AM108 1:00pm

Class 15: Bifurcations

Goals for the day:

1. Relate oscillation in a chemical system to bifurcation.
2. Distinguish between different bifurcations of cycles (saddle node of cycles, saddle node infinite period, homoclinic bifurcation).

Team Problems:

Please work in a pair today. You are welcome to bring in computational tools if you find it helpful.

1. (8.3.1, The Brusselator). Often the BZ-reaction is studied experimentally in something called a continuously stirred tank reactor, and the reactor is set up with constant inflows of new reactant, and a constant outflow. This continuous flow enables the reaction to oscillate without decaying to a non-oscillatory rest state. A relatively large number of chemical species and intermediate reactions are involved in the BZ-reaction, so models that match the experiments can be challenging to analyze mathematically. A number of simplifications of the BZ-reaction have been developed.

   The Brusselator model given here is a toy chemical reaction model that captures some of the qualitative aspects of the BZ-reaction. The kinetics of this reaction are

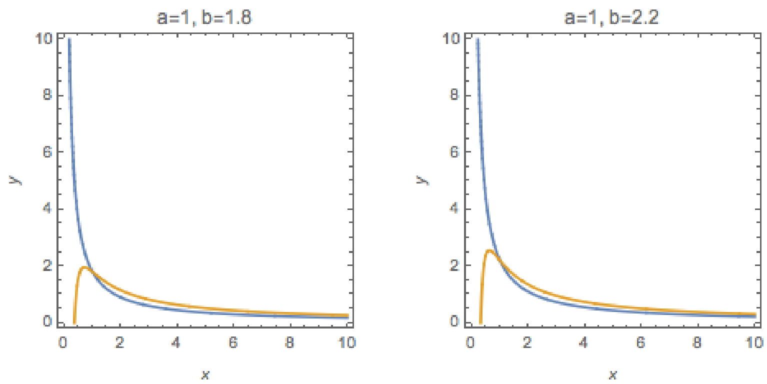
   $$\dot{x} = 1 - (b+1)x + ax^2y$$
   $$\dot{y} = bx - ax^2y$$

   with $a, b > 0$ parameters of the system and $x, y \geq 0$ dimensionless concentration.

   (a) Analyze this system: find the fixed points, classify them as a function of the parameters, identify any bifurcations. If there is an oscillatory bifurcation, find the approximate frequency and period of oscillation close to the bifurcation.

   Given the context of the problem, you should anticipate that there will be onset of oscillation at some parameter set.

   (b) Often we use numerics to determine whether a bifurcation is supercritical or subcritical. If we are able to construct a trapping region, though, the Poincaré Bendixson theorem would allow us to show that we have a stable limit cycle, and classify based on that. Work to construct a trapping region (and to finish identifying the bifurcation). *Nullclines are shown below to help with this. Also note that* $\dot{x} = -\dot{y} + 1 - x$.



2. (8.4.1) Consider the system given by

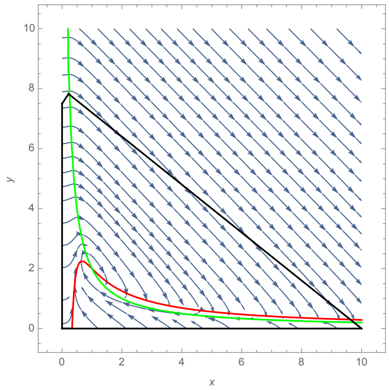
   $$\dot{r} = r(1 - r^2)$$
   $$\dot{\theta} = \mu - \sin\theta$$

   with μ slightly greater than 1, so we are on the verge of an infinite period bifurcation. In lecture we saw the approximate waveform for $x(t)$. Reproduce this argument to approximate the $x(t)$ waveform, and also find the waveform for $y(t)$.

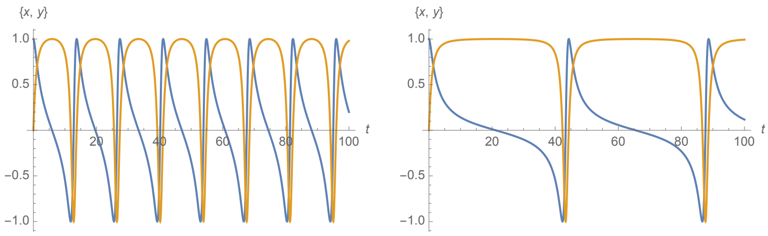
3. (8.4.2) Discuss the bifurcations of the system $\dot{r} = r(\mu - \sin r)$, $\dot{\theta} = 1$ as μ varies.

Some Answers:

1. Fixed point is $(1, \frac{b}{a})$, trace is $b - a - 1$, determinant is $a > 0$ so stability changes with trace at $b_c = a + 1$. (Hopf bifurcation) At $b_c = a + 1$ the Jacobian at the fixed point is $\begin{pmatrix} a & a \\ -a-1 & -a \end{pmatrix}$ with trace $= 0$ and det $= a$, so $a = -(i\omega)^2 = \omega^2$ meaning frequency is $\sqrt{a}$ and period is about $\frac{2\pi}{\sqrt{a}}$. Stable limit cycle so supercritical Hopf.



2. Waveforms for $\mu = 1.1$ and $\mu = 1.01$.



3. f.p. of r in the μr plane for $r \geq 0$ are shown in the figure. There is always a f.p. at $r = 0$. Stability is given by μ so for $\mu < 0$, stable spiral and for $\mu > 0$, unstable spiral. At $\mu = 0$ there must be a bifurcation. This is a supercritical Hopf bifurcation (based on the direction of the branch in the bif'n diagram).

   For $\mu < -1$ the origin is the only fixed point in the μr plane so no limit cycles exist. At $\mu = -1$ an infinite number of limit cycles are born in saddle-node of cycles bifurcations. These occur at $r = 2\pi n$ for $n = 1, 2, \ldots$. These cycles are stable on their upper branch and unstable on their lower (we can reason this out by knowing that the Hopf produces a stable cycle and the rest of the stability must alternate).

   At $\mu = 1$ they disappear in another set of saddle node bifurcations, and these occur at $r = \pi + 2\pi n$ for $n = 0, 1, 2, \ldots$.