AM108 1:00pm

Class 13: self-excited oscillations

1. (Using Mathematica on the van der Pol system)
   Let

$$\dot{x} = \mu(y - F(x))$$
$$\dot{y} = -\frac{1}{\mu}x$$
$$F(x) = \frac{1}{3}x^3 - x, \quad \mu > 0$$

   (a) To enter μ in Mathematica, use the sequence `esc m esc`. Try to use this to type $\alpha, \beta, \mu, \tau, \theta$ in Mathematica.

   (b) To find the fixed points using Mathematica, we can simultaneously solve $\dot{x} = 0, \dot{y} = 0$.
   Example:
```
f = b (y - x) (\[Alpha] + x^2) - x;
g = c - x;
soln = Solve[{f == 0, g == 0}, {x, y}]
```
   Modify these commands to identify the fixed point of the van der Pol system in terms of μ.
   Note the use of == *rather than* = *for these equations.*

   (c) We can compute the Jacobian matrix in Mathematica as follows
```
A = {{D[f, x], D[f, y]}, {D[g, x], D[g, y]}}
```
   To evaluate this matrix at a fixed point,
```
mat = Simplify[A /. soln[[1]]]
```
   i. Find the linearization of the van der Pol system about the fixed point using Mathematica commands.
   ii. What do the `D[f,x]` type commands do?
   iii. How would you enter the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ in Mathematica?

   The simplify command is not needed - it is the /. that does the evaluation of A at the fixed point.

   (d) To classify the fixed point, we can find the trace, determinant, and eigenvalues:
```
Tr[mat]
Det[mat]
Eigenvalues[mat]
```
   Use these to classify the stability of the fixed point and to identify whether it is a node or a spiral.

   (e) To make a phase portrait of the linearized system, we make a phase portrait of the system $\dot{\underline{x}} = A\underline{x}$.
```
StreamPlot[(mat.{{x}, {y}}) /. \[Mu] -> 1, {x, -2, 2}, {y, -2, 2}]
```
   Make phase portraits trying a few different values of μ.

   (f) To make a phase portrait for the nonlinear system:
```
StreamPlot[{{f}, {g}} /. \[Mu] -> 7, {x, -3, 3}, {y, -3, 3},FrameLabel->{"x","y"}]
```
   Go ahead and do this for the van der Pol as well. It isn't so clear what is happening in the phase portrait for this system, so let's add a numerical trajectory.

   (g) To create trajectories of the system choose values for each of the parameters and then integrate numerically using the command `NDSolve`. That command needs each of the diff eqs in the system as well as initial conditions. *Note the use of* == *rather than* = *for these equations.*
```
\[Mu] = 10;
soln = NDSolve[{x'[t] == \[Mu] (y[t] - (1/3 x[t]^3 - x[t])),
   y'[t] == -1/\[Mu] x[t], x[0] == 0.2, y[0] == 0.2}, {x, y}, {t, 0,
   70}]
   Clear[\[Mu]]
```

Note the explicit dependence on t in the equations above.

To plot the solutions:

```
Plot[x[t] /. soln, {t, 10, 70}, AxesLabel->{"t","x"}]
Plot[y[t] /. soln, {t, 10, 70}, AxesLabel->{"t","y"}]
ParametricPlot[{x[t], y[t]} /. soln, {t, 0, 70}, AxesLabel->{"x","y"}]
```

Find a way to identify the approximate period of this oscillation for a few values of μ. In the videos the period was computed to approximately be $2\mu(\frac{3}{2} - \ln 2)$. Compare this to what you measure (and put both values into the chart on the front board).

(h) The `Show` command allows you to put two plots on the same axes.

```
p1 = ParametricPlot[{x[t], y[t]} /. soln, {t, 0, 70},
  AxesLabel -> {"x", "y"}]
p2 = StreamPlot[{{f}, {g}} /. \[Mu] -> 7, {x, -3, 3}, {y, -3, 3},
  FrameLabel -> {"x", "y"}]
Show[p2, p1]
Show[p1, p2]
```

i. What difference does the order of the plots in the command make?

ii. Plot the $\dot{x}$ nullcline on the same plot as the trajectory. Can you see any distance between them?

(i) Find a way to estimate the distance between the limit cycle and the nullcline for a few values of μ.

2. (Time averaging an oscillation over a single period) Find $\left\langle(\sin t)^2\right\rangle$ or $\left\langle(\cos t)^2(\sin t)^2\right\rangle$, where $\langle f\rangle$ denotes the average of a periodic function f over a single cycle. For $f(t)$ a function of period 2π, we can compute this average as $\frac{1}{2\pi}\int_0^{2\pi} f(t)dt$.

