Class 14: Bifurcations

Goals for the day:

1. Identify bifurcations that are familiar from 1D systems in 2D systems.
2. Find and classify a Hopf bifurcation.

Team Problems:

1. (8.1.6) Consider the system

$$\dot{x} = y - 2x$$
$$\dot{y} = \mu + x^2 - y.$$

   (a) Sketch the nullclines.

   (b) Find and classify the bifurcations that occur as μ varies.

   (c) Show that at the point of bifurcation the $\dot{x} = 0$ and $\dot{y} = 0$ nullclines are tangent. Do you expect this to hold in general for a saddle-node bifurcation? What about for a transcritical or pitchfork bifurcation?

2. (8.2.8) Consider the dimensionless predator-prey system:

$$\dot{x} = x(x(1-x) - y)$$
$$\dot{y} = y(x - a), \quad a > 0.$$

   (a) Which variable is representing prey, and which predators?

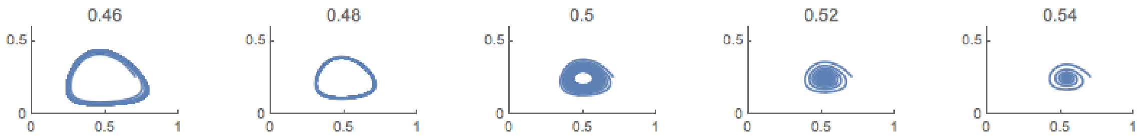
   (b) Find the fixed points of this system. (You can use Mathematica or do this by hand)

   (c) Determine the stability of these fixed points. (You can use Mathematica or do this by hand) *The trace and determinant will be sufficient to classify two of the points. For the third fixed point, drawing the nullclines may help you classify it. Note that your classification will include different cases for different ranges of a.*

   (d) What type of bifurcation occurs when $a = 1$? What about when $a = \frac{1}{2}$?

   (e) Estimate the frequency of limit cycle oscillations for a very close to the bifurcation.

   (f) Is the Hopf bifurcation supercritical or subcritical?



Initial conditions of $(0.7, 0.26)$

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avals = Table[av, {av, 0.46, 0.54, 0.02}];
timeseries =
  Table[NDSolve[{x'[t] == (f /. {x -> x[t], y -> y[t]}),
     y'[t] == (g /. {x -> x[t], y -> y[t], a -> avals[[kv]]}),
     x[0] == 0.7, y[0] == 0.26}, {x[t], y[t]}, {t, 0, 400}], {kv, 1,
    5}];
GraphicsGrid[{Table[
   ParametricPlot[{x[t], y[t]} /. timeseries[[kv]], {t, 0, 400},
    PlotRange -> {{0, 1}, {0, 0.6}}, PlotLabel -> avals[[kv]],
    Ticks -> {{0, 0.5, 1}, {0, 0.5, 1}}], {kv, 1, 5}]}]
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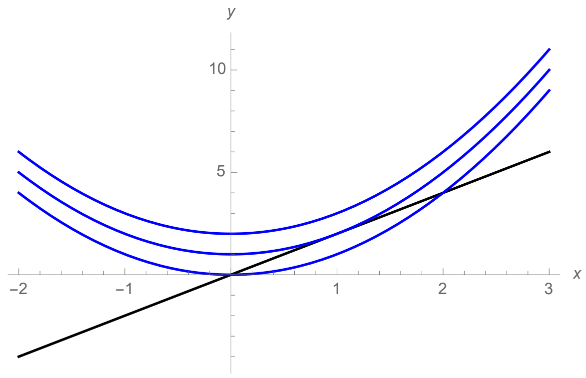
Some Answers:

1. (a) The $\dot{x} = 0$ nullcline is in black and the $\dot{y} = 0$ nullcline is in blue for three different values of μ. These values are below, at, and above the bifurcation point.



(b) To find and classify the bifurcations we need to identify the fixed points and find where the number of fixed points changes. We can see from the nullclines that the black and blue lines intersect in two places for some values of μ, and as μ increases they intersect at a single point and then no points. This means there's a saddle-node bifurcation. So I've classified the bifurcation just from looking at the nullcline picture. To find the bifurcation, I suppose I need to find the point where there's just one fixed point, or, I can find one of the fixed points and identify where it changes stability. Either option would work.

At the fixed points $y = 2x$ and $y = \mu + x^2$ because $y - 2x = 0$ and $\mu + x^2 - y = 0$. This means $2x = \mu + x^2$ so $x^2 - 2x + \mu = 0$. Using the quadratic formula,

$$x_{\pm} = 1 \pm \frac{1}{2}\sqrt{4 - 4\mu} = 1 \pm \sqrt{1 - \mu}.$$

So there is just a single fixed point when $\mu = 1$ and for $\mu > 1$ there are no fixed points. Thus the bifurcation occurs when $\mu = 1$ at the point $(1, 2)$.

(c) The slope of the $\dot{x} = 0$ nullcline of $y = 2x$ is 2. We want to show the other nullcline has the same slope at the point of bifurcation. The other nullcline is $y = x^2 + \mu$ so its slope is given by $2x$ and at the bifurcation point, $x = 1$, so its slope is also 2. The two lines intersect at $(1, 2)$ for $\mu = 1$ and they have the same slope, so they are tangent.

Assume we have two nullclines that are continuous and differentiable. At the bifurcation, they need to intersect in a single point, while just before (or just after) the bifurcation they need to intersect in multiple nearby points. For the normal forms of these bifurcations, the nullclines will be tangent at the bifurcation, so whenever we can transform to the normal form, we expect a tangency.

2. prey: x, predator: y. fixed points: $(0, 0)$, $(1, 0)$, and $(a, a - a^2)$. classification: $(0, 0)$ a saddle for $a > 0$, $(1, 0)$ a saddle ofr $0 < a < 1$, stable for $a > 1$. $(a, a - a^2)$ unstable for $0 < a < \frac{1}{2}$, stable for $\frac{1}{2} < a < 1$ and a saddle of $a > 1$. supercritical Hopf at $a_c = 1/2$. frequency of oscillation is given by the imaginary part of the eigenvalues near a_c so $\omega \approx \frac{1}{2\sqrt{2}}$.