^2)$$
$$\dot{\theta} = 2 - r^2.$$

Sketch the phase portrait for this system.

5. (Poincaré-Bendixson) Let

$$\dot{x} = x - y - x^3$$
$$\dot{y} = x + y - y^3.$$



Construct a trapping region. *Try using a large square centered at the origin and a small circle centered at the origin. Recall* $r\dot{r} = x\dot{x} + y\dot{y}$, $\dot{\theta} = (x\dot{y} - y\dot{x})/r^2$.

6. Consider the dynamical system

$$\dot{r} = r\sin^2\theta(1-r^2)(4-r^2)$$
$$\dot{\theta} = 1.$$

Sketch the phase portrait for this system.

Answers:

1. Find a $g(x,y)$ such that $\frac{\partial g(x,y)y}{\partial x} + \frac{\partial g(x,y)(x-x^3-\mu y)}{\partial y} < 0$ for all (x,y).

2. $f(x) = x(1-x^2)(4-x^2) = 0$ gives the fixed points. These are $x = 0, \pm 1, \pm 2$. We restrict ourselves to $x \geq 0$ so these are $x = 0, 1, 2$.

   For their stability, check the sign of $\frac{df(x)}{dx}$. $\frac{df}{dx} = (1-x^2)(4-x^2) + x(-2x)(4-x^2) + x(1-x^2)(-2x)$.

   At $x = 0$ this is $(1-0)(4-0) = 4$. Because stability alternates we know that $x = 0$ is unstable, $x = 1$ is stable and $x = 2$ is unstable, but let's check these.

   At $x = 1$ this is $1(-2)(4-1) = -6$ so stable as expected.

   At $x = 2$ this is $2(1-4)(-4) = 12$ so unstable as expected.

3. Now we have a system given in polar.

   r is a radius and θ is an angle, both in the xy plane.

   The fixed points for r are as above, with stability as given above. So there is a fixed point at $(0,0)$ that is unstable, there is a limit cycle at radius 1 from the origin, which is stable, and there is a limit cycle at radius 2 from the origin, which is unstable.

4. The radial behavior is the same in this system but now θ is not changing steadily in time. Instead how it changes depends on the radius. At a radius of $r = \sqrt{2}$, $\dot{\theta}$ changes sign, so for $r < \sqrt{2}$, θ increases with time. And for $r > \sqrt{2}$, θ is decreasing with time.

5. For large y, $\dot{y} \approx -y^3$. For large x, $\dot{x} \approx -x^3$. For small x, y, $\dot{x} \approx x - y, \dot{y} \approx y + x, r\dot{r} \approx r^2$.

6. Now the angular behavior is back to being steady, but the speed of motion in the radial direction depends on angle. The limit cycle locations have not changed, but the way trajectories move on the cycles and to get to the cycles now has an angular dependence. Trajectories have slow radial change when θ is near 0 or π, so at those values, the angle keeps changing steadily while the radius is barely changing. The radius changes fastest near $\theta = \pi/2$ and $3\pi/2$, where the radial change is just like the parts above.

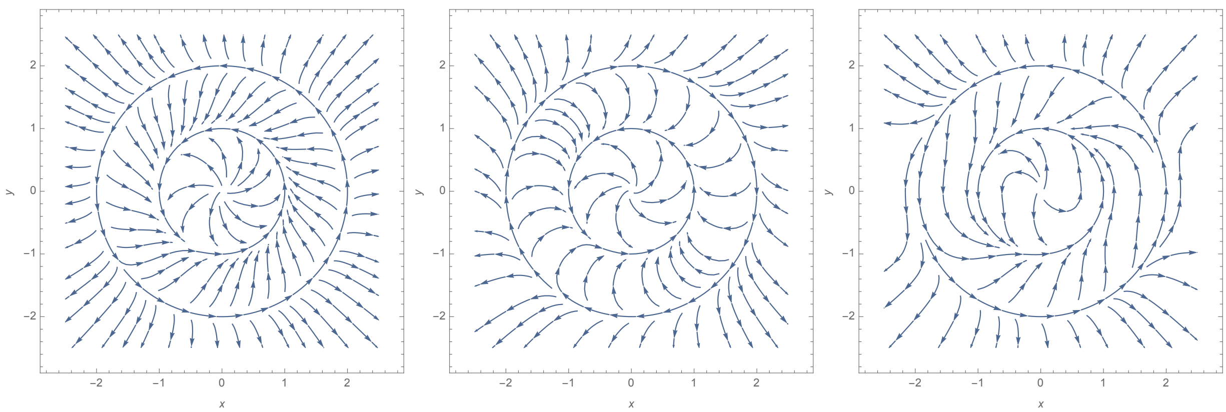


Figure 1: From left to right, we have the phase portraits for 2, 3, and 4