I find that a straightforward way to compute this type of quantity is to use $\cos t = \frac{1}{2}(e^{it} + e^{-it})$ and $\sin t = \frac{1}{2i}(e^{it} - e^{-it})$.

Note that $\cos nt = \frac{1}{2}(e^{int} + e^{-int}$. Also note that $\int_0^{2\pi} \cos nt\ dt = 0$ for $n = 1, 2, 3, ...$

3. After a change of variables the van der Pol system is

$$\dot{x} = \mu(y - \frac{1}{3}x^3 + x)$$
$$\dot{y} = -\frac{1}{\mu}x.$$

Consider the case where $\mu >> 1$.

(a) In this relaxation oscillation, the trajectory is moving very quickly when it jumps between the two parts of the $\dot{x} = 0$ nullcline. Looking at the $\dot{x}$ equation, convince yourself that it is moving at a velocity of order $\mathcal{O}(\mu)$ during the jumps.

(b) The trajectory is traversing a distance of about 3 (this is an order one number) as it jumps. Combine the order of the distance and the order of the velocity to estimate the order of magnitude of the time that it spends jumping.

(c) How close does the trajectory need to be to the $\dot{x} = 0$ nullcline for both $\dot{x}$ and $\dot{y}$ to be the same order of magnitude?

(d) While moving along the nullcline, the trajectory moves about 1 in x and a bit less than 2 in y. It is basically moving on the curve $y = \frac{1}{3}x^3 - x$ (it is not quite on the curve, but it is close to that curve the whole time). The time it spends traversing the curve is

$$\mathcal{O}(\mu^k)$$

for some integer k. To estimate time (just as we did for the oscillators in chapter 4), we set up an integral of the form

$$\int_{x_1}^{x_2} \frac{dt}{dx} dx$$

or something like this. This integral shows us how the time depends on μ. Using

$$\int_{x_1}^{x_2} \frac{dt}{dx} dx$$

doesn't work so well. It puts

$$y - (\frac{1}{3}x^3 - x)$$

in the denominator (so a dependence on x and y, not just x). Plus, that quantity is basically zero, and how far it is from zero depends on μ. We could try again with

$$\int_{y_1}^{y_2} \frac{dt}{dy} dy.$$

This leads to problems, too, because it will be a function of x in the integral. So actually, Steve used

$$\int_{x_1}^{x_2} \frac{dt}{dy}\frac{dy}{dx} dx.$$

(This is an example of persisting until something works, and luckily getting something to work before we run out of options). Use the setup of this final integral to identify k. There is no need to evaluate it. Note that for $\frac{dy}{dx}$ we're thinking of ourselves as, to good approximation, being stuck on the nullcline, so you can compute this directly.

(e) Compare the amount of time spent jumping to the amount of time spend moving along the curve. (We have the timescales of these processes, and not the exact amounts of time, so compare those).

Answer:

1.

2. $\frac{1}{2}$ and $\frac{1}{8}$.

3. (a) When the trajectory is flowing across, $\dot{y}$ is small: $|x| < 3$ or so, and μ is big, so $|\frac{x}{\mu}| < \frac{3}{\mu}$, which is small (specifically order of $\frac{1}{\mu}$. This means the motion is basically horizontal. And it is moving at a speed of $\mu(y - x^3/3 + x)$ in the horizontal direction. Away from the nullcline itself, $y - x^3/3 + x$ is order 1, so μ times it is order μ. It jumps across a distance of maybe 3 as it moves horizontally. This is an order 1 distance. Since *distance/time* = *velocity* the time is *distance/velocity* which is order $\frac{1}{\mu}$. This is a small number. That means it jumps across pretty quickly.

   (b) For $\dot{x}$ and $\dot{y}$ to be the same order we need them to both be order $\frac{1}{\mu}$ so we need $(y - x^3/3 + x)$ to be order $\frac{1}{\mu^2}$ so that when it is multiplied by μ it is order $\frac{1}{\mu}$. So we need to be within order $\frac{1}{\mu^2}$ of the nullcline.

   (c) $\frac{dt}{dy} = -\mu\frac{1}{x}$ and $\frac{dy}{dx} = x^2 - 1$ on the nullcline. So $T = \int_{x_1}^{x_2} -\mu\frac{1}{x}(x^2 - 1)dx$. This is μ multiplied by a number, and the number will be order 1. So this time is order of μ.

   (d) The jump is order of $\frac{1}{\mu}$ and the motion along the curve is order of μ. This means we spend a ton more time on the curve compared to doing the jump.